Clay Mathematics Proceedings
Volume **18**, 2013

Remarks on curve classes on rationally connected varieties

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This note is dedicated to Joe Harris, whose influence on the subject of curves on rationally connected algebraic varieties (among other topics!) is invaluable.

Abstract. We study for rationally connected varieties X the group of degree 2 integral homology classes on X modulo those which are algebraic. We show that the Tate conjecture for divisor classes on surfaces defined over finite fields implies that this group is trivial for any rationally connected variety.

1. Introduction

Let X be a smooth complex projective variety. Define

$$Z^{2i}(X) = \frac{\mathrm{Hdg}^{2i}(X,\mathbb{Z})}{H^{2i}(X,\mathbb{Z})_{alg}}, \tag{1.1}$$

where $\mathrm{Hdg}^{2i}(X,\mathbb{Z})$ is the space of integral Hodge classes on X and $H^{2i}(X,\mathbb{Z})_{alg}$ is the subgroup of $H^{2i}(X,\mathbb{Z})$ generated by classes of codimension i closed algebraic subsets of X.

These groups measure the defect of the Hodge conjecture for integral Hodge classes, hence they are trivial for $i = 0$, 1 and $n = \dim X$, but in general they can be nonzero by [**1**]. Furthermore they are torsion if the Hodge conjecture for *rational* Hodge classes on X of degree $2i$ holds. In addition to the previously mentioned case, this happens when $i = n - 1$, $n = \dim X$, due to the Lefschetz theorem on $(1,1)$-classes and the hard Lefschetz isomorphism (cf. [**23**]). We will call classes in $\mathrm{Hdg}^{2n-2}(X,\mathbb{Z})$ "curve classes", as they are also degree 2 homology classes.

Note that the Kollár counterexamples (cf. [**14**]) to the integral Hodge conjecture already exist for curve classes (that is degree 4 cohomology classes in this case) on projective threefolds, unlike the Atiyah-Hirzebruch examples which work for degree 4 integral Hodge classes in higher dimension.

It is remarked in [**21**], [**23**] that the two groups

$$Z^4(X),\ Z^{2n-2}(X),\ n := \dim X$$

2010 *Mathematics Subject Classification.* Primary 14C25, 14F99, 14K99.

Key words and phrases. Integral Hodge class, integral Tate class, rationally connected varieties.

The author thanks the organizers of the beautiful conference "A celebration of algebraic geometry" for inviting me there. The author also thanks Jean-Louis Colliot-Thélène, Olivier Debarre and János Kollár for useful discussions.

are birational invariants. (For threefolds, this is the same group, but not in higher dimension.) The nontriviality of these birational invariants for rationally connected varieties is asked in [**23**]. Still more interesting is the nontriviality of these invariants for unirational varieties, having in mind the Lüroth problem (cf. [**3**], [**2**], [**4**]).

Concerning the group $Z^4(X)$, Colliot-Thélène and the author proved in [**8**], building on the work of Colliot-Thélène and Ojanguren [**5**], that it can be nonzero for unirational varieties starting from dimension 6. What happens in dimensions 5 and 4 is unknown (the four dimensional case being particularly challenging in our mind), but in dimension 3, there is the following result proved in [**22**]:

THEOREM 1.1. *(Voisin 2006) Let X be a smooth projective threefold which is either uniruled or Calabi-Yau (meaning that K_X is trivial and $H^1(X,\mathcal{O}_X)=0$). Then the group $Z^4(X)$ is equal to 0.*

This result, and in particular the Calabi-Yau case, implies that the group $Z^6(X)$ is also 0 for a Fano fourfold X which admits a smooth anticanonical divisor. Indeed, a smooth anticanonical divisor $j: Y \hookrightarrow X$ is a Calabi-Yau threefold, so that we have $Z^4(Y)=0$ by Theorem 1.1 above. As $H^2(Y,\mathcal{O}_Y)=0$, every class in $H^4(Y,\mathbb{Z})$ is a Hodge class, and it follows that $H^4(Y,\mathbb{Z}) = H^4(Y,\mathbb{Z})_{alg}$. As the Gysin map $j_*: H^4(Y,\mathbb{Z}) \to H^6(X,\mathbb{Z})$ is surjective by the Lefschetz theorem on hyperplane sections, it follows that $H^6(X,\mathbb{Z}) = H^6(X,\mathbb{Z})_{alg}$, and thus $Z^6(X)=0$.

