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Emmy Noether in Bryn Mawr

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Representations of Lie Groups and the Orbit Method

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It was a great honor and pleasure to be invited by the Association for Women Mathematicians, to give an address for Emmy Noether's 100th birthday.

I have chosen here to talk about my own present mathematical interests, the theory of Lie group representations, and I would like on this occasion to point out that many women are now contributing to the development of this field. This fact would have pleased Emmy Noether. Although she was extraordinary, she would not have thought of herself in these terms, she would have been against holding up her name as a yardstick by which to measure all past, present, and future accomplishments of mathematicians, women and men. We want to celebrate her as an ordinary woman, who could find at this time, only in herself, the necessary courage and inner peace to be what she has been: Emmy Noether.

Let G be a Lie group, i.e. G is an analytic manifold with a group structure such that the group operations are analytic. Let us denote by e the identity element of G.

Consider $\mathfrak{g}$ the tangent space to G at e. For every $X \in \mathfrak{g}$, there exists a homomorphism $\mathbf{R} \to G: t \to \exp tX$ such that $(X \cdot \varphi)(e) = (d/dt)\varphi(\exp tX)|_{t=0}$ for every differentiable function φ on G. The map $\exp: \mathfrak{g} \to G$ given by $X \to \exp X$ is called the exponential map.

If $g_0 \in G$, the right translation $R(g_0) \cdot g = gg_0$ defines a right action of G on G. If $X \in \mathfrak{g} = T_e(G)$, we denote by X^* the vector field on G such that $(X^*)_{g_0} = R(g_0)_* \cdot X$. The bracket of two elements of $\mathfrak{g}$ is then defined by the relation $[X, Y]^* = [X^*, Y^*]$, where on the right-hand side the bracket is the bracket of vector fields, i.e. $[X^*, Y^*] \cdot \varphi = X^*Y^* \cdot \varphi - Y^*X^* \cdot \varphi$. This defines a Lie algebra structure on $\mathfrak{g}$.

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Consider the adjoint action of G on itself by inner automorphisms $(\mathrm{Ad}\, g_0) \cdot g = g_0 g g_0^{-1}$. The differential of the map $\mathrm{Ad}\, g_0 \colon G \to G$ at the identity gives rise to a map (still denoted by $\mathrm{Ad}\, g$) on $T_e(G) = \mathfrak{g}$. We have then $g \cdot (\exp tX) g^{-1} = \exp(t\, \mathrm{Ad}\, g \cdot X)$. (We may write $g \cdot X$ for $\mathrm{Ad}\, g \cdot X$.) This action of G on $\mathfrak{g}$ is called the adjoint action. We define also $(\mathrm{ad}\, X) \cdot Y = [X, Y]$, for $X, Y \in \mathfrak{g}$.

I. Examples of Lie Groups

Let us start by giving some examples of Lie groups G. Of course, the most evident example is

1.1. Example 1: $G =$ Vector Space. $G = V$ is a finite dimensional real vector space (with the addition law).

Clearly $\mathfrak{g} = V$, and the exponential map is the identity map.

As G is commutative, the adjoint action of G on $\mathfrak{g}$ is trivial, i.e. $g \cdot v = v$ for every $g \in G$, $v \in \mathfrak{g}$.

A closely related example is the example of a torus.

1.2. Example 2: $G =$ Torus. Let us consider $G = \{z; z \in \mathbf{C}, |z| = 1\}$ (with the multiplicative law).

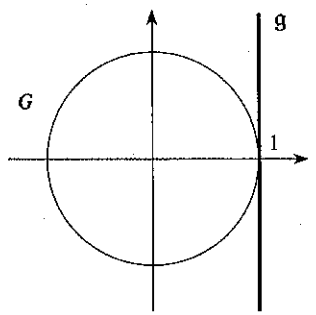

Figure 1

The tangent space $\mathfrak{g}$ at the identity is

$$\mathfrak{g} = i\mathbf{R}.$$

The exponential map is the map $i\theta \to e^{i\theta}$. The adjoint action is trivial.

1.3. More generally, let Γ be a lattice in a vector space V and consider

$$G = T = V/\Gamma.$$

The Lie algebra of G is naturally identified with V.

$$\mathfrak{g} = V.$$

The exponential map $\exp\colon V \to V/\Gamma$ is the natural quotient map. The adjoint action is trivial.

1.4. Let us now consider the "basic" example of a Lie group:

Example 3: The Full Linear Group $\mathrm{GL}(n, \mathbf{R})$. Let

$$G = \mathrm{GL}(n, \mathbf{R}) = \{n \times n \text{ invertible real matrices}\}.$$

As G is an open subset of all $n \times n$ real matrices, we have:

$$\mathfrak{g} = \mathfrak{gl}(n, \mathbf{R}) = \{n \times n \text{ real matrices}\}.$$

The exponential map is the usual matrix exponential

$$\exp X = 1 + X + \frac{X^2}{2!} + \cdots + \frac{X^n}{n!} + \cdots.$$

The Lie algebra structure on $\mathfrak{gl}(n, \mathbf{R})$ is $[A, B] = A \circ B - B \circ A$. The adjoint action is given by conjugation:

$$(\mathrm{Ad}\, g) \cdot A = gAg^{-1}.$$

1.5. For the purpose of this discussion, it will be sufficient to consider here linear groups G. A linear group G is a closed subgroup of $\mathrm{GL}(n, \mathbf{R})$. Thus its Lie algebra $\mathfrak{g}$ is a subalgebra of $\mathfrak{gl}(n, \mathbf{R})$. The corresponding notions of exponential bracket, and adjoint action are therefore the restrictions to $\mathfrak{g}$ and G of the preceding operations, i.e., for:

$$\mathfrak{g} \subset \mathfrak{gl}(n, \mathbf{R}),$$
$$G \subset \mathrm{GL}(n, \mathbf{R}),$$

we still have:

$$\exp X = 1 + X + \frac{X^2}{2!} + \cdots + \frac{X^n}{n!} + \cdots, \qquad X \in \mathfrak{g},$$
$$[A, B] = A \circ B - B \circ A, \qquad A, B \in \mathfrak{g},$$
$$g \cdot X = gXg^{-1}, \qquad g \in G, \quad X \in \mathfrak{g}.$$

Let us consider some examples of such linear groups:

1.6. Example 4: The Heisenberg Group. We consider the group

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}; x, y, z \in \mathbf{R} \right\}.$$

It is easy to see that:

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & p & e \\ 0 & 0 & q \\ 0 & 0 & 0 \end{pmatrix}; p, q, e \in \mathbf{R} \right\}.$$

We write a basis of $\mathfrak{g}$ as follows:

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We have the following relations:

$$[P, Q] = P \circ Q - Q \circ P = E,$$
$$[P, E] = 0,$$
$$[Q, E] = 0.$$

1.7. Example 5: $G = \mathrm{SU}(2)$. Let us consider the vector space $\mathbf{C}^2$ with its usual Hermitian inner product $|z_1|^2 + |z_2|^2$. We consider the group U(2) of all complex linear transformations g of $\mathbf{C}^2$ leaving this inner product stable. To be able later on to illustrate our notions by pictures in $\mathbf{R}^3$, we will consider the subgroup SU(2) of U(2) of elements g in U(2) such that $\det g = 1$. It is easy to see that:

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}; \alpha, \beta \in \mathbf{C}, |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

Hence SU(2) is a compact manifold of dimension 3. The Lie algebra $\mathfrak{su}(2)$ of the group SU(2) consists of 2×2 complex matrices X which are anti-hermitian and have trace zero. Thus:

$$\mathfrak{g} = \mathfrak{su}(2) = \left\{ \begin{pmatrix} ix_3 & -x_1 + ix_2 \\ x_1 + ix_2 & -ix_3 \end{pmatrix}; x_i \in \mathbf{R} \right\}.$$

We remark here for later use that the function $X \mapsto \det X = x_1^2 + x_2^2 + x_3^2$ is invariant under the adjoint action $g \cdot X = gXg^{-1}$ of G on $\mathfrak{g}$.

1.8. Example 6: $\mathrm{SL}(2, \mathbf{R})$. We consider the group of 2×2 real matrices with determinant 1, i.e.

$$G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \mathbf{R}; ad - bc = 1 \right\}.$$

Thus G is a closed subgroup of $\mathrm{GL}(2, \mathbf{R})$. It is easy to see that:

$$\mathfrak{g} = \mathfrak{sl}(2, \mathbf{R}) = \{\text{matrices of trace zero}\} = \left\{ X = \begin{pmatrix} x_1 & x_2 + x_3 \\ x_2 - x_3 & -x_1 \end{pmatrix}, x_i \in \mathbf{R} \right\}.$$

Let us also remark here that the function $X \mapsto \det X = x_3^2 - (x_1^2 + x_2^2)$ is invariant under the adjoint action $g \cdot X = gXg^{-1}$ of G on $\mathfrak{g}$.

II. The Dual of G and the Plancherel Formula

2.1. One of the main objects of representation theory of Lie groups is the study of the dual $\hat{G}$ of the Lie group G or, "equivalently," of the characters of the group G We will now define these notions, allowing some imprecisions from time to time. Let us refer to the useful book [37] for more details.

A representation of G on a vector space V is a homomorphism $g \mapsto T(g)$ of G into the group of linear transformations of V, i.e. we have

$$T(g_1 g_2) = T(g_1) \circ T(g_2),$$
$$T(e) = \mathrm{id}_V.$$

The representation T of G in V is called a unitary representation of G, if V is a Hilbert space, the operators $T(g)$ unitary operators and the map $(g, v) \mapsto T(g) \cdot v$ continuous.

We have an obvious notion of equivalence of unitary representations. Two unitary representations T_1, T_2 of G in Hilbert spaces H_1, H_2 are equivalent if there exists a unitary isomorphism $I: H_1 \to H_2$ such that the following diagram:

$$\begin{array}{ccc} H_1 & \xrightarrow{I} & H_2 \\ \uparrow {\scriptstyle T_1(g)} & & \uparrow {\scriptstyle T_2(g)} \\ H_1 & \xrightarrow{I} & H_2 \end{array}$$

is commutative, for every $g \in G$.

If (T_1, H_1) and (T_2, H_2) are two unitary representations of G, we can form the representation $T = T_1 \oplus T_2$ acting on the direct sum $H = H_1 \oplus H_2$ by $T(g) = T_1(g) \oplus T_2(g)$. A representation T is irreducible if T is not obtained as a direct sum of two representations. Equivalently, (T, H) is irreducible if there exists no proper Hilbert subspace of H invariant under T. As every unitary representation T of G in a Hilbert space H is a "sum" (eventually a "continuous sum") of unitary irreducible representations of G, the essential objects of unitary representation theory are the irreducible representations of G.

By definition, the dual $\hat{G}$ of G is the set of equivalence classes of unitary irreducible representations of G.

2.2. Let us give now an important example of a unitary representation: the regular representation. Consider on G the left invariant Haar measure dg (unique up to a positive scalar multiple). Consider the Hilbert space $L^2(G)$ of dg-square integrable functions on G, i.e.

$$L^2(G) = \left\{ f; \int |f(g)|^2 \, dg < \infty \right\}.$$

The left action $(L(g_0)f)(g) = f(g_0^{-1}g)$ defines a unitary representation of G on $L^2(G)$. This representation is highly reducible, and an important problem is to describe explicitly the decomposition of this representation into irreducible representations.

2.3. Let us now define the notion of the character of a representation. If T is a representation of G in a finite dimensional vector space V, the character of T is the function $\chi_T(g) = \operatorname{tr} T(g)$. It is clear that this function is invariant under conjugation, i.e. $\chi_T(g_0 g g_0^{-1}) = \chi_T(g)$. It is well known that if G is a compact group, the description of $\hat{G}$ is equivalent to the description of all the functions χ_T.

Let us now consider the case of an infinite dimensional representation. Define, for a function φ in $L^1(G)$, the operator $T(\varphi) = \int_G \varphi(g)T(g)\,dg$, i.e.

$$\langle T(\varphi)x, y\rangle = \int_G \varphi(g)\langle T(g)x, y\rangle\,dg.$$

If (T, H) is irreducible and if φ is a C^∞ function with compact support on G, it is often the case (in particular for all our examples) that the operator $T(\varphi)$ has a trace. If e_i is any orthonormal basis of H, we have $\operatorname{tr} T(\varphi) = \sum_i \langle T(\varphi)e_i, e_i\rangle$. Furthermore, the map $\varphi \to \operatorname{tr} T(\varphi)$ happens to be a distribution on G. This distribution (if defined) is called the character of the representation T. The distribution character $\operatorname{tr} T$ depends of the choice of dg. It is clear that, if T is a representation of G in a finite dimensional vector space V, we have

$$\operatorname{tr} T(\varphi) = \int_G (\operatorname{tr} T(g))\varphi(g)\,dg.$$

2.4. A group G is called unimodular if the left Haar measure is also right invariant. Equivalently, G is unimodular, if $|\det \operatorname{Ad} g| = 1$ for every $g \in G$. If G is unimodular, the distribution $\operatorname{tr} T$ (if defined) is an invariant (by the adjoint action) distribution on G, as:

$$\begin{aligned}(\operatorname{tr} T, (\operatorname{Ad} g_0)^{-1}\cdot\varphi) &= \operatorname{tr}\int_G T(g)\varphi(g_0 g g_0^{-1})\,dg\\ &= \operatorname{tr}\int_G T(g_0^{-1} g g_0)\varphi(g)\,dg\\ &\quad\text{as } dg \text{ is left and right invariant,}\\ &= \operatorname{tr}\int_G T(g_0)^{-1}T(g)T(g_0)\varphi(g)\,dg\\ &= \operatorname{tr} T(g_0)^{-1}T(\varphi)T(g_0)\\ &= \operatorname{tr} T(\varphi).\end{aligned}$$

Consider the δ-distribution $\varphi \mapsto \varphi(e)$. It is an invariant distribution on G. The purpose of the Plancherel formula is to express the value of a function φ at e in function of the values of the characters $\operatorname{tr} T(\varphi)$. It is indeed possible to do so, for any "tame" unimodular Lie group (we will consider here only real algebraic

groups, which are "tame" Lie groups), as we have the "abstract Plancherel theorem":

2.5. Theorem (see [14]). *Let G be a tame unimodular group. There exists a measure $d\mu$ (called the Plancherel measure) on $\hat{G}$ such that:*

$$\varphi(e) = \int_{\hat{G}} \operatorname{tr} T(\varphi)\,d\mu(T)$$

for every C^∞ function φ on G with compact support.

