

Equivariant index formulas for orbifolds

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1 Introduction

Let P be a smooth manifold. Let H be a compact Lie group acting on P . We assume that the action of H is infinitesimally free, that is the stabilizer $H(y)$ of any point $y \in P$ is a finite subgroup of H . We write the action of H on the right. The quotient space P/H is an orbifold (if H acts freely, then P/H is a manifold). Reciprocally any orbifold M can be presented this way: for example, one might choose P to be the bundle of orthonormal frames for a choice of a metric on M and $H = O(n)$ if $n = \dim M$. We will assume that there is a compact Lie group G acting on P such that its action commutes with the action of H . We will write the action of G on the left. Then the space P/H is provided with a G -action. Such data (P, H, G) will be our definition of a presented G -orbifold. We will say shortly that P/H is a G -orbifold.

Consider a compact G -orbifold P/H . A tangent vector on P tangent at $y \in P$ to the orbit $H \cdot y$ will be called a vertical tangent vector. Let T_H^*P be the subbundle of T^*P orthogonal to all vertical vectors. We will say that T_H^*P is the horizontal cotangent space. We denote by (y, ξ) a point in T^*P . Consider two $(G \times H)$ -equivariant vector bundles \mathcal{E}^\pm on P . Let $\Gamma(P, \mathcal{E}^\pm)$ be the spaces of smooth sections of \mathcal{E}^\pm . Let

$$\Delta : \Gamma(P, \mathcal{E}^+) \rightarrow \Gamma(P, \mathcal{E}^-)$$

be a $(G \times H)$ -invariant differential operator. Consider the principal symbol $\sigma(\Delta)$ of Δ . The operator Δ is said to be H -transversally elliptic if

$$\sigma(\Delta)(y, \xi_0) : \mathcal{E}_y^+ \rightarrow \mathcal{E}_y^-$$

is invertible for all $\xi_0 \in (T_H^*P)_y - \{0\}$. When Δ is H -transversally elliptic, the equivariant index of Δ is defined as in [1] and is a trace-class virtual representation of $G \times H$. Introduce $(G \times H)$ -invariant metrics on P and on \mathcal{E}^\pm . Let Δ^* be the formal adjoint of Δ . The virtual space $Q(\Delta)$ of H -invariant “solutions” of Δ

$$Q(\Delta) = [(Ker(\Delta))^H] - [(Ker(\Delta^*))^H]$$

is a finite dimensional virtual representation space for G . More generally, we consider $(G \times H)$ -transversally elliptic operators on P . Then the space $Q(\Delta)$ of H -invariant “solutions” of Δ is a trace class virtual representation of G .

Let us first consider the case where Δ is H -transversally elliptic and H acts freely. It is then easy to describe what is the virtual representation $Q(\Delta)$ of G . Since Δ commutes with H , the operator Δ determines a map

$$\Delta^{P/H} : \Gamma(P, \mathcal{E}^+)^H \rightarrow \Gamma(P, \mathcal{E}^-)^H.$$

We have $\Gamma(P, \mathcal{E}^\pm)^H = \Gamma(P/H, \mathcal{E}^\pm/H)$ and $\Delta^{P/H}$ is a G -invariant elliptic operator on P/H . Thus we have, for $s \in G$,

$$\text{Tr } Q(\Delta)(s) = \text{index}(\Delta^{P/H})(s).$$

Let $(P/H)(s)$ be the set of fixed points for the action of s on P/H . The equivariant index formula of Atiyah-Segal-Singer ([2],[4]) allows us to write $\text{index}(\Delta^{P/H})(s)$ as an integral over $T^*(P/H)(s)$. If H acts only infinitesimally freely, we will give an integral formula for $\text{Tr } Q(\Delta)(s)$ generalizing the formula for $\text{index}(\Delta^{P/H})(s)$ in the case of free action.

More generally if Δ is a $(G \times H)$ -transversally elliptic operator on P , we state in Theorem 2 a formula for the character of the trace class virtual representation $Q(\Delta)$ of G in terms of the equivariant cohomology of $T^*(P/H)$. This theorem generalizes the cohomological index formula given in [7], [9] for the equivariant index of G -transversally elliptic operators on compact manifolds to the case of compact orbifolds.

If $G = \{e\}$, we identify $Q(\Delta)$ with an integer. Several authors gave an integral formula for this integer in various degrees of generality. The notion of an orbifold was introduced by Satake who proved a Gauss-Bonnet formula ([16]) for orbifolds. For any H -transversally elliptic operator Δ , a formula for the number $Q(\Delta)$ was given by Atiyah (Corollary 9.12 ,[1]) in the case where H is a torus. When P/H is a complex algebraic variety, \mathcal{F}/H an

holomorphic orbifold bundle on P/H and Δ the $\bar{\partial}$ operator on the space of sections of \mathcal{F}/H , the number $Q(\Delta)$ was computed by Kawasaki [12]. It is the Riemann-Roch number of a sheaf on P/H . For H an arbitrary compact group and any H -transversally elliptic operator Δ , a formula for the number $Q(\Delta)$ was given by Kawasaki [13].

In our case as well as in Kawasaki's proof in [13], Atiyah's algorithm to compute the equivariant index of a H -transversally elliptic operator is a fundamental ingredient. Indeed our proof of the general formula for index of transversally elliptic operators ([9]) relies heavily on Atiyah's results in [1]. Once this general formula is established, it is a pleasant exercise on Fourier inversion for compact groups to deduce the formula given here for G -transversally elliptic operators on orbifolds from our index formula for transversally elliptic operators on manifolds. I feel it is useful to do this exercise in order to extend to symplectic orbifolds the universal formula ([17]) for the character of a quantized representation. In fact, G -orbifolds appear naturally when studying the quantized representation associated to a pre-quantized symplectic manifold M . Let M be a symplectic manifold with Hamiltonian action of $G \times H$. Let \mathcal{L} be a Kostant-Souriau line bundle on M and $\mu : M \rightarrow \mathfrak{h}^*$ be the moment map for the H -action. Consider the space $M_{red} = \mu^{-1}(0)/H$. When 0 is a regular value of μ , the space M_{red} is a symplectic orbifold with a G -action. The quantized representation $Q(M, \mathcal{L})$ is a virtual representation of $G \times H$ constructed as the $(\mathbb{Z}/2\mathbb{Z})$ -graded space of solutions of the \mathcal{L} -twisted Dirac operator on M . If M_{red} is an orbifold, the virtual representation $Q(M_{red}, \mathcal{L}_{red})$ of G can be constructed in a similar way [19]. We give in Proposition 4 an integral formula for the character of the quantized representation $Q(M_{red}, \mathcal{L}_{red})$ of the symplectic orbifold M_{red} .

