Green-Lazarsfeld's conjecture for generic curves of large gonality

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Abstract

We prove that the Green-Lazarsfeld gonality conjecture holds for generic curves of genus g, and gonality d, if $g/3 < d < [(g+3)/2]$.

Denoting by $K_{p,q}(X, L)$ the Koszul cohomology with value in a line bundle L (*see* [5]), Green and Lazarfeld proved the following (*cf.* Appendix to [5]):

Theorem 1. *Let X be a complex manifold, L_1, and L_2 be two vector bundles on X such that $r_1 := h^0(L_1) - 1 \geqslant 1$, and $r_2 := h^0(L_2) - 1 \geqslant 1$. Then $K_{r_1+r_2-1,1}(X, L_1 \otimes L_2) \neq 0$.*

Let now C be a complex smooth projective curve of gonality

$$d := \min\{\deg(L), h^0(L) \geqslant 2\}.$$

Green-Lazarsfeld's theorem, applied to $L_1 = \mathcal{O}_C(D)$, where $\deg(D) = d$, $h^0(C, \mathcal{O}_C(D)) = 2$, and $L_2 = L - D$ with $\deg(L)$ sufficiently large, implies

$$K_{h^0(L)-d-1,1}(C, L) \neq 0.$$

The gonality conjecture predicts that this is optimal, namely:

Conjecture 2 (Green and Lazarsfeld, [6]): *For C a curve of gonality d, and for any line bundle L of sufficiently large degree, we have $K_{h^0(L)-d,1}(C, L) = 0$.*

In spite of the evidence brought by the "$K_{p,1}$ Theorem" of [5], which solves the hyperelliptic case (among other things), after having formulated the gonality conjecture, Green and Lazasfeld shown their mistrust of the statement they had just made. Since then, the conjecture has been almost forgotten, and it took a while untill some new evidence was discovered (*see* [3], [1]). This delay is probably due to the fact that the conjecture did not count among the mathematical highlights of the last years, as almost all the attention in the syzygy theory was focused on the more famous Green conjecture.

The aim of this short note is to mix together the main results of [1], and of [7] in order to verify the Green-Lazarfeld conjecture for generic curves of large gonality. We prove the following:

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Theorem 3. *For any positive integers g and d such that $g/3+1 \leqslant d < [(g+3)/2]$, the gonality conjecture is valid for generic curves of genus g and gonality d.*

Note that for generic curves of genus g the gonality equals $[(g+3)/2]$, and thus our theorem covers all possible, not too small, gonalities, except for the generic gonality.

In the statement of Theorem 3, the word "generic" should be read in the usual sense. The complex curves of fixed genus g, and gonality d, are parametrised by an irreducible subvariety of the moduli space $\mathcal{M}_g$, and a "generic curve" is a curve corresponding to a generic point of this variety. Irreducibility follows from the well-known fact that the closure of this subvariety is actually the closure of the image in $\mathcal{M}_g$ of a Hurwitz scheme and then apply [4].

Finally, we mention that all the notation we use in the sequel is standard, and we refer to [5] for basic facts about Koszul cohomology.

Proof of Theorem 3. We divide the proof in several steps.

Step 1. First of all, we recall the following result from [1]:

Theorem 4. *If L is a nonspecial line bundle on a curve C, which satisfies $K_{n,1}(C,L)=0$, for a positive integer n, then, for any effective divisor E of degree e, we have $K_{n+e,1}(C,L+E)=0$.*

In particular, if $K_{h^0(L)-d,1}(C,L) = 0$ for a nonspecial line bundle L, then $K_{h^0(L')-d,1}(C,L')=0$ for any L' of sufficiently large degree. By means of the Zariski semi-continuity of graded Betti numbers (*see*, for example, [2]), it suffices to exhibit one curve C of genus g, and gonality at most d, and one nonspecial line bundle L on C satisfying $K_{h^0(L)-d,1}(C,L)=0$.

Step 2. Construction of C and L.

A suitable choice of such a curve C is provided by the proof of Corollary 1 of [7]. We start with a $K3$ surface S whose Picard group is cyclic, generated by a line bundle $\mathcal{L}$ of self-intersection $\mathcal{L}^2=4k-2$, where $k=g-d+1$. We denote $\nu=g-2d+2\geqslant 1$. Under these assumptions, as $\nu\leqslant k/2$, we know that there exists an irreducible curve $X\in|\mathcal{L}|$, having exactly ν simple nodes as singular points, and no other singularities, and such that its normalization C is a d-gonal curve, *see* [7].