In the paper [**11**], it was proved more generally that if X is any Fano fourfold, the group $Z^6(X)$ is trivial. Similarly, if X is a Fano fivefold of index 2, the group $Z^8(X)$ is trivial.

These results have been generalized to higher dimensional Fano manifolds of index $n-3$ and dimension ≥ 8 by Enrica Floris [**9**] who proves the following result:

THEOREM 1.2. *Let X be a Fano manifold over $\mathbb{C}$ of dimension $n \geq 8$ and index $n-3$. Then the group $Z^{2n-2}(X)$ is equal to 0: Equivalently, any integral cohomology class of degree $2n-2$ on X is algebraic.*

The purpose of this note is to provide evidence for the vanishing of the group $Z^{2n-2}(X)$, for any rationally connected variety over $\mathbb{C}$. Note that in this case, since $H^2(X,\mathcal{O}_X)=0$, the Hodge structure on $H^2(X,\mathbb{Q})$ is trivial, and so is the Hodge structure on $H^{2n-2}(X,\mathbb{Q})$, so that $Z^{2n-2}(X) = H^{2n-2}(X,\mathbb{Z})/H^{2n-2}(X,\mathbb{Z})_{alg}$. We will first prove the following two results.

PROPOSITION 1.3. *The group $Z^{2n-2}(X)$ is locally a deformation invariant for rationally connected manifolds X.*

Let us explain the meaning of the statement. Consider a smooth projective morphism $\pi: \mathcal{X} \to B$ between connected quasi-projective complex varieties, with n dimensional fibers. Recall from [**15**] that if one fiber $X_b := \pi^{-1}(b)$ is rationally connected, so is every fiber. Let us endow everything with the usual topology. Then the sheaf $R^{2n-2}\pi_*\mathbb{Z}$ is locally constant on B. On any Euclidean open set $U \subset B$ where this local system is trivial, the group $Z^{2n-2}(X_b)$, $b \in U$, is the finite quotient of the *constant* group $H^{2n-2}(X_b,\mathbb{Z})$ by its subgroup $H^{2n-2}(X_b,\mathbb{Z})_{alg}$. To say that $Z^{2n-2}(X_b)$ is locally constant means that on open sets U as above, the subgroup $H^{2n-2}(X_b,\mathbb{Z})_{alg}$ of the constant group $H^{2n-2}(X_b,\mathbb{Z})$ does not depend on b.

It follows from the above result that the vanishing of the group $Z^{2n-2}(X)$ for X a rationally connected manifold reduces to the similar statement for X defined over a number field.

Let us now define an l-adic analogue $Z^{2n-2}(X)_l$ of the group $Z^{2n-2}(X)$ (cf. [**6**], [**7**]). Let X be a smooth projective variety defined over a field K which in the sequel will be either a finite field or a number field. Let $\overline{K}$ be an algebraic closure of K. Any cycle $Z \in CH^s(X_{\overline{K}})$ is defined over a finite extension of K. Let l be a prime integer different from $p = \mathrm{char}\, K$ if K is finite. It follows that the cycle class

$$cl(Z) \in H^{2s}_{et}(X_{\overline{K}}, \mathbb{Q}_l(s))$$

is invariant under an open subgroup of $\mathrm{Gal}(\overline{K}/K)$.

Classes satisfying this property are called Tate classes. The Tate conjecture for finite fields asserts the following:

Conjecture 1.4. *(cf. [**18**] for a recent account) Let X be smooth and projective over a finite field K. The cycle class map gives for any s a surjection*

$$cl : CH^s(X_{\overline{K}}) \otimes \mathbb{Q}_l \to H^{2s}(X_{\overline{K}}, \mathbb{Q}_l(s))_{Tate}.$$

Note that the cycle class map defined on $CH^s(X_{\overline{K}})$ in fact takes values in $H^{2s}(X_{\overline{K}}, \mathbb{Z}_l(s))$, and more precisely in the subgroup $H^{2s}(X_{\overline{K}}, \mathbb{Z}_l(s))_{Tate}$ of classes invariant under an open subgroup of $\mathrm{Gal}(\overline{K}/K)$. We thus get for each i a morphism

$$cl^i : CH^i(X_{\overline{K}}) \otimes \mathbb{Z}_l \to H^{2i}(X_{\overline{K}}, \mathbb{Z}_l(i))_{Tate}.$$

