FREE BOUNDARY MINIMAL SURFACES IN THE UNIT 3-BALL

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ABSTRACT. In a recent paper A. Fraser and R. Schoen have proved the existence of free boundary minimal surfaces Σ_n in B^3 which have genus 0 and n boundary components, for all $n \geq 3$. For large n, we give an independent construction of Σ_n and prove the existence of free boundary minimal surfaces $\tilde{\Sigma}_n$ in B^3 which have genus 1 and n boundary components. As n tends to infinity, the sequence Σ_n converges to a double copy of the unit horizontal (open) disk, uniformly on compacts of B^3 while the sequence $\tilde{\Sigma}_n$ converges to a double copy of the unit horizontal (open) punctured disk, uniformly on compacts of $B^3 - \{0\}$.

1. INTRODUCTION AND STATEMENT OF THE RESULT.

In this paper, we are interested in minimal surfaces which are embedded in the Euclidean 3-dimensional unit open ball B^3 and which meet S^2, the boundary of B^3, orthogonally. Following [2], we refer to such minimal surfaces as *free boundary minimal surfaces*.

Obviously, the horizontal unit disk, which is the intersection of the horizontal plane passing through the origin with the unit 3-ball, is an example of such free boundary minimal surface. Moreover, it is the only free boundary solution of topological disk type, [7]. Let $s_* > 0$ be the solution of

$$s_* \tanh s_* = 1.$$

The so called *critical catenoid* parameterized by

$$(s, \theta) \mapsto \frac{1}{s_* \cosh s_*} \left(\cosh s \, \cos \theta, \cosh s \, \sin \theta, s \right),$$

is another example of such a *free boundary minimal surface*. A. Fraser and M. Li conjectured that it is the only free boundary minimal surface of topological annulus type [1].

Free boundary minimal surfaces arise as critical points of the area among surfaces embedded in the unit 3-ball whose boundaries lie on S^2 but are free to vary on S^2. The fact that the area is critical for variations of the boundary of the surface which are tangent to S^2 translates into the fact that the minimal surface meets S^2 orthogonally.

In a recent paper [3], A. Fraser and R. Schoen have proved the existence of free boundary minimal surfaces Σ_n in B^3 which have genus 0 and n boundary components, for all $n \geq 3$. For large n, these surfaces can be understood as the connected sum of two nearby parallel horizontal disks joined by n boundary bridges

which are close to scaled down copies of half catenoids obtained by diving a catenoid which vertical axis with a plane containing it, which are arranged periodically along the unit horizontal great circle of S^2. Furthermore, as n tends to infinity, these free boundary minimal surfaces converge on compact subsets of B^3 to the horizontal unit disk taken with multiplicity two.

We give here another independent construction of Σ_n, for n large enough. Our proof is very different from the proof of A. Fraser and R. Schoen and is more in the spirit of the proof of the existence of minimal surfaces in S^3 by doubling the Clifford torus by N. Kapouleas and S.-D. Yang [4]. We also prove the existence of free boundary minimal surfaces in B^3 which have genus 1 and n boundary components, for all n large enough.

To state our result precisely, we define P_n to be the regular polygon with n-sides, which is included in the horizontal plane $\mathbb{R}^2 \times \{0\}$ and whose vertices are given by

$$\left(\cos\left(\frac{2\pi j}{n} \right), \, \sin\left(\frac{2\pi j}{n} \right), \, 0 \right) \in \mathbb{R}^3, \qquad \text{for} \qquad j = 1, \ldots, n.$$

We define $\mathfrak{S}_n \subset O(3)$ to be the subgroup of isometries of $\mathbb{R}^3$ which is generated by the orthogonal symmetry with respect to the horizontal plane $x_3 = 0$, the symmetry with respect to coordinate axis Ox_1 and the rotations around the vertical axis Ox_3 which leave P_n globally invariant.

Our main result reads :

Theorem 1.1. *There exists $n_0 \geq 0$ such that, for each $n \geq n_0$, there exists a genus 0 free boundary minimal surface Σ_n and a genus 1 free boundary minimal surface $\tilde{\Sigma}_n$ which are both embedded in B^3 and meet S^2 orthogonally along n closed curves.*

Both surfaces are invariant under the action of the elements of $\mathfrak{S}_n$ and, as n tends to infinity, the sequence Σ_n converges to a double copy of the unit horizontal (open) disk, uniformly on compacts of B^3 while the sequence $\tilde{\Sigma}_n$ converges to a double copy of the unit horizontal (open) punctured disk, uniformly on compacts of $B^3 - \{0\}$.

Even though we do not have a proof of this fact, it is very likely that (up to the action of an isometry of $\mathbb{R}^3$), the surfaces Σ_n coincide with the surfaces already constructed by R. Schoen and A. Fraser. In contrast, the existence of $\tilde{\Sigma}_n$ is new and does not follow from the results in [3]. The parameterization of the free boundary minimal surfaces we construct is not explicit, nevertheless our construction being based on small perturbations of explicitly designed surfaces, it has the advantage to give a rather precise description of the surfaces Σ_n and $\tilde{\Sigma}_n$. Naturally, the main drawback is that the existence of the free boundary minimal surfaces is only guaranteed when n, the number of boundary curves, is large enough.

2. Plan of the paper.

In section 3, we study the mean curvature of surfaces embedded in B^3 which are graphs over the horizontal disk $D^2 \times \{0\}$. In section 4 we analyse harmonic functions which are defined on the unit punctured disk in the Euclidean 2-plane and have log type singularities at the punctures. In section 5 for every $n \in \mathbb{N}$ large enough we construct a family of genus 0 surfaces $\mathcal{S}_n$ and a family of genus 1 surfaces $\tilde{\mathcal{S}}_n$ embedded in B^3 which are approximate solutions to the minimal surface equation,

meet the unit sphere S^2 orthogonally and have n boundary components. In section 6 we consider all embedded surfaces in B^3 which are close to $\mathcal{S}_n$ and $\tilde{\mathcal{S}}_n$ and meet the sphere S^2 orthogonally. In section 7 we analyse the linearised mean curvature operator about $\mathcal{S}_n$ and $\tilde{\mathcal{S}}_n$. Finally, in the last section we explain the Fixed-Point Theorem argument that allows us for n large enough to deform $\mathcal{S}_n$ and $\tilde{\mathcal{S}}_n$ into free boundary minimal surfaces Σ_n and $\tilde{\Sigma}_n$ satisfying the theorem (1.1).

3. The mean curvature operator for graphs in the unit 3-ball

We are interested in surfaces embedded in B^3 which are graphs over the horizontal disk $D^2 \times \{0\}$. To define these precisely, we introduce the following parametrization of the unit ball

$$X(\psi, \phi, x_3) := \frac{1}{\cosh x_3 + \cos \psi} \left(\sin \psi \, e^{i\phi}, \sinh x_3 \right),$$

where $\psi \in (0, \pi/2)$, $\phi \in S^1$ and $x_3 \in \mathbb{R}$. The horizontal disk $D^2 \times \{0\}$ corresponds to $x_3 = 0$ in this parametrization and the unit sphere S^2 corresponds to $\psi = \pi/2$. Also, the leaf $x_3 = x_3^0$ is a constant mean curvature surface (in fact it is a spherical cap) with mean curvature given by

$$H = 2 \sinh x_3^0,$$

(we agree that the mean curvature is the *sum* of the principal curvatures, not the average) moreover, this leaf meets S^2 orthogonally.

In these coordinates, the expression of the Euclidean metric is given by

$$X^* g_{eucl} = \frac{1}{(\cosh x_3 + \cos \psi)^2} \left(d\psi^2 + (\sin \psi)^2 \, d\phi^2 + dx_3^2 \right).$$

We consider the coordinate

$$z = \frac{\sin \psi}{1 + \cos \psi} \, e^{i\phi},$$

which belongs to the unit disk $D^2 \subset \mathbb{C}$. We then define $\mathcal{X}$ by the identity

$$\mathcal{X}(z, x_3) = X(\psi, \phi, x_3),$$

where z and (ψ, ϕ) are related as above. Then

$$\mathcal{X}(z, x_3) = A(z, x_3)(z, B(z) \sinh x_3),$$

where the functions B and A and explicitly given by

$$B(z) = \frac{1}{2} \left(1 + |z|^2 \right), \quad A(z, x_3) := \frac{1}{1 + B(z)(\cosh x_3 - 1)}.$$

In the coordinates $z \in D^2$ and $x_3 \in \mathbb{R}$ the expression of the Euclidean metric is given by

$$\mathcal{X}^* g_{eucl} = A^2(z, x_3) \left(dz^2 + B^2(z) \, dx_3^2 \right).$$

In the next result, we compute the expression of the mean curvature of the graph of a function $z \mapsto u(z)$ in B^3, and by such a graph we mean a surface parametrized by

$$z \in D^2 \mapsto \mathcal{X}(z, u(z)) \in B^3.$$

We have the:

Lemma 3.1. *The mean curvature with respect to the metric $\mathcal{X}^* g_{eucl}$ of the graph of the function u, namely the surface parametrized by $(z, u(z))$, is given by*

$$H(u) = \frac{1}{A^3(u)\,B}\,\mathrm{div}\left(\frac{A^2(u)\,B^2\,\nabla u}{\sqrt{1 + B^2\,|\nabla u|^2}}\right) + 2\sqrt{1 + B^2\,|\nabla u|^2}\,\sinh u,$$

where by definition $A(u) = A(\cdot, u)$. In this expression, the metric used to compute the gradient of u, the divergence and the norm of ∇u is the Euclidean metric on D^2.

Proof. The area form of the surface parametrized by $z = x_1 + i\,x_2 \mapsto (z, u(z))$ is given by

$$da := A^2(u)\,\sqrt{1 + B^2\,|\nabla u|^2}\,dx_1\,dx_2,$$

and hence the area functional is given by

$$\mathrm{Area}(u) := \iint_{D^2} A^2(u)\,\sqrt{1 + B^2\,|\nabla u|^2}\,dx_1\,dx_2.$$

The differential of the area functional at u is given by

$$D\mathrm{Area}\big|_u (v) = \iint_{D^2}\left(-\frac{A^2(u)\,B^2\,\nabla u \cdot \nabla v}{\sqrt{1 + B^2\,|\nabla u|^2}} + 2A(u)\,\partial_{x_3} A(u)\sqrt{1 + B^2|\nabla u|^2}\,v\right)dx_1\,dx_2.$$

But

$$\partial_{x_3} A = -A^2\,B\,\sinh x_3,$$

and hence we conclude that

$$D\mathrm{Area}\big|_u (v) =$$

$$-\iint_{D^2}\left(\mathrm{div}\left(\frac{A^2(u)\,B^2\nabla u}{\sqrt{1 + B^2\,|\nabla u|^2}}\right) + 2\,A^3(u)\,B\,\sqrt{1 + B^2\,|\nabla u|^2}\,\sinh u\right)v\,dx_1\,dx_2.$$

To conclude, observe that the unit normal vector to the surface parametrized by $z \mapsto \mathcal{X}(z, u(z))$ is given by

$$N := \frac{1}{A(u)}\,\frac{1}{\sqrt{1 + B^2\,|\nabla u|^2}}\left(-B\,\nabla u + \frac{1}{B}\,\partial_{x_3}\right),$$

and hence

$$g_{eucl}(N, \partial_{x_3}) = \frac{A(u)\,B}{\sqrt{1 + B^2\,|\nabla u|^2}},$$

so that

$$g_{eucl}(N, \partial_{x_3})\,da = A^3(u)\,B,$$

and the result follows from the first variation of the area formula

$$D\mathrm{Area}_{|u}(v) = -\iint_{D^2} H(u)\,g_{eucl}(N, \partial_{x_3})\,v\,da.$$

This completes the proof of the result. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Using the above Lemma, we obtain the expression of the linearised mean curvature operator about $u = 0$. It reads

$$(3.1) \qquad\qquad L_{gr}\,v = \Delta(Bv) = \Delta\left(\frac{1 + |z|^2}{2}\,v\right),$$

where Δ is the (flat) Laplacian on D^2.

Lemma 3.2. *Take a change of variables in $D^2 : z = r\, e^{i\phi}$, $r \in (0,1)$, $\phi \in S^1$ and a function $u \in \mathcal{C}^1(D^2)$, such that $\frac{\partial u}{\partial r}\big|_{r=1} = 0$. Then the graph of u in B^3 meets the sphere S^2 orthogonally at the boundary.*

Proof. The surface parametrized by $(r, \phi) \mapsto \mathcal{X}(r\, e^{i\phi}, u(r, \phi))$ is embedded in B^3 and meets ∂B^3 at $r = 1$. The result follows from the fact that the tangent vector

$$T_r(1) = \partial_r \, \mathcal{X}(r\, e^{i\phi}, u(r, \phi))\big|_{r=1} = \frac{1}{\cosh^2 u(1, \phi)} \left(e^{i\phi}, \, \sinh u(1, \phi)\right),$$

is collinear to the normal vector

$$N_S = \frac{1}{\cosh u(1, \phi)} \left(e^{i\phi}, \sinh u(1, \phi)\right),$$

to the sphere S^2 at the point $\mathcal{X}(e^{i\phi}, u(1, \phi))$. $\qquad\square$

4. Harmonic functions with singularities defined on the unit disk

Take some number $n \in \mathbb{N}$. Our goal is to construct a graph in B^3 which has bounded mean curvature, is invariant under the transformation $z \mapsto \bar{z}$ and the rotations by $\frac{2\pi}{n}$ and is close to a half-catenoid in small neighbourhoods of the n-th roots of unity

$$z_m = e^{\frac{2\pi i m}{n}} \in \partial D^2, \; m = 1, \ldots n,$$

(and, in the case of the second construction, to a catenoid in a small neighbourhood of $z = 0$).

The parametrisation of a standard catenoid C in $\mathbb{R}^3$ is

$$X^{cat}(s, \theta) = \left(\cosh s \, e^{i\phi}, s\right), \quad (s, \phi) \in \mathbb{R} \times S^1.$$

It may be divided into two pieces $C^\pm$, which can be parametrized by

$$z \in \mathbb{C} \setminus D^2 \mapsto \left(z, \, \pm \log |z| \mp \log 2 + \mathcal{O}(|z|^{-2})\right), \quad \text{as} \quad |z| \to \infty.$$

We would like to find a function Γ_n, which satisfies

$$(4.2) \qquad \begin{cases} L_{gr} \, \Gamma_n = 0 & \text{in} \quad D^2 \; (D^2 \setminus \{0\}) \\[2mm] \partial_r \Gamma_n = 0 & \text{on} \quad \partial D^2 \setminus \{z_1, \ldots, z_n\} \end{cases},$$

and which has logarithmic singularities at $z = z_m$ (and $z = 0$). Notice that the operator L_{gr} in the unit disk with Neumann boundary data has a kernel which consists of the coordinate functions x_1, x_2. This corresponds to tilting the unit disk $D^2 \times \{0\}$ in B^3. The kernel can be eliminated by asking Γ_n to be invariant under the action of a group of rotations around the vertical axis.

Notice also that the constant functions are not in the kernel of L_{gr} : by moving the disk in the vertical direction in the cylinder $D^2 \times \mathbb{R}$ we do not get a minimal but a constant mean curvature surface in B^3.

Take a function G_n, such that $G_n(z^n) = B(z) \, \Gamma_n(z)$. Then the problem (4.2) is equivalent to

$$(4.3) \qquad \begin{cases} \Delta G_n = 0 & \text{in} \quad D^2 \; (D^2 \setminus \{0\}) \\[2mm] \partial_r G_n - \frac{1}{n} G_n = 0 & \text{on} \quad \partial D^2 \setminus \{1\}. \end{cases}$$

We construct G_n explicitly. For all integer $n \geq 2$, we put

$$(4.4) \qquad G_n(z) := -\frac{n}{2} + \mathrm{Re}\left(\sum_{j=1}^{\infty} \frac{n\, z^j}{nj - 1}\right).$$

Writing

$$\frac{1}{nj - 1} = \sum_{k=0}^{\infty} \frac{1}{(nj)^{k+1}},$$

we see that also have the expression

$$(4.5) \qquad G_n(z) := -\frac{n}{2} + \mathrm{Re}\left(\sum_{k=0}^{\infty} \frac{H_k(z)}{n^k}\right),$$

where, for all $k \in \mathbf{N}$, the function H_k is given by

$$(4.6) \qquad H_k(z) := \sum_{j=1}^{\infty} \frac{z^j}{j^{k+1}}.$$

Observe, and this will be useful, that

$$(4.7) \qquad H_0(z) = -\ln(1 - z).$$

Obviously, G_n is harmonic in the open unit disk. Making use of (4.6), we see that, for all $k \geq 1$,

$$\partial_r\left(\mathrm{Re}\, H_k\right) = \mathrm{Re}\, H_{k-1},$$

on ∂D^2, while it follows from (4.7) that

$$\partial_r\left(\mathrm{Re}\, H_0\right) = \frac{1}{2},$$

again on $\partial D^2 - \{1\}$. Therefore, we conclude from (4.5) that

$$n\, \partial_r G_n - G_n = 0,$$

on ∂D^2.

For all integer $n \geq 1$, we define in $D^2 - \{0\}$, the function $\tilde{G}_n$ by

$$(4.8) \qquad \tilde{G}_n(z) := -n - \log|z|.$$

Again $\tilde{G}_n$ is harmonic in $D^2 - \{0\}$ and we also have

$$n\, \partial_r \tilde{G}_n - \tilde{G}_n = 0,$$

on ∂D^2.

To complete this paragraph, we define

$$\Gamma_n(z) := \frac{1}{B(z)}\, G_n(z^n) \qquad \text{and} \qquad \tilde{\Gamma}_n(z) := \frac{1}{B(z)}\, \tilde{G}_n(z^n).$$

By construction, $L_{gr}\, \Gamma_n = 0$ in D^2 and $\partial_r \Gamma_n = 0$ on ∂D^2, away from the n-th roots of unity ; while $L_{gr}\, \tilde{\Gamma}_n = 0$ in $D^2 - \{0\}$ and $\partial_r \tilde{\Gamma}_n = 0$ on ∂D^2.

4.1. **Matching Green's function.** Take two parameters $0 < \varepsilon < 1$ and $0 < \tilde{\varepsilon} < 1$ and consider a catenoid $C_{\tilde{\varepsilon}}$ in $\mathbb{R}^3$, parametrized by

$$X_{\tilde{\varepsilon}}^{cat} : (s, \phi) \in \mathbb{R} \times S^1 \mapsto \left(\tilde{\varepsilon} \cosh s\, e^{i\phi}, \tilde{\varepsilon} s \right),$$

and n half-catenoids $C_{\varepsilon,m}$, $m = 1, \ldots, n$

$$X_{\varepsilon,m}^{cat} : (s, \theta) \in \mathbb{R} \times \left[\frac{\pi}{2}, \frac{3\pi}{2} \right] \mapsto \left(\varepsilon \cosh s\, e^{i\theta} + z_m, \varepsilon s \right),$$

centered at the n-th roots of unity z_m. In a neighbourhood of $z = 0$ ($z = z_m$), we can take a change of variables

$$z = \tilde{\varepsilon} \cosh s\, e^{i\phi}, \quad (z = z_m + \varepsilon \cosh s\, e^{i\theta}), \quad \phi \in S^1, \quad (\theta \in [-\theta_\varepsilon + 2\pi m/n, \theta_\varepsilon + 2\pi m/n])$$

$$s \in [-s_{\tilde{\varepsilon}}, 0] \quad \text{and} \quad s \in [0, s_{\tilde{\varepsilon}}], \quad (s \in [-s_\varepsilon, 0] \quad \text{and} \quad s \in [0, s_\varepsilon]),$$

for certain parameters $s_{\tilde{\varepsilon}}$, $s_\varepsilon \in (0, +\infty)$ and $\theta_\varepsilon \in (0, \pi/2)$. We can parametrize the lower and the upper parts of $C_{\tilde{\varepsilon}}$ and $C_{\varepsilon,m}$ as graphs

$$z \mapsto \left(z, \pm G_{\tilde{\varepsilon}}^{cat} \right), \quad z \mapsto \left(z, \pm G_{\varepsilon,m}^{cat} \right),$$

where in some small neighbourhoods of $z = 0$ ($z = z_m$),

$$G_{\tilde{\varepsilon}}^{cat}(z) = \tilde{\varepsilon} \log \frac{\tilde{\varepsilon}}{2} - \tilde{\varepsilon} \log |z| + \mathcal{O}\left(\tilde{\varepsilon}^3 / |z|^2 \right),$$

$$G_{\varepsilon,m}^{cat}(z) = \varepsilon \log \frac{\varepsilon}{2} - \varepsilon \log |z - z_m| + \mathcal{O}\left(\varepsilon^3 / |z - z_m|^2 \right)$$

Our goal is to find positive parameters τ and $\tilde{\tau}$ and a connection between ε and $\tilde{\varepsilon}$, such that the function

$$z \mapsto \tau\, G_n(z^n) + \tilde{\tau}\, \tilde{G}_n(z^n),$$

would be close to $G_{\tilde{\varepsilon}}^{cat}$ in a neighbourhood of $z = 0$ and to $G_{\varepsilon,m}^{cat}$ in a neighbourhood of $z = z_m$. We denote

$$f_n(z) := \sum_{k=0}^{\infty} \frac{H_k(z^n)}{n^k} = \sum_{k=0}^{\infty} \frac{1}{n^k} \sum_{j=1}^{\infty} \frac{z^{nj}}{j^{k+1}},$$

and remind that $G_n(z^n) = -\frac{n}{2} + \operatorname{Re} f_n(z)$. It is easy to verify the function $f_n(z)$ satisfies

$$\frac{\partial f_n}{\partial z}(z) = -\frac{d}{dz} \log(1 - z^n) + \frac{1}{z} f_n(z),$$

which yields

$$\frac{d}{dz} \left(\frac{f_n}{z} \right) = \frac{n z^{n-2}}{z^n - 1}.$$

We can write

$$\frac{nz^{n-2}}{z^n-1}-\frac{1}{z_m(z-z_m)}=\frac{nz^{n-2}z_m-\sum_{k=0}^{n-1}z^{n-1-k}z_m^k}{z_m(z^n-1)}=$$

$$\frac{\sum_{k=0}^{n-1}\left(-z^{n-1-k}z_m^k+z_mz^{n-2}\right)}{z_m(z^n-1)}=\frac{z^{n-2}(z_m-z)+z_m\sum_{k=2}^{n-1}z^{n-1-k}\left(z^{k-1}-z_m^{k-1}\right)}{z_m(z-z_m)\sum_{k=0}^{n-1}z^{n-1-k}z_m^k}=$$

$$\frac{-z^{n-2}+z_m\sum_{k=2}^{n-1}z^{n-1-k}\sum_{l=0}^{k-2}z^{k-2-l}z_m^l}{z_m\sum_{k=0}^{n-1}z^{n-1-k}z_m^k}:=h_n(z).$$