Let us first remark here that the support of the measure $d\mu$ may be smaller than $\hat{G}$. This already happens for the group $SL(2, \mathbf{R})$. We will denote by $\hat{G}_r$ the support of $d\mu$ and call $\hat{G}_r$ the reduced dual of G.

2.6. Let us come back to our examples and describe accordingly the dual $\hat{G}$ of these groups and the Plancherel formula.

2.7. Example 1: $G = V$.

(a) *The set $\hat{G}$*

Let V be a real finite dimensional vector space and V^* be the dual vector space. If $f \in V^*$, we consider $\chi_f(x) = e^{i\langle f, x\rangle}$. The map $x \to \chi_f(x)$ defines a character of the additive group V. (We also use the word character for a 1-dimensional representation of a group G.) Thus we have:

$$\hat{V} \simeq V^*.$$

(b) *Characters*

Let dx be a Lebesgue measure on V. If φ is a C^∞ function with compact support, we have

$$\operatorname{tr} \chi_f(\varphi) = \int_V \varphi(x)\chi_f(x)\,dx = \int_V \varphi(x)e^{i\langle f, x\rangle}\,dx = \hat{\varphi}(f).$$

Thus $f \mapsto \operatorname{tr} \chi_f(\varphi)$ is the function on V^* given by the Fourier transform of φ.

(c) *The Plancherel formula*

The Plancherel formula is the usual Plancherel inversion formula:

$$\varphi(0) = \int_{V^*} \hat{\varphi}(f)\,df,$$

where df is the dual Lebesgue measure on $V^* = \hat{V}$.

2.8. Example 2: $G = T$.

(a) *The set $\hat{G}$*

Let $G = T = \{z; z \in \mathbf{C}, |z| = 1\}$.

If $n \in \mathbf{Z}$, define $\chi_n(z) = z^n$. We have

$$\hat{G} \simeq \mathbf{Z} \qquad (n \leftrightarrow \chi_n).$$

(b) *Characters*

Let us choose on T the Haar measure giving total mass 1 to T.

If φ is a C^∞ function on T, then

$$\operatorname{tr} \chi_n(\varphi) = \int \varphi(e^{i\theta}) e^{in\theta} \frac{d\theta}{2\pi}$$

is the $(-n)$th Fourier coefficient of the periodic function $\theta \to \varphi(e^{i\theta}) = \sum a_n e^{in\theta}$.

(c) *The Plancherel formula*

The Plancherel formula is deduced immediately from the expansion of a function in its Fourier coefficients:

$$\varphi(1) = \sum \operatorname{tr} \chi_n(\varphi).$$

2.9. More generally, let $T = V/\Gamma$ be a n-dimensional torus. It is clear that every representation of T gives, by composition with the natural projection $V \xrightarrow{\exp} V/\Gamma = T$, a representation of V. Thus $\hat{T}$ is included in V^*.

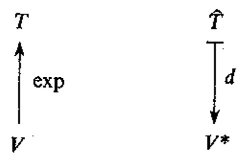

Figure 2

This map is represented for $V = \mathbf{R}$, $\Gamma = 2\pi\mathbf{Z}$, by:

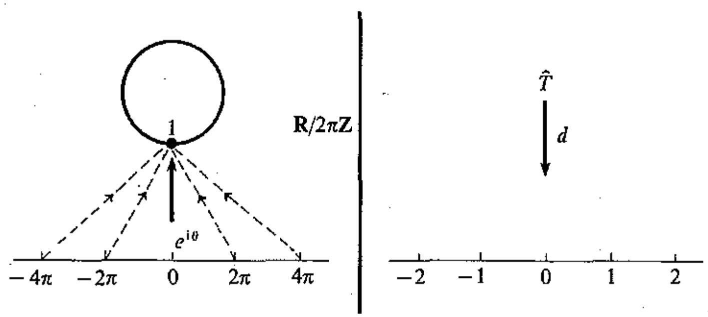

Figure 3

It is obvious that a character χ_f of V gives rise to a character of T, if and only if $\chi_f(x) = e^{i(f,x)} = 1$, for every x in V such that $\exp x = 1$, i.e. for every x in Γ. Thus we have identified $d(\hat{T})$ with the dual lattice Γ^* of Γ in V^*. Therefore,

(a) $$\hat{T} = \Gamma^* = \{f \in V^*; (f, \gamma) \in 2\pi\mathbf{Z}, \text{for every } \gamma \in \Gamma\}.$$

Let us normalize the Lebesgue measure on V such that the volume of the fundamental parallelepiped spanned by Γ is 1. Thus we have, for φ a C^∞ function on V/Γ:

(b) $$\operatorname{tr} \chi_f(\varphi) = \int_{V/\Gamma} \varphi(x) e^{i(f,x)}\, dx, \text{ for } f \in \Gamma^*.$$

The Plancherel formula is:

(c) $$\varphi(0) = \sum_{f \in \Gamma^*} \operatorname{tr} \chi_f(\varphi)$$

for every C^∞ function φ on V/Γ.

Let us point out at this occasion that, by an immediate averaging process, this formula is equivalent to:

(d) The Poisson formula:

$$\sum_{\gamma \in \Gamma} \varphi(\gamma) = \sum_{f \in \Gamma^*} \hat{\varphi}(f)$$

for every C^∞ function φ on V with compact support (or in the Schwartz space of V).

2.10. Example 3: $G = \mathrm{GL}(n, \mathbf{R})$. Let us now consider our basic example of a Lie Group $G = \mathrm{GL}(n, \mathbf{R})$. Now is the time to reveal the truth. The dual $\hat{G}$ of $\mathrm{GL}(n, \mathbf{R})$ is not known. The case $n = 2$, the first work in the representation theory of semi-simple groups, was completed by Valentine Bargmann in 1946 [6]. But since this time the general case seems still out of reach.

2.11. Let us mention some of the recent results on this question of the determination of the unitary dual of real semi-simple Lie groups:

If $n \leq 4$, the dual $\hat{G}$ of $G = \mathrm{GL}(n, \mathbf{R})$ has been determined by Birgit Speh [60].

The dual $\hat{G}$ of any complex semi-simple Lie group of rank two (i.e. $\mathrm{SL}(3, \mathbf{C})$, $\mathrm{Sp}(2, \mathbf{C})$, G_2) has been determined by Michel Duflo [16].

The dual $\hat{G}$ of any semi-simple Lie group of real rank one (i.e. $G = \mathrm{SO}(n, 1)$, $\mathrm{SU}(n, 1)$, $\mathrm{Sp}(n, 1)$ or a real form of F_4) has been determined by Welleda Baldoni-Silva and Dan Barbasch [4], [5].

The dual $\hat{G}$ of $G = \mathrm{SU}(2, 2)$ has been determined by Anthony Knapp and Birgit Speh [38].

Thus we still know very little about unitary representation theory of real semi-simple Lie groups. However, as remarked before, the Plancherel inversion formula for a Lie group does not involve the complete description of $\hat{G}$, but only of the reduced dual $\hat{G}_r$. If G is a real semi-simple Lie group (with finite center), the reduced dual $\hat{G}_r$ of $\hat{G}$ as well as the "concrete" Plancherel measure $d\mu(T)$ on $\hat{G}_r$ has been determined by Harish-Chandra [26]. We will discuss further some of the corresponding results.

2.12. Example 4: G = The Heisenberg Group.

(a) *The set* $\hat{G}$

The description of the dual $\hat{G}$ of the Heisenberg group was the object of the famous theorem of Stone–Von Neumann [1931] on the "Uniqueness for the Schrödinger operators" [47].

Here is the complete list of all equivalence classes of unitary irreducible representations of G.

(1) Let $H = L^2(\mathbf{R})$.

For each λ a non-zero real number, consider the representation T_λ of G in $L^2(\mathbf{R})$ given by:

$$(T_\lambda \exp(tP)\cdot\varphi)(y) = e^{i\lambda ty}\varphi(y),$$
$$(T_\lambda \exp(tQ)\cdot\varphi)(y) = \varphi(y-t),$$
$$(T_\lambda \exp(tE)\cdot\varphi)(y) = e^{i\lambda t}\varphi(y) \quad \text{for } \varphi \in L^2(\mathbf{R}).$$

(2) Let $H = \mathbf{C}$

For each (α, β) a pair of real numbers, consider the character

$$T_{\alpha,\beta}(\exp xP \exp yQ \exp zE) = e^{i\alpha x}e^{i\beta y}.$$

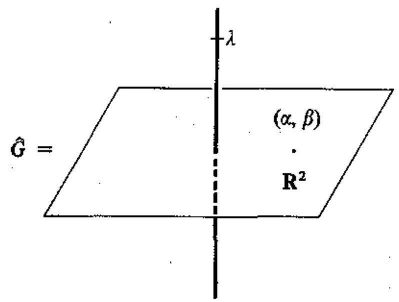

Figure 4

Remark. Differentiating formally the representation T_λ, we obtain

$$dT_\lambda(P) = \frac{d}{dt} T_\lambda \exp(tP)|_{t=0} = i\lambda y,$$

$$dT_\lambda(Q) = \frac{d}{dt} T_\lambda \exp(tQ)|_{t=0} = -\frac{\partial}{\partial y},$$

$$dT_\lambda(E) = \frac{d}{dt} T_\lambda \exp(tE)|_{t=0} = i\lambda.$$

The operators $i\lambda y$, $-\partial/\partial y$ determine a representation of the "Schrodinger operators" by skew-adjoint operators, in particular these operators satisfy the canonical commutation relation: $P \circ Q - Q \circ P = i\lambda\,\mathrm{Id}$.

(b) *Characters*

Let us determine the characters of the corresponding representations. It is not difficult to compute that, for φ a C^∞ function on G with compact support:

$$\operatorname{tr} T_\lambda(\varphi) = \frac{2\pi}{\lambda}\int_{\mathbf{R}} \varphi(\exp zE)e^{i\lambda z}\,dz,$$

with the Haar measure on G given by $dx\,dy\,dz$.

(c) *The Plancherel formula*

We have the Plancherel formula

$$\varphi(e) = \int_{\mathbf{R}} \operatorname{tr} T_\lambda(\varphi)\frac{\lambda\,d\lambda}{(2\pi)^2}.$$

Proof. From the formula given in (b) for the characters, the Plancherel formula is immediately deduced from the usual Fourier inversion formula.

2.13. Example 5: $G = \mathrm{SU}(2)$.

(a) *The set* $\hat{G}$

Let n be a positive integer and consider:

$$V_n = \{\text{Polynomials on } \mathbf{C}^2 \text{ of homogeneous degree } n-1\}$$
(the dimension of the vector space V_n is n).

The group SU(2) acts on $\mathbf{C}^2$, and thus acts on polynomials on $\mathbf{C}^2$ via $(g\cdot P)(z) = P(g^{-1}\cdot z)$. It is clear that this action preserves the space V_n. We have

$$\hat{G} \simeq N$$

by $[n \leftrightarrow (T_n(g)P)(z) = P(g^{-1}z),\ P \in V_n]$.

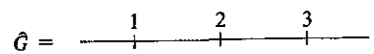

Figure 5

(b) *Characters*

Let $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix}$ be an element of SU(2). The polynomials $z_1^i z_2^j$ $(i + j = n - 1)$ are eigenvectors for the transformation $T_n\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix}$. Thus we can easily compute:

$$\operatorname{tr} T_n\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix} = \sum_{k=0}^{n-1} e^{i(n-1-2k)\theta} = \frac{e^{in\theta} - e^{-in\theta}}{e^{i\theta} - e^{-i\theta}}.$$

(This formula is a special case of the Weyl formula for characters [64].)

(c) *The Plancherel formula*
We have the Plancherel formula

$$\varphi(e) = \sum_{n>0} n \operatorname{tr} T_n(\varphi).$$

A proof of this will be given in Appendix 2.