We can thus introduce the following variant of the groups $Z^{2i}(X)$:

$$Z^{2i}_{et}(X)_l := H^{2i}_{et}(X_{\overline{K}}, \mathbb{Z}_l(i))_{Tate}/\mathrm{Im}\, cl^i.$$

An argument similar to the one used for the proof of Proposition 1.3 will lead to the following result:

Proposition 1.5. *Let X be a smooth rationally connected variety defined over a number field K, with ring of integers $\mathcal{O}_K$. Assume given a projective model $\mathcal{X}$ of X over* $\mathrm{Spec}\, \mathcal{O}_K$. *Fix a prime integer l. Then except for finitely many $p \in \mathrm{Spec}\, \mathcal{O}_K$, the group $Z^{2n-2}_{et}(X)_l$ is isomorphic to the group $Z^{2n-2}_{et}(X_p)_l$.*

In the course of the paper, we will also consider variants $Z^{2n-2}_{rat}(X)$, resp. $Z^{2n-2}_{et,rat}(X)_l$ of the groups $Z^{2n-2}(X)$, resp. $Z^{2n-2}_{et}(X)_l$, obtained by taking the quotient of the group of integral Hodge classes (resp. integral l-adic Tate classes) by the subgroup generated by classes of *rational* curves. This variant is suggested by Kollár's paper (cf. [**16**, Question 3, (1)]). By the same arguments, these groups are also deformation and specialization invariants for rationally connected varieties.

Our last result is conditional but it strongly suggests the vanishing of the group $Z^{2n-2}(X)$ for X a smooth rationally connected variety over $\mathbb{C}$. Indeed, we will prove using the main result of [**19**] and the two propositions above the following consequence of Theorem 1.5:

Theorem 1.6. *Assume Tate's conjecture* 1.4 *holds for degree* 2 *Tate classes on smooth projective surfaces defined over a finite field. Then the group $Z^{2n-2}(X)$ is trivial for any smooth rationally connected variety X over $\mathbb{C}$.*

2. Deformation and specialization invariance

Proof of Proposition 1.3. We first observe that, due to the fact that relative Hilbert schemes parameterizing curves in the fibers of B are a countable union of varieties which are projective over B, given a simply connected open set $U \subset B$ (in the classical topology of B), and a class $\alpha \in \Gamma(U, R^{2n-2}\pi_*\mathbb{Z})$ such that α_t is algebraic for $t \in V$, where V is a smaller nonempty open set $V \subset U$, then α_t is algebraic for any $t \in U$.

To prove the deformation invariance, we just need using the above observation to prove the following:

LEMMA 2.1. *Let $t \in U \subset B$, and let $C \subset X_t$ be a curve and let $[C] \in H^{2n-2}(X_t, \mathbb{Z}) \cong \Gamma(U, R^{2n-2}\pi_*\mathbb{Z})$ be its cohomology class. Then the class $[C]_s$ is algebraic for s in a neighborhood of t in U.*

PROOF. By results of [**15**], there are rational curves $R_i \subset X_t$ with ample normal bundle which meet C transversally at distinct points, and with arbitrary tangent directions at these points. We can choose an arbitrarily large number D of such curves with generically chosen tangent directions at the attachment points. We then know by [**10**, §2.1] that the curve $C' = C \cup_{i \leq D} R_i$ is smoothable in X_t to a smooth unobstructed curve $C'' \subset X_t$, that is $H^1(C'', N_{C''/X_t}) = 0$. This curve C'' then deforms with X_t (cf. [**12**], [**13**, II.1]) in the sense that the morphism from the deformation of the pair (C'', X_t) to B is smooth, and in particular open. So there is a neighborhood of V of t in U such that for $s \in V$, there is a curve $C''_s \subset X_s$ which is a deformation of $C'' \subset X_t$. The class $[C''_s] = [C'']_s$ is thus algebraic on X_s. On the other hand, we have

$$[C''] = [C'] = [C] + \sum_i [R_i].$$

As the R_i's are rational curves with positive normal bundle, they are also unobstructed, so that the classes $[R_i]_s$ also are algebraic on X_s for s in a neighborhood of t in U. Thus $[C]_s = [C'']_s - \sum_i [R_i]_s$ is algebraic on X_s for s in a neighborhood of t in U. The lemma, hence also the proposition, is proved.