The function $h_n(z)$ is continuous in a small neighbourhood of $z=z_m$ and

$$|h_n(z_m)|\leq c\,n,$$

for a constant c which does not depend on n. So, we have

$$\frac{d}{dz}\left(\frac{f_n}{z}\right)+\frac{1}{z_m(z-z_m)}=h_n(z),$$

which yields that in a neighbourhood of $z=z_m$

$$\frac{f_n(z)}{z}+\frac{1}{z_m}\log(z-z_m)=\frac{1}{z_m}\lim_{z\to z_m}\left(f_n(z)+\log(z-z_m)\right)+\int_z^{z_m}h_n(z)dz,$$

where the integral is taken along the segment of the straight line passing from z to z_m and by log we mean the principal value of complex logarithm defined in the unit disc deprived of a segment of a straight line which doesn't pass thought any of the n-th roots of unity. We have

$$\sum_{k=1}^{\infty}\frac{1}{n^k}H_k(z_m^n)=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{n^k\,j^{k+1}}\leq\frac{\pi^2}{6n}.$$

Moreover,

$$\mathrm{Re}\lim_{z\to z_m}\left(-\log(1-z^n)+\log(z-z_m)\right)=-\log|n\,z_m^{n-1}|=-\log n.$$

So, in the neighbourhood of $z=z_m$, we have

$$G_n(z^n)=-\frac{n}{2}+c(n)+\log|z-z_m|+\mathcal{O}(|z-z_m|\log|z-z_m|)+\mathcal{O}(n|z-z_m|),$$

where $|c(n)|\leq c\log n$ for a constant c which does not depend on n and

$$\tau\,G_n(z^n)+\tilde{\tau}\,\tilde{G}_n(z^n)=\begin{cases}-n\left(\tilde{\tau}+\tau/2\right)-\tilde{\tau}n\log|z|+\mathcal{O}(\tau|z|^n),\ \text{as }|z|\to0\\[2mm]-n\left(\tilde{\tau}+\tau/2\right)+\tau\,c(n)-\tau\log|z-z_m|+\\[2mm]\mathcal{O}(\tau|z-z_m|\log|z-z_m|)+\mathcal{O}(\tau n|z-z_m|),\ \text{as }|z-z_m|\to0\end{cases}.$$

We should take $\tau=\varepsilon$ and $n\tilde{\tau}=\tilde{\varepsilon}$. Moreover, we should have

$$-\tilde{\varepsilon}-\frac{\varepsilon n}{2}=\tilde{\varepsilon}\log\frac{\tilde{\varepsilon}}{2}\quad\text{and}\quad-\tilde{\varepsilon}-\frac{\varepsilon n}{2}+\varepsilon\,c(n)=\varepsilon\log\frac{\varepsilon}{2}.$$

This gives us the relation

$$\log \frac{\varepsilon}{\tilde{\varepsilon}} + \frac{\tilde{\varepsilon}}{\varepsilon} - \frac{n}{2}\frac{\varepsilon}{\tilde{\varepsilon}} = -\frac{n}{2} + c(n) + 1,$$

and

$$\frac{\varepsilon}{\tilde{\varepsilon}} = g_n^{-1}(-\frac{n}{2} + c(n) + 1) =: d(n),$$

where

$$g_n(t) \,:\, t \in (0, +\infty) \,\mapsto\, \log t - \frac{n}{2}\,t + \frac{1}{t} \in (-\infty, \infty),$$

is an everywhere decreasing function. Finally, we find

$$\tilde{\varepsilon} = 2\,e^{-1 - \frac{n}{2}\,d(n)} \quad \text{and} \quad \varepsilon = d(n)\,\tilde{\varepsilon}.$$

(In the case, where we do not have a singularity at $z = 0$ we just need to take $\varepsilon = e^{-\frac{n}{2} + c(n)}$). We obtain for all $\beta \in (0, 1)$

$$(4.9) \quad \tau\, G_n(z^n) = \varepsilon \log \frac{\varepsilon}{2|z - z_m|} + \mathcal{O}(\varepsilon^{1-\beta}\,|z - z_m|), \text{ as } |z - z_m| \to 0$$

$$\tau\, G_n(z^n) + \tilde{\tau}\,\tilde{G}_n(z) = \begin{cases} \tilde{\varepsilon} \log \dfrac{\tilde{\varepsilon}}{2|z|} + \mathcal{O}(\tilde{\varepsilon}^{1-\beta}\,|z|^n), \text{ as } |z| \to 0 \\[2ex] \varepsilon \log \dfrac{\varepsilon}{2|z - z_m|} + \mathcal{O}(\varepsilon^{1-\beta}\,|z - z_m|), \quad \text{as } |z - z_m| \to 0 \end{cases}$$

Finally, we put $\mathcal{G}_n(z) = \tau\,G_n(z^n)/B$ and $\tilde{\mathcal{G}}_n(z) = \left(\tilde{\tau}\,\tilde{G}_n(z^n) + \tau\,G_n(z^n)\right)/B$ and

$$\tilde{\mathcal{G}}_n(z) = \begin{cases} 2\,\tilde{\varepsilon} \log \dfrac{\tilde{\varepsilon}}{2|z|} + \mathcal{O}(\tilde{\varepsilon}^{1-\beta}\,|z|^2), \text{ as } |z| \to 0 \\[2ex] \varepsilon \log \dfrac{\varepsilon}{2|z - z_m|} + \mathcal{O}(\varepsilon^{1-\beta}\,|z - z_m|), \text{ as } |z - z_m| \to 0 \end{cases}$$

Remark 4.1. *We can now explain why our construction works only for large n. On one hand, in order to match the graph of the Green's function $\tilde{\mathcal{G}}_n$ with catenoids we need to truncate the cantenoids far enough and scale them by a small enough factor. On the other hand, in the neighbourhood of singularities the constant term of $\tilde{\mathcal{G}}_n$ depends on the number of singularities n and, as constant functions are not in the kernel of the linearised mean curvature operator, this gives the correspondence between the scaling factors of the catenoids and n.*

5. Catenoidal bridges and necks

In this section we explain the construction of the surface $\mathcal{S}_n$, invariant under the action of the group $\mathfrak{S}_n$, which has bounded mean curvature, meets the unit sphere S^2 orthogonally at the boundary and is close to two horizontal disks "glued together" with the help of "catenoidal bridges" in the neighbourhood the n-th roots of unity. We will also denote $\tilde{\mathcal{S}}_n$ the genus 1 surface, obtained from $\mathcal{S}_n$ by attaching a "catenoidal neck" at $z = 0$. We will explain now what we mean by "catenoidal bridges" and "catenoidal neck".

5.1. **Catenoidal bridges.** One of the possible constructions would be to "glue" the graph $\mathcal{X}(z, \mathcal{G}_n(z))$ together with minimal stripes obtained by intersecting euclidean catenoids centered at $z = z_m$ with the unit sphere. The difficulty of this approach is that those stripes would not meet the sphere orthogonally. We prefer to find a way to put half-catenoids in the unit sphere in the orthogonal way loosing the minimality condition.

Remark 5.1. *We describe below the construction of the surface $\tilde{\mathcal{S}}_n$. The construction of the surface $\mathcal{S}_n$ can be obtained by replacing the function $\tilde{\mathcal{G}}_n$ by $\mathcal{G}_n$ which has the same expansion in the terms of ε at $z = z_m$, taking into account that the relation between the parameters n and ε changes.*

We use the notation $\mathbb{C}_-$ for the half-plane $\{z \in \mathbb{C} \,|\, \mathrm{Re}(z) < 0\}$. For $m = 1, \ldots, n$ consider the conformal mappings

$$\lambda_m : \mathbb{C}_- \longrightarrow D^2, \quad \lambda_m(\zeta) = e^{\frac{2i\pi m}{n}} \frac{1 + \zeta}{1 - \zeta}.$$

These mappings transform a half-disk in the $\mathbb{C}_-$ centered at $\zeta = 0$ and of radius $\rho < 1$ to a domain obtained by the intersection of the unit disk D^2 with a disk of radius $\frac{2\rho}{1 - \rho^2}$ and a center at $\frac{1 + \rho^2}{1 - \rho^2} e^{\frac{2i\pi m}{n}}$. Let $(\zeta = \xi_1 + i\xi_2, \xi_3)$ be the coordinates in $\mathbb{C}_- \times \mathbb{R}$, then we define the mapping

$$\Lambda_m : \mathbb{C}_- \times \mathbb{R} \longrightarrow D^2 \times \mathbb{R}, \quad \Lambda_m(\zeta, \xi_3) = (\lambda_m(\zeta), 2\,\xi_3).$$

Consider the half-catenoid $C_{\varepsilon/2}$ in $\mathbb{C}_- \times \mathbb{R}$, parametrized by

$$X_{\varepsilon/2}^{cat} : (\sigma, \theta) \in \mathbb{R} \times \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \mapsto \left(\frac{\varepsilon}{2} \cosh \sigma\, e^{i\theta}, \frac{\varepsilon}{2}\sigma\right)$$

In the regions, where $\sigma > 0$ or $\sigma < 0$ we can take the change of variables

$$\zeta = \frac{\rho}{2}\, e^{i\theta} = \frac{\varepsilon}{2} \cosh \sigma\, e^{i\theta}, \quad \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$$

and, having in mind that the function $\tilde{\mathcal{G}}_n$ defined in $\bar{D}^2 \setminus \{z_1, \ldots, z_n\}$ is invariant under rotations by the angle $\frac{2\pi}{n}$, consider a vertical graph over $\mathbb{C}_-$:

$$(\rho, \theta) \mapsto \left(\frac{\rho}{2}\, e^{i\theta}, \frac{1}{2} \bar{\mathcal{G}}_n(\rho, \theta)\right), \quad \text{where} \quad \bar{\mathcal{G}}_n(\rho, \theta) = \tilde{\mathcal{G}}_n(\lambda_m(\rho/2\, e^{i\theta}))$$

In the neighbourhood of $\rho = 0$, we have $|\lambda_m(\rho/2\, e^{i\theta}) - z_m| = \rho + \mathcal{O}(\rho^2)$. So, using the expansion (4.9) for the function $\tilde{\mathcal{G}}_n$ in the neighbourhood of $z = z_m$, we obtain a similar expansion for $\bar{\mathcal{G}}_n$ in the neighbourhood of 0:

$$\bar{\mathcal{G}}_n(\rho, \theta) = \varepsilon \log \frac{\varepsilon}{2\rho} + \mathcal{O}(\varepsilon^{1-\beta}\rho), \quad \forall \beta \in (0, 1).$$

At the same time the lower and the upper part of the $C_{\varepsilon/2}$ can be seen as graphs of the functions

$$\pm G_{\varepsilon/2}^{cat}(\rho) = \pm \frac{\varepsilon}{2} \log \frac{\varepsilon}{2\rho} + \mathcal{O}(\varepsilon^3/\rho^2).$$

Now take a function $\bar{\Upsilon}$ which is defined in the neighbourhood of $|\zeta| = \varepsilon$ by

$$\bar{\Upsilon}(\rho, \theta) = (1 - \bar{\eta}_\varepsilon(\rho)) \frac{1}{2} \bar{\mathcal{G}}_n(\rho, \theta) + \bar{\eta}_\varepsilon(\rho)\, G_{\varepsilon/2}^{cat}(\rho),$$

where $\bar{\eta}_\varepsilon$ is a cut-off function, such that

$$\bar{\eta}_\varepsilon \equiv 1, \quad \text{for} \quad \varepsilon < \rho < 1/2\,\varepsilon^{2/3}, \quad \bar{\eta}_\varepsilon \equiv 0, \quad \text{for} \quad \rho > \varepsilon^{2/3}.$$

Using this, we can parametrize the surface $\mathcal{S}_n$ in the region

$$\Omega_{cat}^m := \lambda_m \left\{ \varepsilon/2 \, \cosh\sigma \, e^{i\theta} \,:\, \varepsilon\cosh\sigma < 1/2\,\varepsilon^{2/3}, \; \theta \in [\pi/2, 3\pi/2] \right\},$$

by $(\sigma, \theta) \mapsto \mathcal{X} \circ \Lambda_m \left(\varepsilon/2 \, \cosh\sigma \, e^{i\theta}, \frac{\varepsilon}{2}\sigma \right)$ and as a bi-graph

$$\mathcal{X} \left\{ (z, 2\,\Upsilon^m(z)) \cup (z, -2\,\Upsilon^m(z)) \right\},$$

$$\text{for } z \in \Omega_{glu}^m := \lambda_m \left\{ \rho/2 \, e^{i\theta} \,:\, 1/2\,\varepsilon^{2/3} < \rho < 2\,\varepsilon^{2/3}, \; \theta \in [\pi/2, 3\pi/2] \right\},$$

where $\Upsilon^m(z)$ is a function, such that $\Upsilon^m(\lambda_m(\rho/2\,e^{i\theta})) = \bar{\Upsilon}(\rho, \theta)$.

Remark 5.2. *Orthogonality at the boundary*

In a neighbourhood of its m-th component of the boundary the surface $\tilde{\mathcal{S}}_n$ $(\mathcal{S}_n)$ can be seen as the image by the mapping $\mathcal{X} \circ \Lambda_m$ of a surface (which we denote $\bar{S}_n$) contained in the half-space $\mathbb{C}_- \times \mathbb{R}$. Consider a foliation of the half-space by horizontal half-planes. It is clear that every leaf of this foliation is orthogonal to $\partial\mathbb{C}_- \times \mathbb{R}$. Thus, the normal to $\partial\mathbb{C}_- \times \mathbb{R}$ at a point is tangent to the horizontal leaf passing through this point. So, if there existed a tangent vector field along $\bar{S}_n$, horizontal at $\partial\bar{S}_n$, then it would have to be collinear to the normal to $\partial\mathbb{C}_- \times \mathbb{R}$.

On the other hand, the image of the the foliation by horizontal half-planes by the mapping $\mathcal{X} \circ \Lambda_m$ gives a foliation of the unit ball by spherical caps which are orthogonal to S^2 at the boundary. The horizontal vector field tangent to $\partial\mathbb{C}_- \times \mathbb{R}$ is sent by this mapping to a vector field tangent to the sphere and to a spherical cap leaf. The result follows from the fact that the restriction of $\mathcal{X} \circ \Lambda_m$ to horizontal half-planes is conformal.

Finally in our case, the existence of the horizontal tangent vector field follows from the fact that $\partial_\theta X_{\varepsilon/2}^{cat}$ is horizontal and that $\partial_\theta \bar{\mathcal{G}}_n = \partial_\theta G_{\varepsilon/2}^{cat} = \partial_\theta \bar{\eta}_\varepsilon = 0$ at $\theta \in \{\pi/2, 3\pi/2\}$.

Let $\mathcal{H}$ denote the mean curvature of the surface $\tilde{\mathcal{S}}_n$.

Proposition 5.1. *There exists a constant c which does not depend on ε such that in the region*

$$\Omega_{cat}^m = \lambda_m \left\{ \varepsilon/2 \, \cosh\sigma \, e^{i\theta} \,:\, \varepsilon\cosh\sigma < 1/2\,\varepsilon^{2/3}, \; \theta \in [\pi/2, 3\pi/2] \right\},$$

we have

$$\left| \mathcal{H}(\lambda_m(\varepsilon/2 \, \cosh\sigma \, e^{i\theta})) \right| \leq \frac{c}{\cosh\sigma}$$

Proof. The proof consists of calculating the mean curvature of $C_{\varepsilon/2}$ with respect to the ambient metric

$$(\mathcal{X} \circ \Lambda_m)^* g_{eucl}(\zeta, \xi_3) = A^2(\Lambda_m(\zeta, \xi_3)) \left(d\zeta^2 + B^2(\Lambda_m(\zeta, \xi_3)) d\xi_3^2 \right)$$

$$= \frac{4}{\left[|1 - \zeta|^2 + (1 + |\zeta|^2)(\cosh(2\xi_3) - 1) \right]^2} \left(d\zeta^2 + (1 + |\zeta|^2)^2 \, d\xi_3^2 \right)$$

$$= a^2(\zeta, \xi_3) \left(d\zeta^2 + b^2(\zeta) \, d\xi_3^2 \right) =: g_m(\zeta, \xi_3)$$

where $a(\zeta, \xi_3) = \dfrac{2}{|1 - \zeta|^2 + (1 + |\zeta|^2)(\cosh(2\xi_3) - 1)}$ and $b(\zeta) = 1 + |\zeta|^2$.

Let ∇^ε denote the Levi-Civita connection corresponding to this metric. We have the following estimates for the Christoffel symbols in a neighborhood of $(\zeta, \xi_3) = (0, 0)$:

$$\Gamma^1_{11} = -\Gamma^1_{22} = \Gamma^2_{12} = \tfrac{1}{a}\tfrac{\partial a}{\partial \xi_1} = \mathcal{O}(1), \quad \Gamma^2_{11} = -\Gamma^2_{22} = -\Gamma^1_{12} = \tfrac{1}{a}\tfrac{\partial a}{\partial \xi_2} = \mathcal{O}(1)$$

$$\Gamma^1_{13} = \Gamma^2_{23} = \Gamma^3_{33} = \tfrac{1}{a}\tfrac{\partial a}{\partial \xi_3} = \mathcal{O}(\xi_3), \quad \Gamma^3_{11} = \Gamma^3_{22} = -\tfrac{1}{ab^2}\tfrac{\partial a}{\partial \xi_3} = \mathcal{O}(\xi_3),$$

$$\Gamma^1_{33} = -(\tfrac{b^2}{a}\tfrac{\partial a}{\partial \xi_1} + b\tfrac{\partial b}{\partial \xi_1}) = \mathcal{O}(1), \quad \Gamma^2_{33} = -(\tfrac{b^2}{a}\tfrac{\partial a}{\partial \xi_2} + b\tfrac{\partial b}{\partial \xi_2}) = \mathcal{O}(1),$$

$$\Gamma^3_{13} = \tfrac{1}{a}\tfrac{\partial a}{\partial \xi_1} + \tfrac{1}{b}\tfrac{\partial b}{\partial \xi_1} = \mathcal{O}(1), \quad \Gamma^3_{23} = \tfrac{1}{a}\tfrac{\partial a}{\partial \xi_2} + \tfrac{1}{b}\tfrac{\partial b}{\partial \xi_2} = \mathcal{O}(1)$$

$$\Gamma^1_{23} = \Gamma^2_{13} = \Gamma^3_{12} = 0$$

Using $|\zeta| = \varepsilon/2 \cosh\sigma$, $\xi_3 = \varepsilon/2\,\sigma$, and

$$\nabla^\varepsilon_{\partial_p} \partial_q = \partial_p\,\partial_q\,X^{cat}_{\varepsilon/2} + \left[\partial_p X^{cat}_{\varepsilon/2}\right]^i \left[\partial_q X^{cat}_{\varepsilon/2}\right]^j \Gamma^k_{ij}\,\partial_k,$$

where ∂_p and ∂_q stand for ∂_σ or ∂_θ and $\partial_k = \partial_{\xi_k}$, $k = 1, 2, 3$. We get

$$\left|\left[\nabla^\varepsilon_{\partial_p}\partial_q - \partial_p\,\partial_q\,X^{cat}_{\varepsilon/2}\right]^i (\sigma, \theta)\right| \leq c\,\varepsilon^2 \cosh^2\sigma, \quad i = 1, 2$$

$$\left|\left[\nabla^\varepsilon_{\partial_p}\partial_q - \partial_p\,\partial_q\,X^{cat}_{\varepsilon/2}\right]^3 (\sigma, \theta)\right| \leq c\,\varepsilon^2 \cosh\sigma.$$

The unit normal to $C_{\varepsilon/2}$ with respect to the metric g_m is

$$\mathcal{N}(\sigma, \theta) = \dfrac{1}{a\sqrt{\tfrac{b^2}{\cosh^2\sigma} + \tanh^2\sigma}} \left(-\dfrac{b}{\cosh\sigma}\,e^{i\theta}, \tfrac{1}{b}\tanh\sigma\right).$$

Using the the expansions for a and b in the neighbourhood of 0 and fact that the third coordinate of the vector $\partial_p\,\partial_q\,X^{cat}_{\varepsilon/2}$ is zero for all p and q we get the following expression for the second fundamental form :

$$\mathfrak{h}_\varepsilon(\sigma, \theta) = \varepsilon(d\sigma^2 - d\phi^2) + h_\varepsilon(\sigma, \theta),$$

where $\left|(h_\varepsilon)_{pq}(\sigma, \theta)\right| \leq c\,\varepsilon^2 \cosh\sigma$. On the other hand, we can write the expansion of the metric induced on $C_{\varepsilon/2}$ from g_m

$$\mathfrak{g}_\varepsilon(\sigma, \theta) = \varepsilon^2 \cosh^2\sigma(d\sigma^2 + d\phi^2) + g_\varepsilon(\sigma, \theta),$$

where $\left|(g_\varepsilon)_{pq}(\sigma, \theta)\right| \leq c\varepsilon^3 \cosh^3\sigma$.