2 Equivariant index formula on orbifolds

2.1 Differential forms and integration

Let N be a manifold with infinitesimally free action of a compact group H . Let G be a compact Lie group acting on N such that the action of G commutes with the action of H . A differential form $\alpha \in \mathcal{A}(N)$ will be called H -horizontal (or simply horizontal if H is understood) if $\iota(Y_N)\alpha = 0$ for all $Y \in \mathfrak{h}$. A form α on N is called H -basic if α is H -horizontal and H -invariant. If the action of H on N is free, a basic form is the pull-back of a

form on N/H . Thus we will also say that a H -basic differential form α on N is a differential form on N/H . The operator $d_{\mathfrak{g}}$ on G -equivariant differential forms on N is defined as in (Chapter 7, [5]). For $X \in \mathfrak{g}$, we denote by d_X the operator $d - \iota(X_N)$ on forms on N . A G -equivariant differential form on N is called H -basic if, for all $X \in \mathfrak{g}$, the differential form $\alpha(X)$ is H -basic. We will also say that α is a G -equivariant differential form on N/H . The operator $d_{\mathfrak{g}}$ preserves the space of G -equivariant differential forms on N/H .

We identify the bundle of vertical vectors with $N \times \mathfrak{h}$. Choose a $(G \times H)$ -invariant decomposition

$$(1) \quad TN = T_{hor}N \oplus (N \times \mathfrak{h}).$$

This decomposition allows us to identify T_H^*N with T_{hor}^*N .

The decomposition (1) gives us a connection form

$$(2) \quad \theta \in (\mathcal{A}^1(N) \otimes \mathfrak{h})^{H \times G}.$$

We denote by $\Theta \in \mathcal{A}^2(N) \otimes \mathfrak{h}$ the curvature of θ . Let ϕ be a smooth function on \mathfrak{h} , then we define the horizontal form $\phi(\Theta)$ on N using Taylor's expansion of ϕ at 0. If ϕ is invariant, then $\phi(\Theta)$ is basic.

The stabilisers $H(y)$ of points $y \in N$ are finite subgroups of H . The set B of conjugacy classes of stabilisers of elements of N is a partially ordered set. Let N_a be a connected component of N . Then the set $\{H(y), y \in N_a\}$ has a unique minimal element ([10]). This element S_a is referred to as the generic stabiliser on N_a . We consider the generic stabiliser as a locally constant function from N to conjugacy classes of subgroups of H writing $S(y) = S_a$ if $y \in N_a$. Let $|S(y)|$ be the order of $S(y)$. In particular $y \rightarrow |S(y)|$ is a locally constant function on N . We denote this function by $|S|$ (or $|S^N|$ when we need to specify the manifold N). An element $y \in N$ such that $H(y)$ is conjugated to $S(y)$ is called regular. We denote by N_{reg} the set of regular elements. It is a H -invariant open subset of N and N_{reg}/H is a manifold.

Assume the bundle T_{hor}^*N has a H -invariant orientation o . We will then say that N/H is oriented. If N is connected, we define $\dim(N/H)$ to be $\dim N - \dim H$. Otherwise, we consider $\dim(N/H)$ as a locally constant function on N .

A H -basic differential form α defines a differential form on N_{reg}/H . If α is compactly supported on N , then the component $\alpha_{[\dim(N/H)]}$ of exterior degree $\dim(N/H)$ of α is integrable on the oriented manifold N_{reg}/H . By

definition

$$(3) \quad \int_{N/H} \alpha = \int_{N_{reg}/H} \alpha_{[\dim(N/H)]}.$$

Let us give a formula for $\int_{N/H} \alpha$ as an integral over N . Let $n = \dim \mathfrak{h}$. Let E^1, E^2, \dots, E^n be a basis of \mathfrak{h} . We write the connection form $\theta \in \mathcal{A}^1(N) \otimes \mathfrak{h}$ as

$$\theta = \sum_1^n \theta_k E^k.$$

Let E_1, E_2, \dots, E_n be the dual basis of \mathfrak{h}^* . It defines an Euclidean volume form dY on \mathfrak{h} and an orientation $o^{\mathfrak{h}}$ on \mathfrak{h} . We denote by dh the Haar measure on H tangent to dY at the identity of H . Notice that the form

$$v_{o^{\mathfrak{h}}} = (\text{vol}(H, dh))^{-1} \theta_1 \wedge \theta_2 \wedge \dots \wedge \theta_n$$

depends only of θ and $o^{\mathfrak{h}}$.

Assume N/H is oriented. Let $o^{N/H}$ be the corresponding orientation. Then N is oriented. We choose as positive volume form $\omega \wedge v_{o^{\mathfrak{h}}}$ if ω is a positive H -invariant section of $\Lambda^{max} T_H^* N$. We denote this orientation by $o^{N/H} \wedge o^{\mathfrak{h}}$. If α is a basic form on N with compact support, then

$$(4) \quad \int_{N/H} \alpha = \int_N |S| \alpha \wedge v_{o^{\mathfrak{h}}}.$$

In this formula, the orientation on N is the orientation $o^{N/H} \wedge o^{\mathfrak{h}}$.

If $\mathcal{V} \rightarrow N$ is a H -equivariant vector bundle over N with projection p_0 , then the integration over the fiber of a H -basic differential form on \mathcal{V} is a H -basic differential form on N . If α is compactly supported, we have the integration formula

$$(5) \quad \int_{\mathcal{V}/H} \alpha = \int_{N/H} |S^{\mathcal{V}}| / |S^N| (p_0)_* \alpha.$$

Let us define the cotangent bundle to an orbifold N/H . When H acts freely on N , then N/H is a smooth manifold and we have a canonical identification $T^*(N/H) = (T_H^* N)/H$. In our case, the action of H on $T_H^* N$ is infinitesimally free and we define $T^*(N/H)$ as an orbifold by $T^*(N/H) = (T_H^* N)/H$. It is important to notice that the orbifold $T^*(N/H)$ is orientable.