We set $L=K_C(p+q)$, where p, and q are two distinct points of C that lie over a node x of X.

Step 3. Recall the main result of [7].

Theorem 5. *The $K3$ surface S being as above, we have $K_{k,1}(S,\mathcal{L})=0$.*

It follows directly from this result, from the adjunction formula, and from the hyperplane section theorem for Koszul cohomology (*see* [5]) that $K_{k,1}(X,K_X)=0$. Since $k=h^0(C,L)-d$, the proof of our theorem is concluded by the following lemma.

Lemma 6. *Let X be a nodal curve, $C \xrightarrow{f} X$ be the normalization of X, and $p, q \in C$ be two distinct points lying over the same node x of X. Then, for any $n \geqslant 1$, we have a natural injective map $K_{n,1}(C, K_C(p+q)) \subset K_{n,1}(X, K_X)$.*

Proof of Lemma 6. We factor the normalization morphism f through $C \xrightarrow{h} Y \xrightarrow{g} X$, where g is just the smoothing of the node x. We also denote by x_i, $i = 1, ..., \nu - 1$, the other nodes of X, and p_i, q_i the points of C lying over x_i, and we make use of the following natural short exact sequences:

$$0 \to \mathcal{O}_X \to g_*\mathcal{O}_Y \to \mathbb{C}(x) \to 0,$$
$$0 \to \mathcal{O}_Y \to h_*\mathcal{O}_C \to \bigoplus_{i=1}^{\nu-1} \mathbb{C}(x_i) \to 0,$$
$$0 \to \mathcal{O}_X \to f_*\mathcal{O}_C \to \mathbb{C}(x) \oplus \left(\bigoplus_{i=1}^{\nu-1} \mathbb{C}(x_i) \right) \to 0.$$

Firstly, we remark that the natural inclusion

$$H^0(C, K_C(p+q)) \subset H^0(C, f^*K_X) = H^0(C, K_C(p+q+\sum_{i=1}^{\nu-1}(p_i+q_i)))$$

yields an inclusion

$$H^0(C, K_C(p+q)) \subset H^0(Y, g^*K_X) \cong \text{Ker} \left(H^0(C, f^*K_X) \to \bigoplus_{i=1}^{\nu-1} \mathbb{C}(x_i) \right).$$

Furthermore, $H^0\left(C, K_C^{\otimes 2}(2p+2q)\right) \subset H^0\left(Y, g^*K_X^{\otimes 2}\right)$, and thus we obtain an injective map $K_{n,1}(C, K_C(p+q)) \subset K_{n,1}(Y, g^*K_X)$ for any $n \geqslant 1$ (apply [1], Remark 1.1).

Then, we observe that, as $H^1(X, K_X) \cong \mathbb{C}$, and $H^1(Y, g^*K_X) = 0$, the natural morphism $H^0(X, K_X) \to H^0(Y, g^*K_X)$ is an isomorphism. Since $H^0\left(X, K_X^{\otimes 2}\right) \subset H^0\left(Y, g^*K_X^{\otimes 2}\right)$, we get an isomorphism $K_{n,1}(X, K_X) \cong K_{n,1}(Y, g^*K_X)$ for any $n \geqslant 1$, and thus we are done.

Remark 7. In Lemma 6, the fact of working over the complex numbers is not essential. Similar arguments as those used here show that the statement is true for nodal curves over any algebraically closed field.

Remark 8. The use of the results of [7] in the proof of Theorem 3 shades a new light on the relationships between Green's conjecture, and the gonality conjecture, the two statements seemed to be (in a somewhat misterious way) intimately related to each other (*see* also [1]). In fact, Lemma 6 answers partially to the conjecture made in [1].

Remark 9. For the curve C considered in the proof of Theorem 3, the vanishing $K_{h^0(L')-d,1}(C, L') = 0$ predicted by the gonality conjecture holds for any line bundle L' of degree at least $3g$. The same is true for generic d-gonal curves of genus g, where g

and d are as above.

Remark 10. The case $g > (d-1)(d-2)$ has already been treated in [1]. Therefore, from the viewpoint of the Green-Lazarfeld conjecture for generic curves of fixed genus g, and gonality d, for $d \geqslant 6$, there is a gap remaining for $3d-3 < g \leqslant (d-1)(d-2)$ that has to be solved differently. The case of maximal gonality $d = [(g+3)/2]$, which seems to be the most difficult, is also left over.

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