2.14. Example 6: $G = \mathrm{SL}(2, \mathbf{R})$.

(a) *The reduced dual* $\hat{G}_r$
The set $\hat{G}$ has been determined in the classical article of V. Bargmann [6], still the best reference on this subject. The complete list of the unitary irreducible representations of $\mathrm{SL}(2, \mathbf{R})$ consists of "discrete series," "principal series," and "complementary series." Only the first two series contribute to the Plancherel measure. Furthermore, the description of the complementary series is more subtle, so we will list here only the set $\hat{G}_r$. It consists of:

(1) The discrete series
(1.1) Let n be a positive integer. Let us consider the upper half-plane $\mathbf{P}^+ = \{z = x + iy;\, x, y \in \mathbf{R}, y > 0\}$. The group $\mathrm{SL}(2, \mathbf{R})$ acts as a group of holomorphic transformations on $\mathbf{P}^+$ by $z \mapsto g \cdot z = (az + b)/(cz + d)$.
Let n be a positive integer. Consider

$$H_n = \left\{\varphi, \text{ holomorphic functions on } \mathbf{P}^+ \text{ such that } \int_{\mathbf{P}^+} |\varphi|^2 y^{n-1}\, dx\, dy < \infty\right\}.$$

(This Hilbert space would be $\{0\}$, if n were negative.)
For

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbf{R}),$$

define

$$(T_n(g^{-1})\varphi)(z) = (cz + d)^{-(n+1)} \varphi\left(\frac{az + b}{cz + d}\right).$$

The map $g \mapsto T_n(g)$ defines a unitary irreducible representation of G in H_n.
The series of representations (T_n, H_n) $(n \geq 1)$ is called the holomorphic discrete series.

(1.2) Let n be a negative integer. Consider

$$H_n = \left\{\varphi, \text{ antiholomorphic functions on } \mathbf{P}^+ \text{ such that } \int_{\mathbf{P}^+} |\varphi|^2 y^{|n|-1}\, dx\, dy < \infty\right\}.$$

Then, for

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbf{R}),$$

consider

$$(T_n(g^{-1})\varphi)(z) = \overline{(cz + d)}^{-(1-n)} \varphi\left(\frac{az + b}{cz + d}\right).$$

The map $g \mapsto T_n(g)$ defines a unitary irreducible representation of G in H_n.
The series of representations (T_n, H_n) $(n \leq -1)$ is called the antiholomorphic discrete series.

(2) The principal series
Let $H = L^2(\mathbf{R})$, let s be a non-negative real number. Define the representations $T_s^{\pm}$ of G in $L^2(\mathbf{R})$ by:

$$(T_s^+(g^{-1})f)(x) = |cx + d|^{-1+is} f\left(\frac{ax + b}{cx + d}\right),$$

$$(T_s^-(g^{-1})f)(x) = \operatorname{sign}(cx + d)|cx + d|^{-1+is} f\left(\frac{ax + b}{cx + d}\right), \quad \text{if } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

It is easy to verify that T_s^+ and T_s^- are unitary representations of G in $L^2(\mathbf{R})$. Furthermore, T_s^+ and T_s^- are irreducible, except for the representation T_0^- which breaks up into two irreducible pieces. The series of representations (T_s^+, T_s^-) for $s \geq 0$ forms the two principal series of $\mathrm{SL}(2, \mathbf{R})$.
A schematic diagram of $\hat{G}_r$ for $G = \mathrm{SL}(2, \mathbf{R})$ is thus:

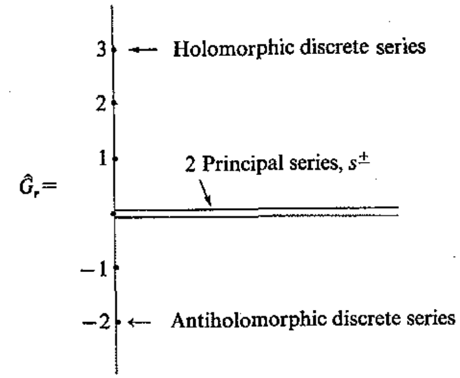

Figure 6

(b) *Characters*
We give here the results on the characters of the representations of $\mathrm{SL}(2, \mathbf{R})$ as proven by Harish-Chandra [24]. First of all, as the case for any real semi-simple

Lie group G, the character distribution $\varphi \to \operatorname{tr} T(\varphi)$ is well defined for every T in $\hat{G}$ and is given by integration against a locally L^1-function Θ_T, i.e.

$$\operatorname{tr}\left(\int_G T(g)\varphi(g)\,dg\right) = \int_G \Theta_T(g)\varphi(g)\,dg$$

for φ a C^∞ function on G with compact support.

Clearly $\Theta_T(g_0 g g_0^{-1}) = \Theta_T(g)$. Thus Θ_T is determined by its restriction to the subsets

$$B = \left\{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}; \theta \in \mathbf{R}\right\},$$

and

$$A = \left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}; a \in \mathbf{R}^*\right\},$$

as almost all (for dg) elements of SL(2, $\mathbf{R}$) are conjugated to an element of one of these two sets [SL(2, $\mathbf{R}$) has two conjugacy classes of Cartan subgroups].

We have the following formulae:

(1) Discrete series

(1.1) Holomorphic discrete series $n \geq 1$

$$\Theta_{T_n}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \frac{e^{in\theta}}{e^{-i\theta} - e^{i\theta}},$$

$$\Theta_{T_n}\begin{pmatrix} \varepsilon e^t & 0 \\ 0 & \varepsilon e^{-t} \end{pmatrix} = \varepsilon^{n+1}\frac{e^{-|nt|}}{|e^t - e^{-t}|} \quad \text{if } \varepsilon = \pm 1.$$

(1.2) Antiholomorphic discrete series $n \leq -1$

$$\Theta_{T_n}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \frac{e^{in\theta}}{e^{i\theta} - e^{-i\theta}},$$

$$\Theta_{T_n}\begin{pmatrix} \varepsilon e^t & 0 \\ 0 & \varepsilon e^{-t} \end{pmatrix} = \varepsilon^{1-n}\frac{e^{-|nt|}}{|e^t - e^{-t}|}.$$

(2) Principal series

$$\Theta_{T_s^\pm}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = 0,$$

$$\Theta_{T_{s^+}}\begin{pmatrix} \varepsilon e^t & 0 \\ 0 & \varepsilon e^{-t} \end{pmatrix} = \frac{e^{ist} + e^{-ist}}{|e^t - e^{-t}|},$$

$$\Theta_{T_{s^-}}\begin{pmatrix} \varepsilon e^t & 0 \\ 0 & \varepsilon e^{-t} \end{pmatrix} = \varepsilon\frac{e^{ist} + e^{-ist}}{|e^t - e^{-t}|}.$$

(c) *The Plancherel formula for G*

We have the formula;

$$2\pi\varphi(e) = \sum_{\substack{n\in\mathbf{Z}\\ n\neq 0}} |n| \operatorname{tr} T_n(\varphi)$$
$$+ \tfrac{1}{2}\int_{\mathbf{R}^+} s \tanh\frac{\pi s}{2} \operatorname{tr} T_s^+(\varphi)\,ds$$
$$+ \tfrac{1}{2}\int_{\mathbf{R}^+} s \coth\frac{\pi s}{2} \operatorname{tr} T_s^-(\varphi)\,ds,$$

which can be deduced from the preceding formula for the characters (see [41]).

2.15. Let G be a general real semi-simple Lie group. We give now a cursory summary of results on unitary representation theory of G. As we pointed out before, only the part $\hat{G}_r$ of $\hat{G}$ is known, while a complete description of $\hat{G}$ is still an unsolved problem. Let us, for example, mention that a most remarkable representation of the symplectic group, the Shale–Weil representation, is a singular unitary representation (singular in the sense that its two components are not in $\hat{G}_r$) and that its existence has not been recaptured by any systematic procedure.

Let us center our attention on $\hat{G}_r$ and the Plancherel formula. The central reference for this topic is the work of Harish-Chandra. As in the case of SL(2, $\mathbf{R}$), we may list representations in $\hat{G}_r$ by series. Let Car G be the set of conjugacy classes of Cartan subgroups of G. There is as many series as elements in Car G: to a conjugacy class of a Cartan subgroup H of G corresponds a series of representations $\{T_i, i \in I_H\}$. The elements of this series may be indexed by a subset I_H of $\hat{H}$. [I_H parametrizes the set of regular characters of H modulo the action of a finite group. For example, in the case of SL(2, $\mathbf{R}$), there are two conjugacy classes of Cartan subgroups, namely the conjugacy class of B and the one of A. The discrete series is indexed by the set of characters of B, except the trivial character, i.e. by $\mathbf{Z} - \{0\}$. The principal series is indexed by $\hat{A} = \mathbf{Z}/2\mathbf{Z} \times \mathbf{R}$ modulo the action $(\varepsilon, s) \to (\varepsilon, -s)$.]

In general, if G has a compact Cartan subgroup B, the corresponding series $\{T_\lambda, \lambda \in \hat{B}, \lambda \text{ regular}\}$, is indexed by a discrete set. The corresponding representation T_λ whose existence was proven "abstractly" by Harish-Chandra occurs as a discrete summand in the regular representation of G in $L^2(G)$. Thus this series is called the discrete series of G. Let T_λ be a representation of the discrete series. Harish-Chandra gave an explicit formula [25] for the character $\Theta_\lambda(g)\,dg$ of T_λ, in the case g is a regular elliptic element of G. This formula is a finite sum over the fixed points for the action of g on G/B and is formally similar to the Atiyah–Bott fixed point formula [1] for the twisted Dirac operator. It was a remarkable result of W. Schmid [52] that, indeed the representation T_λ can be realized in the space of L^2-solutions of the twisted Dirac operator D_λ on G/B.

The other series $\{T_i; i \in I_H\}$ of representations of G are constructed in a simple way from representations of discrete series of reductive subgroups of G.

The Plancherel measure is thus the measure $d\mu(T)$ on the set $\hat{G}_r = \bigcup_{H\in \mathrm{Car}\, G} I_H$, such that

2.16 $$\varphi(e) = \int_{\hat{G}_r} \operatorname{tr} T(\varphi)\, d\mu(T).$$

The explicit formula for $d\mu(T)$ has been determined by Harish-Chandra, and the Plancherel formula proven [26]. May I confess that I never understood the original proof of Harish-Chandra and that I am very grateful to Rebecca Herb to have given recently a more accessible proof?

2.17. For this proof and on its own right, the integrand tr $T(\varphi)$ is worth detailing. It is a difficult question and there have been several attempts to find explicit formulae for it. In an article on this subject [53], W. Schmid went so far as to declare: "For a general group G, it will be very difficult to express the discrete series characters by a completely explicit global formula in closed form—if it can be done at all." But, one should never give up hope and recently Rebecca Herb [27], [31] gave formulae for the locally L^1-function Θ_T defining the distribution character $\Theta_T(g)\, dg$ of a representation T of $\hat{G}_r$. Some of the ingredients for her formulae are related to the work of Diana Shelstad on "Orbital integrals and base change" [54], [57].

When having explicit formulae for the integrand tr $T(\varphi)$ and the Harish-Chandra formula for $d\mu(T)$, it was then (theoretically) simple to reprove 2.17. This was also accomplished by Rebecca Herb for linear semi-simple Lie groups [31], [32].

2.18. We would like now to discuss representation theory of general Lie groups. We will try to give a glimpse on some beautiful and deep results on general Lie groups and show how a large part of the specific results we have described here fit in the general theory of the "orbit method." However, as shown exemplarily by the case of $G = GL(n, \mathbf{R})$, it would be too much to hope that a single mode of explanation will lead to a total understanding of $\hat{G}$.

III. The Orbit Method

Let us consider a general Lie group G. What kind of parameters should we look for to describe $\hat{G}$? It was A. A. Kirillov [35], who discovered universal parameters for $\hat{G}$, whatever the Lie group G is. This idea is very simple and is referred to as the "orbit method."

3.1. Let G be a general Lie group, $\mathfrak{g}$ the Lie algebra of G and $\mathfrak{g}^*$ the dual vector space of $\mathfrak{g}$. As G operates on $\mathfrak{g}$ by the adjoint action, G operates on $\mathfrak{g}^*$ in such a way that

$$\langle g\cdot f, g\cdot X\rangle = \langle f, X\rangle, \qquad f\in\mathfrak{g}^*, \quad X\in\mathfrak{g}.$$

This action of G in $\mathfrak{g}^*$ is called the coadjoint action. Let us consider the orbits of G in $\mathfrak{g}^*$ under the coadjoint action of G. Kirillov's idea is that the dual $\hat{G}$ of G should be related to the dual vector space $\mathfrak{g}^*$ of $\mathfrak{g}$, or more exactly, related to the set $\mathfrak{g}^*/G$ of orbits of G in $\mathfrak{g}^*$. Let us quote here the first striking result of Kirillov [35].

3.2. Theorem. *Let G be a simply connected nilpotent Lie group, then $\hat{G}$ is isomorphic to $\mathfrak{g}^*/G$.*

I would like to comment here on this theorem. Once the principle of the correspondence is stated (the irreducible representations associated to orbits are constructed by induction) the proof follows in a straightforward manner from G. W. Mackey theory [43]. However, it is the statement itself which is remarkable. This idea generated many new insights on many aspects of representation theory of Lie groups and Lie algebras.

Let us first take a look at the space of orbits for our examples of nilpotent groups, i.e. Examples 1, 2, and 4. (If G is a connected nilpotent Lie group, with universal covering $\tilde{G}$, the set $\hat{G}$ is a subset of $\hat{\tilde{G}}$. Thus $\hat{G}$ is identified with a subset of G-orbits in $\mathfrak{g}^*$, depending on G.)

3.3. Example 1: G = Vector Space. Let $G = V$.
Then

$$\mathfrak{g} = V,$$
$$\mathfrak{g}^* = V^* \simeq \mathfrak{g}^*/G \simeq \hat{G}.$$

Thus the theorem of Kirillov is true in this case. (Fortunately!)