Indeed the restriction of the canonical 1-form ω^N of T^*N to T_H^*N is a basic 1-form, that is a form on $T^*(N/H)$. We denote it by $\omega^{N/H}$ and refer to it as the canonical 1-form on $T^*(N/H)$. The 2-form $d\omega^{N/H}$ is non degenerate on $T_{hor}(T_H^*N)$. We will choose on $T^*(N/H)$ the symplectic orientation given by $-d\omega^{N/H}$.

2.2 Index formula

Let $M = P/H$ be a compact G -orbifold. Consider two $(G \times H)$ -equivariant vector bundles \mathcal{E}^\pm on P . Let

$$\Delta : \Gamma(P, \mathcal{E}^+) \rightarrow \Gamma(P, \mathcal{E}^-)$$

be a $(G \times H)$ -invariant differential operator. We assume that Δ is a $(G \times H)$ -transversally elliptic operator on P . We will give an integral formula for $\text{Tr} Q(\Delta)$ in terms of the equivariant cohomology of T^*M . We need some definitions.

Let \mathcal{E} be a H -equivariant bundle over P . If ∇ is a H -invariant connection on \mathcal{E} , we define its moment $\mu \in \Gamma(P, \text{End}(\mathcal{E}))\mathfrak{h}^*$ and the equivariant curvature of ∇ as in (Chapter 7, [5]). Our conventions for characteristic classes will be those of [11]. They differ slightly from those of [5]. In particular, if $F(Y)$ ($Y \in \mathfrak{h}$) is the equivariant curvature of ∇ , the equivariant Chern character will be $\text{ch}(\mathcal{E}, \nabla)(Y) = \text{Tr}(e^{F(Y)})$.

We will say that ∇ is a H -horizontal connection if $\mu(Y) = 0$ for all $Y \in \mathfrak{h}$. It is always possible to choose an horizontal connection on \mathcal{E} . This can be done as follows. Consider a connection form $\theta \in \mathcal{A}^1(P)\mathfrak{h}$ for the action of H on P . Let ∇ be a H -invariant connection on \mathcal{E} with moment $\mu \in \Gamma(P, \text{End}(\mathcal{E}))\mathfrak{h}^*$. Then the contraction (μ, θ) is a $\text{End}(\mathcal{E})$ -valued 1-form on P . Define $\nabla' = \nabla + (\mu, \theta)$. Then ∇' is horizontal.

If \mathcal{E} is a $(G \times H)$ -equivariant vector bundle on P , it is always possible to choose on \mathcal{E} a $(G \times H)$ -invariant horizontal connection ∇ . Then the equivariant Chern character of (\mathcal{E}, ∇) is a G -equivariant basic form on P . An important example in the following is the case of a trivial vector bundle $[V_\tau] = P \times V_\tau$ where V_τ is a representation space of H . Let us denote also by τ the infinitesimal representation of \mathfrak{h} in V_τ . It is easy to see that $d + \tau(\theta)$ is an horizontal connection with equivariant Chern character the basic equivariant form $\text{ch}([V_\tau])(X) = \text{Tr}(\tau(\exp \Theta(X)))$ where, for $X \in \mathfrak{g}$, $\Theta(X) = -(\theta, X_P) + \Theta$ is the equivariant curvature.

If $(s, u) \in G \times H$, the manifold

$$P(s, u) = \{p \in P; sp = pu\}$$

is a $(G(s) \times H(u))$ -manifold, where $G(s)$ is the centralizer of $s \in G$ and $H(u)$ the centralizer of $u \in H$. The group $H(u)$ acts infinitesimally freely on $P(s, u)$. We denote by $M(s, u)$ the orbifold $P(s, u)/H(u)$. If γ is conjugated to u , the orbifold $M(s, \gamma)$ is diffeomorphic to $M(s, u)$.

Consider the horizontal bundle $T_{hor}P(s, u) \subset T_{hor}P|_{P(s, u)}$ and the horizontal normal bundle

$$T_{hor, P(s, u)}P = T_{hor}P|_{P(s, u)}/T_{hor}P(s, u).$$

The vector bundles $T_{hor}P(s, u)$ and $T_{hor, P(s, u)}P$ are $(G(s) \times H(u))$ -equivariant vector bundles on $P(s, u)$.

Define $T_{M(s, u)}M$ to be the orbifold bundle $(T_{hor, P(s, u)}P)/H(u)$ over $M(s, u)$. If M is a G -manifold, then $T_{M(s, u)}M$ is the normal bundle to $M(s, u)$ in M .

Let ∇ be a $(G \times H)$ -invariant horizontal connection on $T_{hor}P$. Then ∇ induces $H(u)$ -horizontal connections ∇_0 on $T_{hor}P(s, u)$ and ∇_1 on $T_{hor, P(s, u)}P$. Let $R_0(X)$, $R_1(X)$ be the $G(s)$ -equivariant curvatures of ∇_0 and ∇_1 . On $P(s, u)$ the action of (s, u) induces an endomorphism $g(s, u)$ of the bundle $T_{hor, P(s, u)}P$. Define the $G(s)$ -equivariant closed forms on $P(s, u)/H(u)$

$$(6) \quad J(M(s, u))(X) = \det\left(\frac{e^{R_0(X)/2} - e^{-R_0(X)/2}}{R_0(X)}\right)$$

and

$$(7) \quad D_{(s, u)}(T_{M(s, u)}M)(X) = \det(1 - g(s, u)e^{R_1(X)})$$

for $X \in \mathfrak{g}(s)$.

We denote by p_0 the projection $T_H^*P \rightarrow P$. We denote by σ_0 the restriction of the principal symbol σ of Δ to T_H^*P . Let $\nabla^{\mathcal{E}^\pm}$ be horizontal connections on \mathcal{E}^\pm . Consider the superconnection $\mathbb{A}_0(\sigma_0)$ on $p_0^*\mathcal{E} = p_0^*\mathcal{E}^+ \oplus p_0^*\mathcal{E}^-$ defined by:

$$\mathbb{A}_0(\sigma_0) = \begin{pmatrix} p_0^*\nabla^{\mathcal{E}^+} & i\sigma_0^* \\ i\sigma_0 & p_0^*\nabla^{\mathcal{E}^-} \end{pmatrix}.$$