Finally, $\left|\mathcal{H}(\varepsilon/2 \cosh\sigma\,e^{i\theta})\right| = \left|\mathrm{tr}\left(\mathfrak{g}_\varepsilon^{-1}\mathfrak{h}_\varepsilon\right)(\sigma, \theta)\right| \leq \dfrac{c}{\cosh\sigma}.$

$\square$

5.2. **Catenoidal neck at $z = 0$.** In a small neighborhood of $z = \tilde{\varepsilon}$ take the change of variables $z = r\, e^{i\phi}$, $\phi \in S^1$. Then, in the neighbourhood of $z = 0$, we have

$$B(r\, e^{i\phi}) = B(r) = \frac{1}{2} + \mathcal{O}(r^2).$$

We remind that that $\tau\, G_n(z^n) + \tilde{\tau}\, \tilde{G}_n(z^n) = \tilde{\mathcal{G}}_n\, B$ is a function whose graph is close to the lower part of the euclidean catenoid scaled a factor $\tilde{\varepsilon}$. Then, the graph of $\tilde{\mathcal{G}}_n$ is close to the lower part of the surface $\tilde{C}_{\tilde{\varepsilon}}$, parametrized by

$$\tilde{X}^{cat}_{\tilde{\varepsilon}} \, : \, (s, \phi) \in \mathbb{R} \times S^1 \mapsto (\tilde{\varepsilon} \cosh s\, e^{i\phi}, 2\, \tilde{\varepsilon} s).$$

Let us define a cut-off function $r \mapsto \eta^0_{\tilde{\varepsilon}}(r)$, such that

$$\eta^0_{\tilde{\varepsilon}}(r) \equiv 1 \quad \text{for} \quad r \in \left(0, 1/2\, \tilde{\varepsilon}^{1/2}\right), \quad \eta^0_{\tilde{\varepsilon}}(r) \equiv 0 \quad \text{for} \quad r > 2\, \tilde{\varepsilon}^{1/2}.$$

Taking the change of variables: $\tilde{\varepsilon} \cosh s\, e^{i\phi} = z = r\, e^{i\phi}$, for $s > 0$ or $s < 0$ we can parametrize the lower and the upper part of $\tilde{C}_{\tilde{\varepsilon}}$ as vertical graphs

$$z \mapsto (z, \pm 2\, G^{cat}_{\tilde{\varepsilon}}),$$

where

$$G^{cat}_{\tilde{\varepsilon}}(r) = \frac{\tilde{\varepsilon}}{2} \log\left(\frac{\tilde{\varepsilon}}{2r}\right) + \mathcal{O}\left(\frac{\tilde{\varepsilon}^3}{r^2}\right).$$

We define the function

$$\tilde{\Upsilon} : \bar{D}^2 \setminus \{0, z_1, \ldots, z_m\} \longrightarrow \mathbb{R},$$

$$\tilde{\Upsilon}(r, \phi) = (1 - \eta^0_{\tilde{\varepsilon}}(r))\, \tilde{\mathcal{G}}_n(r\, e^{i\phi}) + 2\, \eta^0_{\tilde{\varepsilon}}\, G^{cat}_{\varepsilon}(r),$$

and parametrized $\tilde{S}_n$ in the region

$$\Omega^0_{cat} := \left\{ \tilde{\varepsilon} \cosh s\, e^{i\phi} \, : \, \tilde{\varepsilon} \cosh s < 1/2\, \tilde{\varepsilon}^{1/2}, \phi \in S^1 \right\},$$

by $(s, \phi) \mapsto \mathcal{X} \circ \tilde{X}^{cat}_{\tilde{\varepsilon}}(s, \phi)$ and as a bi-graph :

$$(r, \phi) \mapsto \mathcal{X}\left\{ (r\, e^{i\phi}, \tilde{\Upsilon}(r, \phi)) \cup (r\, e^{i\phi}, -\tilde{\Upsilon}(r, \phi)) \right\},$$

$$\text{for} \quad z \in \Omega^0_{glu} := \left\{ r\, e^{i\phi} \, : \, 1/2\, \tilde{\varepsilon}^{1/2} < r < 2\, \tilde{\varepsilon}^{1/2}, \, \phi \in S^1 \right\}.$$

We use the notations :

$$B(s) = B(\tilde{\varepsilon} \cosh s) = \frac{1}{2}(1 + \tilde{\varepsilon}^2 \cosh^2 s) \quad \text{and}$$

$$A(s) = A(\tilde{\varepsilon} \cosh s, 2\, \tilde{\varepsilon} s) = \frac{1}{1 + B(s)\, (\cosh(2\, \tilde{\varepsilon} s) - 1)}.$$

Proposition 5.2. *There exists a constant c which does not depend on ε such that in the region*

$$\Omega^0_{cat} = \left\{ \tilde{\varepsilon} \cosh s\, e^{i\phi} \, : \, \tilde{\varepsilon} \cosh s < 1/2\, \tilde{\varepsilon}^{1/2}, \phi \in S^1 \right\},$$

the mean curvature of the surface $\tilde{S}_n$ satisfies

$$\left| \mathcal{H}(\tilde{\varepsilon} \cosh s\, e^{i\phi}) \right| \leq c\, \tilde{\varepsilon}^{1-\beta}, \quad \forall \beta \in (0, 1).$$

Proof. As in the proposition 5.1 we would like to calculate the mean curvature of $\tilde{C}_\varepsilon$ with respect to the ambient metric

$$\mathcal{X}^* g_{eucl}(z, x_3) = A^2(z, x_3) \left(dz^2 + B^2(z) dx_3^2 \right) =: \tilde{g}(z, x_3).$$

We denote by $\tilde{\nabla}^{\tilde{\varepsilon}}$ the Levi-Civita connection corresponding to this metric. Then, in the neighborhood of $(z, x_3) = (0, 0)$ the Cristoffel symbols satisfy :

$$\tilde{\Gamma}_{11}^1 = -\tilde{\Gamma}_{22}^1 = \tilde{\Gamma}_{12}^2 = \tfrac{1}{A} \tfrac{\partial A}{\partial x_1} = \mathcal{O}(|z|x_3), \quad \tilde{\Gamma}_{11}^2 = -\tilde{\Gamma}_{22}^2 = -\tilde{\Gamma}_{12}^1 = \tfrac{1}{A} \tfrac{\partial A}{\partial x_2} = \mathcal{O}(|z|x_3)$$

$$\tilde{\Gamma}_{13}^1 = \tilde{\Gamma}_{23}^2 = \tilde{\Gamma}_{33}^3 = \tfrac{1}{A} \tfrac{\partial A}{\partial x_3} = \mathcal{O}(x_3), \quad \tilde{\Gamma}_{11}^3 = \tilde{\Gamma}_{22}^3 = -\tfrac{1}{AB^2} \tfrac{\partial A}{\partial x_3} = \mathcal{O}(x_3),$$

$$\tilde{\Gamma}_{33}^1 = -(\tfrac{B^2}{A} \tfrac{\partial A}{\partial x_1} + B \tfrac{\partial B}{\partial x_1}) = \mathcal{O}(|z|), \quad \tilde{\Gamma}_{33}^2 = -(\tfrac{B^2}{A} \tfrac{\partial A}{\partial x_2} + B \tfrac{\partial B}{\partial x_2}) = \mathcal{O}(|z|),$$

$$\tilde{\Gamma}_{13}^3 = \tfrac{1}{A} \tfrac{\partial A}{\partial x_1} + \tfrac{1}{B} \tfrac{\partial B}{\partial x_1} = \mathcal{O}(|z|), \quad \tilde{\Gamma}_{23}^3 = \tfrac{1}{A} \tfrac{\partial A}{\partial x_2} + \tfrac{1}{B} \tfrac{\partial B}{\partial x_2} = \mathcal{O}(|z|)$$

$$\tilde{\Gamma}_{23}^1 = \tilde{\Gamma}_{13}^2 = \tilde{\Gamma}_{12}^3 = 0$$

Using $|z| = \tilde{\varepsilon} \cosh s$, $x_3 = 2\tilde{\varepsilon} s$, and

$$\tilde{\nabla}^{\tilde{\varepsilon}}_{\partial_p} \partial_q = \partial_p \partial_q \tilde{X}^{cat}_{\tilde{\varepsilon}} + [\partial_p \tilde{X}^{cat}_{\tilde{\varepsilon}}]^i [\partial_q \tilde{X}^{cat}_{\tilde{\varepsilon}}]^j \tilde{\Gamma}^k_{ij} \partial_k,$$

where ∂_p and ∂_q stand for ∂_s or ∂_t, we get

$$\left| [\tilde{\nabla}^{\tilde{\varepsilon}}_{\partial_p} \partial_q - \partial_p \partial_q \tilde{X}^{cat}_{\tilde{\varepsilon}}]^i(s, \theta) \right| \le c \tilde{\varepsilon}^3 \cosh^3 s, \quad i = 1, 2$$

$$\left| [\tilde{\nabla}^{\tilde{\varepsilon}}_{\partial_p} \partial_q - \partial_p \partial_q \tilde{X}^{cat}_{\tilde{\varepsilon}}]^3(s, \theta) \right| \le c \tilde{\varepsilon}^{3-\beta} \cosh^2 s, \quad \forall \beta \in (0, 1).$$

The normal vector field to $\tilde{C}_{\tilde{\varepsilon}}$ with respect to the metric $\mathcal{X}^* g_{eucl}$ is

$$\tilde{\mathcal{N}}(s, \phi) = \frac{1}{A \sqrt{\frac{4B^2}{\cosh^2 s} + \tanh^2 s}} \left(-\frac{2B}{\cosh s} e^{i\phi}, \frac{1}{B} \tanh s \right).$$

As the third coordinate of the vector $\partial_p \partial_q \tilde{X}^{cat}_{\tilde{\varepsilon}}$ is zero for all p and q, we get the following expression for the second fundamental form :

$$\tilde{\mathfrak{h}}_{\tilde{\varepsilon}}(s, \phi) = \tilde{\varepsilon} (ds^2 - d\phi^2) + \tilde{h}_{\tilde{\varepsilon}}(s, \phi),$$

where $\left| \left(\tilde{h}_{\tilde{\varepsilon}} \right)_{pq} (s, \phi) \right| \le c \tilde{\varepsilon}^{3-\beta} \cosh^2 s$. On the other hand the metric induced on $C_{\tilde{\varepsilon}}$ from $\mathcal{X}^* g_{eucl}$ can be written as

$$\tilde{\mathfrak{g}}_{\tilde{\varepsilon}}(s, \phi) = \tilde{\varepsilon}^2 \cosh^2 s (ds^2 + d\phi^2) + \tilde{g}_{\tilde{\varepsilon}}(s, \phi),$$

where $\left| (\tilde{g}_{\tilde{\varepsilon}})_{pq} (s, \phi) \right| \le c \tilde{\varepsilon}^{4-\beta} \cosh^2 s$. Finally,

$$\left| \mathcal{H}(\tilde{\varepsilon} \cosh s \, e^{i\phi}) \right| = \left| \text{tr} \left(\tilde{\mathfrak{g}}_{\tilde{\varepsilon}}^{-1} \tilde{\mathfrak{h}}_{\tilde{\varepsilon}} \right) (s, \phi) \right| \le c \tilde{\varepsilon}^{1-\beta}, \quad \forall \beta \in (0, 1).$$

$\square$

5.3. **The graph region.** Away from the catenoidal bridges and the catenoidal neck, that is in the region

$$\Omega_{gr} := \left\{ z \in D^2 \; : \; 2\,\tilde{\varepsilon}^{1/2} < |z| < 1 \right\} \setminus \bigcup_{m=1}^{n} \lambda_m \left\{ \zeta \in \mathbb{C}_- \; : \; |\zeta| < 2\,\varepsilon^{2/3} \right\},$$

we parametrize the surface $\tilde{\mathcal{S}}_n$ as a bi-graph

$$\mathcal{X}\left\{ (z, \tilde{\mathcal{G}}_n(z)) \cup (z, -\tilde{\mathcal{G}}_n(z)) \right\}.$$

Proposition 5.3. *There exists a constant c which does not depend on ε, such that in the region*

$$\Omega_{gr} \cup \Omega_{glu}^0 \bigcup_{m=1}^{n} \Omega_{glu}^m$$

$$= \left\{ z \in D^2 \; : \; 1/2\,\tilde{\varepsilon} < |z| < 1 \right\} \setminus \bigcup_{m=1}^{n} \lambda_m \left\{ \zeta \in \mathbb{C}_- \; : \; |\zeta| < 1/2\,\varepsilon^{2/3} \right\},$$

the mean curvature $\mathcal{H}$ of $\tilde{\mathcal{S}}_n$ satisfies

$$(5.10) \qquad |\mathcal{H}(z)| \leq c\,\varepsilon^{3-\beta} \left(\frac{1}{|z|^4} + \sum_{m=1}^{n} \frac{1}{|z - z_m|^4} \right), \qquad \forall \beta \in (0,1).$$

Proof. According to the lemma (3.1), the mean curvature of the graph $\mathcal{X}\left(z, u(z)\right)$, $u \in \mathcal{C}^2(D^2)$ satisfies :

$$H(u) = \frac{1}{A^3(u)\,B} \operatorname{div}\left(\frac{A^2(u)\,B^2\,\nabla u}{\sqrt{1 + B^2\,|\nabla u|^2}} \right) + 2\,\sqrt{1 + B^2\,|\nabla u|^2}\,\sinh u$$

$$= \frac{2}{A^2(u)} \frac{B\nabla u \nabla A(u)}{\sqrt{1 + B^2|\nabla u|^2}} + \frac{2}{A(u)} \frac{\nabla B \nabla u}{\sqrt{1 + B^2|\nabla u|^2}} + \frac{1}{A(u)} \frac{B\Delta u}{\sqrt{1 + B^2|\nabla u|^2}}$$

$$- \frac{1}{A(u)} \frac{B^2 \nabla B \nabla u |\nabla u|^2}{(1 + B^2|\nabla u|^2)^{3/2}} - \frac{1}{2A(u)} \frac{B^3 \operatorname{Hess}_u(\nabla u, \nabla u)}{(1 + B^2|\nabla u|^2)^{3/2}} + 2\sqrt{1 + B^2|\nabla u|^2}\,\sinh u$$

$$= \Delta\,(Bu) + P_3(u, \nabla u, \nabla^2 u)$$

where P_3 is a bounded nonlinear function which can be decomposed in entire series in u and the components of ∇u and $\nabla^2 u$ for $\|u\|_{\mathcal{C}^1} \leq 1$ with terms of lowest order 3 and where the components of ∇u appear with an even power and the components of $\nabla^2 u$ only with the power 1.

In the region Ω_{gr} the surface $\tilde{\mathcal{S}}_n$ is parametrized as a graph of one of the functions $\pm\tilde{\mathcal{G}}_n$, where

$$B(z)\,\tilde{\mathcal{G}}_n(z) = -\frac{\varepsilon n}{2} + \varepsilon \operatorname{Re}(f_n(z)) + \tilde{\varepsilon}\log|z| + \tilde{\varepsilon},$$

Studying the behaviour of the function $f_n(z)$ one can easily verify that

$$\left|\tilde{\mathcal{G}}_n(z)\right| \leq c\varepsilon \left(|\log \varepsilon| + |\log |z|| + \sum_{m=1}^{n} |\log |z - z_m||\right),$$

$$\left|\nabla \tilde{\mathcal{G}}_n(z)\right| \leq c\varepsilon \left(\frac{1}{|z|} + \sum_{m=1}^{n} \frac{1}{|z - z_m|}\right),$$

$$\left|\nabla^2 \tilde{\mathcal{G}}_n(z)\right| \leq c\varepsilon \left(\frac{1}{|z|^2} + \sum_{m=1}^{n} \frac{1}{|z - z_m|^2}\right)$$

As $\Delta\left(B\tilde{\mathcal{G}}_n\right) = 0$, analysing carefully the terms in $P_3(\tilde{\mathcal{G}}_n, \nabla\tilde{\mathcal{G}}_n, \nabla^2\tilde{\mathcal{G}}_n)$ one can see that (5.10) is true in Ω_{gr}.

In the regions Ω_{glu}^m the surface $\tilde{\mathcal{S}}_n$ is parametrized as a graph of one of the functions $\pm 2\,\Upsilon^m$, where

$$\Upsilon^m(\lambda_m(\rho/2\,e^{i\theta})) = \bar{\Upsilon}(\rho,\theta) = (1 - \bar{\eta}_\varepsilon(\rho))\frac{1}{2}\,\bar{\mathcal{G}}_n(\rho,\theta) + \bar{\eta}_\varepsilon(\rho)\,G_{\varepsilon/2}^{cat}(\rho).$$

In the neighbourhood of $\zeta = 0$ we have:

$$\nabla_z = \frac{1}{2}\,\nabla_\zeta(1 + \mathcal{O}(|\zeta|)), \quad \nabla_z^2 = \frac{1}{4}\,\nabla_\zeta^2(1 + \mathcal{O}(|\zeta|)).$$

Using that $\left|\lambda_m(\rho/2\,e^{i\theta}) - z_m\right| = \rho + \mathcal{O}(\rho^2)$, we obtain

$$\bar{\mathcal{G}}_n \sim G_{\varepsilon/2}^{cat} = \mathcal{O}(\varepsilon\log\varepsilon), \quad \bar{\mathcal{G}}_n - G_{\varepsilon/2}^{cat} = \mathcal{O}(\varepsilon^{1-\beta}\rho),$$

$$\left|\nabla\bar{\mathcal{G}}_n\right| \sim \left|\nabla G_{\varepsilon/2}^{cat}\right| = \mathcal{O}\left(\frac{\varepsilon}{\rho}\right) \quad \left|\nabla(\bar{\mathcal{G}}_n - G_{\varepsilon/2}^{cat})\right| = \mathcal{O}(\varepsilon^{1-\beta}),$$

$$\left|\nabla^2\bar{\mathcal{G}}_n\right| \sim \left|\nabla^2 G_{\varepsilon/2}^{cat}\right| = \mathcal{O}\left(\frac{\varepsilon}{\rho^2}\right), \quad \left|\nabla^2(\bar{\mathcal{G}}_n + G_{\varepsilon/2}^{cat})\right| = \mathcal{O}(\varepsilon^{1/3-\beta})$$

We introduce the function

$$\bar{B}(\rho,\theta) = B\left(\lambda_m\left(\frac{\rho}{2}\,e^{i\theta}\right)\right) = \frac{1 + \left|\left(1 + \frac{\rho}{2}\,e^{i\theta}\right)/\left(1 - \frac{\rho}{2}\,e^{i\theta}\right)\right|}{2} = 1 + \mathcal{O}(\rho).$$

We have

$$\partial_\rho\bar{B} \sim \partial_\rho^2 B \sim \partial_\rho\partial_\theta\bar{B} = \mathcal{O}(1), \quad \partial_\theta\bar{B} \sim \partial_\theta^2\bar{B} = \mathcal{O}(\rho).$$

On the other hand, the cut-off function $\bar{\eta}_\varepsilon$ satisfies

$$\bar{\eta}_\varepsilon = \mathcal{O}(1), \quad \frac{d\bar{\eta}_\varepsilon}{d\rho} = \mathcal{O}\left(\frac{1}{\rho}\right), \quad \frac{d^2\bar{\eta}_\varepsilon}{d\rho^2} = \mathcal{O}\left(\frac{1}{\rho^2}\right).$$

Using that

$$\nabla\bar{\Upsilon} = \frac{1}{2}(1 - \bar{\eta}_\varepsilon)\,\nabla\bar{\mathcal{G}}_n + \bar{\eta}_\varepsilon\nabla G_{\varepsilon/2}^{cat} + \nabla\bar{\eta}_\varepsilon\left(G_{\varepsilon/2}^{cat} - \frac{1}{2}\bar{\mathcal{G}}_n\right),$$

$$\nabla^2 \bar{\Upsilon} = \frac{1}{2}(1 - \bar{\eta}_\varepsilon) \, \nabla^2 \bar{\mathcal{G}}_n + \bar{\eta}_\varepsilon \nabla^2 G^{cat}_{\varepsilon/2}$$

$$+ 2\nabla\bar{\eta}_\varepsilon \left(\nabla \left(G^{cat}_{\varepsilon/2} - \frac{1}{2}\bar{\mathcal{G}}_n \right) \right)^t + \nabla^2 \bar{\eta}_e \left(G^{cat}_{\varepsilon/2} - \frac{1}{2}\bar{\mathcal{G}}_n \right),$$

we get the estimates

$$\left| \nabla \bar{\Upsilon} \right| = \mathcal{O}\left(\frac{\varepsilon^{1-\beta}}{\rho} \right), \quad \left| \nabla^2 \bar{\Upsilon} \right| = \mathcal{O}\left(\frac{\varepsilon^{1-\beta}}{\rho^2} \right),$$

and

$$P_3(\Upsilon) = \mathcal{O}\left(\frac{\varepsilon^{3-\beta}}{\rho^4} \right) = \mathcal{O}(\varepsilon^{1/3-\beta}),$$

for all $\beta \in (0,1)$. We also have

$$\Delta_z = \frac{1}{4} \left| 1 - \zeta^2 \right| \Delta_\zeta,$$

where Δ_z and Δ_ζ are Laplacian operators in coordinates z and ζ. So,

$$\Delta \left(\bar{B} \, G^{cat}_{\varepsilon/2} \right) = G_{\varepsilon/2} \, \Delta \bar{B} + 2\nabla \bar{B} \, \nabla G^{cat}_{\varepsilon/2} + \bar{B} \, \Delta G^{cat}_{\varepsilon/2} = \mathcal{O}\left(\frac{\varepsilon^3}{\rho^4} \right).$$

Putting this calculations together, we check that in Ω^m_{glu} the mean curvature of $\tilde{\mathcal{S}}_n$ satisfies

$$\mathcal{H} = \mathcal{O}\left(\varepsilon^{1/3-\beta} \right), \quad \forall \beta \in (0,1).$$

An identical proof shows that in Ω^0_{glu} we have $\mathcal{H} = \mathcal{O}(\varepsilon^{1-\beta})$. $\qquad\square$

6. PERTURBATIONS OF $\tilde{\mathcal{S}}_n$

Recall that the surface $\tilde{\mathcal{S}}_n$ can be seen as an image by the mapping $\mathcal{X}$ of a surface $\tilde{S}_n$ (constructed in the previous paragraph) which is contained in the unit cylinder $D^2 \times \mathbb{R}$. We would like to calculate the mean curvature of small perturbations of $\tilde{\mathcal{S}}_n$ and to this end we calculate the mean curvature of small perturbations of the surface $\tilde{S}_n$ with respect to the metric $\tilde{g} = \mathcal{X}_* g_{eucl}$.