3.4. Example 2: $G = T$. Let $G = T = V/\Gamma$.
Then

$$\mathfrak{g} = V,$$
$$\mathfrak{g}^* = V^* = \mathfrak{g}^*/G,$$
$$\hat{G} \simeq \Gamma^* \subset \mathfrak{g}^*.$$

3.5. Example 3: The Heisenberg Group. Recall that

$$G = \left\{\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}; x, y, z\in\mathbf{R}\right\},$$

$$\mathfrak{g} = \left\{\begin{pmatrix}0 & p & e\\ 0 & 0 & q\\ 0 & 0 & 0\end{pmatrix}; p, q, e\in\mathbf{R}\right\},$$

with basis P, Q, E.

It is easy to compute the adjoint action of G on $\mathfrak{g}$:

$$\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & p & e\\ 0 & 0 & q\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}^{-1} = \begin{pmatrix}0 & p & e+qx-py\\ 0 & 0 & q\\ 0 & 0 & 0\end{pmatrix}.$$

The coadjoint action of G in $\mathfrak{g}^*$ is:

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}^{-1} \cdot (\alpha P^* + \beta Q^* + \lambda E^*) = (\alpha - \lambda y)P^* + (\beta + \lambda x)Q^* + \lambda E^*.$$

Thus we see that the orbit of the point λE^* for $\lambda \neq 0$ is the 2-dimensional plane defined by $(f, E) = \lambda$, while the points $\alpha P^* + \beta Q^*$ are 0-dimensional orbits. We obtain the following pictures of $\mathfrak{g}^*$ and $\mathfrak{g}^*/G$:

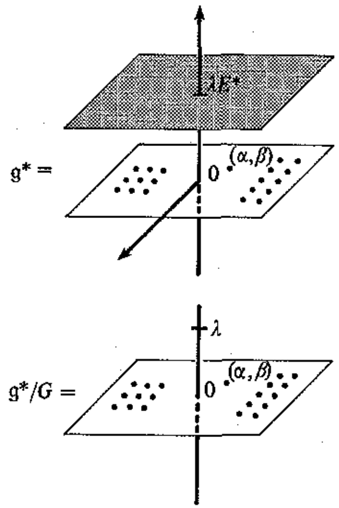

Figure 7

Note that the element $f = \alpha P^* + \beta Q^*$ of $\mathfrak{g}^*$ is a 1-dimensional character of the Lie algebra $\mathfrak{g}$ (i.e. $f[X, Y] = 0$, for every $X, Y \in \mathfrak{g}$). As expected, the Kirillov correspondence assigns to this point f of $\mathfrak{g}^*$ the character of G given by $T_f(\exp X) = e^{i(f, X)}$ which is the representation $T_{\alpha,\beta}$ of Section II, Example 4. The Kirillov correspondence assigns to the orbit of the point λE^* the representation T_λ. Thus we see clearly by comparison of 2.12, Figure 3 and 3.5, Figure 6, how $\mathfrak{g}^*/G$ becomes a natural set of parameters for $\hat{G}$.

3.6. This striking result for nilpotent groups leads to the following question. Is there also for any general Lie group G a relation between $\hat{G}$ and $\mathfrak{g}^*/G$? The answer is "yes." An explanation of the relation calls to my mind many diverse thoughts including analogies, conjectures, and expectations, and it is not possible to give such a simple answer in the general case as in the nilpotent case. I will select here the analytic aspect of these relations given by the Kirillov universal character formula, leaving out many other equally rich aspects, such as the algebraic or

geometric ones. For example, the power of Kirillov's orbit method for the study of enveloping algebras was foreshadowed by the work of J. Dixmier [13]. See the book [15] by Dixmier and, among other recent articles, those of Colette Moeglin [44], [45] and Moeglin and Rentschler [46].

3.7. Let $\mathfrak{g}$ be a Lie algebra. Consider the analytic function

$$j(X) = \det\left(\frac{e^{\operatorname{ad} X/2} - e^{-\operatorname{ad} X/2}}{\operatorname{ad} X}\right)$$

on the Lie algebra $\mathfrak{g}$, and define $j^{1/2}$ the analytic square root of j (defined at least in a neighborhood of 0).

Kirillov [36] conjectured the following universal formula for characters: Let G be a tame unimodular Lie group. For almost every representation T in $\hat{G}_r$, there exists an orbit $\mathcal{O}_T$ (of maximal dimension) of G in $\mathfrak{g}^*$ such that we have the equality of generalized functions (at least in a neighborhood of 0),

$$\operatorname{tr} T(\exp X) j^{1/2}(X) = \int_{\mathcal{O}_T} e^{i\langle \xi, X\rangle}\, d\mu(\xi),$$

where $d\mu(\xi)$ is a G-invariant measure on $\mathcal{O}$. (The normalization of $d\mu(\xi)$ will be made precise later on.)

For this equality to hold, we should have:

$$\operatorname{tr} \int_{\mathfrak{g}} T(\exp X) j^{1/2}(X)\varphi(X)\, dX = \int_{\mathcal{O}_T} \left(\int_{\mathfrak{g}} e^{i\langle \xi, X\rangle}\varphi(X)\, dX\right) d\mu(\xi)$$

at least, for every C^∞ function φ on $\mathfrak{g}$ supported in a small neighborhood of 0.

3.8. The Kirillov character formula defines conjecturally a map $T \to \mathcal{O}_T$ from $\hat{G}_r$ to $\mathfrak{g}^*/G$. Of course, in Example 1: $G = V$, the map $f \in V^* \to \chi_f(x) = e^{i(f, x)}$ is also the one compatible with this universal character formula.

What is the image of $\hat{G}_r$ under this map? As it is obvious from the case of compact groups ($\hat{G}$ being a discrete set), not every orbit corresponds to a representation of G. The corresponding orbit should satisfy some integrality conditions, which appear naturally when considering the inverse problem: How to construct from an orbit $\mathcal{O}$ of the coadjoint representation a "natural" representation $T_{\mathcal{O}}$ associated to $\mathcal{O}$ having the prescribed character formula. This inverse problem is referred to as the "quantization" of an orbit and, from my point of view, has no completely satisfactory answer. The reader should consult the book [23] by Guillemin and Sternberg for insights on possible "quantization methods."

3.9. Let us now discuss these integrability conditions on $\mathcal{O}$, which arise already naturally in a preliminary step, the Kostant–Souriau prequantization of the orbit $\mathcal{O}$ [40], [58].

Let $\mathcal{O}$ be an orbit of the coadjoint representation. If $f \in \mathcal{O}$, we have $\mathcal{O} = G \cdot f = G/G(f)$. The stabilizer $G(f)$ of f has Lie algebra $\mathfrak{g}(f)$

$$\mathfrak{g}(f) = \{X \in \mathfrak{g}, f([X, Y]) = 0, \text{ for every } Y \in \mathfrak{g}\}.$$

We can define an alternate non-degenerate 2-form σ_f on the tangent space $\mathfrak{g}\cdot f = \mathfrak{g}/\mathfrak{g}(f)$ to the orbit $\mathcal{O}$ at the point f by the formula $\sigma_f(X\cdot f, Y\cdot f) = f([X, Y])$. This way, we obtain a 2-form σ on $\mathcal{O}$. This form σ gives to $\mathcal{O}$ the structure of a symplectic manifold. If $\dim \mathcal{O} = 2d$, the term $\sigma^d/(d!\,(2\pi)^d)$ of maximal degree of $e^{\sigma/2\pi}$ defines the canonical Liouville measure $dm_{\mathcal{O}}$ on $\mathcal{O}$.

Consider the map $X \to f(X)$ on $\mathfrak{g}(f)$. Clearly $f([X, Y]) = 0$ for $X, Y \in \mathfrak{g}(f)$. We introduce:

$$K(f) = \{\chi, \text{ characters of } G(f) \text{ such that } \chi(\exp X) = e^{i(f,X)} \text{ for } X \in \mathfrak{g}(f)\}.$$

We say that an orbit is integral if $K(f) \neq \varnothing$. (If $G(f)$ is simply connected, $K(f)$ consists of one element.) In particular, f must satisfy the integrality conditions: $(1/2\pi)(f, X) \in \mathbf{Z}$ for all $X \in \mathfrak{g}(f)$ such that $\exp X = e$.

For each character $\chi \in K(f)$, we can construct a line bundle $\mathscr{L}_\chi \to \mathcal{O}$ with $\mathscr{L}_\chi = G \times \mathbf{C}/G(f)$, where $u \in G(f)$ acts on $G \times \mathbf{C}$ by $(g, z)\cdot u = (gu, \chi_f(u)^{-1}z)$. It is not difficult to see that the first Chern class of this line bundle $\mathscr{L}_\chi$ is $\sigma/2\pi$.

Let us say here that the universal formula has some formal analogy with the index formula for the twisted Dirac operator. For example, if G is compact and $\mathcal{O}$ admits a spin structure, the universal formula for $X = 0$ gives us an integral formula for the dimension of the representation $T_{\mathcal{O}}$

$$\dim T_{\mathcal{O}} = \int_{\mathcal{O}} \frac{\sigma^d}{(2\pi)^d d!},$$

which coincides with the index formula ([2]) for the twisted (by $\mathscr{L}_\chi$) Dirac operator D_χ on $\mathcal{O}$. (Recall here that $\mathcal{O}$ was supposed to be of maximal dimension. It follows then from [21] that the tangent bundle to $\mathcal{O}$ is a trivial element of K-theory, thus the term $\mathscr{A}$ contributing to the index formula for D_χ is here equal to 1.) Furthermore, Nicole Berline and I [8], have shown that it is indeed possible to give an integral formula for the equivariant index of a connected compact group of transformations of an elliptic complex over a compact manifold, generalizing Kirillov's universal character formula.

In another direction, when $\mathcal{O}$ is not compact but still admits a G-invariant spin structure, the formula of Connes and Moscovici [12] for the L^2-index of D_χ is a precise analogue of Kirillov's formula for $X = 0$. Thus, at least for orbits of maximal dimension with compact stabilizers and spin structures, all these indications would lead us to discover (as Christopher Columbus "discovered" America) the importance of the twisted Dirac operator on orbits to construct the "quantized" representation $T_{\mathcal{O}}$. However, if $\mathcal{O}$ is a general orbit of a group G in $\mathfrak{g}^*$, there is no canonical construction of a representation $T_{\mathcal{O}}$ (even, for some $\mathcal{O}$'s, no construction whatsoever [63]), nor are there sufficiently powerful theorems on unicity of various methods of quantizations. (When G is solvable, the "uniqueness of the quantized orbit" is elucidated in numerous cases. Let us quote the uniqueness theorem of Auslander and Kostant on "independence of positive polarizations" [3] and the uniqueness theorems of Penney [48] and Rosenberg [50] on the realization of $T_{\mathcal{O}}$ in L^2-cohomology spaces.)

3.11. The preceding discussion stressed the importance for the representation theory of G of a subset of orbits satisfying integrality conditions.

Let G be a real algebraic group. By a profound generalization of the result of Kirillov on nilpotent groups, Michel Duflo [17] was able to construct a set X of parameters for $\hat{G}_r$. (When G is semi-simple, this construction is based on Harish-Chandra's work. In case G solvable, a set analogous to X was introduced by Pukanszky [49].) Duflo defines the notion of G-admissible orbits, which is an appropriate modification (see Appendix 1) of the notion of G-integral orbits. In all the examples of the text, the set $\mathfrak{g}_a^*$ of admissible orbits coincide with the set of integral orbits. The set X is given together with a map $d: X \to \mathfrak{g}_a^*/G$ having finite fibers; M. S. Khalgui [34] proved under some very general hypothesis that the universal formula for characters is valid and the map d is indeed the map $T \to \mathcal{O}_T$. We will describe briefly X in Appendix 1. In the examples of this text, we can describe X as follows: Consider the set

$$P = \{(f, \chi);\ G\cdot f \text{ of maximal dimension, } \chi \in K(f)\}.$$

The group G acts on P. The set P/G is fibered naturally over the set of integral orbits of maximal dimension by the map $d(G\cdot(f, \chi)) = G\cdot f$. We may take for X the set P/G (or a subset with complement of measure 0, as X is defined up to a set of measure 0 for the Plancherel measure $d\mu(T)$).

3.12. Let us describe now for all our examples this fibering and indicate how to prove its compatibility with the universal character formula, i.e. let us relate the character tr T with the Fourier transform of the corresponding orbits $\mathcal{O}_T$.

3.13. Example 4: The Heisenberg Group. The set X is the set of orbits $\mathcal{O}_\lambda$ of maximal dimension where $\mathcal{O}_\lambda = G\cdot\lambda E^*$, $\lambda \neq 0$. Let us consider the character tr T_λ of the representation T_λ (2.12). We will verify that according to the universal character formula (j in this example is identically 1)

$$\operatorname{tr} T_\lambda(\exp X) = \int_{\mathcal{O}_\lambda} e^{i\langle\xi, X\rangle}\frac{\sigma_\lambda}{2\pi},$$

where σ_λ is the canonical 2-form on the orbit $\mathcal{O}_\lambda$.