□

REMARK 2.2. There is an interesting variant of the group $Z^{2n-2}(X)$, which is suggested by Kollár (cf. [**16**]) given by the following groups:

$$Z^{2n-2}_{rat}(X) := H^{2n-2}(X, \mathbb{Z})/\langle [C],\, C \text{ rational curve in } X \rangle.$$

Here, by a rational curve, we mean an irreducible curve whose normalization is rational. These groups are torsion for X rationally connected, as proved by Kollár ([**13**, Theorem 3.13 p 206]). It is quite easy to prove that they are birationally invariant.

The proof of Proposition 1.3 gives as well the following result (already noticed by Kollár [**16**]) :

VARIANT 2.3. *If $\mathcal{X} \to B$ is a smooth projective morphism with rationally connected fibers, the groups $Z^{2n-2}_{rat}(\mathcal{X}_t)$ are local deformation invariants.*

Let us give one application of Proposition 1.3 (or rather its proof) and/or its variant 2.3. Let X be a smooth projective variety of dimension $n + r$, with $n \geq 3$ and let $\mathcal{E}$ be an ample vector bundle of rank r on X. Let $C_1, \ldots, C_k$ be smooth curves in X whose cohomology classes generate the group $H^{2n+2r-2}(X, \mathbb{Z})$. For $\sigma \in H^0(X, \mathcal{E})$, we denote by X_σ the zero locus of σ. When $\mathcal{E}$ is generated by sections, X_σ is smooth of dimension n for general σ.

THEOREM 2.4. *1) Assume that the sheaves $\mathcal{E} \otimes \mathcal{I}_{C_i}$ are generated by global sections for $i = 1, \ldots, k$. Then if X_σ is smooth rationally connected for general σ, the group $Z^{2n-2}(X_\sigma)$ vanishes for any σ such that X_σ is smooth of dimension n.*

2) Under the same assumptions as in 1), assume the curves $C_i \subset X$ are rational. Then if X_σ is smooth rationally connected for general σ, the group $Z^{2n-2}_{rat}(X_\sigma)$ vanishes for any σ such that X_σ is smooth of dimension n.

PROOF. 1) Let $j_\sigma : X_\sigma \to X$ be the inclusion map. Since $n \geq 3$ and $\mathcal{E}$ is ample, by Sommese's theorem [**20**], the Gysin map $j_{\sigma *} : H^{2n-2}(X_\sigma, \mathbb{Z}) \to H^{2n+2r-2}(X, \mathbb{Z})$ is an isomorphism. It follows that the group $H^{2n-2}(X_\sigma, \mathbb{Z})$ is a constant group. In order to show that $Z^{2n-2}(X_\sigma)$ is trivial, it suffices to show that the classes $(j_{\sigma *})^{-1}([C_i])$ are algebraic on X_σ since they generate $H^{2n-2}(X_\sigma, \mathbb{Z})$. Since the X_σ's are rationally connected, Theorem 1.3 tells us that it suffices to show that for each i, there exists a $\sigma(i)$ such that $X_{\sigma(i)}$ is smooth n-dimensional and that the class $(j_{\sigma(i)*})^{-1}([C_i])$ is algebraic on $X_{\sigma(i)}$.

It clearly suffices to exhibit one smooth $X_{\sigma(i)}$ containing C_i, which follows from the following lemma:

LEMMA 2.5. *Let X be a variety of dimension $n + r$ with $n \geq 2$, $C \subset X$ be a smooth curve, $\mathcal{E}$ be a rank r vector bundle on X such that $\mathcal{E} \otimes \mathcal{I}_C$ is generated by global section. Then for a generic $\sigma \in H^0(X, \mathcal{E} \otimes \mathcal{I}_C)$, the zero set X_σ is smooth of dimension n.*

PROOF. The fact that X_σ is smooth of dimension n away from C is standard and follows from the fact that the incidence set $(\sigma, x) \in \mathbb{P}(H^0(X, \mathcal{E} \otimes \mathcal{I}_C)) \times (X \setminus C)$, $\sigma(x) = 0\}$ is smooth of dimension $n + N$, where $N := \dim \mathbb{P}(H^0(X, \mathcal{E} \otimes \mathcal{I}_C))$. It thus suffices to check the smoothness along C for generic σ.