Let, as before, $\tilde{\mathcal{N}}$ denote the unit normal vector field to $\tilde{C}_{\tilde{\varepsilon}}$ with respect to the metric

$$\tilde{g}(z, x_3) = A^2(z, x_3) \left(dz^2 + B^2(z) \, dx_3^2 \right).$$

and take a function $w \in \mathcal{C}^2(\tilde{S}_n)$ small enough, invariant under rotations by the angle $\frac{2\pi}{n}$ and the transformation $z \mapsto \bar{z}$. We denote $\tilde{S}_n(w)$ the surface parametrized by

$$(s, \phi) \in \mathbb{R} \times S^1 \mapsto \tilde{X}^{cat}_{\tilde{\varepsilon}}(s, \phi) + w(s, \phi)\tilde{\mathcal{N}}(s, \phi),$$

in region Ω^{cat}_0. Furthermore, in the region

$$\Omega_{gr} \cup \Omega^0_{glu} = \left\{ z \in D^2 \, : \, 1/2 \, \tilde{\varepsilon}^{1/2} < |z| < 1 \right\} \setminus \overset{n}{\underset{m=1}{\cup}} \lambda_m \left\{ \zeta \in \mathbb{C}_- \, : \, |\zeta| < 2 \, \varepsilon^{2/3} \right\},$$

we parametrize $S_n(w)$ by

$$z \mapsto \left(z, \pm \tilde{\Upsilon}(z) \right) \pm w(z) \tilde{\Xi}_{\tilde{\varepsilon}}(z), \quad \text{where} \quad \tilde{\Xi}_{\tilde{\varepsilon}} = (1 - \eta^0_{\tilde{\varepsilon}}) \, \partial_{x_3} + \eta^0_{\tilde{\varepsilon}} \frac{1}{2} \tilde{\mathcal{N}},$$

where $\eta^0_{\tilde{\varepsilon}}(|z|)$ is the cut-off function defined in the paragraph 6.2. Notice that in Ω^0_{glu}

$$\|\tilde{\Xi}_{\tilde{\varepsilon}}\|_{\tilde{g}} \sim \|\partial_{x_3}\|_{\tilde{g}} = \frac{1}{2} + \mathcal{O}(\tilde{\varepsilon}).$$

In the neighbourhood of the n-th root of unity z_m the surface $\tilde{S}_n$ can be seen as an image by the mapping Λ_m of a surface $\bar{S}_n$ contained in $\mathbb{C}_- \times \mathbb{R}$. We put $\bar{w} = w \circ \lambda_m \in \mathcal{C}^2(\bar{S}_n)$ and parametrize $\tilde{S}_n(w)$ in the region Ω_{cat}^m by

$$(\sigma, \theta) \in \mathbb{R} \times [\pi/2, 3\pi/2] \mapsto \Lambda_m \left(X_{\varepsilon/2}^{cat}(\sigma, \theta) + \frac{a}{2} \bar{w} \mathcal{N}(\sigma, \theta) \right),$$

where $\mathcal{N}$ is the unit normal vector field to the half-catenoid $C_{\varepsilon/2}$ with respect to the metric

$$g_m(\zeta, \xi_3) = (\Lambda_m \circ \mathcal{X})_* g_{eucl} = a^2(\zeta, \xi_3) \left(d\zeta^2 + b^2(\zeta) \, d\xi_3^2 \right).$$

In the regions Ω_{glu}^m, we parametrize $\tilde{S}_n(w)$ by

$$\zeta \mapsto \Lambda_m \left((\zeta, \pm \bar{\Upsilon}(\zeta)) \pm \bar{w}(z) \, \bar{\Xi}_\varepsilon(\zeta) \right), \quad \text{where} \quad \bar{\Xi}_{\tilde{\varepsilon}} = \frac{1}{2} \left((1 - \bar{\eta}_\varepsilon) \, \partial_{\xi_3} + \bar{\eta}_\varepsilon \, a\mathcal{N} \right),$$

and $\bar{\eta}_\varepsilon(|\zeta|)$ be the cut-off function, introduced in the paragraph 6.1. Notice that in Ω_{glu}^m, we have

$$\|\Xi_{\tilde{\varepsilon}}\|_{g_m} \sim \frac{1}{2} \|\partial_{\xi_3}\|_{g_m} = 1 + \mathcal{O}(\varepsilon^{4/3}).$$

Remark 6.1. *We multiply the vector field $\mathcal{N}$ by the factor a in order to make the vector field*

$$\partial_\theta \left(a\mathcal{N} \right)(\zeta) \, \big|_{\theta \in \left\{ \frac{\pi}{2}, \frac{3\pi}{2} \right\}},$$

horizontal. In this case, the condition sufficient for $\tilde{S}_n(w)$ to be orthogonal to the unit sphere is

$$\partial_\theta \left(\bar{w} \right) \big|_{\theta \in \left\{ \frac{\pi}{2}, \frac{3\pi}{2} \right\}} = 0.$$

Notation 6.1. *Let Ω denote a coordinate domain we work in. From now on, when we don't need a more detailed information, we use the following notations :*

- *L for any bounded second order linear differential operator defined in Ω (in other words $L\,w$ is a linear combination of w and the components of ∇w and $\nabla^2 w$ with coefficients which are bounded functions in Ω, where ∇ and ∇^2 are the gradient and the Hessian in the chosen coordinates).*

- *$Q^k(w, \nabla w, \nabla^2 w)$, $k \in \mathbb{N}$, for any nonlinear function, which can be decomposed in entire series with terms of lowest order k, and where the components of $\nabla^2 w$ appear only with power 1. We will also use the notation $Q^k(w)$ for brevity.*

- *Let $\gamma \in \mathcal{C}^\infty(\Omega)$ be a positive real function. We denote $L^\gamma w$ and $Q^{k,\gamma}(w)$ functions which share the same properties as $L\,w$ and $Q^k(w)$ with the only difference that the components of the gradient and the Hessian of w are calculated with respect to the metric $\gamma^{-2} g_{eucl}$.*

  For example, if we work in the coordinates (r, ϕ) and take $\gamma = r$, then L^γ will be a linear differential operator in $r^2 \partial_r^2$, ∂_ϕ^2, $r\partial_{r\phi}^2$, $r\partial_r$ and ∂_ϕ.

6.1. **Mean curvature of the perturbed graph.** In the region

$$\Omega_{gr} = \{ z \in D^2 \, : \, 2\,\tilde{\varepsilon}^{1/2} < |z| < 1 \} \setminus \bigcup_{m=1}^{n} \lambda_m \left\{ \zeta \in \mathbb{C}_- \, : \, |\zeta| < 2\,\varepsilon^{2/3} \right\},$$

we suppose that $\|w\|_{\mathcal{C}^2(\Omega_{gr})} < 1$. Then, the mean curvature of $\tilde{S}_n(w)$ satisfies

$$\mathcal{H}(w) = \mathcal{H}(0) + \Delta(B\,w) + P_3 \left(\tilde{\mathcal{G}}_n + w \right).$$

Analysing carefully the the terms of P_3 and the expansion of the function $\tilde{\mathcal{G}}_n$, we get

(6.11)
$$\mathcal{H}(w) = \mathcal{H}(0) + \Delta(B\,w)$$

$$+ \max\left\{ \left|\nabla\tilde{\mathcal{G}}_n\right|^2, \left|\nabla^2\tilde{\mathcal{G}}_n\right|\left|\nabla\tilde{\mathcal{G}}_n\right|, \tilde{\mathcal{G}}_n^2, \left|\tilde{\mathcal{G}}_n\right|\left|\nabla^2\tilde{\mathcal{G}}_n\right|, \left|\tilde{\mathcal{G}}_n\right|\left|\nabla\tilde{\mathcal{G}}_n\right| \right\} L\,w$$

$$+ \max\left\{ \left|\tilde{\mathcal{G}}_n\right|, \left|\nabla\tilde{\mathcal{G}}_n\right|, \left|\nabla^2\tilde{\mathcal{G}}_n\right| \right\} Q_2(w) + Q_3(w)$$

We introduce the weight functions

$$\gamma_0(z) = |z|, \quad \gamma_m(z) = |z - z_m|, \; m = 1, \ldots, n \quad \text{and} \quad \gamma(z) = |z| \prod_{m=1}^{n} |z - z_m|.$$

Then, (using the relation between ε and $\tilde{\varepsilon}$) (6.11) can be written as

$$\mathcal{H}(w) = \mathcal{H}(0) + \Delta(B\,w) + \frac{\varepsilon^{2-\beta}}{\gamma^4} L_z^\gamma\,w + \frac{\varepsilon^{1-\beta}}{\gamma^4} Q_z^{2,\gamma}(w) + \frac{\varepsilon^{-\beta}}{\gamma^4} Q_z^{3,\gamma}(w).$$

for all $\beta \in (0,1)$. (We use the lower index to indicate the coordinate system we work in). On the other hand, in the neighbourhood of $z = 0$ we have

$$\mathcal{H}(w) = \mathcal{H}(0) + \Delta(B\,w) + \frac{\tilde{\varepsilon}^2}{|z|^4} L_z^{\gamma_0}\,w + \frac{\tilde{\varepsilon}}{|z|^4} Q_z^{2,\gamma_0}(w) + \frac{1}{|z|^4} Q_z^{3,\gamma_0}(w),$$

and in the neighbourhood of $z = z_m$

$$\mathcal{H}(w) = \mathcal{H}(0) + \Delta(B\,w) + \frac{\varepsilon^{2-\beta}}{|z - z_m|^4} L_z^{\gamma_m}\,w + \frac{\varepsilon^{1-\beta}}{|z - z_m|^4} Q_z^{2,\gamma_m}(w) + \frac{\varepsilon^{-\beta}}{|z - z_m|^4} Q_z^{3,\gamma_m}(w).$$

for all $\beta \in (0,1)$ (where we used $\dfrac{1}{\gamma} < \dfrac{n}{\gamma_m}$ in the neighborhood of z_m).

6.2. Mean curvature of the perturbed neck. In the region

$$\Omega_{cat}^0 = \{z \in D^2 : \tilde{\varepsilon} < |z| < 1/2\,\tilde{\varepsilon}^{1/2}\},$$

the surface $\tilde{S}_n(w)$ is parametrized as a normal graph around $\tilde{C}_{\tilde{\varepsilon}}$ for the function $u = 1/2\,w$. We suppose that

(6.12)
$$\left\| \frac{w}{\tilde{\varepsilon}\cosh s} \right\|_{\mathcal{C}^2((-s_*,s_*)\times S^1)} \leq 1, \quad \tilde{\varepsilon}\cosh s_* = 1/2\,\tilde{\varepsilon}^{1/2}$$

The tangent space to $\tilde{S}_n(w)$ is spanned by the vector fields

$$\tilde{T}_s(u) = \tilde{T}_s + \partial_s u\,\tilde{\mathcal{N}} + u\,\partial_s\tilde{\mathcal{N}}, \quad \tilde{T}_\phi(u) = \tilde{T}_\phi + \partial_\phi u\,\tilde{\mathcal{N}} + u\,\partial_\phi\tilde{\mathcal{N}},$$

and let us choose functions $\nu, \varkappa, \mu \in \mathcal{C}^\infty(\mathbb{R})$, such that $\nu(0) = \varkappa(0) = \mu(0) = 0$ and

$$\tilde{\mathcal{N}}(u) = \tilde{\mathcal{N}} + \nu(u)\,\tilde{\mathcal{N}} + \varkappa(u)\,T_s + \mu(u)\,\tilde{T}_\phi,$$

is the normal unit vector field to $\tilde{S}(w)$. We have

(6.13)
$$\tilde{g}(u)\left(\tilde{\mathcal{N}}(u), \tilde{T}_s(u)\right) = 0, \quad \tilde{g}(u)\left(\tilde{\mathcal{N}}(u), \tilde{T}_\phi(u)\right) = 0, \quad \tilde{g}(u)\left(\tilde{\mathcal{N}}(u), \tilde{\mathcal{N}}(u)\right) = 1.$$

where $\tilde{g}(u)$ is the scalar product corresponding to the metric $\tilde{g}$ taken along $\tilde{S}(w)$. Using the expression for $\tilde{g}$, we get

$$\tilde{g}(u)(s,\phi) - \tilde{g}(s,\phi) = \left(\tilde{\varepsilon}^{1-\beta} L_{s,\phi}\,u + Q_{s,\phi}^2(u)\right)\tilde{g}(s,\phi) + \left(\tilde{\varepsilon}\,L_{s,\phi}\,u + \frac{1}{\cosh^2 s} Q_{s,\phi}^2(u)\right) dx_3^2,$$

and from (6.13), we deduce that

$$\nu(u) = \tilde{\varepsilon}^{1-\beta} \, L_{s,\phi} \, u + Q^2_{s,\phi}(u)$$

$$\varkappa(u) = -\frac{1}{\tilde{\varepsilon}^2 \cosh^2 s} \partial_s u + \frac{1}{\cosh^2 s} \, L_{s,\phi} \, u + \frac{\tilde{\varepsilon}^{1-\beta}}{\tilde{\varepsilon}^2 \cosh^2 s} \, Q^2_{s,\phi}(u)$$

$$\mu(u) = -\frac{1}{\tilde{\varepsilon}^2 \cosh^2 s} \partial_\phi u + \frac{1}{\cosh^2 s} \, L_{s,\phi} \, u + \frac{\tilde{\varepsilon}^{1-\beta}}{\tilde{\varepsilon}^2 \cosh^2 s} \, Q^2_{s,\phi}(u),$$

where we used the fact that $dx_3^2(\tilde{T}_s, \tilde{\mathcal{N}})$ and $dx_3^2(\tilde{T}_\phi, \tilde{\mathcal{N}})$ can be bounded by a constant times $\tilde{\varepsilon}$ and the estimate

$$\left| \tilde{g}(\tilde{T}_p, \tilde{T}_q) - \tilde{\varepsilon}^2 \cosh^2 s \, \delta_{pq} \right| \leq c \, \varepsilon^3,$$

where $\tilde{T}_p$ and $\tilde{T}_q$ stand for $\tilde{T}_s$ or $\tilde{T}_\phi$. We can write the normal vector field to $\tilde{S}(w)$ in the form

$$\tilde{\mathcal{N}}(u) = \tilde{\mathcal{N}} - \frac{1}{\tilde{\varepsilon}^2 \cosh^2 s} \left(\partial_s u \, \tilde{T}_s + \partial_\phi u \, \tilde{T}_\phi \right)$$

$$+ \left[\tilde{\varepsilon}^{1-\beta} \, L_{s,\phi} \, u + Q^2_{s,\phi}(u) \right]^\perp + \left[\frac{\tilde{\varepsilon}}{\cosh s} \, L_{s,\phi} \, u + \frac{\tilde{\varepsilon}^{-\beta}}{\cosh s} \, Q^2_{s,\phi}(u) \right]^T$$

where $[*]^\perp$ and $[**]^T$ denote a normal and a tangent vector fields of norm $*$ and $**$.

We denote $\tilde{\nabla}^{\tilde{\varepsilon}}(u)$ and $\tilde{\Gamma}^k_{ij}(u)$ the Levi-Civita connection and the Cristoffel symbols corresponding to the metric $\tilde{g}$ and taken along the surface $\tilde{S}(w)$. Then, we have

$$\tilde{\Gamma}^k_{ij}(u) = \tilde{\Gamma}^k_{ij} + L_{s,\phi} \, u + Q^2_{s,\phi}(u),$$

$$\tilde{\nabla}^{\tilde{\varepsilon}}_{\partial_p} \partial_q(u) = \tilde{\nabla}^{\tilde{\varepsilon}}_{\partial_p} \partial_q + \partial_p \partial_q u \, \tilde{\mathcal{N}} + \partial_p u \, \partial_q \tilde{\mathcal{N}} + \partial_q u \, \partial_p \tilde{\mathcal{N}} + u \, \partial_p \partial_q \tilde{\mathcal{N}}$$

$$+ \tilde{\varepsilon}^2 \cosh^2 s \, L_{s,\phi} \, u + \tilde{\varepsilon} \cosh s \, Q^2_{s,\phi}(u)$$

where ∂_p and ∂_q stand for ∂_ϕ or ∂_s. This allows us to find the second fundamental form of the surface $\mathcal{S}_n(w)$:

$$\left(\tilde{\mathfrak{h}}_{\tilde{\varepsilon}}(u) \right)_{pq} = \tilde{g}(u) \left(\nabla^{\tilde{\varepsilon}}_{\partial_p} \partial_q(u), \tilde{\mathcal{N}}(u) \right).$$

Note that

$$\left| \tilde{g}(\partial_p \tilde{\mathcal{N}}, \partial_q \tilde{\mathcal{N}}) - \frac{1}{\cosh^2 s} \, \delta_{pq} \right| \leq c \, \tilde{\varepsilon}.$$

Putting all the estimates together, we obtain

$$\tilde{\mathfrak{h}}_{\tilde{\varepsilon}}(u)(s, \phi) = \tilde{\mathfrak{h}}_\varepsilon(s, \phi) + \begin{pmatrix} \partial_s^2 u & \partial_s \partial_\theta u \\ \partial_s \partial_\phi u & \partial_\phi^2 u \end{pmatrix}$$

$$- \frac{u}{\cosh^2 s} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \tanh s \begin{pmatrix} -\partial_s u & \partial_\phi u \\ \partial_\phi u & \partial_s u \end{pmatrix}$$

$$+ \left(\tilde{\varepsilon}^{2-\beta} \, L_{s,\phi} \, u + \tilde{\varepsilon}^2 \cosh^2 s \, L_{s,\phi} \, u + \frac{1}{\tilde{\varepsilon} \cosh^2 s} \, Q^2_{s,\phi}(u) \right) \tilde{\mathfrak{h}}_0(s, \phi)$$

where $\tilde{\mathfrak{h}}_0$ is a bounded symmetric 2-form, which does not depend on $\tilde{\varepsilon}$. On the other hand, the first fundamental form $\tilde{\mathfrak{g}}_{\tilde{\varepsilon}}(u)$, which corresponds to the metric induced on $\mathcal{S}_n(w)$ from $\tilde{g}$, satisfies

$$\tilde{\mathfrak{g}}_{\tilde{\varepsilon}}(u) = \tilde{\mathfrak{g}}_{\tilde{\varepsilon}} - 2u\,\tilde{\mathfrak{h}}_{\tilde{\varepsilon}} + Q^2_{s,\phi}(u).$$

This yields

$$\frac{\det(\tilde{\mathfrak{g}}_{\tilde{\varepsilon}}(u))}{\det(\tilde{\mathfrak{g}}_{\tilde{\varepsilon}})}\tilde{\mathfrak{g}}_{\tilde{\varepsilon}}^{-1}(u) = \tilde{\mathfrak{g}}_{\tilde{\varepsilon}}^{-1} + \frac{2\varepsilon u}{\tilde{\varepsilon}^4 \cosh^4 s}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$+ \left(\frac{1}{\tilde{\varepsilon}\cosh^4 s}\,L_{s,\phi}u + \frac{1}{\tilde{\varepsilon}^4 \cosh^4 s}Q^2_{s,\phi}(u) \right)\tilde{\mathfrak{g}}_0$$

where $\tilde{\mathfrak{g}}_0$ is a bounded 2-form. Going back to $w = 2\,u$, we obtain
(6.14)
$$\mathcal{H}(w) = \mathcal{H}(0) + \frac{1}{2}\frac{1}{\tilde{\varepsilon}^2 \cosh^2 s}\left(\partial_s^2 + \partial_\phi^2 + \frac{2}{\cosh s} \right)w$$

$$+ \left(1 + \frac{\tilde{\varepsilon}^{-\beta}}{\cosh^2 s} \right)L_{s,\phi}\,w + \frac{1}{\tilde{\varepsilon}^3 \cosh^4 s}\,Q^2_{s,\phi}(w) + \frac{1}{\tilde{\varepsilon}^4 \cosh^4 s}\,Q^3_{s,\phi}(w).$$

6.3. Mean curvature of the perturbed bridges. In the region

$$\Omega^m_{cat} = \lambda_m\{\zeta \in \mathbb{C}_- \; : \; \varepsilon < |\zeta| < 1/2\,\varepsilon^{2/3}\},$$

the surface $\tilde{S}_n(w)$ is parametrized as the image by the mapping Λ_m of the normal graph about C_ε for the function $\bar{u} = a\,\bar{w}$, scaled by the factor $\frac{1}{2}$. We suppose that

(6.15) $$\left\| \frac{\bar{w}}{\varepsilon \cosh \sigma} \right\|_{\mathcal{C}^2((-\sigma_*,\sigma_*)\times[\pi/2,3\pi/2])} \le 1, \quad \varepsilon \cosh \sigma_* = 1/2\,e^{2/3}$$

Our goal is to calculate the mean curvature of $S(\bar{w})$ with respect to the metric

$$g_m = a^2(d\zeta^2 + b^2 d\xi_3^2).$$

The computation is very similar to the one we have done in the previous paragraph and we only need to change several estimates. The scalar product along $S(\bar{w})$ satisfies

$$g_m(\bar{u})(\sigma,\theta) - g_m(\sigma,\theta) = \left(L_{\sigma,\theta}\,\bar{u} + Q^2_{\sigma,\theta}(\bar{u}) \right)g_m(\sigma,\theta) + \left(\varepsilon L_{\sigma,\theta}\,\bar{u} + \frac{1}{\cosh^2 \sigma}\,Q^2_{\sigma,\theta}(\bar{u}) \right)d\tau^2.$$

Then, the normal vector field to $\mathcal{S}_n(\bar{w})$ can be written as

$$\mathcal{N}(\bar{u}) = \mathcal{N} - \frac{1}{\varepsilon^2 \cosh^2 \sigma}\,(\partial_\sigma \bar{u}\,T_\sigma + \partial_\theta \bar{u}\,T_\theta)$$

$$+ \left[L_{\sigma,\theta}\,\bar{u} + Q^2_{\sigma,\theta}(\bar{u}) \right]^\perp + \left[\frac{\varepsilon}{\cosh \sigma}\,L_{\sigma,\theta}\,\bar{u} + \frac{1}{\varepsilon \cosh \sigma}\,Q^2_{\sigma,\theta}(\bar{u}) \right]^T,$$

and the components of the Levi-Civita connection are

$$\nabla^\varepsilon_{\partial_\alpha}\partial_\beta(\bar{u}) = \nabla^\varepsilon_{\partial_\alpha}\partial_\beta + \partial_\alpha\partial_\beta \bar{u}\,\mathcal{N} + \partial_\alpha \bar{u}\,\partial_\beta \mathcal{N} + \partial_\beta \bar{u}\,\partial_\alpha \mathcal{N} + \bar{u}\,\partial_\alpha\partial_\beta \mathcal{N}$$

$$+ \varepsilon \cosh \sigma\,L_{\sigma,\theta}\,\bar{u} + Q^2_{\sigma,\theta}(\bar{u}).$$