Indeed, for φ a Schwartz function on $\mathfrak{g}$, we have: (2.12(b))

$$\operatorname{tr}\left(\int T_\lambda(\exp X)\varphi(X)\,dX\right) = \frac{2\pi}{\lambda}\int e^{i\lambda z}\varphi(zE)\,dz,$$

while the second member is

$$\int_{\mathbf{R}^2}\left(\int_{\mathbf{R}^3} e^{i(\lambda E^* + pP^* + qQ^*, xP + yQ + zE)}\varphi(x, y, z)\,dx\,dy\,dz\right)\frac{1}{2\pi\lambda}\,dp\,dq$$
$$= \frac{2\pi}{\lambda}\int e^{i\lambda z}\varphi(0, 0, z)\,dz,$$

by the usual Fourier inversion formula.

3.14. For a simply connected nilpotent group N, the function j is identically 1, and the set X coincides with the set of orbits of maximal dimension. Recall that Kirillov theorem (3.2) gives a description of $\hat{G}$ as $\mathfrak{g}^*/G$. The corresponding character formula

$$\operatorname{tr} T(\exp X) = \int_{\mathcal{O}_T} e^{i\langle \xi, X\rangle} \frac{\sigma^d}{(2\pi)^d d!}$$

holds in fact for every representation T in $\hat{G}$.

More generally, if the representation $T_{\mathcal{O}}$ can be constructed via Mackey induction, the universal character formula holds under certain conditions and can be proven easily ([42]).

Unfortunately, for a general Lie group G, as stressed in 2.10, there is no parametrization of the entire set $\hat{G}$, nor systematic construction of a representation $T_{\mathcal{O}}$ corresponding to an admissible orbit $\mathcal{O}$, if $\mathcal{O}$ is not of maximal dimension. Furthermore, even if $T_{\mathcal{O}}$ is "given" to us, the universal character formula for tr $T_{\mathcal{O}}$ would have to be modified ([33], [8]).

3.15. We now consider the:

EXAMPLE 5: $G = SU(2)$. We have

$$\mathfrak{g} = \mathfrak{su}(2) = \left\{\begin{pmatrix} ix_3 & -x_1 + ix_2 \\ x_1 + ix_2 & -ix_3 \end{pmatrix}, x_i \in \mathbf{R}\right\}.$$

We identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via the G-invariant bilinear form $(X, Y) \to -\frac{1}{2}\operatorname{Tr}(XY)$. Recall that the function $X \to \det X$ is invariant by the adjoint action of G on $\mathfrak{g}$. Thus the orbits of G in $\mathfrak{g}^*$ are the spheres $x_1^2 + x_2^2 + x_3^2 = r^2$.

If

$$f = \begin{pmatrix} i\lambda & 0 \\ 0 & -i\lambda \end{pmatrix} \in \mathfrak{g}^* \qquad (\lambda \neq 0),$$

we have

$$\mathfrak{g}(f) = \left\{\begin{pmatrix} i\theta & 0 \\ 0 & -i\theta \end{pmatrix}; \theta \in \mathbf{R}\right\},$$

$$G(f) = \left\{\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}; \theta \in \mathbf{R}\right\}.$$

The form f is integral if $\chi_f\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = e^{i\lambda\theta}$ is well defined on $G(f)$, i.e. if $\lambda \in \mathbf{Z} - \{0\}$. As $G(f)$ is connected, $K(f)$ is either empty or, if f is integral, it is the set with the one element χ_f.

Thus the set $\mathfrak{g}_G^*$ of integral orbits of maximal dimension coincides with all spheres S_n with positive integral radius. We may picture the set $\mathfrak{g}_G^*$ and X by:

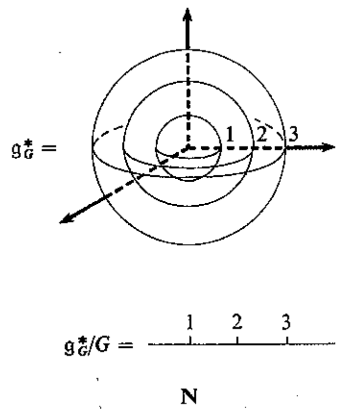

Figure 8

As we compare (2.13), Figure 5 with Figure 8, we recovered our description of $\hat{G} = \{T_n; n \in N\}$.

Let us describe the relation of the character of T_n with the Fourier transform of the measure on the sphere S_n. It is easy to see that the function $j(X)$ has an analytic square root on all of $\mathfrak{g}$ and that

$$j^{1/2}\begin{pmatrix} \theta & 0 \\ 0 & -\theta \end{pmatrix} = \frac{e^{i\theta} - e^{-i\theta}}{2i\theta}.$$

The universal character formula asserts that

$$\operatorname{tr} T_n\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} j^{1/2}\begin{pmatrix} \theta & 0 \\ 0 & -\theta \end{pmatrix} = \int_{x_1^2+x_2^2+x_3^2=n^2} e^{ix_3\theta} \frac{\sigma}{2\pi},$$

i.e.

$$\frac{e^{in\theta} - e^{-in\theta}}{e^{i\theta} - e^{-i\theta}} \frac{e^{i\theta} - e^{-i\theta}}{2i\theta} = \int_{x_1^2+x_2^2+x_3^2=n^2} e^{ix_3\theta} \frac{\sigma}{2\pi}.$$

Thus we have to prove:

$$\frac{e^{in\theta}}{2i\theta} - \frac{e^{-in\theta}}{2i\theta} = \int_{x_1^2+x_2^2+x_3^2=n^2} e^{ix_3\theta} \frac{\sigma}{2\pi},$$

for $\sigma/2\pi$ the canonical Liouville measure on the orbit S_n of SU(2).

It is immediate to verify this formula using spherical coordinates (see Appendix 2).

3.16. For a compact Lie group G, the universal character formula is equivalent to a well-known formula of Harish-Chandra, established long before. Developing an earlier idea of R. Bott [10], Nicole Berline and I [7] have given a simpler proof of a more general formula of Duistermaat–Heckmann for torus actions on symplectic manifolds [22].

We may explain the idea of the method on the preceding Example 5 as follows: Consider the action of the one-parameter group $e^{i\theta}$ on the orbit S_n. It has two fixed points, the point $p^+ = (0, 0, n)$ and the point $p^- = (0, 0, -n)$.

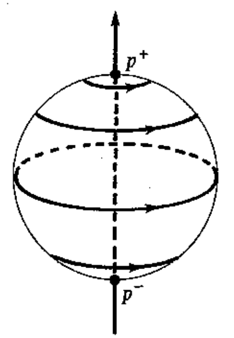

Figure 9

It is easy to see that the form to be integrated is exact, except at these two points and that the two terms $e^{in\theta}/2i\theta$ and $e^{-in\theta}/2i\theta$ comes from a calculus of residues at these two points.

3.17. Let us finally consider our

EXAMPLE 6: $G = \mathrm{SL}(2, \mathbf{R})$. We have

$$\mathfrak{g} = \mathfrak{sl}(2, \mathbf{R})$$
$$= \left\{\begin{pmatrix} x_1 & x_2 + x_3 \\ x_2 - x_3 & -x_1 \end{pmatrix}; x_i \in \mathbf{R}\right\}.$$

We identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via the G-invariant bilinear form $(X, Y) \to \frac{1}{2}\operatorname{tr}(XY)$. Recall that the function $\det X = x_3^2 - (x_1^2 + x_2^2)$ is invariant by the action of G on $\mathfrak{g}^*$. From this, it follows that the orbits of G in $\mathfrak{g}^*$ are:

(1) (a) The upper sheet $x_3 \geq 0$ of the two-sheeted hyperboloid

$$x_3^2 - (x_1^2 + x_2^2) = \lambda^2, \qquad \lambda \neq 0.$$

A typical element of this orbit is

$$f = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}, \qquad \lambda > 0.$$

(b) The lower sheet $x_3 \leq 0$ of the two-sheeted hyperboloid

$$x_3^2 - (x_1^2 + x_2^2) = \lambda^2, \qquad \lambda \neq 0.$$

A typical element of this orbit is

$$f = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}, \qquad \lambda < 0.$$

We shall denote by $\mathcal{O}_\lambda^d$ the orbit of the element $f = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}$.

(2) The one-sheeted hyperboloid $x_3^2 - (x_1^2 + x_2^2) = -s^2 (s \neq 0)$.

A typical element of this orbit is $f = \begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix}$. We denote by $\mathcal{O}_s^p$ the orbit of the element $\begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix}$.

(3) The point $\{0\}$ and the two connected components of the light cone

$$(x_3^2 - x_1^2 - x_2^2 = 0, x \neq 0).$$

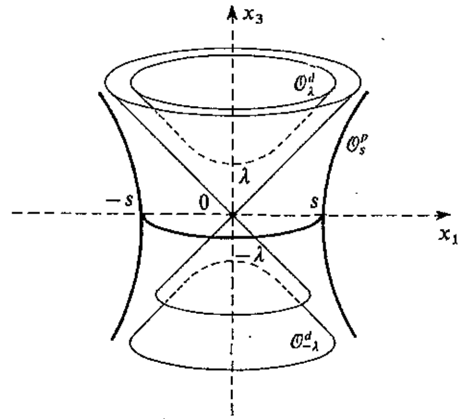

Figure 10

We determine now the set of integral orbits:

(1) For

$$f = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix} \in \mathfrak{g}^* \qquad (\lambda \neq 0),$$

we have:

$$\mathfrak{g}(f) = \left\{\begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}; \theta \in \mathbf{R}\right\},$$
$$G(f) = \left\{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}; \theta \in \mathbf{R}\right\}.$$

The form f is integral if

$$\chi_f\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = e^{i\lambda\theta}$$

is well defined on $G(f)$, i.e. if $\lambda \in \mathbf{Z} - \{0\}$. In this case ($\lambda$ integer), as $G(f)$ is connected, $K(f)$ consists of one element, the character χ_f.

(2) For

$$f = \begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix} \in \mathfrak{g}^*,$$

$$\mathfrak{g}(f) = \left\{\begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}; t \in \mathbf{R}\right\},$$

$$G(f) = \left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}; a \in \mathbf{R} - \{0\}\right\}.$$

Every f of this form is integral, and for any s, the set $K(f)$ consists of two elements, the characters χ_s^+ and χ_s^- with

$$\chi_s^+\begin{pmatrix} \varepsilon e^t & 0 \\ 0 & \varepsilon e^{-t} \end{pmatrix} = e^{ist}, \qquad \varepsilon = \pm 1,$$

$$\chi_s^-\begin{pmatrix} \varepsilon e^t & 0 \\ 0 & \varepsilon e^{-t} \end{pmatrix} = \varepsilon e^{ist}.$$

We denote by $\mathfrak{g}_G^*$ the set of integral orbits of maximal dimension of the sets (1) and (2) (i.e. we omit the light cone). Thus $\mathfrak{g}_G^*$ consists of the orbits:

$$\mathcal{O}_n^d = G\cdot\begin{pmatrix} 0 & n \\ -n & 0 \end{pmatrix}, \qquad n > 0,$$

$$\mathcal{O}_n^d = G\cdot\begin{pmatrix} 0 & n \\ -n & 0 \end{pmatrix}, \qquad n < 0,$$

$$\mathcal{O}_s^p = G\cdot\begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix}, \qquad s \neq 0.$$

Thus we obtain Figure 11 for $\mathfrak{g}_G^*$ and $\mathfrak{g}_G^*/G$.

The set $X = \hat{G}_r$ is given by X^{irr}/G with $X^{\text{irr}} = \{(f, \chi), f \in \mathfrak{g}_G^*, \chi \in K(f)\}$. Thus X is fibered over $\mathfrak{g}_G^*/G$ with fibers consisting of the one point T_n over the orbit $\mathcal{O}_n^d$ (d for discrete) and of the two points $T_s^\pm$ over the orbit $\mathcal{O}_s^p$ (p for principal). The reader may then compare Figure 10 with Figure 5 (2.14) to visualize the fibering.

The universal character formula for the representation T_n associated to the orbit $\mathcal{O}_n$ is equivalent to the equality of generalized functions ($n > 0$):

$$\text{“}\frac{e^{in\theta}}{2i\theta} = -\int_{\substack{x_3^2-(x_1^2+x_2^2)=n^2 \\ x_3\geq 0}} e^{ix_3\theta}\,\frac{\sigma}{2\pi}\text{”},$$

when $\sigma/2\pi$ is the canonical 2-form on $\mathcal{O}_n$.

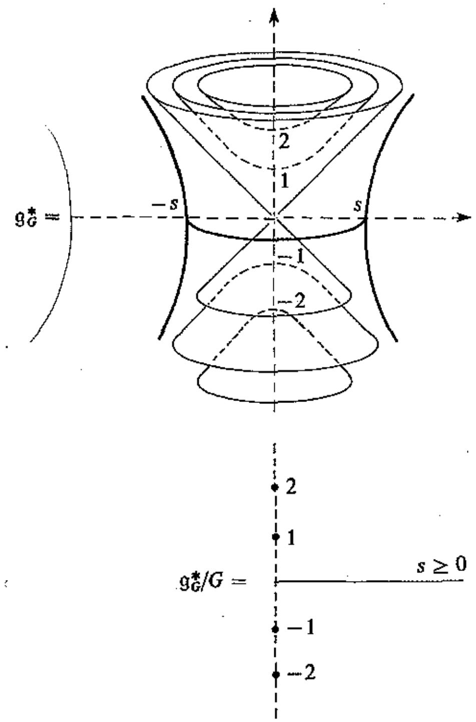

Figure 11

3.18. The validity of this formula or more generally the validity of the universal character formula for a semi-simple Lie group was established by W. Rossmann [51]: This fundamental result gave legitimacy to the claims of universality for the orbit method.