This is checked by observing that since $\mathcal{E} \otimes \mathcal{I}_C$ is generated by global sections, its restriction $\mathcal{E} \otimes N^*_{C/X}$ is also generated by global sections. This implies that for each point $c \in C$, the condition that X_σ is singular at c defines a codimension n closed algebraic subset P_c of $P := \mathbb{P}(H^0(X, \mathcal{E} \otimes \mathcal{I}_C))$, determined by the condition that $d\sigma_c : N_{C/X,c} \to \mathcal{E}_c$ is not surjective. Since $\dim C = 1$, the union of the P_c's cannot be equal to P if $n \geq 2$. □

This concludes the proof of 1) and the proof of 2) works exactly in the same way. □

REMARK 2.6. (Added in proof.) After this paper was accepted, it has been proved by Runpu Zong [**24**] that every curve on a rationally connected variety over $\mathbb{C}$ is algebraically equivalent, hence in particular cohomologous, to a (non-effective) integral sum of rational curves. This shows that the groups $Z^{2n-2}(X)$ and $Z^{2n-2}(X)_{rat}$ are in fact isomorphic for rationally connected n-folds X over $\mathbb{C}$.

Let us finish this section with the proof of Proposition 1.5.

PROOF OF PROPOSITION 1.5. Let $p \in \operatorname{Spec} \mathcal{O}_K$, with residue field $k(p)$. Assume $\mathcal{X}_p$ is smooth. For l prime to $\operatorname{char} k(p)$, the (adequately constructed) specialization map

$$H^{2n-2}_{et}(X_{\overline{K}}, \mathbb{Z}_l(n-1)) \to H^{2n-2}_{et}(\mathcal{X}_{\overline{p}}, \mathbb{Z}_l(n-1)) \tag{2.1}$$

is then an isomorphism (cf. [**17**, Chapter VI, §4]).

Observe also that since $X_{\overline{K}}$ is rationally connected, the rational étale cohomology group $H^{2n-2}_{et}(X_{\overline{K}}, \mathbb{Q}_l(n-1))$ is generated over $\mathbb{Q}_l$ by curve classes. Hence the same is true for $H^{2n-2}_{et}(\mathcal{X}_{\overline{p}}, \mathbb{Q}_l(n-1))$. Thus the whole cohomology groups

$$H^{2n-2}_{et}(X_{\overline{K}}, \mathbb{Z}_l(n-1)),\ H^{2n-2}_{et}(\mathcal{X}_{\overline{p}}, \mathbb{Z}_l(n-1))$$

consist of Tate classes, and (2.1) gives an isomorphism

$$H^{2n-2}_{et}(X_{\overline{K}}, \mathbb{Z}_l(n-1))_{Tate} \to H^{2n-2}_{et}(\mathcal{X}_{\overline{p}}, \mathbb{Z}_l(n-1))_{Tate}. \tag{2.2}$$

In order to prove Proposition 1.5, it thus suffices to prove the following:

LEMMA 2.7. *1) For almost every $p \in \operatorname{Spec} \mathcal{O}_K$, the fiber $\mathcal{X}_{\overline{p}}$ is smooth and separably rationally connected.*

2) If $\mathcal{X}_{\overline{p}}$ is smooth and separably rationally connected, for any curve $C_{\overline{p}} \subset \mathcal{X}_{\overline{p}}$, the inverse image $[C_{\overline{p}}]_{\overline{K}} \in H^{2n-2}_{et}(X_{\overline{K}}, \mathbb{Z}_l(n-1))$ of the class $[C_{\overline{p}}] \in H^{2n-2}_{et}(\mathcal{X}_{\overline{p}}, \mathbb{Z}_l(n-1))$ via the isomorphism (2.2) is the class of a 1-cycle on $X_{\overline{K}}$.

PROOF. 1) When the fiber $\mathcal{X}_p$ is smooth, the separable rational connectedness of $\mathcal{X}_{\overline{p}}$ is equivalent to the existence of a smooth rational curve $C_{\overline{p}} \cong \mathbb{P}^1_{\overline{k(p)}}$ together with a morphism $\phi : C_{\overline{p}} \to \mathcal{X}_{\overline{p}}$ such that the vector bundle $\phi^* T_{\mathcal{X}_{\overline{p}}}$ on $\mathbb{P}^1_{\overline{k(p)}}$ is a direct sum $\oplus_i \mathcal{O}_{\mathbb{P}^1_{\overline{k(p)}}}(a_i)$ where all a_i are positive. Equivalently

$$H^1(\mathbb{P}^1_{\overline{k(p)}}, \phi^* T_{\mathcal{X}_{\overline{p}}}(-2)) = 0. \tag{2.3}$$

The smooth projective variety $X_{\overline{K}}$ being rationally connected in characteristic 0, it is separably rationally connected, hence there exist a finite extension K' of K, a curve C and a morphism $\phi : C \to X$ defined over K', such that $C \cong \mathbb{P}^1_{K'}$ and $H^1(\mathbb{P}^1_{K'}, \phi^* T_{X_{K'}}(-2)) = 0$.

