

# Multiplicities Formula for Geometric Quantization. Part I

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## 1 Introduction

Let  $G$  be a compact Lie group with Lie algebra  $\mathfrak{g}$  acting on a compact symplectic manifold  $M$  by a Hamiltonian action. If  $X \in \mathfrak{g}$ , we denote by  $X_M$  the vector field on  $M$  induced by the action of  $G$ . We denote by  $\sigma$  the symplectic form on  $M$  and by  $\mu : M \rightarrow \mathfrak{g}^*$  the moment map. To simplify, we will assume in this article that  $M$  has a  $G$ -invariant spin structure. We will show in the appendix how to remove this assumption.

Let us assume that  $M$  is prequantized and let  $\mathcal{L}$  be the Kostant-Souriau line bundle on  $M$ . We denote by  $R(G)$  the ring of virtual finite dimensional representations of  $G$ . An element of  $R(G)$  is thus a difference of two finite dimensional representations of  $G$ . We associate to  $(M, \mathcal{L})$  a virtual representation  $Q(M, \mathcal{L}) \in R(G)$  of  $G$  constructed as follows: Choose a  $G$ -invariant Riemannian structure on  $M$ . Let  $\mathcal{S}^\pm$  be the half-spin bundles over  $M$  determined by the spin structure and the symplectic orientation of  $M$ . Let  $\Gamma(M, \mathcal{S}^\pm \otimes \mathcal{L})$  be the spaces of smooth sections of  $\mathcal{S}^\pm \otimes \mathcal{L}$ . Consider the twisted Dirac operator

$$D_{\mathcal{L}}^+ : \Gamma(M, \mathcal{S}^+ \otimes \mathcal{L}) \rightarrow \Gamma(M, \mathcal{S}^- \otimes \mathcal{L}).$$

This is an elliptic operator commuting with the action of  $G$ . We define a virtual representation  $Q(M, \mathcal{L})$  of  $G$  by the formula:

$$Q(M, \mathcal{L}) = (-1)^{\dim M/2} ([Ker D_{\mathcal{L}}^+] - [Coker D_{\mathcal{L}}^+]).$$

The virtual representation  $Q(M, \mathcal{L})$  so obtained is independent of the choice of the Riemannian structure on  $M$ . If  $M$  and  $\mathcal{L}$  have  $G$ -invariant complex

structures, then  $Q(M, \mathcal{L})$  (apart from a shift of parameters) is the direct image of the sheaf  $\mathcal{O}(\mathcal{L})$  of holomorphic sections of  $\mathcal{L}$  by the map  $M \rightarrow \text{point}$ . In the differentiable category, we employ as in Atiyah-Hirzebruch [3] the Dirac operator to define the direct image  $Q(M, \mathcal{L}) \in R(G) = K_G(\text{point})$  of  $\mathcal{L} \in K_G(M)$ . If the group  $G$  is trivial, then  $Q(M, \mathcal{L}) \in \mathbb{Z}$  is the index of the operator  $D_{\mathcal{L}}^+$ . We call this number the Riemann-Roch number of  $(M, \mathcal{L})$ .

We are interested in describing the decomposition of  $Q(M, \mathcal{L})$  in irreducible representations of  $G$ . Let  $G = T$  be a torus. Let  $P \subset \mathfrak{t}^*$  be the lattice of weights of  $T$ . We have a decomposition

$$Q(M, \mathcal{L}) = \sum_{\xi \in iP} n(\xi, M, \mathcal{L}) e_{i\xi}$$

where  $n(\xi, M, \mathcal{L}) \in \mathbb{Z}$  and is non zero for a finite number of  $\xi$  in the lattice  $iP \subset \mathfrak{t}^*$ .

The number  $n(\xi, M, \mathcal{L})$  is called the multiplicity of the weight  $e_{i\xi}$  in  $Q(M, \mathcal{L})$ . The function  $\xi \mapsto n(\xi, M, \mathcal{L})$  defined on the lattice  $iP \subset \mathfrak{t}^*$  is the Fourier transform of the character  $\text{Tr} Q(M, \mathcal{L})$  of  $T$ . It is the quantum analogue of the function over  $\mu(M) \subset \mathfrak{t}^*$  equal to the density  $\mu_*(\beta_M)$  of the pushforward by  $\mu$  of the Liouville measure  $\beta_M$  of  $M$ . The value of the locally polynomial density  $\mu_*(\beta_M)$  on a regular value  $\xi \in \mathfrak{t}^*$  of  $\mu$  is equal to the symplectic volume of the reduced fiber  $M_\xi = T \backslash \mu^{-1}(\xi)$  ([12]). Let  $\xi \in iP \cap \mu(M)$ . Assume furthermore that  $T$  acts freely on  $\mu^{-1}(\xi)$ . Then  $\mathcal{L}_\xi = T \backslash (\mathcal{L}|_{\mu^{-1}(\xi)})$  is a Kostant-Souriau line bundle on the reduced symplectic manifold  $M_\xi$  and  $M_\xi$  inherits a spin structure from  $M$ . Guillemin- Sternberg [17] conjectured that the value of  $n(\xi, M, \mathcal{L})$  at a regular value  $\xi \in iP \cap \mu(M)$  is the Riemann-Roch number  $Q(M_\xi, \mathcal{L}_\xi)$ . They prove this conjecture in the holomorphic case under some positivity assumptions on  $\mathcal{L}$ .