It is possible to generalize the argument sketched in 3.16 to prove Rossmann's formula [7]:

Let us remark, in the case of SL(2, **R**), that the action of the one-parameter group $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ on $\mathcal{O}_n$ has only one fixed point, the point $p_0 = (0, 0, n)$.

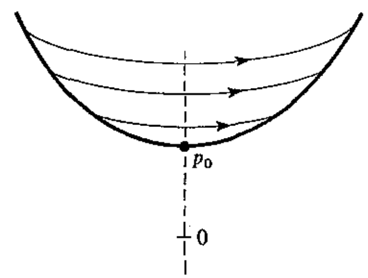

Figure 12

As in the case of SU(2), the form to be integrated is exact except at this point. Thus there is only one residue to be calculated which leads to the term $-e^{in\theta}/2i\theta$.

Similarly, the character formula $\Theta_\lambda(g)$ for the representation T_λ of the discrete series has a simple geometric interpretation as a fixed point formula, for the elements g belonging to the elliptic set. However, the general formulae of Rebecca Herb for arbitrary regular elements g of G are not yet reducible to a simple geometric interpretation.

3.19. We have related in Examples 1–6 the set $\hat{G}_r$ with the geometric set X, and the distribution tr T with Fourier transforms of canonical measures on orbits. Thus, due to the work of Michel Duflo, we see that the set $\hat{G}_r$ is described adequately. It is still, however, an open question to determine explicitly for a general Lie group G the corresponding Plancherel measure on X.[1] When G is a simply connected nilpotent Lie group, the set $\hat{G}$ is merely the set of orbits $\mathfrak{g}^*/G$. Each orbit $\mathcal{O}$ has a canonical measure $dm_{\mathcal{O}}$. Let dg be a Haar measure on G, dX the corresponding measure on $\mathfrak{g}$, df the dual Haar measure on $\mathfrak{g}^*$. The usual Fourier inversion formula on the vector space

$$\varphi(0) = \int_{\mathfrak{g}^*} \hat{\varphi}(f)\, df,$$

yield immediately to the Plancherel formula:

$$\tilde{\varphi}(e) = \varphi(0) = \int_{\mathfrak{g}^*/G} \left(\int_{\mathcal{O}} \hat{\varphi}\, dm_{\mathcal{O}} \right) dp(\mathcal{O}) = \int_{\mathfrak{g}^*/G} \operatorname{tr}(T_{\mathcal{O}}, \tilde{\varphi})\, dp(\mathcal{O}),$$

with $\tilde{\varphi}$ the C^∞-function on G such that $\varphi = \tilde{\varphi} \cdot \exp$, and $dp = df/dm_{\mathcal{O}}$ the quotient measure of df by the canonical measures $dm_{\mathcal{O}}$. A similar formula holds for a solvable simply connected type I unimodular Lie group G. (Pukanszky [49], Charbonnel [11]).

In general, as we have seen, orbits $\mathcal{O}_T$ corresponding to representations must satisfy some non-empty integrality conditions and the corresponding set $\mathfrak{g}^*_G = \{\bigcup \mathcal{O}_T, T \in \hat{G}_r\}$ is not dense in $\mathfrak{g}^*$.

The case of a torus $T = V/\Gamma$ is instructive. In this case, $\mathfrak{g} = V$ and $\mathfrak{g}^* = V^*$, while the set of orbits corresponding to representations of T is the discrete subset $\Gamma^* \subset V^*$. Let dx be the Euclidean measure on V giving measure 1 to the fundamental parallelepiped on Γ. The Poisson summation formula is:

$$\sum_{\gamma \in \Gamma} \varphi(\gamma) = \sum_{\gamma^* \in \Gamma^*} \varphi(\gamma^*),$$

that we may rewrite as:

$$\sum_{\substack{\gamma \in \mathfrak{g} \\ \exp \gamma = e}} \varphi(\gamma) = \sum_{\gamma^* \in G \subset \mathfrak{g}^*} \varphi(\gamma^*).$$

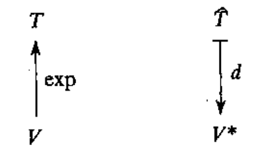

Figure 13

[1] The Plancherel measure is now determined (M. Duflo, special year in Lie groups representations, Maryland, 1983).

3.20. Consider for a general Lie group G the maps $\exp: \mathfrak{g} \to G$ and $d: \hat{G}_r \to \mathfrak{g}^*/G$, (we write $d(T) = \mathcal{O}_T$) and the subsets

$$\mathfrak{g}_G = \{X \in \mathfrak{g}; \exp X = e\} \quad \text{of } \mathfrak{g},$$

$$\mathfrak{g}^*_G = \bigcup \mathcal{O}_T, \quad T \in \hat{G}_r \qquad \text{of } \mathfrak{g}^*.$$

Recall that $\mathfrak{g}^*_G$ is determined in purely geometric terms by some integrality conditions (see Appendix 1). In the examples given in this text, $\mathfrak{g}^*_G$ may be taken as the set of integral orbits of maximum dimension. I conjectured [61] that a similar "Poisson formula" relates the sets $\mathfrak{g}_G$ and $\mathfrak{g}^*_G$. Let us formulate this conjecture in these terms (see Appendix 1 for a more precise formulation): "There exists a G-invariant positive generalized function v_G on $\mathfrak{g}^*$ of support $\mathfrak{g}^*_G$ whose Fourier transform is a distribution n_G, such that the support of n_G is contained in $\mathfrak{g}_G$ and n_G coincides with $\delta(0)$ near the origin."

In the case of the torus T, v_G is the δ-function of the lattice Γ^* and n_G is the δ-distribution of the lattice Γ.

Of course if G is a simply connected nilpotent Lie group, we can take $n_G = \delta(0)$ and $v_G = 1$. I proved that this conjecture holds for G semi-simple linear [62], and Duflo proved that this conjecture holds for G any complex Lie group [19].

3.21. Let us now examine briefly in Examples 5 and 6 the content of this conjecture. As it will be clear on the examples, the form of the generalized function v_G is prescribed by the Plancherel formula of G.

3.22. Example 5: $G = SU(2)$. Recall that the orbits of G in $\mathfrak{g}$ ($\simeq \mathfrak{g}^*$) are the spheres $S_r = \{x_1^2 + x_2^2 + x_3^2 = r^2\}$ ($r \geq 0$).

As

$$\exp\begin{pmatrix} i\theta & 0 \\ 0 & -i\theta \end{pmatrix} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

the set $\mathfrak{g}_G$ consists of all orbits S_r of radius $r = (0, 2\pi, 4\pi, 6\pi, \ldots)$.

Recall that $\mathfrak{g}^*_G$ consists of all orbits S_r of non-zero integral radius $(1, 2, 3, \ldots)$.

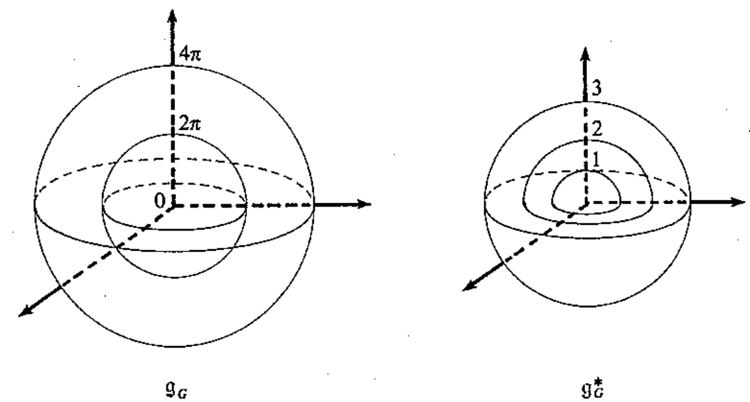

Figure 14

Let, for $a \in \mathbf{R}$, Θ_a be the G-invariant function on $\mathfrak{g}^*$ such that

$$\Theta_a\begin{pmatrix} i\lambda & 0 \\ 0 & -i\lambda \end{pmatrix} = \tfrac{1}{2}(e^{i\lambda a} + e^{-i\lambda a}).$$

Let v_G be the G-invariant generalized function on $\mathfrak{g}^*$ given by $v = \sum_{a \in 2\pi\mathbf{Z}} \Theta_a$, i.e.:

$$v_G\begin{pmatrix} i\lambda & 0 \\ 0 & -i\lambda \end{pmatrix} = \sum_n e^{2i\pi n\lambda}.$$

It is not difficult to prove (see Appendix 2) that for $df = (1/4\pi)\, d\xi_1\, d\xi_2\, d\xi_3$ on $\mathfrak{g} \simeq \mathfrak{g}^*$

$$\int_{\mathfrak{g}^*} v_G(f)\varphi(f)\, df = \sum_{n \in \mathbf{N}} n\left(\int_{S_n} \varphi \frac{\sigma_n}{2\pi}\right),$$

where $\sigma_n/2\pi$ is the canonical Liouville measure on S_n. Thus $v_G(f)df$ is a positive measure supported on $\mathfrak{g}_G^*$ (this measure is clearly derived from the Plancherel formula on $\hat{G}$). The proof of this equality follows from the usual Poisson summation formula:

$$\sum_n e^{2i\pi nx}\, dx = \sum \delta(n).$$

In Appendix 2, we will compute the Fourier transform n_G of v_G and show that n_G satisfies the required properties of the conjecture, in particular that n_G is a distribution of support $\mathfrak{g}_G$.

3.23. Example 6: $G = \mathrm{SL}(2, \mathbf{R})$. Denote by $\mathcal{O}_\lambda^d$ the orbit of the element $\begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}$ and by $\mathcal{O}_s^p$ the orbit of the element $\begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix}$. As

$$\exp\begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

the set $\mathfrak{g}_G$ consists of all orbits $\mathcal{O}_a^d$, for $a \in 2\pi\mathbf{Z}$. Recall that the set $\mathfrak{g}_G^*$ consists of the orbits $\mathcal{O}_n^d$, for n non-zero integer and of all the orbits $\mathcal{O}_s^p$ ($s \neq 0$), see Figure 15.

Define, for $a \in \mathbf{R}$, Θ_a as being the G-invariant function on $\mathfrak{g}^*$ such that

$$\Theta_a\begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix} = e^{i\lambda a}, \qquad \Theta_a\begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix} = e^{-|as|}.$$

(Θ_a is defined by this relation except on the light cone, but, as Θ_a is bounded, Θ_a defines unambiguously a generalized function on $\mathfrak{g}^*$.)

Define $v_G = \sum_{a \in 2\pi\mathbf{Z}} \Theta_a$. It is not difficult to prove (see Appendix 3) that for $df = (1/4\pi)\, dx_1\, dx_2\, dx_3$ on $\mathfrak{g} \simeq \mathfrak{g}^*$

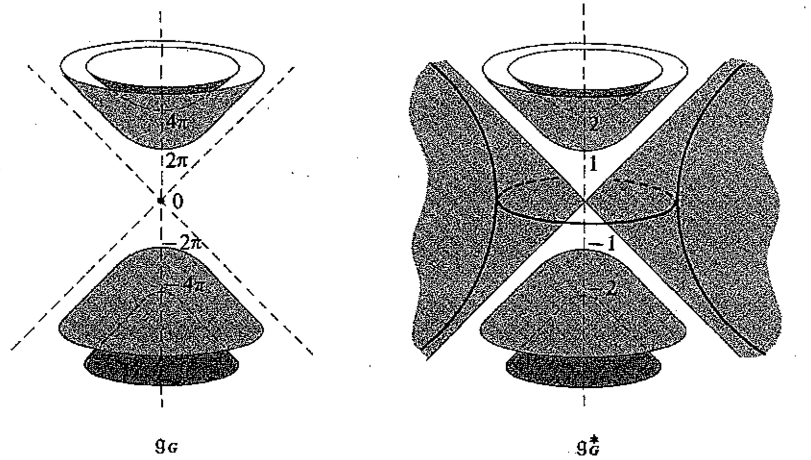

Figure 15

$$\text{3.24} \quad \int_{\mathfrak{g}^*} v_G(f)\varphi(f)\, df = \sum_{n \in \mathbf{Z}} |n| \left(\int_{\mathcal{O}_n^d} \varphi \frac{\sigma}{2\pi}\right) + \int_0^\infty s \coth \pi s \left(\int_{\mathcal{O}_s^p} \varphi \frac{\sigma}{2\pi}\right) ds.$$

Then $v_G(f)\, df$ is a positive measure supported on $\mathfrak{g}_G^*$. (This measure is clearly derived from the image of the Plancherel measure on $\hat{G}_r$ by the map $\hat{G}_r \to \mathfrak{g}_G^*/G$. Recall that the fiber of this map above $\mathcal{O}_s$ consists of two points T_s^+ and T_s^- with respective weights in the Plancherel measure $\frac{1}{2}s$ than $(\pi s/2)$ and $\frac{1}{2}s \coth(\pi s/2)$ and that $\frac{1}{2}$ than $(\pi s/2) + \frac{1}{2}\coth(\pi s/2) = \coth \pi s$.)