The first and the second fundamental forms satisfy :

$$\frac{\det(\mathfrak{g}_\varepsilon(\bar{u}))}{\det(\mathfrak{g}_\varepsilon)}\mathfrak{g}_\varepsilon^{-1}(\bar{u}) = \mathfrak{g}_\varepsilon^{-1} + \frac{\varepsilon\bar{u}}{\varepsilon^4\cosh^4\sigma}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$+\left(\frac{1}{\varepsilon\cosh^4\sigma}\,L_{\sigma,\theta}\,\bar{u} + \frac{1}{\varepsilon^4\cosh^4\sigma}\,Q^2_{\sigma,\theta}(\bar{u})\right)\mathfrak{g}_0,$$

$$\mathfrak{h}_\varepsilon(\bar{u})(\sigma,\theta) = \mathfrak{h}_\varepsilon(\sigma,\theta) + \begin{pmatrix} \partial^2_\sigma\bar{u} & \partial_\sigma\partial_\theta\bar{u} \\ \partial_\sigma\partial_\theta\bar{u} & \partial^2_\theta\bar{u} \end{pmatrix} - \frac{\bar{u}}{\cosh^2\sigma}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$+\tanh\sigma\begin{pmatrix} -\partial_\sigma\bar{u} & \partial_\theta\bar{u} \\ \partial_\theta\bar{u} & \partial_\sigma\bar{u} \end{pmatrix} + \left(\varepsilon\cosh\sigma\,L_{\sigma,\theta}\,\bar{u} + \frac{1}{\varepsilon\cosh^2\sigma}\,Q^2_{\sigma,\theta}(\bar{u})\right)\mathfrak{h}_0(\sigma,\theta),$$

where $\mathfrak{h}_0$ and $\mathfrak{g}_0$ are bounded symmetric 2-forms which do not depend on ε. Finally, going back to $\bar{w} = \frac{1}{a}\,\bar{u}$, we get

(6.16)
$$\begin{aligned}\mathcal{H}(\bar{w}) \;&= \mathcal{H}(0) + \frac{1}{\varepsilon^2\cosh^2\sigma}\left(\partial^2_\sigma + \partial^2_\theta + \frac{2}{\cosh^2\sigma}\right)\bar{w} \\ &+ \frac{1}{\varepsilon\cosh\sigma}L_{\sigma,\theta}\,\bar{w} + \frac{1}{\varepsilon^3\cosh^4\sigma}\,Q^2_{\sigma,\theta}(\bar{w}) + \frac{1}{\varepsilon^4\cosh^4\sigma}\,Q^3_{\sigma,\theta}(\bar{w}).\end{aligned}$$

6.4. **Mean curvature of the perturbed "gluing regions".** Let M be a smooth hypersurface in a smooth Riemannian manifold endowed with a metric g. Take w a small smooth function and V_1 and V_2 two smooth vector fields on M. Let $\mathcal{H}^i(w)$ denote the mean curvature of the hypersurfaces obtained by perturbation of M in the direction V_i, $i = 1, 2$. We have the following result:

Lemma 6.1. *The following relation holds*
$$D\mathcal{H}^2\big|_{w=0}(v) = D\mathcal{H}^1\big|_{w=0}(\tau\,v) + g(\nabla_M\mathcal{H}(0), \mathcal{T})$$

where $\tau = \frac{|V_2^\perp|}{|V_1^\perp|}$, and $\mathcal{T} = V_2^T - \tau\,V_1^T$, and where $V_i^\perp$ and V_i^T denote the orthogonal projections of V_i on the normal and the tangent bundle of M.

Proof. This lemma is a simple generalisation of the result proven in [9] where the case when one of the vector fields V_i is a unit normal to M is treated. The proof consists of applying the implicit function theorem to the equation

$$p + t\,V_1(p) = q + s\,V_2(q), \quad p, q \in M, \quad t, s \in \mathbb{R},$$

expressing locally p and t as functions of q and s:

$$p = \Phi(q, s) \quad \text{and} \quad t = \Psi(q, s),$$

with $\Phi(q, 0) = q$ and $\Psi(q, 0) = 0$. We obtain then

$$\partial_s\Psi(\cdot, 0)[V_1]^\perp = [V_2]^\perp \quad \text{and} \quad \partial_s\Phi(\cdot, 0) = [V_2]^T - \partial_s\Psi(\cdot, s)[V_1]^T.$$

Moreover, we have

$$D\mathcal{H}^1\big|_{w=0}(\partial_s\Psi(\cdot, 0)\,v) + \nabla\mathcal{H}(0)\cdot\partial_s\Phi\,v = D\mathcal{H}^2\big|_{w=0}(v),$$

and the result follows. $\qquad\square$

Now let us return to the surface $\tilde{\mathcal{S}}_n$. Making use of the proof of the proposition 3.5, one can see that in the region Ω_{glu}^0 the components of $\nabla^{\tilde{g}}\mathcal{H}$ are bounded by a constant times $\tilde{\varepsilon}^{1/2-\beta}$. Moreover, using the expression obtained in the lemma 3.1 for the normal vector field to the surface parametrized as a graph of the function $\tilde{\Upsilon}$, we get

$$[\tilde{\bar{\Xi}}_{\tilde{\varepsilon}}]^N/[\partial_{x_3}]^N = 1 + \mathcal{O}(\tilde{\varepsilon}) \quad \text{and} \quad [\tilde{\bar{\Xi}}_{\tilde{\varepsilon}}]^T \sim [\tilde{\partial}_{x_3}]^T = \mathcal{O}(\tilde{\varepsilon}).$$

Therefore, lemma (6.1) with $V_1 = \partial_{x_3}$ and $V_2 = \tilde{\bar{\Xi}}_{\tilde{\varepsilon}}$ yields

$$\mathcal{H}(w) = \mathcal{H}(0) + \Delta(B\,w) + L_z^{\gamma_0}\,w + \tilde{\varepsilon}^{-1}\,Q_z^{2,\gamma_0}(w) + \tilde{\varepsilon}^{-2}Q_z^{3,\gamma_0}.$$

in Ω_{glu^0}. Similarly, in Ω_{glu}^m taking $V_1 = \partial_{\xi_3}$ and $V_2 = \bar{\bar{\Xi}}_\varepsilon$, and using the fact that the components of the gradient of $\mathcal{H}$ are bounded by a constant times $\varepsilon^{-1/3-\beta}$ and the fact that

$$[\bar{\Xi}_\varepsilon]^N/[\partial_{\xi_3}]^N = 1 + \mathcal{O}(\varepsilon^{4/3}) \quad \text{and} \quad [\bar{\Xi}_\varepsilon]^T \sim [\tilde{\partial}_{\xi_3}]^T = \mathcal{O}(\varepsilon^{4/3}),$$

we get

$$\mathcal{H}(w) = \mathcal{H}(0) + \Delta(B\,w) + \varepsilon^{-2/3-\beta}\,L_z^{\gamma_m}\,w + \varepsilon^{-5/3-\beta}\,Q_z^{2,\gamma_m}(w) + \varepsilon^{-8/3-\beta}\,Q_z^{3,\gamma_m}(w),$$

for all positive $\beta \in (0,1)$.

7. Linear analysis in the puncture disk

We would like to analyse the Laplace operator subject to the Robin boundary data:

$$(7.17) \qquad \begin{cases} \Delta\,w = f \quad \text{in} \quad D^2 \setminus \{0\} \quad (\text{or } D^2) \\ \partial_r w - w = 0 \quad \text{on} \quad S^1 \setminus \{z_1, \ldots, z_n\} \end{cases}$$

where f is a given function whose regularity and properties will be stated shortly. In what follows we suppose that we work in the domain $D^2 \setminus \{0\}$. The case of the entire open disk D^2 can be treated in an analogous manner with certain simplifications.

First of all, we take f even with respect to the angular variable and, for a given $n \geq 2$, invariant under rotations by the angle $\frac{2\pi}{n}$. With this assumption, the operator associated to (7.17) does not have any bounded kernel and hence, the solvability of (7.17) follows from classical arguments. For example, if $f \in \mathcal{C}^{0,\alpha}(\overline{D^2})$ we get the existence of $w \in \mathcal{C}^{2,\alpha}(\overline{D^2})$ solution w of (7.17). Moreover,

$$|w|_{\mathcal{C}^{2,\alpha}(\overline{D^2})} \leq C\left(\|w\|_{\mathcal{C}^0(\overline{D^2})} + \|f\|_{\mathcal{C}^{0,\alpha}(\overline{D^2})}\right)$$

We would like to understand what happens if we allow f to have singularities at 0 and/or z_m, $m = 1, \ldots, n$.

We define the weighted spaces we will work in. As before we set

$$\gamma(z) = |z| \prod_{m=1}^n |z - z_m|,$$

and we assume that we are given $\nu \in \mathbb{R}$. We say that a function $u \in L_{loc}^\infty(D^2)$ belongs to the space $L_\nu^\infty(D^2)$ if

$$\|\gamma^{-\nu}u\|_{L^\infty(D^2)} < \infty.$$

Let us use the notation D_*^2 for the open punctured disc $D^2\backslash\{0\}$. The space $\mathcal{C}_\nu^{k,\alpha}(D_*^2)$ is defined to be the space of functions $u \in \mathcal{C}_{loc}^{k,\alpha}(D_*^2)$ for which the following norm is finite

$$\|u\|_{\mathcal{C}_\nu^{k,\alpha}(D_*^2)} \quad := \quad \|\gamma^{-\nu}\,u\|_{\mathcal{C}^{k,\alpha}(D_*,\,\gamma^{-2}\,g_{eucl})}.$$

Observe that, on the right hand side, we do not use the Euclidean metric to calculate the gradient of a function but rather a singular metric $\gamma^{-2}\,g_{eucl}$. As a consequence, a function u belongs to $\mathcal{C}_\nu^{k,\alpha}(D_*^2)$ if

$$\sup_{z\in D_*^2}\left|\gamma^{-\nu}\,u\right| + \sum_{i=1}^k \sup_{z\in D_*^2}\left|\gamma^{-\nu+i}\,\nabla^i u\right| +$$

$$\sup_{z,z'\in D_*^2}\left\{\frac{|\gamma^{-\nu+k+\alpha}(z)\,\nabla^k u(z) - \gamma^{-\nu+k+\alpha}(z')\nabla^k u(z')|}{|z-z'|^\alpha}\right\} < \infty$$

Like in the section 4, instead of the problem (7.17), we can consider an equivalent problem defined in $\overline{D^2}\setminus\{0,1\}$. Take the change of variables $z \mapsto z^n$ and notice that

$$|z|^2\,\Delta(z) = n^2\,|z|^{2n}\,\Delta(z^n).$$

We take a function F in $D^2\setminus\{0\}$, such that

$$F(z^n) = \frac{1}{n^2}\,|z|^{2-2n}f(z).$$

Consider the problem:

(7.18)
$$\begin{cases} \Delta\,W = F \quad \text{in} \quad D^2\setminus\{0\} \\[2mm] \partial_r W - \dfrac{1}{n}W = 0 \quad \text{on} \quad S^1\setminus\{1\} \end{cases}$$

We define the space $L_{\nu_0,\nu_1}^\infty(D^2)$ as the space of functions $U \in L_{loc}^\infty(D^2)$ for which

$$\||z|^{-\nu_0}|z-1|^{-\nu_1}\,U\|_{L^\infty(D^2)} < \infty.$$

Notice, if we take $f \in L_{\nu-2}^\infty(D^2)$, then $F \in L_{\nu/n-2,\nu-2}^\infty(D^2)$ and

$$\|F\|_{L_{\nu/n-2,\nu-2}^\infty(D^2)} = \frac{1}{n^2}\,\|f\|_{L_{\nu-2}^\infty(D^2)}.$$

Proposition 7.1. *Assume that $\nu \in (0,1)$. Then, there exists a constant $C > 0$ and, for all $n \geq 2$, for all F, such that $|z|^{-\nu/n+2}\,F \in L^\infty(D^2)$, there exist a unique function Ψ_0 and a unique constant c_0^*, such that $W_0 := \Psi_0 + n\,c_0^*$ is a solution to (7.18) and*

$$\|\,|z|^{-\nu/n}\,\Psi_0\,\|_{L^\infty(D^2)} + |c_0^*| \leq C\,\|\,|z|^{-\nu/n+2}\,F\,\|_{L^\infty(D_*^2)}.$$

Proof. First, let us assume that F does not depend on the angular variable ϕ. In this case, (7.18) reduces to a second order ordinary differential equation which can be solved explicitly.

$$\Psi_0^{rad}(r) = \int_0^r \frac{1}{s}\int_0^s t\,F(t)\,dt\,ds, \quad W_0^{rad} = \Psi_0^{rad} + n\,c_0^*$$

$$c_0^* = -\int_0^1 s\,F(s)\,ds + \frac{1}{n}\int_0^r \frac{1}{s}\int_0^s t\,F(t)\,dt\,ds$$

With little work, one checks that the result is indeed correct in this spacial case.

Furthermore, we claim that, if we restrict our attention to the space of functions for which

$$\int_{S^1} F(re^{i\phi})\, r\, d\phi = 0$$

for all $r \in (0,1)$, then there exists a function W_0^{mean} such that

$$\||z|^{-\nu/n} W_0^{mean}\|_{L^\infty(D^2)} \leq C \||z|^{-\nu/n+2} F\|_{L^\infty(D^2)}$$

for a constant C independent of n. We construct W_0^{mean} as a limit of solutions to the Poisson's equation in annulus-type domains with mixed boundary date.

More precisely, take $\epsilon \in (0,1)$ and let us denote A_ϵ the annulus $D^2 \setminus D^2(\epsilon)$. For a fixed n let $W_{\epsilon,n}$ be the solution to the problem

(7.19)
$$\begin{cases} \Delta W_{\epsilon,n} = F \quad \text{in} \quad A_\epsilon, \\[2mm] \partial_r W_{\epsilon,n} - \frac{1}{n} W_{\epsilon,n} = 0 \quad \text{on} \quad S^1, \quad W_{\epsilon,n} = 0 \quad \text{on} \quad S^1(\epsilon). \end{cases}$$

There exists a constant $C(\epsilon, n)$ which depends on ϵ and n and such that

$$\|W_{\epsilon,n}\|_{L^\infty(A_\epsilon)} \leq C(\epsilon, n) \|F\|_{L^\infty(A_\epsilon)}$$

Changing the constant $C(\epsilon, n)$, we can rewrite this as follows

(7.20)
$$\||z|^{-\nu/n} W_{\epsilon,n}\|_{L^\infty(A_\epsilon)} \leq C(\epsilon, n) \||z|^{-\nu/n+2} F\|_{L^\infty(A_\epsilon)}$$

If the constant $C(\epsilon, n) = C(n)$ didn't depend on ϵ, then for every $\epsilon_0 \in (0,1)$, and for all $\epsilon < \epsilon_0$ we would have

$$\|W_{\epsilon,n}\|_{L^\infty\left(A_{\varepsilon_0/2}\right)} \leq C(n) \||z|^{-\nu/n+2} F\|_{L^\infty(D_*^2)}.$$

Then, by elliptic regularity theory, changing the constant $C(n)$ if necessary, we would have

$$\|\nabla W_{\epsilon,n}\|_{L^\infty(A_{\epsilon_0})} \leq C(n) \||z|^{-\nu/n+2} F\|_{L^\infty(D_*^2)}$$

Thus, when ϵ tends to 0, the sequence $W_{\epsilon,n}$ would admit a subsequence converging on compact sets of D_*^2 to a function W_n, a solution of (7.18) for a fixed n, such that

$$\||z|^{-\nu/n} W_n\|_{L^\infty(D^2)} \leq C(n) \||z|^{-\nu/n+2} F\|_{L^\infty(D^2)}$$

The fact that the constant $C(\varepsilon, n)$ doesn't depend on ϵ can be proven by an argument by contradiction. We suppose, that there exists a sequence of parameters ϵ_j and a sequence of points z_j such that

$$\||z|^{-\nu/n} W_{j,n}\|_{L^\infty(A_j)} \leq 1, \quad W_{j,n}(z_j) = |z_j|^{\nu/n},$$
$$\text{and} \quad \Delta W_{j,n} = F_{j,n}, \quad \||z|^{2-\nu/n} F_{j,n}\|_{L^\infty(A_j)} \underset{j\to\infty}{\to} 0$$

where $W_{j,n} := \dfrac{1}{C(\epsilon_j, n)} W_{\epsilon_j,n}$, $F_{j,n} := \dfrac{1}{C(\epsilon_j, n)} F$ and $A_j = A_{\epsilon_j}$.

We suppose first that the sequence z_j converges to a point $z_\infty \in D_*^2$. We denote

$$\mathcal{W}_{j,n}(z) = W_{j,n}\left(|z_j|\, z\right) |z_j|^{-\nu/n},$$

then, for every j, we have

$$\mathcal{W}_{j,n}\left(z_j/|z_j|\right) = 1.$$

The sequence $\mathcal{W}_{j,n}$ admits a subsequence converging on compact sets to a function $\mathcal{W}_n$ which is a solution to

$$\begin{cases} \Delta \mathcal{W}_n = 0 \quad \text{in} \quad D^2_* \\ \partial_r \mathcal{W}_n - \frac{1}{n}\mathcal{W}_n = 0 \quad \text{on} \quad S^1 \end{cases}.$$

Moreover, we have $|\mathcal{W}_n(z)| \leq |z|^{\nu/n}$ and $\mathcal{W}_n\left(\frac{z_\infty}{|z_\infty|}\right) = 1$. Using the fact that $\mathcal{W}_n$ has no radial part and that the problem (7.18) has no bounded kernel, we get a contradiction.

When the sequence of points z_j tends to 0 at the same time as $\frac{|z_j|}{\epsilon_j} \underset{j\to\infty}{\to} 0$ we obtain a sequence of functions $\mathcal{W}_{j,n}$, which admits a subsequence converging on compact sets to a function $\mathcal{W}_n$ which is a solution to the problem

$$\Delta \mathcal{W}_n = 0, \quad \text{in} \quad \mathbb{R}^2 \setminus \{0\}, \quad |\mathcal{W}_n| \leq c\,|z|^{\nu/n},$$

which implies $W_n \equiv 0$ and contradicts the fact that $\mathcal{W}_{j,n}\left(\frac{z_j}{|z_j|}\right) = 1$ for all j.

It remains to deal with the case when $z_j \underset{j\to\infty}{\to} 0$ and $\frac{|z_j|}{\epsilon_j} \underset{j\to\infty}{\to} a$, where a is a constant strictly greater than 1. In this case $\mathcal{W}_{n,j}$ admits a subsequence converging on compact sets to a function $\mathcal{W}_n$, which is a solution to

$$\begin{cases} \Delta \mathcal{W}_n = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus D^2(a) \\ \mathcal{W}_n = 0 \quad \text{on} \quad S^1(a) \end{cases}$$

and such that $|\mathcal{W}_n| \leq c\,|z|^\nu$. Once again, this implies $\mathcal{W}_n \equiv 0$ and gives a contradiction.

Finally, the case when $z_j \underset{j\to\infty}{\to} 0$ and $\frac{|z_j|}{\epsilon_j} \underset{j\to\infty}{\to} 1$ doesn't happen. For every j we have

$$\begin{cases} \Delta W_{j,n} = F_{j,n} \quad \text{in} \quad \mathbb{D}^2(2\epsilon_j) \setminus D^2(\epsilon_j) \\ W_{j,n} = 0 \quad \text{on} \quad S^1(\epsilon_j) \end{cases}$$

Moreover,

$$|F_{j,n}| \leq \epsilon_j^{\nu/n-2} \quad \text{and} \quad |W_{j,n}| \leq \epsilon_j^{\nu/n}.$$

Then in the subsets of $\mathbb{D}^2(2\epsilon_j) \setminus D^2(\epsilon_j)$ we have $|\nabla W_{j,n}| \leq c\,\epsilon_j^{\nu/n-1}$. This implies that in the neighbourhood of $|z| = \epsilon_j$, we have

$$|W_{j,n}| \leq C\,\epsilon_j^{\nu/n-1}\left(|z| - \epsilon_j\right).$$

At $z = z_j$ this yields $\frac{|z_j|}{\epsilon_j} - 1 \geq C$, which is not possible starting from a certain j.

Similarly, we can prove that the constant $C(n)$ in (7.20) does not depend on n. If it were not the case we could define a sequence of function $\tilde{W}_n$ and a sequence of points z_n, such that $\tilde{W}_n\left(z_n/|z_n|\right) = 1$. Then $\tilde{W}_n$ would admit a subsequence converging on compact sets to a function $\tilde{W}$, which is harmonic in a unit disk and has homogeneous Neumann boundary data. Using that $\int_{D^2} \tilde{W}\,dx_1\,dx_2 = 0$, we get the contradiction.