Then the equivariant Chern character $\text{ch}_{s, u}(\mathbb{A}_0(\sigma_0))(X)$ is a $G(s)$ -equivariant form on the space $(T_{hor}^*P(s, u))/H(u) = T^*M(s, u)$. Thus we can define

a $G(s)$ -equivariant closed basic differential form on $T_{hor}^*P(s, u)$ given for $X \in \mathfrak{g}(s)$ small by

$$(8) \quad I(s, u, \sigma_0)(X) = \frac{\text{ch}_{s,u}(\mathbb{A}_0(\sigma_0))(X)}{J(M(s, u))(X)D_{s,u}(T_{M(s,u)}M)(X)}.$$

For $X = 0$, we write

$$(9) \quad I(s, u, \sigma_0) = I(s, u, \sigma_0)(0).$$

Assume first that Δ is H -transversally elliptic. Then the restriction σ_0 of the principal symbol of Δ is homogeneous of positive order on each fiber of the vector bundle T_{hor}^*P . Furthermore $\sigma_0(y, \xi_0)$ is invertible when ξ_0 is not zero. Thus for $X \in \mathfrak{g}(s)$, the form $\text{ch}_{s,u}(\mathbb{A}_0(\sigma_0))(X)$ is rapidly decreasing on $T_{hor}^*P(s, u)$ (this is seen as in [7]) so that $I(s, u, \sigma_0)(X)$ can be integrated over $T^*M(s, u)$.

For $s \in G$, we denote by $C(s)$ the set of elements $\gamma \in H$ such that $P(s, \gamma) \neq \emptyset$. Then $C(s)$ is invariant by conjugacy and the set $(C(s)) = C(s)/Ad(H)$ is a finite set. Let $M(s, \gamma)$ be the orbifold $P(s, \gamma)/H(\gamma)$. We denote by $S(s, \gamma)$ the generic stabiliser for the action of $H(\gamma)$ on $P(s, \gamma)$. The functions $\dim M(s, \gamma)$ and $|S(s, \gamma)|$ are locally constant functions on $P(s, \gamma)$.

Theorem 1 . *Let $M = P/H$ be an orbifold. Let Δ be a $(G \times H)$ -invariant differential operator on P . Assume that Δ is H -transversally elliptic. Then, for each $s \in G$, the trace of the virtual finite dimensional representation $Q(\Delta)$ of G satisfies the formula*

$$\text{Tr } Q(\Delta)(s \exp X) =$$

$$\sum_{\gamma \in (C(s))} \int_{T^*M(s, \gamma)} (2i\pi)^{-\dim M(s, \gamma)} |S(s, \gamma)|^{-1} \frac{\text{ch}_{s, \gamma}(\mathbb{A}_0(\sigma_0))(X)}{J(M(s, \gamma))(X)D_{s, \gamma}(T_{M(s, \gamma)}M)(X)}$$

for X small in $\mathfrak{g}(s)$.

Assume now that Δ is only $(G \times H)$ -transversally elliptic. Let ω^M be the canonical 1-form of T^*M . Similarly we obtain canonical 1-forms on $\omega^{M(s, \gamma)}$ on $T^*M(s, \gamma)$. Define then

$$I^\omega(s, \gamma, \sigma_0)(X) = \frac{e^{-id_X \omega^{M(s, \gamma)}} \text{ch}_{s, \gamma}(\mathbb{A}_0(\sigma_0))(X)}{J(M(s, \gamma))(X)D_{s, \gamma}(T_{M(s, \gamma)}M)(X)}.$$

Then the form $I^\omega(s, \gamma, \sigma_0)(X)$ is a $G(s)$ -equivariant form on $T^*M(s, \gamma)$ which can be integrated in $\mathfrak{g}(s)$ -mean ([8]).

The formula for $\text{Tr } Q(\Delta)$ given in Theorem 1 for Δ a H -transversally elliptic operator has to be modified to obtain a meaningful formula in the case of a $(G \times H)$ -transversally elliptic operator Δ where $\text{Tr } Q(\Delta)$ is only a generalized function on G . The next theorem extends the cohomological formula for the index of G -transversally elliptic operators on manifolds ([8],[9]) to the case of G -transversally elliptic operators on orbifolds.

Theorem 2 . *Let $M = P/H$ be an orbifold. Let Δ be a $(G \times H)$ -invariant differential operator on P . Assume that Δ is $(G \times H)$ -transversally elliptic. Then, for each $s \in G$, the trace of the virtual trace class representation $Q(\Delta)$ of G satisfies the equality*

$$\text{Tr } Q(\Delta)(s \exp X) = \sum_{\gamma \in (C(s))} \int_{T^*M(s, \gamma)} (2i\pi)^{-\dim M(s, \gamma)} |S(s, \gamma)|^{-1} \frac{e^{-id_X \omega^{M(s, \gamma)}} \text{ch}_{s, \gamma}(\mathbb{A}_0(\sigma_0))(X)}{J(M(s, \gamma))(X) D_{s, \gamma}(T_{M(s, \gamma)} M)(X)}$$

as an equality of generalised functions on a neighborhood of 0 in $\mathfrak{g}(s)$.

Remark 2.1 *If Δ is only pseudo-differential, the formula above holds provided we choose a “good” representative σ_0 ([8]) of the symbol of Δ .*

Before proving these theorems, let us write more explicitly the formula of Theorem 1 in the case where $G = \{e\}$. Then we must consider the set $C(e)$ of elements $\gamma \in H$ such that the set $P(\gamma) = \{p \in P, p\gamma = p\}$ is not empty. We define $M(\gamma) = P(\gamma)/H(\gamma)$. The formula obtained for the number $Q(\Delta) = \dim(\text{Ker}(\Delta))^H - \dim(\text{Ker } \Delta^*)^H$ is thus Kawasaki’s formula:

$$(10) \quad Q(\Delta) = \sum_{\gamma \in (C(e))} \int_{T^*M(\gamma)} (2i\pi)^{-\dim M(\gamma)} |S(\gamma)|^{-1} \frac{\text{ch}_\gamma(\mathbb{A}_0(\sigma_0))}{J(M(\gamma)) D_\gamma(T_{M(\gamma)} M)}.$$

Let us give two examples where this formula is easily seen to be true.