The proof of the formula 3.24 relies on the two identities:

$$\sum_{n \in \mathbf{Z}} e^{2i\pi n\lambda}\, d\lambda = \sum \delta(n),$$

$$\sum_{n \in \mathbf{Z}} e^{-|2\pi ns|} = \coth \pi s \quad \text{for } s \neq 0.$$

Let us remark that the left-hand side of these two identities involves the formulas for the characters of the discrete series T_n, respectively, on the element $\begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}$ and on the element $\begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix}$. The proof of the conjecture for general linear semi-simple Lie groups is based on similar equalities between the discrete series constants and the Plancherel functions ([62]).

The proof of this conjecture for non-linear semi-simple Lie groups would lead to some better understanding of the relation between character formulae, integral orbits, and the Plancherel measure on G.

I hope many mathematicians, women and men, will continue to work on the topics touched upon here.

APPENDIX 1

Duflo's Parametrization of $\hat{G}_r$

Let $\mathcal{O}$ be an orbit of the co-adjoint representation. If $f \in \mathcal{O}$, we have $\mathcal{O} = G \cdot f = G/G(f)$. The stabilizer $G(f)$ of f has Lie algebra $\mathfrak{g}(f)$. The 2-form σ_f on the tangent space $\mathfrak{g} \cdot f = \mathfrak{g}/\mathfrak{g}(f)$ to the orbit $\mathcal{O}$ at the point f is invariant by $\tilde{G}(f)$. Thus we obtain a morphism i_f from $G(f)$ to $\mathrm{Sp}(\mathfrak{g}/\mathfrak{g}(f))$. Consider the 2-fold cover $\tilde{G}(f) \to G(f)$ image by the map i_f of the canonical 2-fold cover $\mathrm{Mp}(\mathfrak{g}/\mathfrak{g}(f)) \to \mathrm{Sp}(\mathfrak{g}/\mathfrak{g}(f))$ of the symplectic group by the metaplectic group. Let (ε, e) be the reciprocal image of $e \in G(f)$ in $\tilde{G}(f)$. We denote by

$$X(f) = \{\tau, \text{irreducible representations of } G(f) \text{ in } V_\tau \text{ such that}$$
$$(1)\ \tau(\exp X) = e^{i\langle f, X\rangle}\, \mathrm{Id}_{V_\tau} \text{ for } X \in \mathfrak{g}(f);$$
$$(2)\ \tau(\varepsilon) = -\mathrm{Id}_{V_\tau}\}.$$

An element $f \in \mathfrak{g}^*$ is called admissible if $X(f)$ is not empty. An element $f \in \mathfrak{g}^*$ is called regular if $\mathcal{O} = G \cdot f$ is of maximal dimension.

Let G be a real algebraic group. The set $X(f)$ consists of a finite number of finite dimensional representations of $\tilde{G}(f)$. Let f be regular, then the connected component $G(f)^0$ of $G(f)$ is commutative and is the direct product of its semi-simple part $S(f)$ with its unipotent part $U(f)$. Call f strongly regular, if $S(f)$ is of maximal dimension (among the subgroups $S(f)$, for f regular). The conjugacy classes of the subgroups $S(f)$, f strongly regular, are in finite number. (If G is a complex group, the subgroups $S(f)$, f strongly regular, are in the same conjugacy class. This is not the case of the subgroups $U(f)$ [39]). If G is a semi-simple group, the subgroup $S(f)$, for f strongly regular, is a Cartan subgroup of G.

Denote by

$$\mathfrak{g}_G^* = \{f \in \mathfrak{g}^*, f \text{ admissible}, f \text{ strongly regular}\},$$
$$\mathfrak{g}_G = \{X \in \mathfrak{g}; \exp X = e\}.$$

Define:

$$X^{\mathrm{irr}} = \{(f, \tau); f \in \mathfrak{g}_G^*, \tau \in X(f)\}.$$

The group G acts on X^{irr}. Define $X = X^{\mathrm{irr}}/G$. Duflo constructed an application $(f, \tau) \mapsto T_{f,\tau}$ of X^{irr} in $\hat{G}$, which induces an injective application d of X into $\hat{G}$. Finally, under some hypothesis which are probably automatically satisfied, Duflo proved that the Plancherel measure $d\mu(T)$ is concentrated on $d(X)$. Thus we may consider that the problem of determining $\hat{G}_r$ is entirely solved by these results.

Let us relate this parametrization X of $\hat{G}_r$ with the Kirillov character formula. Let σ be the canonical 2-form on the orbit $\mathcal{O}$ of dimension $2d$. Consider $e^{\sigma/2\pi}$ and its term $(1/d!)(\sigma^d/(2\pi)^d)$ of maximal degree. Let G be unimodular, $f \in \mathfrak{g}_G^*$ such that $\mathcal{O} = G \cdot f$ is closed, $\tau \in X(f)$, then M. S. Khalgui proved that indeed:

$$\mathrm{tr}\, T_{f,\tau}(\exp X) j(X)^{1/2} = \int_{\mathcal{O}} (\dim \tau) e^{i\langle \xi, X\rangle} \frac{1}{d!} \frac{\sigma^d}{(2\pi)^d},$$

as an equality of generalized functions in a neighborhood of O. [34].

(For $\mathfrak{g}$ semi-simple, a corner-stone case, this was due to Rossman [51]. For $\mathfrak{g}$ solvable, it was proven by Duflo [9].) Thus the map $(f, \tau) \mapsto f$ of X^{irr} into $\mathfrak{g}_G^*$ induces the Kirillov map $T \mapsto \mathcal{O}_T$ from $\hat{G}_r$ to $\mathfrak{g}_G^*/G$.

Let us now formulate my conjecture on the Poisson–Plancherel formula. Let dg be a Haar measure on G. Let dX be the corresponding Euclidean measure on $\mathfrak{g}$: i.e. for φ supported in a small neighborhood of 0 (where the exponential map is a diffeomorphism) $\int_G \tilde{\varphi}(g)\, dg = \int_{\mathfrak{g}} \varphi(X) j(X)\, dX$, with $\tilde{\varphi}$ the function on G such that $\varphi = \tilde{\varphi} \cdot \exp$. Let df be the dual measure to dX on $\mathfrak{g}^*$, i.e. df is such that

$$\varphi(0) = \int_{\mathfrak{g}^*} \left(\int_{\mathfrak{g}} e^{i\langle f, X\rangle} \varphi(X)\, dX \right) df = \int_{\mathfrak{g}^*} \hat{\varphi}(f)\, df.$$

Let $d\mu(f, \tau)$ be the Plancherel measure on X. Denote by $\tilde{\mu}$ the measure on $\mathfrak{g}_G^*/G$ image of the measure $(\dim \tau)\, d\mu(f, \tau)$.

Let $\hat{\mu}$ be the positive measure on $\mathfrak{g}_G^*$ such that:

$$\int_{\mathfrak{g}_G^*} \alpha(f)\, d\hat{\mu}(f) = \int_{\mathfrak{g}_G^*/G} \left(\int_{\mathcal{O}} \alpha(f)\, dm_{\mathcal{O}}(f) \right) d\tilde{\mu}(\mathcal{O}).$$

Let $\tilde{\varphi}$ be a function on G supported in a small neighborhood of e and φ the function on $\mathfrak{g}$ such that $\varphi = \tilde{\varphi} \circ \exp$. We have

$$\begin{aligned}(\mathrm{tr}\, T_{f,\tau}, \tilde{\varphi}) &= \mathrm{tr} \int_G T_{f,\tau}(g) \tilde{\varphi}(g)\, dg \\ &= \mathrm{tr} \int_{\mathfrak{g}} T_{f,\tau}(\exp X) \tilde{\varphi}(\exp X) j(X)^{1/2} j(X)^{1/2}\, dX \\ &= (\dim \tau) \int_{\mathcal{O} = G \cdot f} (\varphi j^{1/2})^\wedge(l)\, dm_{\mathcal{O}}(l).\end{aligned}$$

Thus, from the Plancherel formula on G, we obtain

$$\begin{aligned}\varphi(0) = \tilde{\varphi}(e) &= \int_X \mathrm{tr}(T_{f,\tau}, \tilde{\varphi})\, d\mu(f, \tau) \\ &= \int_X (\dim \tau) \int_{\mathcal{O} = G \cdot f} (\varphi j^{1/2})^\wedge(l)\, dm_{\mathcal{O}}(l)\, d\mu(f, \tau) \\ &= \int_{\mathfrak{g}_G^*/G} \left(\int_{\mathcal{O}} (\varphi j^{1/2})^\wedge(l)\, dm_{\mathcal{O}}(l) \right) d\tilde{\mu}(\mathcal{O}) \\ &= \int_{\mathfrak{g}_G^*} (\varphi j^{1/2})^\wedge(f)\, d\hat{\mu}(f).\end{aligned}$$

Let v_G be the positive generalized function on $\mathfrak{g}^*$ concentrated on $\mathfrak{g}_G^*$ such that $d\hat{\mu}(f) = v_G(f)\, df$. We then see from this formula that, if n_G is the Fourier transform of v_G, then n_G is a distribution on $\mathfrak{g}$ which coincides with the Dirac measure at 0.

I formulated the following conjecture (in somewhat more timid terms; I am indebted to Michel Duflo [20] for the present reformulation):

Let G be a Lie group with Lie algebra $\mathfrak{g}$. There exists a tempered distribution n_G on $\mathfrak{g}$ having the following properties:

(1) *n_G is G-invariant;*

(2) n_G *coincides with the Dirac measure in a neighborhood of* 0; *the support of* n_G *is contained in* $\mathfrak{g}_G$;
(3) n_G *is of positive type and its Fourier transform is a generalized positive function* v_G *concentrated on* $\mathfrak{g}_G^*$;
(4) *the function* $j(X)^{1/2}$ *admits an analytic square root in a neighborhood of the support of* n_G *and* $j(X)^{1/2}n_G$ *is a measure.*

(It is clear that it is also expected, from the preceding discussion, when G is a type I unimodular group and X the Duflo set of parameters for $\hat{G}_r$ that $v_G\, df = d\hat{\mu}(f)$.)

APPENDIX 2

Complements on SU(2)

1. Normalization of Measures

Let

$$G = \left\{\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}; |\alpha|^2 + |\beta|^2 = 1\right\} = \mathrm{SU}(2),$$

$$\mathfrak{g} = \left\{\begin{pmatrix} ix_3 & -x_1 + ix_2 \\ x_1 + ix_2 & -ix_3 \end{pmatrix}; x_i \in \mathbf{R}\right\}$$

the Lie algebra of G with corresponding basis J_1, J_2, J_3.

The map $g \mapsto (\alpha, \beta)$ identifies G with

$$S_3 = \{(y_1 + iy_2, y_3 + iy_4); y_1^2 + y_2^2 + y_3^2 + y_4^2 = 1\}.$$

Let dg be the Haar measure on G giving total mass 1 to G. If μ is the surface measure on S_3, then $dg = \mu/2\pi^2$. The Haar measure is left and right invariant.

We consider the corresponding Euclidean measure dX on $\mathfrak{g}$ (i.e. $dg = j(X)\, dX$). We have:

$$dX = \frac{dx_1\, dx_2\, dx_3}{2\pi^2}.$$

2. Weyl Integration Formula

Proposition. *Let f be a continuous function on G, then*:

$$\int_G f(g)\, dg = \frac{1}{4\pi}\int_0^{2\pi} |e^{i\theta} - e^{-i\theta}|^2 \int_G f\left(g\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix}g^{-1}\right) dg\, d\theta.$$

Proof. Recall that an element g of G can be conjugated to an element

$$\left[\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix}; \quad 0 \le \theta \le \pi.\right.$$

$$\left.\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix} \text{ is itself conjugated to } \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta}\end{pmatrix} \text{ by } w = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\right].$$

Let $T = \left\{\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix}; \theta \in \mathbf{R}\right\}$ and consider $P = G/T$. We denote by $\dot{g}$ the image of g on G/T. There exists a unique G-invariant measure $d\dot{g}$ on G/T such that $\int_{G/T} f(\dot{g})\, d\dot{g} = \int_G f(\dot{g})\, dg$. Identify the tangent space at $\dot{e}$ with $\mathbf{R}J_1 \oplus \mathbf{R}J_2$. The corresponding volume form ω at $\dot{e}$ is such that $\omega(J_1 \Lambda J_2) = 1/\pi$. The map $c(g, \theta) = g\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix}g^{-1}$ is a diffeomorphism of $P \times]0, \pi[$ with the subset $G_r' = \{g \in G; g \neq (1, -1)\}$ of G. Thus there exists a measure $\mu(x, \theta)$ on $P \times]0, \pi[$, such that

$$\int f(g)\, dg = \int_{P \times]0,\pi[} f(c(x, \theta))\mu(x, \theta).$$

Using the Ad G-invariance of dg, we see that $\mu(x, \theta) = J(\theta)\, d\theta\, d\dot{g}$. To compute $J(\theta)$, we need then to compute the Jacobian of c at the point $(\dot{e}, \theta)$. In coordinates (y_1, y_2, y_3, y_4) for $S_3 \simeq G$, it is immediate to see that

$$c_*(J_1) = \frac{d}{d\varepsilon}\begin{pmatrix} 1 & -\varepsilon \\ \varepsilon & 1\end{pmatrix}\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix}\begin{pmatrix} 1 & \varepsilon \\ -\varepsilon & 1\end{pmatrix}\Bigg|_{\varepsilon=0}$$

$$= 2\sin\theta \frac{\partial}{\partial y_4},$$

$$c_*(J_2) = -2\sin\theta \frac{\partial}{\partial y_3}.$$

Therefore

$$c_*(dg) = |e^{i\theta} - e^{-i\theta}|^2 \frac{1}{2\pi}\omega \Lambda\, d\theta,$$

and we obtain the formula. □

3. Plancherel Formula for G

Theorem. *Let dg be the Haar measure on G giving total mass* 1 *to G, then*:

$$\varphi(e) = \sum_{n=0} n \operatorname{tr} T_n(\varphi) \quad \textit{for every } \varphi, C^\infty \textit{ function on } G.$$