We choose a model

$$\Phi : \mathcal{C} \cong \mathbb{P}^1_{\mathcal{O}_{K'}} \to \mathcal{X}'$$

of C and ϕ defined over a Zariski open set of $\operatorname{Spec} \mathcal{O}_{K'}$. By upper-semi-continuity of cohomology, the vanishing (2.3) remains true after restriction to almost every closed point $p \in \operatorname{Spec} \mathcal{O}_{K'}$, which proves 1).

2) The proof is identical to the proof of Proposition 1.3: we just have to show that the curve $C_{\overline{p}} \subset \mathcal{X}_{\overline{p}}$ is algebraically equivalent in $\mathcal{X}_{\overline{p}}$ to a difference $C''_{\overline{p}} - \sum_i R_{i,\overline{p}}$, where each curve $C''_{\overline{p}}$, resp. $R_{i,\overline{p}}$ (they are in fact defined over a finite extension $k(p)'$ of $k(p)$), lifts to a curve C'', resp. R_i in $X_{K'}$ for some finite extension K' of K.

Assuming the curves $C''_{\overline{p}}$, $R_{i,\overline{p}}$ are smooth, the existence of such a lifting is granted by the condition $H^1(C''_{\overline{p}}, N_{C''_{\overline{p}}/\mathcal{X}_{\overline{p}}}) = 0$, resp. $H^1(R_{i,\overline{p}}, N_{R_{i,\overline{p}}/\mathcal{X}_{\overline{p}}}) = 0$.

Starting from $C \subset \mathcal{X}_{\overline{p}}$ where $\mathcal{X}_{\overline{p}}$ is separably rationally connected over $\overline{p}$, we obtain such curves $C''_{\overline{p}}$, $R_{i,\overline{p}}$ as in the previous proof, applying [**10**, §2.1]. □

The proof of Proposition 1.5 is finished. □

Again, this proof leads as well to the proof of the specialization invariance of the l-adic analogues $Z^{2n-2}_{et,rat}(X)_l$ of the groups $Z^{2n-2}_{rat}(X)$ introduced in Remark 2.2.

VARIANT 2.8. *Let X be a smooth rationally connected variety defined over a number field K, with ring of integers $\mathcal{O}_K$. Assume given a projective model $\mathcal{X}$ of X over $\operatorname{Spec} \mathcal{O}_K$. Fix a prime integer l. Then for any $p \in \operatorname{Spec} \mathcal{O}_K$ such that $\mathcal{X}_{\overline{p}}$ is smooth separably connected, the group $Z^{2n-2}_{et,rat}(X)_l$ is isomorphic to the group $Z^{2n-2}_{et,rat}(X_p)_l$.*

3. Consequence of a result of Chad Schoen

In [**19**], Chad Schoen proves the following theorem:

THEOREM 3.1. *Let X be a smooth projective variety of dimension n defined over a finite field k of characteristic p. Assume that the Tate conjecture holds for degree 2 Tate classes on smooth projective surfaces defined over a finite extension of k. Then the étale cycle class map:*

$$cl : CH^{n-1}(X_{\overline{k}}) \otimes \mathbb{Z}_l \to H^{2n-2}(X_{\overline{k}}, \mathbb{Z}_l(n-1))_{Tate}$$

is surjective, that is $Z^{2n-2}_{et}(X)_l = 0$.

In other words, the Tate conjecture 1.4 for degree 2 *rational* Tate classes implies that the groups $Z^{2n-2}_{et}(X)_l$ should be trivial for all smooth projective varieties defined over finite fields. This is of course very different from the situation over $\mathbb{C}$ where the groups $Z^{2n-2}(X)$ are known to be possibly nonzero.

REMARK 3.2. There is a similarity between the proof of Theorem 3.1 and the proof of Theorem 1.1. Schoen proves that given an integral Tate class α on X (defined over a finite field), there exist a smooth complete intersection surface $S \subset X$ and an integral Tate class β on S such that $j_{S*}\beta = \alpha$, where j_S is the inclusion of S in X. The result then follows from the fact that if the Tate conjecture holds for degree 2 rational Tate classes on S, it holds for degree 2 integral Tate classes on S.