1) Assume H is a finite group. Then the dimension of the space $Q(\Delta)$ is evidently given by the average of the equivariant index

$$Q(\Delta) = |H|^{-1} \sum_{\gamma \in H} \text{index}(\Delta)(\gamma)$$

Using the equivalent expression given in [7] of the Atiyah-Segal-Singer formula ([2],[4]), we have

$$index(\Delta)(\gamma) = \int_{T^*P(\gamma)} (2i\pi)^{-\dim P(\gamma)} \frac{\text{ch}_\gamma(\mathbb{A}_0(\sigma_0))}{J(P(\gamma))D_\gamma(T_{P(\gamma)}P)}.$$

In particular $index(\Delta)(\gamma)$ is zero if γ does not belong to $C(e)$. Let $\gamma \in C(e)$. In this case $T^*M(\gamma) = T^*P(\gamma)/H(\gamma)$. On each connected component of $P(\gamma)$, the map $T^*P(\gamma) \rightarrow T^*P(\gamma)/H(\gamma)$ is a cover of order $|H(\gamma)/S(\gamma)|$ and by definition, for α a differential form on $P(\gamma)$

$$\int_{T^*M(\gamma)} (2i\pi)^{-\dim M(\gamma)} \alpha = \int_{T^*P(\gamma)} |H(\gamma)|^{-1} |S(\gamma)| (2i\pi)^{-\dim P(\gamma)} \alpha.$$

Rewriting the set $C(e)$ as union of conjugacy classes, we see that the formula for $Q(\Delta)$ is indeed just the average of the Atiyah-Segal-Singer formula.

2) Assume H acts freely on P . Then $C(e) = \{e\}$. Let $M = P/H$. The restriction σ_0 of σ to T_H^*P determines an elliptic symbol still denoted by σ_0 on $T^*M = T_H^*P/H$ which is the principal symbol of $\Delta^{P/H}$. We have $Q(\Delta) = index(\Delta^{P/H})$. The formula (10) for $Q(\Delta)$ as an integral over T^*M of an equivariant characteristic class agrees with the Atiyah-Singer formula for the index of $\Delta^{P/H}$ in function of its principal symbol .

Proof.

Let us now prove Theorem 1 and Theorem 2. We give only the proof of the first theorem as both proofs are very similar to the proof of the Frobenius reciprocity for free actions (Theorem 26, [9]). We give the main steps. Define

$$v(s, \gamma, \sigma_0)(X) = \int_{T^*M(s, \gamma)} (2i\pi)^{-\dim M(s, \gamma)} |S(s, \gamma)|^{-1} \frac{\text{ch}_{s, \gamma}(\mathbb{A}_0(\sigma_0))(X)}{J(M(s, \gamma))(X)D_{s, \gamma}(T_{M(s, \gamma)}M)(X)}.$$

We must prove that

$$(11) \quad \text{Tr } Q(\Delta)(s \exp X) = \sum_{\gamma \in C(s)} v(s, \gamma, \sigma_0)(X).$$

Consider the virtual character $index(\Delta)$ of $G \times H$. Let \hat{H} be the set of classes of irreducible finite dimensional representations of H . For $\tau \in \hat{H}$, consider the operator

$$\Delta \phi I_{V_\tau} : \Gamma(P, \mathcal{E}^+) \phi V_\tau \rightarrow \Gamma(P, \mathcal{E}^-) \phi V_\tau.$$

For $\tau \in \hat{H}$, let $[V_\tau]$ be the trivial bundle on P with fiber V_τ . We have

$$\Gamma(P, \mathcal{E}^\pm) \circ V_\tau = \Gamma(P, \mathcal{E}^\pm \circ [V_\tau]).$$

We denote by Δ^τ the operator $\Delta \circ I_{V_\tau}$. It has symbol $\sigma_\tau = \sigma \circ I_{p^*[V_\tau]}$. The map $\Gamma(P, \mathcal{E}^\pm) \circ V_\tau \circ V_\tau^* \rightarrow \Gamma(P, \mathcal{E}^\pm)$ given by $(\phi \circ f) \mapsto (\phi, f)$ for $f \in V_\tau^*$ and ϕ in $\Gamma(P, \mathcal{E}^\pm) \circ V_\tau$ induces an isomorphism from $(\Gamma(P, \mathcal{E}^\pm) \circ V_\tau)^H \circ V_\tau^*$ to the isotypic space of type τ^* in $\Gamma(P, \mathcal{E}^\pm)$. By definition the trace of the action of G in $[(\text{Ker}(\Delta \circ I_{V_\tau})^H)] - [(\text{Ker}(\Delta^* \circ I_{V_\tau})^H)]$ is $Q(\Delta^\tau)$. Thus we see that

$$\text{index}(\Delta)(s, h) = \sum_{\tau \in \hat{H}} \text{Tr } Q(\Delta^\tau)(s) \text{Tr } \tau^*(h).$$

To verify Equation 11 for $Q(\Delta)$ it is sufficient to verify, for each $s \in G$ and $X \in \mathfrak{g}(s)$ small, that we have the equality of generalised functions of H

$$(12) \quad \text{index}(\Delta)(s \exp X, h) = \sum_{\tau \in \hat{H}} \sum_{\gamma \in (C(s))} v(s, \gamma, \sigma_0^\tau)(X) \text{Tr } \tau^*(h).$$

To simplify formulas, we compute only for $X = 0$. We write $v(s, \gamma, \sigma_0^\tau)$ for $v(s, \gamma, \sigma_0^\tau)(0)$.

Let $u \in H$ and let ϕ be a H -invariant test function on H with support in a small neighborhood of the conjugacy class of u . In particular, we assume that if $\gamma \in (C(s))$ is not conjugated to u , the support of ϕ does not intersect the orbit of γ . Let $\mathfrak{h}(u)$ be the Lie algebra of $H(u)$. Let

$$(13) \quad v_1(\phi) = \int_H \text{index}(\Delta)(s, h) \phi(h) dh$$

and

$$(14) \quad v_2(\phi) = \sum_{\tau \in \hat{H}} \sum_{\gamma \in (C(s))} v(s, \gamma, \sigma_0^\tau) \int_H \text{Tr } \tau^*(h) \phi(h) dh.$$

We need to verify the equality

$$(15) \quad v_1(\phi) = v_2(\phi)$$

Let us first state the main technical lemma. Let N be the manifold

$$N = P \times \mathfrak{h}^*.$$

We denote by $f : P \times \mathfrak{h}^* \rightarrow \mathfrak{h}^*$ the second projection. We consider the 1-form

$$\nu = (\theta, f)$$

on N . We choose a basis E_1, E_2, \dots, E_n of \mathfrak{h}^* . This determines the form $v_{o\mathfrak{h}}$ on P . We write $f = \sum f^i E_i$. We denote by $df = df^1 \wedge df^2 \wedge \dots \wedge df^n$. We denote by p_1 the projection of $N = P \times \mathfrak{h}^*$ on P with fiber \mathfrak{h}^* . The integration over the fiber is defined once chosen an orientation on each fiber. We use the orientation given by df . Furthermore the integration over the fiber is defined with conventions of signs as in [5]: if $p : P \rightarrow B$ is an oriented fibration $p_*(\alpha \wedge p^*\beta) = p_*(\alpha) \wedge \beta$ if α is a form on P and β a form on B .