$\square$

Let us fix a cut-off function χ defined in the unit disk D^2 which is identically equal to 1 in a neighbourhood of $z = 1$ and to 0 in a neighbourhood of $z = 0$. We define deficiency spaces

$$\mathfrak{D}_n = \operatorname{span}\{n\} \quad \text{and} \quad \mathfrak{D}_\chi = \operatorname{span}\{\chi\}$$

Proposition 7.2. *Assume that $\nu \in (0,1)$. Then, there exists a constant $C > 0$ and, for all $n \geq 2$, for all $F \in L^\infty_{\nu/n-2,\nu-2}(D^2)$ there exist a unique function $\Psi \in L^\infty_{\nu/n,\nu}(D^2)$ and unique constants c_0^* and c_1^*, such that $W := \Psi + n\, c_0^* + c_1^*\, \chi$ is a solution to (7.18) and such that*

$$\|W\|_{L^\infty_{\nu/n,\nu}(D^2)\oplus\mathfrak{D}_n\oplus\mathfrak{D}_\chi} < C\, \|F\|_{L^\infty_{\nu/n-2,\nu-2}(D^2)}$$

Proof. We take the conformal mapping

$$\lambda : \mathbb{C}_- \longrightarrow D^2, \quad \lambda(\zeta) = \frac{1+\zeta}{1-\zeta}.$$

which sends a half-disk in $\mathbb{C}_-$ centered at 0 and of radius $\rho \in (0,1)$ to the intersection of the unit disk D^2 with the disk of radius $r_\rho = \frac{2\rho}{1-\rho^2}$ centered at $c_\rho = 1+\frac{2\rho^2}{1-\rho^2}$. For example, for $\rho = \frac{1}{3}$, we get $r_{\frac{1}{3}} = \frac{3}{4}$ and $c_{\frac{1}{3}} = \frac{5}{4}$ and for $a = \frac{1}{5}$, we get $r_{\frac{1}{5}} = \frac{5}{12}$ and $c_{\frac{1}{5}} = \frac{13}{12}$.

We define a cut-off function $\bar\chi \in \mathcal{C}^\infty(\mathbb{R}^2)$, such that

$$\bar\chi(\zeta) = \bar\chi(|\zeta|), \quad \bar\chi \equiv 0 \quad \text{for} \quad |\zeta| \geq 1/3 \quad \text{and} \quad \bar\chi \equiv 1 \quad \text{for} \quad |\zeta| \leq 1/5$$

and put $\chi(z) = \bar\chi\left(|\lambda^{-1}(z)|\right)$. Then, we have $\partial_r\chi|_{r=1} = 0$ and

$$\chi(z) \equiv 0 \quad \text{for} \quad |z - 5/4| \geq 3/4 \quad \text{and} \quad \chi \equiv 1 \quad \text{for} \quad |z - 13/12| \leq 5/12,$$

We decompose

$$F(z) = F_0(z) + F_1(z) = (1 - \chi(z))\, F(z) + \chi(z)\, F(z).$$

Then, we have

$$\|\,|z|^{-\nu/n+2}F_0\,\|_{L^\infty(D^2)} \leq \|F\|_{L^\infty_{\nu/n-2,\nu-2}(D^2)},$$

$$\|\,|z-1|^{-\nu+2}F_1\,\|_{L^\infty(D^2)} \leq \|F\|_{L^\infty_{\nu/n-2,\nu-2}(D^2)}$$

We define $\overline{F}(\zeta) = F(\lambda(\zeta))$. Remark that

$$\Delta_z = \frac{|1-\zeta|^2}{4}\, \Delta_\zeta,$$

and consider the problem

$$\begin{cases} \Delta\overline{W}_1 = \frac{4}{|1-\zeta|^2}\, \overline{F}_1(\zeta) & \text{in} \quad \mathbb{C}_- \cap D^2_*(1/3) \\[2mm] \partial_{\xi_1}\overline{W}_1 = 0 & \text{on} \quad \partial\mathbb{C}_- \cap D^2_*(1/3), \\[2mm] \overline{W}_1 = 0 & \text{on} \quad \mathbb{C}_- \cap \partial D^2_*(1/3) \end{cases} \qquad .$$

We extend $\overline{F}_1$ by symmetry to $D^2_*(1/3)$ and consider the problem

(7.21)
$$\begin{cases} \Delta\overline{W}_1 = \hat{F}_1 & \text{in} \quad D^2_*(1/3) \\[2mm] \overline{W}_1 = 0 & \text{on} \quad S^1(1/3) \end{cases}$$

where $\hat{F}_1 = \frac{4}{|1-\zeta|^2}\,\overline{F}_1(\zeta)$. Automatically, the restriction of $\overline{W}_1$ to $\mathbb{C}_- \cap D^2_*(1/3)$ satisfies $\partial_{\xi_1}\overline{W}_1 = 0$ at $\xi_1 = 0$.

The existence and the properties of $\overline{W}_1$ are obtained in the same way as the existence and the properties of W_0 in the proposition (7.1). We suppose first that the function $\hat{F}_1$ doesn't depend on the angular variable and depends only on $|\zeta| = \rho$. Then, the function

$$\overline{\Psi}_1^{rad}(\rho) = \int_0^\rho \frac{1}{s}\int_0^s t\,\hat{F}_1(t)\,dt\,ds, \quad \overline{W}_1^{rad} = \overline{\Psi}_1^{rad}(\rho) + c_1^*,$$

$$c_1^* = -\int_0^{1/3} \frac{1}{s}\int_0^s t\,\hat{F}_1(t)\,dt\,ds$$

satisfies (7.21) and using that $\|\,|\zeta|^{-\nu+2}\,\hat{F}\,\|_{L^\infty(D^2(1/3))} \le \|F\|_{L^\infty_{\nu/n-2,\nu-2}(D^2)}$, we get

$$\|\,|\zeta|^{-\nu}\,\overline{\Psi}_1\,\|_{L^\infty(D^2)} + |c_1^*| \le C\,\|F\|_{L^\infty_{\nu/n-2,\nu-2}(D^2)}.$$

On the other hand, if

$$\int_{S^1} \hat{F}_1(\rho,\theta)\,\rho\,d\theta = 0, \quad \text{for all} \quad \rho \in (0,1),$$

using the same argument as in the previous proposition, one finds a function $\overline{W}_1^{mean}$, which satisfies (7.21) and such that

$$\|\,|\zeta|^{-\nu}\,\overline{W}_1^{mean}\,\|_{L^\infty(D^2(1/3))} \le C\,\|F\|_{L^\infty_{\nu/n-2,\nu-2}(D^2)}$$

Finally, we put

$$\overline{W}_1 := \overline{W}_1^{mean} + W_1^{rad}, \quad \text{and} \quad W_1 := \overline{W}_1 \circ \lambda^{-1}.$$

The function $\chi\,W_1$ is defined in a neighbourhood of $z = 1$ and can be extended by zero to the entire punctured unit disc D^2_*. We have

$$\begin{cases} \Delta(\chi\,W_1) = F_1 + 2\nabla\chi\,\nabla W_1 + W_1\,\Delta\chi \quad \text{in} \quad D^2_* \\ \partial_r(\chi\,W_1) = 0 \quad \text{on} \quad S^1 \setminus \{1\} \end{cases}$$

The function $\nabla\chi\,\nabla W_1 + W_1\,\Delta\chi$ belongs to $L^\infty_{\nu/n-2,\nu-2}(D^2)$ and has compact support, since is identically zero in the neighbourhood of $z = 0$ and $z = 1$. According to the proposition (7.1) we can find a function W_0 which satisfies

$$\begin{cases} \Delta W_0 = F_0 - 2\nabla\chi\,\nabla W_1 - W_1\,\Delta\chi \quad \text{in} \quad D^2_* \\ \partial_r W_0 - \frac{1}{n}W_0 = 0 \quad \text{on} \quad S^1 \setminus \{1\} \end{cases}$$

By the elliptic regularity in weighted spaces we have

$$\|\,|z-1|^{-\nu}\,W_1\,\|_{L^\infty(D^2)} \le C\|\,|z-1|^{-\nu+2}\,F_1\|_{L^\infty(D^2)},$$

$$\|\,|z-1|^{-\nu+1}\,\nabla W_1\,\|_{L^\infty(D^2)} \le C\|\,|z-1|^{-\nu+2}\,F_1\|_{L^\infty(D^2)}$$

Then,

$$\|\,|z|^{-\nu/n+2}\,(F_0 - 2\nabla\chi\,\nabla W_1 - W_1\,\Delta\chi)\,\|_{L^\infty(D^2)} \le C\|F\|_{L^\infty_{\nu/n-2,\nu-2}(D^2)}.$$

So, we can write $W_0 = \Psi_0 + c_0^*\,n$, where

$$\|\,|z|^{-\nu/n}\,\Psi_0\,\|_{L^\infty(D^2)} + |c_0^*| \le C\|F\|_{L^\infty_{\nu/n-2,\nu-2}(D^2)}.$$

The function

$$W_{almost} := W_0 + \chi W_1,$$

satisfies the problem

$$\begin{cases} \Delta W_{almost} = F \quad \text{in} \quad D_*^2 \\ \partial_r W_{almost} - \frac{1}{n} W_{almost} = -\frac{1}{n} \chi W_1 \quad \text{on} \quad S^1 \setminus \{1\} \end{cases}$$

Take the function

$$h(z) := \frac{|z|^2 - 1}{2n} \chi W_1(z).$$

then,

$$\partial_r h - \frac{1}{n} h = \frac{1}{n} \chi W_1 \quad \text{at} \quad r = 1 \quad \text{and} \quad \Delta h = \frac{|z|^2 - 1}{2n} \chi F_1 + \frac{2r}{n} \partial_r (\chi W_1) + \frac{2}{n} \chi W_1.$$

Consider the operator

$$G_h : L^\infty_{\nu/n-2,\nu-2} \to L^\infty_{\nu/n,\nu} \oplus \mathfrak{D}_n \oplus \mathfrak{D}_\chi, \quad G_h(f) = W_h := W_{almost} + h.$$

We have

$$\Delta \circ G_h = Id + R_h, \quad R_h : L^\infty_{\nu/n-2,\nu-2} \longrightarrow L^\infty_{\nu/n-2,\nu-2}$$

$$R_h(f) = \Delta h, \quad \|R_h\| \leq \frac{1}{n}$$

Finally, we define a continuous linear operator $G = G_h \circ (Id + R_h)^{-1}$ and the function $W = G(f)$, the unique solution to (7.18) which can be written in the form

$$W = \Psi + n c_0^* + c_1^* \chi \in L^\infty_{\nu/n,\nu}(D^2) \oplus \mathfrak{D}_n \oplus \mathfrak{D}_\chi.$$

$\square$

Now we can go back to the initial problem (7.17). Take $f(z) = n^2 |z|^{2n-2} F(z^n)$ and put $w(z) = W(z^n)$. Then, $w \in L^\infty_\nu(D^2) \oplus \mathfrak{D}_n \oplus \mathfrak{D}_{\chi_n}$ and can be written in the form

$$w = \psi + n c_0^* + c_1^* \chi_n(z), \quad \chi_n(z) = \chi(z^n), \quad \|\gamma^{-\nu} \psi\|_{L^\infty(D^2)} \leq C \|\gamma^{-\nu+2} f\|_{L^\infty(D^2)}.$$

Finally, if we take $f \in \mathcal{C}^{0,\alpha}_{\nu-2}(D_*^2)$, then by classical arguments of the elliptic theory in Hölder weighted spaces $\psi \in \mathcal{C}^{0,\alpha}_\nu(D_*^2)$ and there exists a constant C such that

$$\|\psi\|_{C^{2,\alpha}_\nu(D_*^2)} \leq C \|f\|_{C^{0,\alpha}_{\nu-2}(D_*^2)}.$$

8. LINEAR ANALYSIS AROUND THE CATENOIDAL BRIDGES

To analyse the linearised mean curvature operator in the neighbourhood of the catenoidal bridges we consider the following problem

(8.22)
$$\begin{cases} \bar{L}_{cat} w = f \quad \text{in} \quad \mathbb{R} \times \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \\ \partial_\theta w = 0 \quad \text{on} \quad \mathbb{R} \times \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\} \end{cases}$$

where $\bar{L}_{cat} = \partial_\sigma^2 + \partial_\theta^2 + \frac{2}{\cosh^2 \sigma}$, $(\sigma, \theta) \in \mathbb{R} \times \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

Lemma 8.1. *Assume that $\delta \in (-1,0) \cup (0,1)$. The subspace of $(\cosh \sigma)^\delta \mathcal{C}^{2,\alpha} \left(\mathbb{R} \times \left[\frac{\pi}{2}, \frac{3\pi}{2} \right] \right)$ that is invariant by $(\sigma, \theta) \mapsto (\sigma, -\theta)$ and $(\sigma, \theta) \mapsto (-\sigma, \theta)$ and solves*

$$\begin{cases} \bar{L}_{cat}\, w = 0 & in \quad \mathbb{R} \times \left[\frac{\pi}{2}, \frac{3\pi}{2} \right] \\ \partial_\theta w = 0 & on \quad \mathbb{R} \times \{ \frac{\pi}{2}, \frac{3\pi}{2} \} \end{cases}$$

is trivial when $\delta \in (-1,0)$ and is one dimensional and spanned by $\sigma \tanh \sigma - 1$ when $\delta \in (0,1)$.

Proof. We decompose w in Fourier series

$$w(\sigma, \theta) = \sum_{j \in \mathbb{Z}} w_j(\sigma) e^{ij\theta}.$$

then the functions w_j are solutions of the ordinary equation

$$\left(\partial_\sigma^2 - j^2 + \frac{2}{\cosh^2 \sigma} \right) w_j = 0.$$

These solutions are asymptotic either to $(\cosh \sigma)^j$ or to $(\cosh \sigma)^{-j}$. By hypothesis, the solution is bounded by a constant times $(\cosh \sigma)^\delta$ and $|\delta| < 1$, so the solution has to be asymptotic to $(\cosh \sigma)^{-j}$, and then the solution is bounded. On the other hand, $-(j)^2 + \frac{2}{\cosh^2 \sigma} \leq 0$, so the maximum principle assures that $w_j = 0$, for all $j \geq 2$.

Observe that the imposed symmetry $(\sigma, \theta) \mapsto (\sigma, -\theta)$ and the boundary condition imply $w_1 = 0$. When $j = 0$, w_0 is the solution of the ordinary equation

$$\left(\partial_\sigma^2 + \frac{2}{\cosh^2 \sigma} \right) w_0 = 0.$$

By direct computations, we can see that $\tanh \sigma$ and $\sigma \tanh \sigma - 1$ are two independent solutions. The only solution symmetric with respect to the horizontal plane is $\sigma \tanh \sigma - 1$ and it belongs $(\cosh \sigma)^\delta \mathcal{C}^{2,\alpha}(\mathbb{R} \times \left[\frac{\pi}{2}, \frac{3\pi}{2} \right])$ only when $\delta \in (0,1)$. $\qquad \square$

The next step is to prove that, under some hypothesis, there exists a right inverse of the problem (9.25) and it is bounded.

Proposition 8.1. *Assume that $\delta \in (-1,0) \cup (0,1)$. Then given $f \in (\cosh \sigma)^\delta \mathcal{C}(\mathbb{R} \times \left[\frac{\pi}{2}, \frac{3\pi}{2} \right])$, such that $f(\sigma, \theta) = f(-\sigma, \theta) = f(\sigma, -\theta)$ there exists a unique constant d_1^* and a unique function $v \in (\cosh \sigma)^\delta \mathcal{C}^{2,\alpha}(\mathbb{R} \times \left[\frac{\pi}{2}, \frac{3\pi}{2} \right])$ such that the function $w = v + d_1^*$ solves*

(8.23)
$$\begin{cases} \left(\partial_\sigma^2 + \partial_\theta^2 + \frac{2}{\cosh^2 \sigma} \right) w = f, & in \quad \mathbb{R} \times \left[\frac{\pi}{2}, \frac{3\pi}{2} \right] \\ \partial_\theta w = 0, & on \quad \mathbb{R} \times \{ \frac{\pi}{2}, \frac{3\pi}{2} \} \end{cases}$$

and $w(\sigma, \theta) = w(-\sigma, \theta) = w(\sigma, -\theta)$. Moreover, we have

(8.24) $\| (\cosh \sigma)^{-\delta} w \|_{\mathcal{C}^{2,\alpha}\left(\mathbb{R} \times \left[\frac{\pi}{2}, \frac{3\pi}{2} \right] \right)} + |d_1^*| \leq C \, \| (\cosh \sigma)^{-\delta} f \|_{\mathcal{C}^{0,\alpha}\left(\mathbb{R} \times \left[\frac{\pi}{2}, \frac{3\pi}{2} \right] \right)}$

Proof. Let us extand the function f by symmetry to the entire unit cylinder $\mathbb{R} \times S^1$. Then, there exists a function w, which satisfies

$$\left(\partial_\sigma^2 + \partial_\theta^2 + \frac{2}{\cosh^2 \sigma} \right) w = f \quad \text{in} \quad \mathbb{R} \times S^1,$$

and $\quad w = v + d_1^*, \quad \|(\cosh \sigma)^{-\delta} v\|_{\mathcal{C}^{2,\alpha}(\mathbb{R} \times S^1)} + |d_1^*| \le C \, \|(\cosh \sigma)^{-\delta} f\|_{\mathcal{C}^{0,\alpha}(\mathbb{R} \times S^1)}$

This fact follows from the construction given by R. Mazzeo, F. Pacard and D. Pollack in [6]. Here, we give a short sketch of their proof for the sake of completeness. Let first f be a function whose Fourier series in θ is given by

$$f(\sigma, \theta) = \sum_{|j| > 2} f_j(\sigma) \, e^{ij\theta}.$$

Then, for every $t \in \mathbb{R}$, there exists a function $v_t = \sum_{|j| > 2} v_j^t(\sigma) \, e^{ij\theta}$, a unique solution of the problem

$$\left(\frac{d^2}{ds^2} - j^2 + \frac{2}{\cosh^2 \sigma} \right) v_j^t = f_j \quad \text{in} \quad |\sigma| < t, \quad v_j^t(\pm t) = 0, \quad j \ge 2.$$

One can prove this using the maximum principal and the method of sub- and supersolutions, taking $\dfrac{1}{j^2 - 2 - \delta} (\cosh \sigma)^\delta$ as a barrier function. Taking a sum over $|j| > 2$, we get a function v_t such that $\bar{L}_{cat} v_t = f$ and $v_t(\pm t) = 0$. By the Schauder's elliptic theory, there exists a constant C such that

$$\|(\cosh \sigma)^{-\delta} v\|_{\mathcal{C}^{2,\alpha}((-t,t) \times S^1)} \le C \|(\cosh \sigma)^{-\delta} f\|_{\mathcal{C}^{0,\alpha}((-t,t) \times S^1)}.$$

Moreover the constant C does not depend on t, which can be proven by contradiction, using the same argument in the proposition (7.1). Finally, the sequence v_t admits a subsequence which converges to a function v on compact subsets of $\mathbb{R} \times \left[\frac{\pi}{2}, \frac{3\pi}{2} \right]$ as t tends to infinity and such that (8.24) is true.

In the case when $f = f_0(\sigma) + f_{\pm 1}(\sigma) \, e^{\pm i\theta}$, we can construct a solution explicitly, taking

$$w_{\pm 1}(\sigma) = \cosh^{-1} \sigma \int_0^\sigma \cosh^2 t \int_0^t \cosh^{-1} \xi \, f_{\pm 1}(\xi) \, d\xi \, dt.$$

and

$$w_0(\sigma) = \tanh \sigma \int_0^\sigma \tanh^{-2} t \int_0^t \tanh \xi \, f_0(\xi) \, d\xi \, dt$$

Remark, that for $|f_j(\sigma)| \le (\cosh \sigma)^\delta$ for $j = 0, \pm 1$ there exist constants d and d_1^*, such that

$$w_0 + d \, (1 - s \tanh s) = v_0 + d_1^*, \quad |v_0| \le c \, (\cosh \sigma)^\delta$$

moreover

$$|w_{\pm 1}| \le c \, (\cosh \sigma)^\delta$$

The estimates for derivatives of w_0 and $w_{\pm 1}$ are obtained by Schauder's theory. For all $\delta \in (-1, 1)$ we have

$$\|(\cosh \sigma)^{-\delta} (v_0 + w_{\pm 1})\|_{\mathcal{C}^{2,\alpha}(\mathbb{R} \times S^1)} + |d_1^*| \le \|(\cosh \sigma)^{-\delta} f\|_{\mathcal{C}^{0,\alpha}(\mathbb{R} \times S^1)}$$

Finally, by symmetry the restriction of w to $\mathbb{R} \times \left[\frac{\pi}{2}, \frac{3\pi}{2} \right]$ satisfies $\partial_\theta w|_{\{\frac{\pi}{2}, \frac{3\pi}{2}\}} = 0$. $\qquad \square$

9. Linear analysis around the catenoidal neck

In this section in order study the linearised mean curvature operator around the catenoidal neck we consider the equation

$$(9.25) \qquad L_{cat} w = f \quad \text{in} \quad \mathbb{R} \times S^1,$$

where $L_{cat} = \partial_s^2 + \partial_\phi^2 + \frac{2}{\cosh^2 s}$.

We restrict our attention to functions which are even in the variables ϕ and s and invariant under rotations by the angle $\frac{2\pi}{n}$. Given $f \in (\cosh s)^\delta \mathcal{C}^{0,\alpha}(\mathbb{R} \times S^1)$, such that

$$f(s, \phi) = f(-s, \phi) = f(s, -\phi) = f(s, \phi + \pi/n).$$

we define $F(s, \phi) = \frac{1}{n^2} f(\frac{s}{n}, \frac{\phi}{n})$ and consider the problem

$$(9.26) \qquad L_{cat}^n W = \left(\partial_s^2 - j^2 + \frac{2}{n^2 \cosh^2 \frac{s}{n}} \right) W = F.$$

We prove the following two lemmas:

Lemma 9.1. *Assume that $\delta \in (-1, 0) \cup (0, 1)$. The subspace of $(\cosh \frac{s}{n})^\delta L^\infty (\mathbb{R} \times S^1)$ which is invariant by $(s, \phi) \mapsto (s, -\phi)$ and $(s, \phi) \mapsto (-s, \phi)$ and solves*

$$L_{cat}^n W = 0 \quad in \quad \mathbb{R} \times S^1,$$

is trivial for $\delta \in (-1, 0)$ and is one dimensional and spanned by $\frac{s}{n} \tanh \frac{s}{n} - 1$ for $\delta \in (0, 1)$.