I prove that for X a uniruled or Calabi-Yau, and for $\beta \in Hdg^4(X, \mathbb{Z})$ there exist surfaces $S_i \overset{j_{S_i}}{\hookrightarrow} X$ (in an adequately chosen linear system on X) and integral Hodge classes $\beta_i \in Hdg^2(S_i, \mathbb{Z})$ such that $\alpha = \sum_i j_{S_i*}\beta$. The result then follows from the Lefschetz theorem on $(1,1)$-classes applied to the β_i.

We refer to [**7**] for some comments on and other applications of Schoen's theorem, and conclude this note with the proof of the following theorem (cf. Theorem 1.6 of the introduction).

THEOREM 3.3. *Assume Tate's conjecture* 1.4 *holds for degree* 2 *Tate classes on smooth projective surfaces defined over a finite field. Then the group $Z^{2n-2}(X)$ is trivial for any smooth rationally connected variety X over $\mathbb{C}$.*

PROOF. We first recall that for a smooth rationally connected variety X, the group $Z^{2n-2}(X)$ is equal to the quotient $H^{2n-2}(X, \mathbb{Z})/H^{2n-2}(X, \mathbb{Z})_{alg}$, due to the fact that the Hodge structure on $H^{2n-2}(X, \mathbb{Q})$ is trivial. In fact, we have more precisely

$$H^{2n-2}(X, \mathbb{Q}) = H^{2n-2}(X, \mathbb{Q})_{alg}$$

by hard Lefschetz theorem and the fact that

$$H^2(X, \mathbb{Z}) = H^2(X, \mathbb{Z})_{alg}$$

by the Lefschetz theorem on $(1,1)$-classes.

Next, in order to prove that $Z^{2n-2}(X)$ is trivial, it suffices to prove that for each l, the group $Z^{2n-2}(X) \otimes \mathbb{Z}_l = H^{2n-2}(X, \mathbb{Z}_l)/(\mathrm{Im}\, cl) \otimes \mathbb{Z}_l$ is trivial.

We apply Proposition 1.3 which tells as well that over $\mathbb{C}$, the group $Z^{2n-2}(X) \otimes \mathbb{Z}_l$ is locally deformation invariant for families of smooth rationally connected varieties. Note that our smooth projective rationally connected variety X is the fiber X_t of a smooth projective morphism $\phi : \mathcal{X} \to B$ defined over a number field, where

$\mathcal{X}$ and B are quasiprojective, geometrically connected and defined over a number field. By local deformation invariance, the vanishing of $Z^{2n-2}(X)\otimes\mathbb{Z}_l$ is equivalent to the vanishing of $Z^{2n-2}(X_{t'})\otimes\mathbb{Z}_l$ for any point $t'\in B(\mathbb{C})$. Taking for t' a point of B defined over a number field, $X_{t'}$ is defined over a number field. Hence it suffices to prove the vanishing of $Z^{2n-2}(X)\otimes\mathbb{Z}_l$ for X rationally connected defined over a number field L.

We have

$$Z^{2n-2}(X)\otimes\mathbb{Z}_l = H^{2n-2}(X,\mathbb{Z}_l)/(\operatorname{Im} cl)\otimes\mathbb{Z}_l,$$

and by the Artin comparison theorem (cf. [**17**, Chapter III,§3]), this is equal to

$$\frac{H^{2n-2}_{et}(X,\mathbb{Z}_l(n-1))}{(\operatorname{Im} cl)\otimes\mathbb{Z}_l} = Z^{2n-2}_{et}(X)_l$$

since $H^{2n-2}_{et}(X,\mathbb{Z}_l(n-1))$ consists of Tate classes. Hence it suffices to prove that for X rationally connected defined over a number field and for any l, the group $Z^{2n-2}_{et}(X)_l$ is trivial.

We now apply Proposition 1.5 to X and its reduction X_p for almost every closed point $p\in\operatorname{Spec}\mathcal{O}_L$. It follows that the vanishing of $Z^{2n-2}_{et}(X)_l$ is implied by the vanishing of $Z^{2n-2}_{et}(X_p)_l$. According to Schoen's theorem 3.1, the last vanishing is implied by the Tate conjecture for degree 2 Tate classes on smooth projective surfaces. □

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