The following lemma is obtained as Proposition 28 of [9].

Lemma 3 *If ϕ is a test function on \mathfrak{h} , we have*

$$(2i\pi)^{-\dim H} (p_1)_* \left(\int_{\mathfrak{h}} e^{-id_Y \nu} \phi(Y) dY \right) = (-1)^{n(n+1)/2} (\text{vol } H, dh) v_{o\mathfrak{h}} \phi(\Theta).$$

Let us return to the proof of the identity (15).

We first compute $v_1(\phi)$. The generalised function $\text{index}(\Delta)$ can be computed as a special case of the index formula for $(G \times H)$ -transversally elliptic operators. Let, for $Y \in \mathfrak{h}(u)$,

$$J_{\mathfrak{h}(u)}(Y) = \det_{\mathfrak{h}(u)} \frac{e^{\text{ad } Y/2} - e^{-\text{ad } Y/2}}{\text{ad } Y}.$$

Using Weyl integration formula, we have

(16)

$$v_1(\phi) = \text{vol}(H/H(u)) \int_{\mathfrak{h}(u)} \text{index}(\Delta)(s, ue^Y) \phi(ue^Y) J_{\mathfrak{h}(u)}(Y) \det_{\mathfrak{h}/\mathfrak{h}(u)} (1 - ue^Y) dY.$$

Let $p : T^*P \rightarrow P$ the projection. Define on the super-bundle $p^*\mathcal{E} = p^*\mathcal{E}^+ \oplus p^*\mathcal{E}^-$ the superconnection

$$\mathbb{A}(\sigma) = \begin{pmatrix} p^*\nabla^{\mathcal{E}^+} & i\sigma^* \\ i\sigma & p^*\nabla^{\mathcal{E}^-} \end{pmatrix}.$$

Let $T^*P = T_{hor}^*P \oplus P \times \mathfrak{h}^*$. We can assume by homotopy the symbol σ of Δ of the form $\sigma(y, \xi) = \sigma_0(y, \xi_0)$ where ξ_0 is the projection of ξ on $(T_{hor}^*P)_y$.

We choose on TP the direct sum of an horizontal connection on $T_{hor}P$ and of the trivial connection on $P \times \mathfrak{h}$.

Let ω^P be the canonical 1-form on T^*P . Its restriction to $N = P \times \mathfrak{h}^*$ is the 1-form $\nu = (\theta, f)$.

Let $(s, u) \in G \times H$. The index formula for Δ gives in particular for $Y \in \mathfrak{h}(u)$ sufficiently small:

$$index(\Delta)(s, ue^Y) = \int_{T^*P(s,u)} (2i\pi)^{-\dim P(s,u)} \frac{e^{-id_Y \omega^P|_{T^*P(s,u)}} \text{ch}_{s,u}(\mathbb{A}(\sigma))(Y)}{J(P(s, u))(Y) D_{s,u}(T_{P(s,u)}P)(Y)}.$$

The restriction of the connection form θ to $P(s, u)$ is valued in $\mathfrak{h}(u)$ and is a connection form for the $H(u)$ -action on $P(s, u)$. We have $T^*P(s, u) = T_{hor}^*P(s, u) \oplus P(s, u) \times \mathfrak{h}(u)^*$. Thus the bundle $T^*P(s, u)$ projects on $N(s, u) = P(s, u) \times \mathfrak{h}^*(u)$ as well as on $T_{hor}^*P(s, u)$. We still denote by α the pull-back to $T^*P(s, u)$ of a form α on $N(s, u)$ and by β the pull-back to $T^*P(s, u)$ of a form β on $T_{hor}^*P(s, u)$. For our choices of connections and symbols, we have

$$\text{ch}_{s,u}(\mathbb{A}(\sigma))(Y) = \text{ch}_{s,u}(\mathbb{A}_0(\sigma_0))$$

$$J(P(s, u))(Y) = J(M(s, u))J_{\mathfrak{h}(u)}(Y)$$

$$D_{s,u}(T_{P(s,u)}P)(Y) = D_{s,u}(T_{M(s,u)}M) \det_{\mathfrak{h}/\mathfrak{h}(u)}(1 - ue^Y).$$

Thus we obtain

$$index(\Delta)(s, ue^Y)J_{\mathfrak{h}(u)}(Y) \det_{\mathfrak{h}/\mathfrak{h}(u)}(1 - ue^Y) =$$

$$\int_{T^*P(s,u)} (2i\pi)^{-\dim P(s,u)} \frac{e^{-id_Y \omega^P|_{T^*P(s,u)}} \text{ch}_{s,u}(\mathbb{A}_0(\sigma_0))}{J(M(s, u))D_{s,u}(T_{M(s,u)}M)}.$$

Let $(y, \xi) \in T^*P(s, u) = T_{hor}^*P(s, u) \oplus P(s, u) \times \mathfrak{h}(u)^*$. If $\xi = \xi_0 + f$ with $\xi_0 \in (T_{hor}^*P(s, u))_y$ and $f \in \mathfrak{h}(u)^*$, the Chern character $\text{ch}_{s,u}(\mathbb{A}_0(\sigma_0))$ is rapidly decreasing with respect of the variable ξ_0 . The factor $e^{-id_Y \omega^P|_{T^*P(s,u)}}$ integrated against a test function of $Y \in \mathfrak{h}(u)$ is rapidly decreasing in the variable f . A transgression argument similar to those proven in [8] allows us to replace ω^P in $t\nu + (1-t)\omega_P$, with $t \in [0, 1]$. Then we have also

$$index(\Delta)(s, ue^Y)J_{\mathfrak{h}(u)}(Y) \det_{\mathfrak{h}/\mathfrak{h}(u)}(1 - ue^Y) =$$

$$\int_{T^*P(s,u)} (2i\pi)^{-\dim P(s,u)} e^{-id_Y \nu|_{T^*P(s,u)}} \frac{\text{ch}_{s,u}(\mathbb{A}_0(\sigma_0))}{J(M(s,u))D_{s,u}(T_{M(s,u)}M)}.$$