Proof. The proof of this lemma is analogous to the proof of the lemma (8.1) and uses the maximum principal for the Fourier modes $j \geq 1$ and the symmetry with respect to the horizontal plain for $j = 0$. $\qquad \square$

Proposition 9.1. *Assume that $\delta \in (-1, 0) \cup (0, 1)$. Then, given a function $F \in (\cosh \frac{s}{n})^\delta L^\infty (\mathbb{R} \times S^1)$, such that $F(s, \phi) = F(-s, \phi) = F(s, -\phi)$, there exist a unique constant d_0^* and a unique function $V \in (\cosh \frac{s}{n})^\delta L^\infty (\mathbb{R} \times S^1)$ such that the function $W = V + d_0^*$ solves*

$$(9.27) \qquad \left(\partial_s^2 + \partial_\phi^2 + \frac{2}{n^2 \cosh^2 \frac{s}{n}} \right) W = F,$$

and there exists a constant C, which does not depend on n, such that

$$(9.28) \qquad \| \left(\cosh \frac{s}{n} \right)^{-\delta} W \|_{L^\infty(\mathbb{R} \times S^1)} + |d_0^*| \leq C \| \left(\cosh \frac{s}{n} \right)^{-\delta} F \|_{L^\infty(\mathbb{R} \times S^1)}.$$

Proof. We decompose both F and W in Fourier series

$$F = \sum_{j \in \mathbb{Z}} F_j(s) e^{ij\phi}, \quad \text{and} \quad W = \sum_{j \in \mathbb{Z}} W_j(s) e^{ij\phi}.$$

First, let $F(s, \phi) = \sum_{|j| > 1} F_j(s) e^{i\phi j}$. Then, for every $t \in \mathbb{R}$, using the method introduced in [6], we can solve

$$\left(\partial_s^2 - j^2 + \frac{2}{n^2 \cosh^2 \frac{s}{n}} \right) V_j^t = F_j, \quad V_j^t(\pm t) = 0,$$

by the maximum principal taking $\dfrac{1}{j^2 n^2 - 2 - \delta} \left(\cosh \frac{s}{n}\right)^\delta$ as a barrier function. When t tends to infinity, we get a sequence of functions which admits a subsequence converging on compact sets of $\mathbb{R} \times S^1$ which satisfies (9.27) and (9.28). When $F(s, \phi) = F_0(s)$ we find explicitly

$$W_0(s) = \tanh \frac{s}{n} \int_0^{\frac{s}{n}} \tanh^{-2} t \int_0^t \tanh \xi \, F_0(n\xi) \, d\xi \, dt.$$

Like in the proposition 9.1 there exist a function $V_0 \in \left(\cosh \frac{s}{n}\right)^\delta L^\infty(\mathbb{R} \times S^1)$ and a constant d_0^*, such that the function $W_0 = V_0 + d_0^*$ satisfies (9.27) and (9.28).

$\square$

Remark now that the function $v(s, \phi) = V(ns, n\phi)$ is invariant under rotations by the angle $\frac{2\pi}{n}$ and satisfies

$$L_{cat} \, v = f, \quad \text{and} \quad \|(\cosh s)^{-\delta} v\|_{L^\infty(\mathbb{R} \times S^1)} \leq C \, \|(\cosh s)^{-\delta} f\|_{L^\infty(\mathbb{R} \times S^1)}.$$

Finally, by the Schauder's theory, if $f \in (\cosh s)^\delta \mathcal{C}^{0,\alpha}(\mathbb{R} \times S^1)$, then $v \in (\cosh s)^\delta \mathcal{C}^{2,\alpha}(\mathbb{R} \times S^1)$ and

$$\|(\cosh s)^{-\delta} v\|_{\mathcal{C}^{2,\alpha}(\mathbb{R} \times S^1)} \leq C \, \|(\cosh s)^{-\delta} f\|_{\mathcal{C}^{0,\alpha}(\mathbb{R} \times S^1)}$$

10. The Fixed Point Theorem argument

We parametrize $\tilde{\mathcal{S}}_n$ by the following sub-domain of the unit disc:

$$\Omega_\varepsilon = \{z \in D^2 : \tilde{\varepsilon} \leq |z| \leq 1\} \setminus \bigcup_{m=1}^n \lambda_m \{\zeta \in \mathbb{C}_- : |\zeta| \leq \frac{\varepsilon}{2}\}.$$

Take a real number $\nu \in (0, 1)$. We denote $\mathcal{E}_{\nu,n}^{k,\alpha}$ the Banach space which is a subspace of $\mathcal{C}_\nu^{k,\alpha}(\Omega_\varepsilon)$ invariant under the transformation $z \mapsto \bar{z}$ and the rotations by the angle $\frac{\pi}{n}$. Remark, that when we use the change of variables

$$z = \tilde{\varepsilon} \cosh s \, e^{i\phi} \quad \text{or} \quad z = \lambda_m(1/2 \, \varepsilon \cosh \sigma \, e^{i\theta}),$$

the functions

$$(s, \phi) \mapsto w(\tilde{\varepsilon} \cosh s \, e^{i\phi}) \quad \text{and} \quad (\sigma, \theta) \mapsto w(\lambda_m(1/2 \, \varepsilon \cosh \sigma \, e^{i\theta})),$$

belong to the functional spaces

$$(\tilde{\varepsilon} \cosh s)^\nu \mathcal{C}^{2,\alpha}((-s_*, s_*) \times S^1) \quad \text{and} \quad (\varepsilon \cosh \sigma)^\nu \mathcal{C}^{2,\alpha}((-\sigma_*, \sigma_*) \times [\pi/2, 3\pi/2]).$$

Putting together the results of the section 6, for every function $w \in \mathcal{E}_{n,\nu}^{2,\alpha}$ small enough we can construct a surface $\tilde{\mathcal{S}}_n(w)$ which is close to $\tilde{\mathcal{S}}_n$ and whose mean curvature can be expressed as

$$\mathcal{H}(w) = \mathcal{H}(0) + \mathcal{L} \, w + Q(w),$$

where $\mathcal{H}(0)$ is the mean curvature of $\tilde{\mathcal{S}}_n$, $\mathcal{L}$ is a linear differential operator, which has the form

$$\mathcal{L} = \begin{cases} L_{gr} + \frac{\varepsilon^{2-\beta}}{\gamma^4} \, L_z^\gamma & \text{in} \quad \Omega_{gr} \cup \Omega_{glu}^0 \bigcup_{m=1}^n \Omega_{glu}^m \\ L_{cat} + (1 + \frac{\varepsilon^{2-\beta}}{\gamma^2}) \, L_{s,\phi} & \text{in} \quad \Omega_{cat}^0 \\ \left(\bar{L}_{cat} + \frac{\varepsilon^{-\beta}}{\gamma} \, L_{\sigma,\theta}\right) (\ \circ \lambda_m) & \text{in} \quad \Omega_{cat}^m \end{cases}$$

and Q is the nonlinear part of $\mathcal{H}(w)$ which can be written in the form

$$Q(w) = \frac{\varepsilon^{1-\beta}}{\gamma^4}\, Q_z^{2,\gamma}(w) + \frac{\varepsilon^{-\beta}}{\gamma^4} Q_z^{3,\gamma}(w),$$

where the properties of L^γ, Q_2^γ and Q_3^γ are described in the section 6. First, we verify that

$$\|\gamma^2\, \mathcal{H}(0)\|_{\mathcal{C}_\nu^{0,\alpha}(\Omega_\varepsilon)} \le c\,\varepsilon^{5/3-\beta-\nu}, \quad \forall \beta \in (0,1).$$

It follows from the fact that away from 0 and the n-th roots of unity, where $\tilde{S}_n$ is parametrized as a graph of one of the functions $\pm\tilde{\mathcal{G}}_n$, the mean curvature satisfies

$$\mathcal{H}(0) = \left| P_3(\tilde{\mathcal{G}}_n) \right|,$$

and its norm is bounded by a constant times $\varepsilon^{3-\nu-\beta}$. On the other hand, in the gluing regions the mean curvature is bounded by a constant times $\varepsilon^{3-\beta}/\gamma^4$ and in the catenoidal regions by ε/γ.

Secondly, there exist constants $c \in \mathbb{R}$ and $p \in \mathbb{N}$, such that

$$\|\gamma^2\, Q(w)\|_{\mathcal{C}_\nu^{0,\alpha}(\Omega_\varepsilon)} \le c\,\varepsilon^{2/3-p(\alpha+\nu+\beta)}\,\|w\|_{\mathcal{C}_\nu^{2,\alpha}(\Omega_\varepsilon)}, \quad \text{for} \quad \|w\|_{\mathcal{C}_\nu^{2,\alpha}(\Omega_\varepsilon)} \le c\,\varepsilon^{5/3-\nu-\beta}.$$

The surface $\tilde{\mathcal{S}}_n(w)$ is minimal if and only if

$$\mathcal{L}\,w = -\mathcal{H}(0) - Q(w).$$

If $\mathcal{L}$ is an invertible linear continuous operator then function w should satisfy

(10.29) $$w = -\mathcal{L}^{-1}\left(\mathcal{H}(0) + Q(w)\right) = \mathcal{A}(w)$$

If we show that there exists an open ball $\mathcal{B} \subset \mathcal{E}_{n,\nu}^{k,\alpha}$ such that $\mathcal{A} : \mathcal{B} \longrightarrow \mathcal{B}$ is a contraction mapping, then by the Banach Fixed Point theorem, there will exist a unique function w_n a solution to (10.29) such that $\tilde{\Sigma}_n = \tilde{\mathcal{S}}_n(w_n)$ and $\Sigma_n = \mathcal{S}_n(w_n)$ are free boundary minimal surfaces in B^3.

10.1. **Inverse Linear Operator.** We would like to find a linear operator

$$\mathcal{M} : \mathcal{E}_{n,\nu}^{0,\alpha} \longrightarrow \mathcal{E}_{n,\nu}^{2,\alpha}, \quad \text{such that} \quad \gamma^2\,\mathcal{L} \circ \mathcal{M}(f) = f.$$

Take a partition of unity on the unit disk D^2 :

$$\varphi_i \in \mathcal{C}^\infty(D^2), \quad \text{such that} \quad \sum_{i=0}^n \varphi_i = 1, \quad \text{and} \quad \varphi_i = \delta_{ij} \quad \text{in} \quad U_i \supset z_i,$$

where $z_0 = 0$ and z_m, $m = 1,\ldots,n$ are the n-th roots of unity and U_i are small neighborhoods of z_i. Given function $f \in \mathcal{E}_{n,\nu}^{0,\alpha}$ we can decompose it as

$$f = \sum_{i=0}^n \varphi_i\, f = \sum_{i=0}^n f_i, \quad \text{supp}(f_i) \subset U_i.$$

Let us fix the coordinates (s, ϕ) and (σ, θ) and take $s_{\tilde{\varepsilon}} \in \mathbb{R}_+$ such that

$$\tilde{\varepsilon}\cosh s_{\tilde{\varepsilon}} = 1.$$

We can parametrize two copies of the unit punctured disk D^2 by

$$z_+ = r_+\, e^{i\phi}, \quad \text{where} \quad r_+ = e^{s-s_{\tilde{\varepsilon}}}, \quad s \in (-\infty, s_{\tilde{\varepsilon}}) \quad \text{and}$$
$$z_- = r_-\, e^{i\phi}, \quad \text{where} \quad r_- = e^{-s-s_{\tilde{\varepsilon}}}, \quad s \in (-s_{\tilde{\varepsilon}}, +\infty)$$

Remark that we can also parametrize two copies of the punctured half-plane $\mathbb{C}_-$ by

$$\zeta_+ = \rho_+ \, e^{i\theta}, \quad \text{where} \quad \rho_+ = e^{\sigma}, \quad \sigma \in (-\infty, \infty) \quad \text{and}$$

$$\zeta_- = \rho_- \, e^{i\theta}, \quad \text{where} \quad \rho_- = e^{-\sigma}, \quad \sigma \in (-\infty, +\infty)$$

and take the conformal mappings $\lambda_m^{\pm} : \mathbb{C}_- \longrightarrow D^2$ given by

$$\lambda_m^{\pm}(\zeta_\pm) = e^{\frac{2\pi i m}{n}} \frac{1 + \zeta_\pm}{1 - \zeta_\pm}.$$

We define the cut-off functions $\vartheta_0 \in \mathcal{C}^\infty ([-s_{\tilde{\varepsilon}}, s_{\tilde{\varepsilon}}])$ and $\bar{\vartheta} \in \mathcal{C}^\infty(\mathbb{R})$ such that

$$\vartheta \equiv 1, \quad \text{for} \quad s > 1, \quad \vartheta \equiv 0, \quad \text{for} \quad s < -1$$

$$\bar{\vartheta} \equiv 1, \quad \text{for} \quad \sigma > 1, \quad \bar{\vartheta} \equiv 0, \quad \text{for} \quad \sigma < -1$$

Take $f_0(s, \phi) = f_0(\tilde{\varepsilon} \cosh s \, e^{i\phi})$ and $\bar{f}_*(\sigma, \theta) = f_m \circ \lambda_m(\varepsilon \cosh \sigma \, e^{i\theta})$. We can decompose

$$f_0(s, \phi) = f_0^+(s, \phi) + f_0^-(s, \phi) = \vartheta_0(s) \, f_0(s, \phi) + (1 - \vartheta_0(s)) \, f_0(s, \phi), \quad \text{and}$$

$$\bar{f}_*(\sigma, \theta) = \bar{f}_*^+(\sigma, \theta) + \bar{f}_*^-(\sigma, \theta) = \bar{\vartheta}(\sigma) \, \bar{f}_*(\sigma, \theta) + (1 - \bar{\vartheta}(\sigma))\bar{f}_*(\sigma, \theta)$$

We can extend the function f_0^+ by zero to the interval $(-\infty, s_{\tilde{\varepsilon}})$. It defines a function $\breve{f}_0^+$ on the unit punctured disc, parametrized by the variable z_+. In the same manner, we can extend the function f_0^- by zero to to interval $(s_{\tilde{\varepsilon}}, +\infty)$ and that defines a function $\breve{f}_0^-$ on the unit punctured disk parametrized by z_-. We have

$$\breve{f}_0^{\pm}(r_\pm \, e^{i\phi}) = f_0^{\pm}(\pm \log r_\pm + s_{\tilde{\varepsilon}}, \phi).$$

We also define the functions $\mathring{f}_*^{\pm}$ on the half-plane $\mathbb{C}_-$ by $\mathring{f}_*^{\pm}(\rho_\pm \, e^{i\theta}) = \bar{f}_*^{\pm}(\pm \log \rho_\pm, \theta)$. Finally, we put

$$\breve{f}_m^{\pm}(z_\pm) = \mathring{f}_*^{\pm} \circ \left(\lambda_m^{\pm}\right)^{-1} \quad \text{and} \quad \breve{f}^{\pm} = \sum_{i=0}^{n} \breve{f}_i^{\pm},$$

each of the functions $\breve{f}^{\pm}$ on one of the two copies of $\overline{D}^2 \setminus \{0, z_1, \ldots, z_n\}$.

First approximate solution: Using the results of the section 7, we find functions $\breve{w}_{gr}^{\pm} \in \mathcal{E}_{2,\alpha}^{n,\nu} \oplus \mathfrak{D}_n \oplus \mathfrak{D}_{\chi_n}$, solutions to the problems

$$\begin{cases} \gamma^2(z_\pm) \, \Delta \left(B(z_\pm) \, \breve{w}_{gr}^{\pm}\right) = \breve{f}_\pm \quad \text{in} \quad D_*^2 \\ \partial_{r_\pm} \breve{w}_{gr}^{\pm} = 0 \quad \text{in} \quad S^1 \setminus \{z_1, \ldots, z_n\} \end{cases}$$

We have

$$\breve{w}_{gr}^{\pm}(z_\pm) = \breve{\psi}^{\pm}(z_\pm) + n \, c_0^* + c_1^* \, \chi_n(z_\pm), \quad \text{and}$$

$$\|\breve{w}_{gr}^{\pm}\|_{\mathcal{C}_\nu^{2,\alpha}(D_*^2) \oplus \mathfrak{D}_n \oplus \mathfrak{D}_{\chi_n}} \leq C \, \|f\|_{\mathcal{C}_\nu^{2,\alpha}(D_*^2)}$$

We put

$$\psi^+(s, \phi) = \breve{\psi}^+(e^{s - s_{\tilde{\varepsilon}}} \, e^{i\phi}) \in (\tilde{\varepsilon} \cosh s)^{\nu} \mathcal{C}^{2,\alpha}((-\infty, s_{\tilde{\varepsilon}}) \times S^1),$$

$$\psi^-(s, \phi) = \breve{\psi}^-(e^{-s - s_{\tilde{\varepsilon}}} \, e^{i\phi}) \in (\varepsilon \cosh s)^{\nu} \mathcal{C}^{2,\alpha}((-s_{\tilde{\varepsilon}}, +\infty) \times S^1),$$

and $\bar{\psi}^{\pm} = \psi^{\pm} \circ \lambda_m^{\pm} \in (\varepsilon \cosh s)^{\nu} \mathcal{C}^{2,\alpha}(\mathbb{R} \times \left[\frac{\pi}{2}, \frac{3\pi}{2}\right])$.

First estimate of the error: We would like now to analyse the behaviour of the function
$$h := \gamma^2 \mathcal{L} \left(\psi^+ + \psi^-\right) - f$$
Remark, that the change of variables $z = \tilde{\varepsilon} \cosh s \, e^{i\phi}$ transforms
$$\Delta_z \rightsquigarrow \frac{1}{\tilde{\varepsilon}^2 \cosh^2 s} \left(\coth^2 s \, \partial_s^2 + \coth s \, \partial_s (1 - \coth^2 s) + \partial_\phi^2\right)$$
On the other hand, the change of variables $z_{\pm} = e^{\pm s - s_{\varepsilon}} \, e^{i\phi}$ transforms
$$\Delta_{\pm z} \rightsquigarrow e^{\mp 2(s - s_{\varepsilon})} \left(\partial_s^2 + \partial_\phi^2\right)$$
In this section we will denote as c any positive constant which does not depend on ε. Moreover, $|e^{\pm s - s_{\varepsilon}} - \varepsilon \cosh s| \leq \frac{c\varepsilon}{\cosh s}$. Using once the again the partition of unity, we decompose
$$h = \sum_{i=0}^{n} \phi_i \, h = \sum_{i=0}^{n} h_i, \quad \bar{h}_* = h_m \circ \lambda_m.$$

Regarding h_0 as a function in variables (s, ϕ) we can extend it by 0 to $\mathbb{R} \times S^1$. Similarly, regarding $\bar{h}_*$ as a function of (σ, θ) we can extend it by 0 to $\mathbb{R} \times \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$. Fix $\delta \in (-1, 0)$. Then, there exists a universal constant C, such that
$$\|(\cosh s)^{-\delta} h_0\|_{\mathcal{C}^{0,\alpha}(\mathbb{R} \times S^1)} \leq C \, \|f\|_{\mathcal{C}_\nu^{0,\alpha}(\Omega_\varepsilon)},$$

$$\|(\cosh \sigma)^{-\delta} \bar{h}_*\|_{\mathcal{C}^{0,\alpha}(\mathbb{R} \times \left[\frac{\pi}{2}, \frac{3\pi}{2}\right])} \leq C \, \|f\|_{\mathcal{C}_\nu^{0,\alpha}(\Omega_\varepsilon)}$$

Help of the linear analysis around the catenoids: Using the results of the sections 7 and 8, we can find functions w_{cat}^0 and $\bar{w}_{cat}^*$, such that
$$\frac{\gamma^2}{2 \, \tilde{\varepsilon}^2 \cosh^2 s} \, L_{cat} \, w_{cat}^0 = h_0 \quad \text{and} \quad \frac{\bar{\gamma}^2}{\varepsilon^2 \cosh^2 \sigma} \, \bar{L}_{cat} \, \bar{w}_{cat}^* = \bar{h}_*, \quad \partial_\theta \, \bar{w}_{cat}^* = 0,$$
where
$$L_{cat} = \partial_s^2 + \partial_\phi^2 + \frac{2}{\cosh^2 s} \quad \text{and} \quad \bar{L}_{cat} = \partial_\sigma^2 + \partial_\theta^2 + \frac{2}{\cosh^2 \sigma}$$
and $\bar{\gamma} = \gamma \circ \lambda_m$. We can write $w_{cat}^0 = v_{cat}^0 + d_0^*$, and $\bar{w}_{cat}^* = \bar{v}_{cat}^* + d_1^*$, where
$$\|(\cosh s)^{-\delta} v_{cat}^0\|_{\mathcal{C}^{2,\alpha}(\mathbb{R} \times S^1)} + |d_0^*| \leq C \, \|(\cosh s)^{-\delta} h_0^{\varepsilon}\|_{\mathcal{C}^{0,\alpha}(\mathbb{R} \times S^1)} \quad \text{and}$$

$$\|(\cosh \sigma)^{-\delta} \bar{v}_{cat}^*\|_{\mathcal{C}^{2,\alpha}(\mathbb{R} \times \left[\frac{\pi}{2}, \frac{3\pi}{2}\right])} + |d_1^*| \leq \|(\cosh \sigma)^{-\delta} \bar{h}_*\|_{\mathcal{C}^{0,\alpha}(\mathbb{R} \times \left[\frac{\pi}{2}, \frac{3\pi}{2}\right])}$$

We also denote $w_{cat}^m = \bar{w}_{cat}^* \circ \lambda_m$ and $v_{cat}^m = \bar{v}_{cat}^*$. We have
$$\partial_r w_{cat}^m|_{r=1} = \partial_r v_{cat}^m|_{r=1} = 0.$$

Cut-off functions: Let, as before, $\eta_{\tilde{\varepsilon}}^0 \in \mathcal{C}^{\infty}(D^2)$ denote a cut-off function, such that
$$\eta_{\tilde{\varepsilon}}^0(z) = \eta_{\tilde{\varepsilon}}^0(|z|), \quad \eta_{\tilde{\varepsilon}}^0(z) \equiv 1 \quad \text{for} \quad |z| < 1/2 \, \tilde{\varepsilon}^{1/2} \quad \text{and} \quad \eta_{\tilde{\varepsilon}}^0(z) \equiv 0 \quad \text{for} \quad |z| > 2 \, \tilde{\varepsilon}^{1/2}.$$
and $\bar{\eta}_\varepsilon \in \mathcal{C}^{\infty}(\mathbb{C}_-)$ the cut-off function in $\mathbb{C}_-$, such that
$$\bar{\eta}_\varepsilon(\zeta) = \bar{\eta}_\varepsilon(|\zeta|), \quad \bar{\eta}_\varepsilon(\zeta) \equiv 1 \quad \text{for} \quad |\zeta| < 1/2 \, \varepsilon^{2/3} \quad \text{and} \quad \bar{\eta}_\varepsilon(\zeta) \equiv 0 \quad \text{for} \quad |\zeta| > 2 \, \varepsilon^{2/3}.$$

We put $\eta_\varepsilon^m := \bar\eta_\varepsilon \circ \lambda_m^{-1}$.