Recently, E. Witten [31] suggested a formula relating integrals over  $M \times \mathfrak{g}$  of some closed equivariant cohomology classes  $\alpha(X)$  on  $M$  and integrals over the reduced fiber  $M_\xi$ . Witten's formula has been proved by Jeffrey-Kirwan [19]. We realised [29] that the same idea of Witten can be employed modulo some elaboration to the proof of the formula  $n(\xi, M, \mathcal{L}) = Q(M_\xi, \mathcal{L}_\xi)$  for multiplicities. The similarity between Jeffrey-Kirwan-Witten formula and multiplicities formulas was also noticed independently and from a different view point by V.Guillemin [14] and E. Meinrenken [22]. In particular E. Meinrenken [22] proved the formula  $n(\xi, M, \mathcal{L}) = Q(M_\xi, \mathcal{L}_\xi)$  including the case of locally free actions, where the definition of  $Q(M_\xi, \mathcal{L}_\xi)$  has to be suitably modified. Note that a simple proof of the formula  $n(\xi, M, \mathcal{L}) = Q(M_\xi, \mathcal{L}_\xi)$

is obtained in the case of an  $S^1$ -action by Jeffrey-Kirwan [20] using their residue formula.

Our proof relies directly on the universal formula for the character of  $Q(M, \mathcal{L})$ . Let us explain briefly our technique which can be applied as well to other index problems. Let us consider an open  $G$ -invariant subset  $U$  with smooth boundary  $B$  of a compact spin manifold  $M$ . Let  $\lambda$  be a  $G$ -invariant 1-form on  $M$  such that the map  $\mu_\lambda : M \rightarrow \mathfrak{g}^*$  defined by  $\mu_\lambda(X) = (\lambda, X_M)$  does not vanish at any point of the boundary  $B$ . Let  $\mathcal{E}$  be a  $G$ -equivariant complex vector bundle over  $M$ . In [9], the formula of Atiyah-Segal-Singer for the equivariant index  $Q(M, \mathcal{E})$  of the twisted Dirac operator  $D_{\mathcal{E}}$  is reformulated in a neighbourhood of the identity of  $G$  in terms of the equivariant cohomology of  $M$  as follows :

$$\mathrm{Tr} Q(M, \mathcal{E})(\exp X) = (2i\pi)^{-\dim M/2} \int_M \frac{\mathrm{ch}(\mathcal{E})(X)}{J^{1/2}(M)(X)}.$$

Here  $\mathrm{ch}(\mathcal{E})(X)$  is the equivariant Chern character of  $\mathcal{E}$  and  $J^{-1/2}(M)(X)$  is the equivariant  $\hat{A}$ -genus of  $M$ . The class  $J(M)(X)$  is invertible for  $X$  in a neighbourhood of 0 in  $\mathfrak{g}$ . Normalisations are as in [13]. Consider for  $t \in \mathbb{R}$  the  $C^\infty$  functions  $TQ(U, \mathcal{E}, \lambda, t)$  on a neighbourhood of  $0 \in \mathfrak{g}$  defined by:

$$TQ(U, \mathcal{E}, \lambda, t)(X) = (2i\pi)^{-\dim M/2} \int_U e^{-itd_{\mathfrak{g}}\lambda(X)} \frac{\mathrm{ch}(\mathcal{E})(X)}{J^{1/2}(M)(X)}$$

where  $d_{\mathfrak{g}} = d - \iota(X_M)$  is the equivariant differential.

We conjecture that there exists an (infinite-dimensional) trace class virtual representation  $Q(U, \mathcal{E}, \lambda)$  of  $G$  such that we have the identity of generalised functions on a neighbourhood of 0 in  $\mathfrak{g}$ :

$$\mathrm{Tr} Q(U, \mathcal{E}, \lambda)(\exp X) = \lim_{t \rightarrow \infty} TQ(U, \mathcal{E}, \lambda, t)(X)$$

(and similar identities near any point  $s \in G$ ).