We denote by ν_0 the restriction of ν to $P(s,u) \times \mathfrak{h}(u)^*$. Consider the fibration $p_1^u : T^*P(s,u) \mapsto T_{hor}^*P(s,u)$ with fiber $\mathfrak{h}(u)^*$. Using notation (9), we thus have

$$\text{index}(\Delta)(s, ue^Y) J_{\mathfrak{h}(u)}(Y) \det_{\mathfrak{h}/\mathfrak{h}(u)}(1 - ue^Y) = \int_{T_{hor}^*P(s,u)} (2i\pi)^{-\dim P(s,u)} (p_1^u)_*(e^{-id_Y \nu_0}) I(s, u, \sigma_0).$$

Let Θ_0 be the restriction of Θ to $P(s,u)$. The function $Y \mapsto \phi(u \exp Y)$ is a $H(u)$ -invariant function on $\mathfrak{h}(u)$ and the form $\phi(u \exp \Theta_0)$ is a basic form on $P(s,u)$. Applying Lemma 3 to the manifold $P(s,u) \times \mathfrak{h}(u)^*$ and integration formula (16), we obtain:

$$v_1(\phi) = \epsilon \text{vol}(H, dh) \int_{T_{hor}^*P(s,u)} (2i\pi)^{-\dim M(s,u)} v_{\mathfrak{h}(u)} \phi(u \exp \Theta_0) I(s, u, \sigma_0)$$

where ϵ is a sign.

Finally applying Formula 4 to the basic form $\phi(u \exp \Theta_0) I(s, u, \sigma_0)$ we obtain

(17)

$$v_1(\phi) = \text{vol}(H, dh) \int_{T^*M(s,u)} |S(s,u)|^{-1} (2i\pi)^{-\dim M(s,u)} \phi(u \exp \Theta_0) I(s, u, \sigma_0).$$

(A check of orientations shows that the sign ϵ disappears.)

We now compute $v_2(\phi)$. Define

$$v_2(\gamma, \phi) = \sum_{\tau \in \hat{H}} v(s, \gamma, \sigma_0^\tau) \int_H \text{Tr} \tau^*(h) \phi(h) dh.$$

Let $\tau \in \hat{H}$. Let us compute

$$v(s, \gamma, \sigma_0^\tau) = \int_{T^*M(s,\gamma)} (2i\pi)^{-\dim M(s,\gamma)} |S(s,\gamma)|^{-1} \frac{\text{ch}_{s,\gamma}(\mathbb{A}_0(\sigma_0^\tau))}{J(M(s,\gamma))D_{s,\gamma}(T_{M(s,\gamma)}M)}.$$

We have

$$\text{ch}_{s,\gamma}(\mathbb{A}_0(\sigma_0^\tau)) = \text{ch}_{s,\gamma}(\mathbb{A}_0(\sigma_0)) \text{ch}_{s,\gamma}([V_\tau]).$$

For the horizontal connection $d+\tau(\theta)$ on $[V_\tau]$, we have $\text{ch}_{s,\gamma}([V_\tau]) = \text{Tr}(\tau(\gamma \exp \Theta_0))$. Thus

$$v(s, \gamma, \sigma_0^\tau) = \int_{T^*M(s,\gamma)} (2i\pi)^{-\dim M(s,\gamma)} \frac{I(s, \gamma, \sigma_0)}{|S(s, \gamma)|} \text{Tr}(\tau(\gamma \exp \Theta_0)).$$

We obtain

$$\begin{aligned} v_2(\gamma, \phi) &= \\ & \int_{T^*M(s,\gamma)} (2i\pi)^{-\dim M(s,\gamma)} \frac{I(s, \gamma, \sigma_0)}{|S(s, \gamma)|} \left(\sum_{\tau \in \hat{H}} \text{Tr} \tau(\gamma \exp \Theta_0) \left(\int_H \text{Tr} \tau^*(h) \phi(h) dh \right) \right) \\ &= \text{vol}(H, dh) \int_{T^*M(s,\gamma)} (2i\pi)^{-\dim M(s,\gamma)} \frac{I(s, \gamma, \sigma_0)}{|S(s, \gamma)|} \phi(\gamma \exp \Theta_0), \end{aligned}$$

using the Fourier inversion formula.

The basic form $\phi(\gamma \exp \Theta_0)$ depends of the Taylor expansion of ϕ at $\gamma \in H$. Recall that ϕ vanishes on a neighborhood of γ if γ is not conjugated to u . Thus only the class (u) makes a non zero contribution to $v_2(\phi) = \sum_{\gamma \in \mathcal{C}(s)} v_2(\gamma, \phi)$ and we obtain

(18)

$$v_2(\phi) = \text{vol}(H, dh) \int_{T^*M(s,u)} (2i\pi)^{-\dim M(s,u)} |S(s, u)|^{-1} I(s, u, \sigma_0) \phi(u \exp \Theta_0).$$

Comparing Formulae 17 and 18, we obtain Formula 15. ■

3 Quantization on orbifolds

We here consider the special case of Dirac operators. Consider the case where P has a $(G \times H)$ -invariant metric and where T_H^*P is a $(G \times H)$ -equivariant oriented even dimensional bundle with spin-structure. Let

$$TP = T_{hor}P \oplus P \times \mathfrak{h}$$

be the orthogonal decomposition of the tangent bundle. We identify T_H^*P with $T_{hor}P$ with the help of the metric. Let \mathcal{S}_{hor} be the spin bundle for $T_{hor}P$. Choose a $(G \times H)$ -invariant orientation o on $T_{hor}P$. The orientation

o determines a $\mathbb{Z}/2\mathbb{Z}$ -gradation $\mathcal{S}_{hor} = \mathcal{S}_{hor}^+ \oplus \mathcal{S}_{hor}^-$. If $v \in (T_{hor}P)_y$, then the Clifford multiplication $c(v)$ is an odd operator on $(\mathcal{S}_{hor})_y$. Let \mathcal{F} be a $(G \times H)$ -equivariant Hermitian vector bundle on P . Let $\mathcal{S}_{hor} \otimes \mathcal{F}$ be the twisted horizontal spin bundle. With the help of a choice of a $(G \times H)$ -invariant unitary connection $\nabla = \nabla^+ \oplus \nabla^-$ on $\mathcal{S}_{hor} \otimes \mathcal{F} = \mathcal{S}_{hor}^+ \otimes \mathcal{F} \oplus \mathcal{S}_{hor}^- \otimes \mathcal{F}$, we may define the formally self-adjoint ‘‘horizontal’’ Dirac operator $D_{hor, \mathcal{F}}$ by