Furthermore, we introduce the cut-functions $\varkappa_{\tilde\varepsilon}^0 \in \mathcal{C}^\infty(D^2)$ and $\bar\varkappa_\varepsilon \in \mathcal{C}^\infty(\mathbb{C}_-)$, such that

$$\varkappa_{\tilde\varepsilon}^0(z) = \varkappa_{\tilde\varepsilon}^0(|z|), \quad \varkappa_{\tilde\varepsilon}^0(z) \equiv 1 \quad \text{for} \quad |z| < 2\,\tilde\varepsilon^{1/2} \quad \text{and} \quad \varkappa_{\tilde\varepsilon}^0(z) \equiv 0 \quad \text{for} \quad |z| > 3\,\tilde\varepsilon^{1/2}$$

$$\bar\varkappa_\varepsilon(\zeta) = \bar\varkappa_\varepsilon(|\zeta|), \quad \bar\varkappa_\varepsilon(\zeta) \equiv 1 \quad \text{for} \quad |\zeta| < 2\,\varepsilon^{2/3} \quad \text{and} \quad \bar\varkappa_\varepsilon(\zeta) \equiv 0 \quad \text{for} \quad |\zeta| > 3\,\varepsilon^{2/3}$$

and $\varkappa_\varepsilon^m := \bar\varkappa_\varepsilon \circ \lambda_m^{-1}$.

Let $I_{\tilde\varepsilon}^+ = [s_{\tilde\varepsilon}^g, S_{\tilde\varepsilon}^g]$ and $I_{\tilde\varepsilon}^- = [-S_{\tilde\varepsilon}^g, -s_{\tilde\varepsilon}^g]$ be the two subintervals of $[-s_{\tilde\varepsilon}, s_{\tilde\varepsilon}]$ where we glue together the graph of the functions $\pm\tilde{\mathcal{G}}_n$ with the catenoidal neck.

Similarly, let $J_\varepsilon^+ = [\sigma_{\tilde\varepsilon}^g, \Sigma_{\tilde\varepsilon}^g]$ and $J_\varepsilon^- = [-\Sigma_{\tilde\varepsilon}^g, -\sigma_{\tilde\varepsilon}^g]$ be the two intervals where we glue together the graph of the functions $\pm\tilde{\mathcal{G}}_n$ with the half-catenoidal necks. We put

$$R_{\tilde\varepsilon} = e^{-s_{\tilde\varepsilon}^g - s_{\tilde\varepsilon}}, \quad r_{\tilde\varepsilon} = e^{-S_{\tilde\varepsilon}^g - s_{\tilde\varepsilon}} \quad \text{and} \quad \mathcal{P}_{\tilde\varepsilon} = e^{-\sigma_{\tilde\varepsilon}^g}, \quad \rho_{\tilde\varepsilon} = e^{-\Sigma_{\tilde\varepsilon}^g}.$$

We denote $\xi_{\tilde\varepsilon}^0 \in \mathcal{C}^\infty(D^2)$ a cut-off function such that

$$\xi_{\tilde\varepsilon}^0(z) = \xi_{\tilde\varepsilon}^0(|z|), \quad \xi_{\tilde\varepsilon}^0(z) \equiv 1 \quad \text{for} \quad |z| > R_{\tilde\varepsilon}, \quad \text{and} \quad \xi_{\tilde\varepsilon}^0(z) \equiv 0 \quad |z| < r_{\tilde\varepsilon}.$$

In the same manner we define the cut-off function $\bar\xi_\varepsilon \in \mathcal{C}^\infty(\mathbb{C})$, such that

$$\bar\xi_\varepsilon(\zeta) = \bar\xi_\varepsilon(|\zeta|), \quad \bar\xi_\varepsilon(\zeta) \equiv 1 \quad \text{for} \quad |\zeta| > \mathcal{P}_\varepsilon, \quad \text{and} \quad \bar\xi_\varepsilon(\zeta) \equiv 0 \quad |\zeta| < \rho_{\tilde\varepsilon},$$

and we put

$$\xi_\varepsilon^m := \bar\xi_\varepsilon \circ \lambda_m^{-1}.$$

Finally, we define the cut-off function $\xi_\varepsilon \in \mathcal{C}^\infty(D^2)$: $\xi_\varepsilon := \xi_{\tilde\varepsilon}^0 \prod\limits_{m=1}^n \xi_\varepsilon^m$, and also the functions

$$\xi_\varepsilon^\pm \in \mathcal{C}^\infty([-s_{\tilde\varepsilon}, s_{\tilde\varepsilon}] \times S^1): \quad \xi_\varepsilon^\pm(s, \phi) = \xi_\varepsilon(e^{\pm s - s_{\tilde\varepsilon}}\, e^{i\phi}).$$

Notice, that ξ_ε^+ is a function which is equal to 1 on the part of the surface, which is parametrized as a graph of the function $-\tilde{\mathcal{G}}_n$, the parts, where we glue this graph with upper parts of the catenoidal neck and the half-catenoidal bridges and also on the neck and the bridges them selves. It is identically equal to zero on the part of the surface, which is parametrized as a graph of the function $\tilde{\mathcal{G}}_n$. In the same manner, one can easily deduce the properties of ξ_ε^-.

Regular terms: Consider the function

$$w_{reg} := \xi_\varepsilon^+ \, \psi_{gr}^+ + \xi_\varepsilon^- \, \psi_{gr}^- + \eta_{\tilde\varepsilon}^0 \, v_{cat}^0 + \sum_{m=1}^n \eta_\varepsilon^m \, v_{cat}^m.$$

Deficiency terms: We define the functions

$$u_0(s) := 1 - s \tanh s, \quad \bar u(\sigma) := 1 - \sigma \tanh \sigma, \quad u_m := \bar u \circ \lambda_m^{-1},$$

and take Γ_n and $\tilde\Gamma_n$ the functions lying in the kernel of the operator L_{gr} and defined section 4. We also denote

$$\Gamma_n^\pm(s, \phi) = \Gamma_n(e^{\pm s - s_{\tilde\varepsilon}}\, e^{i\phi}), \quad \tilde\Gamma_n^\pm(s, \phi) = \tilde\Gamma_n(e^{\pm s - s_{\tilde\varepsilon}}\, e^{i\phi}).$$

Consider the function

$$\kappa_\varepsilon(s,\phi) := \varkappa_{\tilde\varepsilon}^0 \left(a_0\, u_0 + d_0^*\right) + \sum_{m=1}^n \varkappa_\varepsilon^m \left(a_1\, u_m + d_1^*\right)$$

$$+ \left(1 - \varkappa_{\tilde\varepsilon}^0 - \sum_{m=1}^n \varkappa_\varepsilon^m\right)\Big[c_0^*\, n + c_1^*\, \chi_n + b_0 \left(\xi_\varepsilon^+\, \tilde\Gamma_n^+ + \xi_\varepsilon^-\, \tilde\Gamma_n^-\right)$$

$$+ b_1 \left(\xi_\varepsilon^+\, \Gamma_n^+ + \xi_\varepsilon^-\, \Gamma_n^-\right)\Big]$$

where the constants a_0, a_1, b_0 and b_1 are chosen in such a way that the norm of κ and its derivatives would be small in $\Omega_\kappa^0 \underset{m=1}{\overset{n}{\cup}} \Omega_\kappa^m$, where

$$\Omega_\kappa^0 := \operatorname{supp}(\varkappa_{\tilde\varepsilon}^0) \cap \operatorname{supp}(1 - \varkappa_{\tilde\varepsilon}^0), \quad \text{and} \quad \Omega_\kappa^m := \operatorname{supp}(\varkappa_\varepsilon^m) \cap \operatorname{supp}(1 - \varkappa_\varepsilon^m).$$

More precisely, in Ω_κ^0 we have

$$\xi_\varepsilon^+\, \tilde\Gamma_n^+ + \xi_\varepsilon^-\, \tilde\Gamma_n^- = \begin{cases} -2n + 2n\, s_{\tilde\varepsilon} - 2n\, s + \mathcal{O}(\varepsilon), & \text{when} \quad s > 0 \\ -2n + 2n\, s_{\tilde\varepsilon} + 2n\, s + \mathcal{O}(\varepsilon), & \text{when} \quad s < 0 \end{cases}$$

On the other hand,

$$u_0(s) = \begin{cases} 1 - s + \mathcal{O}(\tilde\varepsilon), & \text{for} \quad s > 0 \\ 1 + s + \mathcal{O}(\tilde\varepsilon) & \text{for} \quad s < 0 \end{cases},$$

and

$$\xi_\varepsilon^+\, \Gamma_n^+ + \xi_\varepsilon^-\, \Gamma_n^- = -\frac{n}{2} + \mathcal{O}(\tilde\varepsilon), \quad \chi_n(s,\phi) \equiv 0, \quad \varkappa_\varepsilon^m \equiv 0$$

This gives us the first equation on a_0, a_1, b_0, b_1:

(10.30) $$a_0 = 2n\, b_0, \quad c_0^*\, n - \frac{b_1\, n}{2} - 2n\, b_0 + 2n\, b_0\, s_{\tilde\varepsilon} = 2n\, b_0 + d_0^*.$$

Similarly, in Ω_κ^m we have for all $\beta \in (0,1)$

$$\xi_\varepsilon^+\, \Gamma_n^+ + \xi_\varepsilon^-\, \Gamma_n^- = \begin{cases} -\dfrac{n}{2} + c(n) - \sigma + \mathcal{O}(\varepsilon^{2/3-\beta}), & \text{when} \quad \sigma > 0 \\ -\dfrac{n}{2} + c(n) + \sigma + \mathcal{O}(\varepsilon^{2/3-\beta}), & \text{when} \quad \sigma < 0 \end{cases}$$

$$\bar u(\sigma) = \begin{cases} 1 - \sigma + \mathcal{O}(\varepsilon^{2/3}), & \text{for} \quad \sigma > 0 \\ 1 + \sigma + \mathcal{O}(\varepsilon^{2/3}) & \text{for} \quad \sigma < 0 \end{cases},$$

and

$$\tilde\Gamma_n(s,\phi) = -n + \mathcal{O}(\varepsilon^{2/3}), \quad \chi_n(s,\phi) \equiv 1, \quad \varkappa_{\tilde\varepsilon}^0 \equiv 0.$$

This gives us the second equation

(10.31) $$a_1 = b_1, \quad c_0^*\, n + c_1^* + b_1(c(n) - \frac{n}{2}) - b_0\, n = b_1 + d_0^*.$$

Then, the system (10.30) and (10.31) has a unique solution.

Second approximate solution: As an approximate solution, we take the function

$$w_{app} := w_{reg} + \kappa.$$

We would like to estimate the norm in $\mathcal{E}_{n,\nu}^{2,\alpha}$ of the function

$$\gamma^2 \, \mathcal{L} \, w_{app} - f.$$

We do it separately in the regions Ω_{cat}^0, Ω_{glu}^0, Ω_κ^0, Ω_{gr}, Ω_κ^m, Ω_{glu}^m and Ω_{cat}^m.

(1) In Ω_{gr}^+, where the surface $\tilde{\mathcal{S}}_n$ is parametrized as a graph of the function $-\tilde{\mathcal{G}}_n$ we have

$$\eta_{\tilde{\varepsilon}}^0 \equiv \eta_\varepsilon^m \equiv 0, \quad \text{and} \quad \xi_\varepsilon^+ \equiv 1, \quad \xi_\varepsilon^- \equiv 0.$$

We see that

$$f = f^+, \quad w_{app} = \psi_{gr}^+ + \kappa, \quad \text{and} \quad \gamma^2 \, \mathcal{L} = \gamma^2 \, L_{gr} + \frac{\varepsilon^{2-\beta}}{\gamma^2} \, L^\gamma$$

$$\kappa = c_0^* \, n + c_1^* \, \chi_n + b_0 \, \tilde{\Gamma}_n^+ + b_1 \, \Gamma_n^+ \quad \text{in} \quad \Omega_\kappa^c \quad \text{and} \quad \kappa = \mathcal{O}(\tilde{\varepsilon}) \quad \text{in} \quad \Omega_\kappa \cap \Omega_{gr}^+.$$

Then,

$$\|\gamma^2 \, \mathcal{L} \, w_{app} - f\|_{\mathcal{C}_\nu^{0,\alpha}(\Omega_{gr}^+)} \leq c \, \varepsilon^{2/3-\beta-\nu} \|f\|_{\mathcal{C}_\nu^{0,\alpha}(\Omega_\varepsilon)}, \quad \forall \beta \in (0,1).$$

Naturally, the same estimate is true in Ω_{gr}^-, where the surface is parametrized as a graph of $\tilde{\mathcal{G}}_n$ and where

$$\eta_{\tilde{\varepsilon}}^0 \equiv \eta_\varepsilon^m \equiv 0, \quad \text{and} \quad \xi_\varepsilon^- \equiv 1, \quad \xi_\varepsilon^+ \equiv 0,$$

and $f = f^-$, $w_{app} = \psi_{gr}^- + \kappa$.

(2) In the region $\Omega_{glu}^{0,+}$, where we glue together the upper part of the catenoidal neck with the graph of the function $-\tilde{\mathcal{G}}_n$ we have

$$\eta_\varepsilon^m \equiv 0, \quad \xi_\varepsilon^+ \equiv 1 \quad \text{and}$$

$$f = f^+, \quad w_{app} = \psi_{gr}^+ + \xi_\varepsilon^- \, \psi_{gr}^- - \eta_{\tilde{\varepsilon}}^0 \, v_{cat}^0 + \kappa,$$

$$\kappa = a_0 \, u_0 + d_0^* \quad \text{and} \quad \gamma^2 \, \mathcal{L} = \gamma^2 \, L_{gr} + \varepsilon^{1-\beta} \, L^{\gamma_0}.$$

We obtain for ε small enough

$$\|\gamma^2 \, \mathcal{L} \, w_{app} - f\|_{\mathcal{C}_\nu^{0,\alpha}(\Omega_{glu}^{0,+})}$$

$$\leq c \left(\|\psi_{gr}^-\|_{\mathcal{C}_\nu^{2,\alpha}(\Omega_{glu}^{+,0})} + \|v_{cat}^0\|_{\mathcal{C}_\nu^{2,\alpha}(\Omega_{glu}^{+,0})} + \|\kappa\|_{\mathcal{C}_\nu^{2,\alpha}(\Omega_{glu}^{+,0})} \right) \leq c \, \varepsilon^{1/2|\delta|-\nu-\beta} \|f\|_{\mathcal{C}_\nu^{0,\alpha}(\Omega_\varepsilon)}$$

By symmetry the same estimate is true in the region $\Omega_{glu}^{0,-}$, where we glue the lower part of the catenoidal neck with the graph of the function $\tilde{\mathcal{G}}_n$ and where

$$\eta_\varepsilon^m \equiv 0, \quad \xi_\varepsilon^- \equiv 1 \quad \text{and}$$

$$f = f^-, \quad w_{app} = \psi_{gr}^- + \xi_\varepsilon^+ \, \psi_{gr}^+ - \eta_{\tilde{\varepsilon}}^0 \, v_{cat}^0 + \kappa, \quad \kappa = a_0 \, u_0 + d_0^*$$

(3) In the regions $\Omega_{glu}^{m,+}$ where we glue together the upper part of the half-catenoidal bridges with the graph of the function $-\tilde{\mathcal{G}}_n$ we have

$$\eta_{\tilde{\varepsilon}}^0 \equiv 0, \quad \xi_\varepsilon^+ \equiv 1 \quad \text{and}$$

$$f = f^+, \quad w_{app} = \psi^+_{gr} + \xi^-_\varepsilon \, \psi^-_{gr} - \eta^m_{\tilde\varepsilon} \, v^m_{cat} + \kappa$$

$$\kappa = a_1 \, u_m + d^*_1 \quad \text{and} \quad \gamma^2 \, \mathcal{L} = \gamma^2 \, L_{gr} + \varepsilon^{2/3-\beta} \, L^{\gamma_m}.$$

We obtain

$$\| \gamma^2 \, \mathcal{L} \, w_{app} - f \|_{\mathcal{C}^{0,\alpha}_\nu(\Omega^{m,+}_{glu})}$$

$$\leq c \left(\|\psi^-_{gr}\|_{\mathcal{C}^{2,\alpha}_\nu(\Omega^{+,m}_{glu})} + \|v^m_{cat}\|_{\mathcal{C}^{2,\alpha}_\nu(\Omega^{+,m}_{glu})} + \|\kappa\|_{\mathcal{C}^{2,\alpha}_\nu(\Omega^{+,m}_{glu})} \right) \leq c \, \varepsilon^{2/3|\delta|-\nu-\beta} \, \|f\|_{\mathcal{C}^{0,\alpha}_\nu(\Omega_\varepsilon)}$$

Once again, by symmetry the same estimate is true in $\Omega^{m,-}_{glu}$.

(4) In the catenoidal region Ω^0_{cat}, where the surface $\tilde{\mathcal{S}}_n$ coincides with the catenoidal neck, we have

$$\eta^m_\varepsilon \equiv 0, \quad \xi^+_\varepsilon \equiv \xi^-_\varepsilon \equiv 1.$$

$$f = f^+ + f^-, \quad w_{app} = \psi^-_{gr} + \psi^+_{gr} - w^0_{cat}, \quad \gamma^2 \, \mathcal{L} = \frac{\gamma^2}{2\, \tilde\varepsilon^2 \cosh^2 s} \, L_{cat} + \varepsilon^{1-\beta} \, L$$

We obtain

$$\| \gamma^2 \, \mathcal{L} \, w_{app} - f \|_{\mathcal{C}^{0,\alpha}_\nu(\Omega^0_{cat})} \leq c \, \varepsilon^{1-\nu-\beta} \, \|f\|_{\mathcal{C}^{0,\alpha}_\nu(\Omega_\varepsilon)}.$$

(5) Finally, in the region Ω^m_{cat}, where the surface $\tilde{\mathcal{S}}_n$ coincides with the m-th half-catenoidal neck, we have

$$\eta^0_\varepsilon \equiv 0, \quad \xi^+_\varepsilon \equiv \xi^-_\varepsilon \equiv 1 \quad \text{and}$$

$$f := f^+ + f^-, \quad w_{app} = \psi^-_{gr} + \psi^+_{gr} - w^m_{cat}, \quad \gamma^2 \, \mathcal{L}(\ \circ \lambda_m) = \frac{\gamma^2}{\varepsilon^2 \cosh^2 \sigma} \, \bar{L}_{cat} + \varepsilon^{2/3-\beta} \, L$$

We obtain

$$\| \gamma^2 \, \mathcal{L} \, w_{app} - f \|_{\mathcal{C}^{0,\alpha}_\nu(\Omega^m_{cat})} \leq c \, \varepsilon^{2/3-\nu-\beta} \, \|f\|_{\mathcal{C}^{0,\alpha}_\nu(\Omega_\varepsilon)}.$$

Now let us introduce the operator $\mathcal{M}_{app} : \mathcal{C}^{0,\alpha}_\nu(\Omega_\varepsilon) \longrightarrow C^{2,\alpha}_\nu(\Omega_\varepsilon)$, such that

$$\mathcal{M}_{app}(f) = w_{app} \quad \text{and} \quad \|\mathcal{M}_{app}\| \leq c \, \varepsilon^{-\nu-\beta}, \quad \forall \beta \in (0,1).$$

On the other hand, if we take δ sufficiently close to -1 and ν and β small enough, then the operator

$$\mathcal{R}_{app} := \gamma^2 \, \mathcal{L} \circ \mathcal{M}_{app} - \text{Id} : \mathcal{C}^{0,\alpha}_\nu(\Omega_\varepsilon) \longrightarrow \mathcal{C}^{0,\alpha}_\nu(\Omega_\varepsilon),$$

satisfies

$$\|R_{app}\| \ll 1.$$

So, $\text{Id} + \mathcal{R}_{app}$ is an invertible operator. We denote

$$\mathcal{M}_{exact} := \mathcal{M}_{app} \circ (\text{Id} + \mathcal{R}_{app})^{-1}, \quad \text{and} \quad \gamma^2 \, \mathcal{L} \circ \mathcal{M}_{exact} = \text{Id}.$$

10.2. **Conclusion.** Using the results of the previous subsection, we conclude that there exist constants $c \in \mathbb{R}_+$ and $p, q \in \mathbb{N}$, such that

$$\| \mathcal{M}_{exact} \left(\gamma^2 \, \mathcal{H}(0) \right) \|_{\mathcal{E}_{n,\nu}^{2,\alpha}} \leq \varepsilon^{5/3 - q(\nu + \beta)},$$

and

$$\| \mathcal{M}_{exact} \left(\gamma^2 \, Q(w) \right) \|_{\mathcal{E}_{n,\nu}^{2,\alpha}} \leq \varepsilon^{1/3 - p(\alpha + \nu + \beta)} \, \| w \|_{\mathcal{E}_{n,\nu}^{2,\alpha}}.$$

This yields that for α, ν, β and ε small enough

$$\| \mathcal{M}_{exact} \left(\gamma^2 \, Q(w_1) \right) - \mathcal{M}_{exact} \left(\gamma^2 \, Q(w_2) \right) \|_{\mathcal{E}_{n,\nu}^{2,\alpha}} \leq \frac{1}{2} \, \| w_1 - w_2 \|_{\mathcal{E}_{n,\nu}^{2,\alpha}},$$

and $\mathcal{M}_{exact} \left(\gamma^2 \, Q(\cdot) \right)$ is a contracting mapping in the ball

$$\mathcal{B}_\varepsilon := \left\{ w \in \mathcal{E}_{n,\nu}^{2,\alpha} \; : \; \| w \|_{\mathcal{E}_{n,\nu}^{2,\alpha}} \leq \varepsilon^{5/3 - q(\nu + \beta)} \right\}.$$

And the result of this article follows from the Banach fixed point theorem.

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