Proof. We have

$$\operatorname{tr} T_n(\varphi) = \int_G \operatorname{tr} T_n(g)\varphi(g)\, dg$$

$$= \frac{1}{4\pi}\int_0^{2\pi} |e^{i\theta} - e^{-i\theta}|^2 \operatorname{tr} T_n\left(g\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix}g^{-1}\right)\varphi\left(g\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix}g^{-1}\right) dg\, d\theta$$

$$= \frac{1}{4\pi}\int_0^{2\pi} (e^{-i\theta} - e^{i\theta})(e^{in\theta} - e^{-in\theta}) \int_G \varphi\left(g\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix}g^{-1}\right) dg\, d\theta.$$

Consider

$$(k\varphi)(\theta) = (e^{-i\theta} - e^{i\theta}) \int_G \varphi\left(g\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix}g^{-1}\right) dg\, d\theta.$$

Then

$$(k\varphi)'(0) = -2i\varphi(e).$$

Let us calculate:

$$\sum_{n>0} n \operatorname{tr} T_n(\varphi) = \frac{1}{4\pi}\sum_{n>0}\int_0^{2\pi}(k\varphi)(\theta)n(e^{in\theta} - e^{-in\theta})\, d\theta$$

$$= \frac{1}{2\pi}\sum_{n\in\mathbf{Z}}\int_0^{2\pi}(k\varphi)(\theta)n(e^{in\theta} - e^{-in\theta})\, d\theta$$

$$= \frac{1}{4\pi}\sum_{n\in\mathbf{Z}}\int_0^{2\pi}(k\varphi)(\theta)ne^{in\theta}\, d\theta$$

as $(k\varphi)(\theta)$ is odd,

$$= \frac{1}{4\pi}\sum_{n\in\mathbf{Z}} i\int_0^{2\pi}(k\varphi)'(\theta)e^{in\theta}\, d\theta$$

$$= \frac{i}{2}(k\varphi)'(0) = \varphi(e). \quad \square$$

q.e.d.

4. Kirillov Character Formula

We want to prove the formula (3.15)

$$\frac{e^{irt} - e^{-irt}}{2it} = \int_{x_1^2+x_2^2+x_3^2=r^2} e^{ix_3t}\,\frac{\sigma_r}{2\pi}.$$

Let μ be the surface measure of S_r. At the point $(0, 0, r) = rJ_3$,

$$\left|\mu\left(\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_2}\right)\right| = 1.$$

The tangent vector generated by the infinitesimal action of J_1 at the point rJ_3 is given by $[J_1, rJ_3] = 2rJ_2$, while the tangent vector generated by the infinitesimal action of J_2 is $-2rJ_1$. By definition,

$$\sigma\left(2r\frac{\partial}{\partial x_2}, -2r\frac{\partial}{\partial x_1}\right) = (rJ_3^*, [J_1, J_2]).$$

Thus $\sigma/2\pi = (1/4\pi r)\mu$. By considering spherical coordinates on S_r, the second member is:

$$\frac{1}{4\pi r}\iint e^{i(r\cos\varphi t)}r^2 \sin\varphi\, d\varphi\, d\theta,$$

which leads immediately to the above formula.

5. The Poisson–Plancherel Formula

Define, for φ a Schwartz function on $\mathfrak{g}$,

$$(K\varphi)(t) = \int_{S_t}\varphi\frac{\sigma}{2\pi} \qquad \text{for } t > 0,$$

$$= -\int_{S_{|t|}}\varphi\frac{\sigma}{2\pi} \quad \text{for } t < 0.$$

In spherical coordinates,

$$(K\varphi)(t) = \frac{t}{4\pi}\iint\varphi(t\cos\theta\sin\varphi, t\sin\theta\sin\varphi, t\cos\varphi)\sin\varphi\, d\varphi\, d\theta.$$

It is immediate to see that $(K\varphi)(t)$ can be extended as a C^∞-odd function of t and that:

$$\int\varphi(x_1, x_2, x_3)\, dx_1\, dx_2\, dx_3 = 2\pi\int_{-\infty}^{\infty} t(K\varphi)(t)\, dt.$$

Define

$$(M_t, \varphi) = \left(\frac{\partial}{\partial t}K\varphi\right)(t).$$

It follows from the expression of $K\varphi(t)$ in spherical coordinates that $(M_0, \varphi) = \varphi(0)$.

Recall that the set $\mathfrak{g}_G$ consists of all the spheres S_t with

$$t \in 2\pi\mathbf{Z} \qquad (t \geq 0).$$

Define

$$(n_G, \varphi) = \sum_{a\in 2\pi\mathbf{Z}}(M_a, \varphi).$$

Recall (3.22) that Θ_a is the G-invariant function on $\mathfrak{g}^*$ such that

$$\Theta_a\begin{pmatrix} i\lambda & 0 \\ 0 & -i\lambda\end{pmatrix} = \tfrac{1}{2}(e^{i\lambda a} + e^{-i\lambda a})$$

and that v_G is the G-invariant generalized function $\sum_{a\in 2\pi\mathbf{Z}}\Theta_a$. Let us verify that the couple (n_G, v_G) verifies the Poisson–Plancherel formula. This will follow from the

Proposition. (a) *n_G is a distribution of support $\mathfrak{g}_G$ and n_G coincides with ∂ in a neighborhood of* 0.

(b) $\hat{M}_a = \Theta_a$,

(c) *If dg is the Haar measure on G giving total mass* 1 *to G, dX the Euclidean measure on $\mathfrak{g}$ such that $dg = j(X)\, dX$, df the dual measure on $\mathfrak{g}^*$, then, for φ a Schwartz function on* $\mathfrak{g}^*$,

$$\int_{\mathfrak{g}^*}\varphi(f)v_G(f)\, df = \sum_{n\in N} n\int_{S_n}\varphi\frac{\sigma_n}{2\pi}.$$

Proof. (a) is clear.

(b) We have to verify that $(M_a, e^{i\langle\xi, X\rangle}) = \Theta_a(\xi)$. By G-invariance, it is sufficient to verify this for $\xi = \begin{pmatrix} i\lambda & 0 \\ 0 & -i\lambda \end{pmatrix}$, i.e. that $(M_a, e^{i\lambda x_3}) = \frac{1}{2}(e^{i\lambda a} - e^{-i\lambda a})$. But

$$K(e^{i\lambda x_3})(a) = \int_{S_a} e^{i\lambda x_3} \frac{\sigma_a}{2\pi}$$
$$= \frac{e^{i\lambda a} - e^{-i\lambda a}}{2i\lambda}, \quad \text{by 4.}$$

thus

$$M_a(e^{i\lambda x_3}) = \frac{\partial}{\partial a} K(e^{i\lambda x_3})(a) = \tfrac{1}{2}(e^{i\lambda a} + e^{-i\lambda a}).$$

(c) We have seen that $dX = dx_1\, dx_2\, dx_3/2\pi^2$, thus $df = d\xi_1\, d\xi_2\, d\xi_3/4\pi$.

By definition of v_G:

$$\int_{\mathfrak{g}^*} \varphi(f) v_G(f)\, df = \sum_{a \in 2\pi\mathbf{Z}} \int \Theta_a(f)\varphi(f)\, df$$
$$= \sum_{a \in 2\pi\mathbf{Z}} \frac{1}{2} \int_{-\infty}^{\infty} tK(\Theta_a \varphi)(t)\, dt$$
$$= \sum_{a \in 2\pi\mathbf{Z}} \frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{iat} + e^{-iat}}{2} t(K\varphi)(t)\, dt$$
$$= \sum_{n \in \mathbf{Z}} \frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{2i\pi nt} + e^{-2i\pi nt}}{2} t(K\varphi)(t)\, dt$$
$$= \sum_{n \in \mathbf{Z}} \frac{1}{2} \int_{-\infty}^{\infty} e^{2i\pi nt} t(K\varphi)(t)\, dt$$

as $t(K\varphi)(t)$ is an even function of t

$$= \sum_{n \in \mathbf{Z}} \tfrac{1}{2} n(K\varphi)(n)$$

from the usual Poisson summation formula,

$$= \sum_{n > 0} n \int \varphi \frac{\sigma_n}{2\pi}. \quad \square$$

q.e.d.

APPENDIX 3

Complement on the Poisson–Plancherel Formula for SL(2, R)

Recall that SL(2, $\mathbf{R}$) has two conjugacy classes of Cartan subalgebras $\mathfrak{b}$ and $\mathfrak{a}$ with

$$\mathfrak{b} = \left\{\begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix}; t \in \mathbf{R}\right\}, \qquad \mathfrak{a} = \left\{\begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix}; s \in \mathbf{R}\right\}.$$

We denote by $\mathcal{O}_t^d$ the conjugacy class of the element $\begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix}$, $\mathcal{O}_s^p = \mathcal{O}_{-s}^p$ the conjugacy class of the element $\begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix}$. Each orbit $\mathcal{O}$ has a canonical Liouville measure $dm_{\mathcal{O}}$. Define, for φ a C^∞ function on $\mathfrak{g}$,

$$(K_{\mathfrak{b}}\varphi)(t) = \int_{\mathcal{O}_t^d} \varphi\, dm_T \qquad \text{for } t > 0,$$
$$= -\int_{\mathcal{O}_t^d} \varphi\, dm_t \qquad \text{for } t < 0,$$
$$(K_{\mathfrak{a}}\varphi)(s) = \int_{\mathcal{O}_s^p} \varphi\, dm_s.$$

We have the integration formula:

$$\int \varphi(x_1, x_2, x_3)\, dx_1\, dx_2\, dx_3 = 4\pi \int_{-\infty}^{\infty} t(K_{\mathfrak{b}}\varphi)(t)\, dt + 4\pi \int_0^{\infty} |s|(K_{\mathfrak{a}}\varphi)(s)\, ds.$$

Define:

$$(M_t, \varphi) = \frac{\partial}{\partial t}(K_{\mathfrak{b}}\varphi)(t) \quad \text{for } t \neq 0.$$

It is not difficult to see that (M_t, φ) can be extended to a continuous function of t and that $(M_0, \varphi) = \varphi(0)$.

Let us now define n_G and v_G. Recall that

$$\mathfrak{g}_G = \bigcup_{a \in 2\pi\mathbf{Z}} \mathcal{O}_a^d,$$
$$\mathfrak{g}_G^* = \left(\bigcup_{n \in \mathbf{Z} - \{0\}} \mathcal{O}_n^d\right) \cup \bigcup_{s \neq 0} \mathcal{O}_s^p.$$

Define

$$(n_G, \varphi) = \sum_{a \in 2\pi\mathbf{Z}} (M_a, \varphi).$$

Recall that Θ_a is the G-invariant function on $\mathfrak{g}^*$ such that

$$\Theta_a\begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix} = e^{ia\lambda}$$
$$\Theta_a\begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix} = e^{-|as|},$$

and that

$$v_G = \sum_{a \in 2\pi\mathbf{Z}} \Theta_a.$$

Define

$$(n_G, \varphi) = \sum_{a \in 2\pi \mathbf{Z}} (M_a, \varphi).$$

Lemma.

$$(v_G, \varphi\, df) = 4\pi \sum_{n \in \mathbf{Z}} |n| \left(\int_{\mathcal{O}_n^d} \varphi \frac{\sigma}{2\pi} \right) + 4\pi \int_0^\infty s \coth \pi s \left(\int_{\mathcal{O}_s^p} \varphi \frac{\sigma}{2\pi} \right) ds.$$

Proof. By definition

$$\int v_G(f)\varphi(f)\, df = \sum_{a \in 2\pi \mathbf{Z}} \int \Theta_a(f)\varphi(f)\, df$$

$$= \sum_{a \in 2\pi \mathbf{Z}} 4\pi \int_{-\infty}^{\infty} tK_b(\Theta_a \varphi)(t)\, dt + 4\pi \int_0^\infty sK_a(\Theta_a \varphi)(s)\, ds$$

$$= \sum_{a \in 2\pi \mathbf{Z}} 4\pi \int_{-\infty}^{\infty} e^{iat} t(K_b \varphi)(t)\, dt + 4\pi \int_0^\infty e^{-|as|} s(K_a \varphi)(s)\, ds.$$

Now the lemma follows from the usual Poisson summation formula on $\mathbf{R}$ ($t(K_b\varphi)(t)$ is a continuous function of t) and from the formula, for $s > 0$

$$\sum_{n \in \mathbf{Z}} e^{-2\pi|ns|} = 2 \sum_{n > 0} e^{-2\pi ns} + 1$$

$$= 2 \sum_{n \geq 0} e^{-2\pi ns} - 1$$

$$= \frac{2}{1 - e^{-2\pi s}} - 1$$

$$= \frac{1 + e^{-2\pi s}}{1 - e^{-2\pi s}}$$

$$= \coth \pi s.$$

To prove the conjecture, it would remain to prove that

$$\hat{M}_a = \Theta_a.$$

This follows from Rossmann's formula [51] and the recurrence relation [25] for discrete series constants. □

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