$$D_{hor, \mathcal{F}} = \sum_i c(e_i) \nabla_{e_i}$$

where e_i runs over an orthonormal basis of $T_{hor}P$. We have $D_{hor, \mathcal{F}} = D_{hor, \mathcal{F}}^+ \oplus D_{hor, \mathcal{F}}^-$ with

$$D_{hor, \mathcal{F}}^+ : \Gamma(P, \mathcal{S}_{hor}^+ \otimes \mathcal{F}) \rightarrow \Gamma(P, \mathcal{S}_{hor}^- \otimes \mathcal{F})$$

and

$$D_{hor, \mathcal{F}}^- : \Gamma(P, \mathcal{S}_{hor}^- \otimes \mathcal{F}) \rightarrow \Gamma(P, \mathcal{S}_{hor}^+ \otimes \mathcal{F}).$$

Clearly the operators $D_{hor, \mathcal{F}}^\pm$ are H -transversally elliptic operators and commute with the natural action of G . The principal symbol of $D_{hor, \mathcal{F}}^+$ is given by

$$\sigma(D_{hor, \mathcal{F}}^+)(y, \xi) = c^+(\xi_0) \otimes I_{\mathcal{F}_y}$$

where ξ_0 is the projection of $\xi \in (T^*P)_y$ on $(T_H^*P)_y$. We define

$$Q^o(P/H, \mathcal{F}) = (-1)^{\dim M/2} Q(D_{hor, \mathcal{F}}^+).$$

When H acts freely, this coincides with the quantization assignment defined in [17]. We generalize to this case the universal formula for the virtual representation $Q^o(P/H, \mathcal{F})$ ([6], [17], [18]).

Consider the vector bundle $T_H^*P \rightarrow P$ with projection p_0 . We have chosen a $(G \times H)$ -invariant orientation o of T_H^*P .

The horizontal connection ∇_0 of T_H^*P determines a connection on \mathcal{S}_{hor} . Consider on the equivariant bundle \mathcal{F} an horizontal connection. Then $\text{ch}_{s,u}(\mathcal{F})$ is a $G(s)$ -equivariant form on $M(s, u)$.

Consider the pull-back of $\mathcal{S}_{hor} \otimes \mathcal{F}$ to T^*P . Then

$$\mathbb{A}(\sigma) = -\mathbf{c}_0 \otimes I_{p^* \mathcal{F}} + p^* \nabla^{\mathcal{S}_{hor} \otimes \mathcal{F}}$$

where \mathbf{c}_0 is the odd bundle endomorphism of $p^* \mathcal{S}_{hor}$ given by $\mathbf{c}_0(y, \xi) = c(\xi_0)$ where c is the Clifford action of $(T_H^*P)_y$ on $(\mathcal{S}_{hor})_y$ and ξ_0 the projection of ξ on $(T_H^*P)_y$.

Let \mathbf{B} be the superconnection on $p_0^*(\mathcal{S}_{hor}) \rightarrow T_{hor}^*P$ defined by:

$$(19) \quad \mathbf{B} = -\mathbf{c}_0 + p_0^*\nabla^{\mathcal{S}_{hor}}.$$

Let $(s, u) \in G \times H$. We have for $X \in \mathfrak{g}(s)$

$$\text{ch}_{s,u}(\mathbb{A}(\sigma))(X) = \text{ch}_{s,u}(\mathbf{B})(X) \text{ch}_{s,u}(\mathcal{F})(X).$$

Consider the bundle $T_{hor}^*P(s, u) \rightarrow P(s, u)$. It is a $(G(s) \times H(u))$ even-dimensional equivariant orientable vector bundle (see Lemma 6.10, [5]).

Let us choose an orientation o' on the vector bundle $T_{hor}^*P(s, u) \rightarrow P(s, u)$. The rank of this vector bundle is $\dim M(s, u)$. If $U_{o'}^{s,u}$ is the Thom form of the vector bundle $T_{hor}^*P(s, u) \rightarrow P(s, u)$, we have

$$i^{\dim M/2} \text{ch}_{s,u}(\mathbf{B})(X) =$$

$$\epsilon((s, u), o, o') (-2\pi)^{\dim M(s,u)/2} J^{1/2}(T^*M(s, u))(X) D_{s,u}^{1/2}(T_{M(s,u)}^*M)(X) U_{o'}^{s,u}(X)$$

where $\epsilon((s, u), o, o')$ is a sign. This follows from [14] (see also Chapter 7, [5]). The equation determines the sign $\epsilon((s, u), o, o')$. Here the generic stabilizer of the action of $H(u)$ on $T_{hor}^*P(s, u)$ is equal to the generic stabiliser $S(s, u)$ for the action of $H(u)$ on $M(s, u)$. Thus integrating over the fibers the formula of Theorem 1 for the index of $D_{hor, \mathcal{F}}^+$ and using Formula 5 we obtain the following proposition which is the analogue of the equivariant Hirzebruch-Riemann-Roch theorem in the form given in ([6], [18]).

Proposition 4 *Let $M = P/H$ be an even-dimensional orbifold such that $T_{hor}P$ is a $(G \times H)$ -oriented spin vector bundle with orientation o . Let \mathcal{F} be a $(G \times H)$ -equivariant complex vector bundle on P . Then*

$$\text{Tr } Q^o(P/H, \mathcal{F})(s \exp X) =$$

$$i^{-\dim M/2} \sum_{\gamma \in (C(s))} \int_{M(s, \gamma), o'} (2\pi)^{-\dim M(s, \gamma)/2} |S(s, \gamma)|^{-1} \frac{\epsilon((s, \gamma), o, o') \text{ch}_{s, \gamma}(\mathcal{F})(X)}{J^{1/2}(M(s, \gamma))(X) D_{s, \gamma}^{1/2}(T_{M(s, \gamma)}M)(X)}$$

for X small in $\mathfrak{g}(s)$.

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