In this article we prove the following theorem ( Theorem 17) :

Let  $T$  be a torus with Lie algebra  $\mathfrak{t}$ . Assume that there exists  $S \in \mathfrak{t}$  such that  $(\lambda, S_M) > 0$  on the boundary  $B$  of  $U$ . Then there exists a trace class virtual representation  $Q(U, \mathcal{E}, \lambda)$  of  $T$  such that

$$\mathrm{Tr} Q(U, \mathcal{E}, \lambda)(\exp X) = \lim_{t \rightarrow \infty} TQ(U, \mathcal{E}, \lambda, t)(X).$$

Furthermore, the fixed point formula of Atiyah-Segal-Singer is valid in the generalised sense. Let  $M^T$  be the submanifold of fixed points for the

action of the torus  $T$ . Our hypothesis implies that  $M^T \cap B = \emptyset$ . Consider the subset  $M^T \cap U$ . Then we have the fixed point formula for  $g = \exp X$

$$\mathrm{Tr} Q(U, \mathcal{E}, \lambda)(g) = \int_{M^T \cap U} \frac{\mathrm{ch}(\mathcal{E})(X)}{J^{1/2}(M^T)} D_{0^+(S)}^{-1}(g)$$

where  $D(g)$  is an holomorphic function on  $T_{\mathbb{C}}$  with value differential forms on  $M^T$  and  $D_{0^+(S)}^{-1}(g)$  is the boundary value of the holomorphic function  $D^{-1}(g)$  on an open subset of  $T_{\mathbb{C}}$  determined by  $S$ . Thus the invariant form  $\lambda$  allows us to construct a trace class virtual representation with character formula given by the Atiyah-Segal-Singer fixed point formula on  $U$  suitably interpreted as a generalised function.

Let us return to the case of an Hamiltonian manifold  $M$  under an action of  $G = S^1$ . Let us indicate briefly how to write  $Q(M, \mathcal{L})$  as a sum of three infinite dimensional trace class (virtual) representations related to the geometry of the moment map  $\mu : M \rightarrow \mathfrak{g}^*$ . We consider  $0 \in \mathfrak{g}^*$  and we assume  $G$  acts freely on  $\mu^{-1}(0)$ .

Choosing a basis  $E$  of  $\mathfrak{g}$ , we consider the map  $f = \mu(E)$  from  $M$  to  $\mathbb{R}$ . Choose  $r$  a small positive real number. Let

$$M_0 = \{x; |f(x)| < r\}, \quad M_+ = \{x; f(x) > r\}, \quad M_- = \{x; f(x) < -r\}.$$

Let  $(\cdot, \cdot)$  be a  $G$ -invariant metric on  $M$  and let  $\lambda(\cdot) = \mu(E)(E_M, \cdot)$ . Then  $\lambda$  is a  $G$ -invariant 1-form on  $M$ . The value  $\lambda(E_M) = \mu(E) \|E_M\|^2$  is strictly positive on the boundary of  $M_+$  and strictly negative on the boundary of  $M_-$ . We can thus construct with the help of the 1-form  $\lambda$  virtual representations  $Q_{\pm}(M, \mathcal{L}) = Q(M_{\pm}, \mathcal{L}, \lambda)$  with character formula given by Atiyah-Segal-Singer fixed point formula on  $M_{\pm}$  expanded in the outer directions: it follows that the virtual character  $Q_+(M, \mathcal{L})$  is of the form  $Q_+(M, \mathcal{L}) = \sum_{n>0} a_n e^{in\theta}$ . Similarly  $Q_-(M, \mathcal{L}) = \sum_{n<0} a_n e^{in\theta}$ . In particular neither  $Q_+(M, \mathcal{L})$  nor  $Q_-(M, \mathcal{L})$  contains the trivial representation of  $G = S^1$ .

We denote  $f^{-1}(0)$  by  $P$ . Consider the principal fibration  $q : P \rightarrow M_{red} = G \backslash P$  with structure group  $G = S^1$ . Let  $\mathcal{L}_{red} = G \backslash (\mathcal{L}|_P)$ . Consider, for all  $n \in \mathbb{Z}$  the characters  $\chi_n(\exp \theta E) = e^{in\theta}$  of  $G$  and let  $\mathcal{T}_n$  be the associated line bundles on  $G \backslash P = M_{red}$ . Then  $Q(M_{red}, \mathcal{L}_{red} \otimes \mathcal{T}_n)$  is a positive or negative integer. Define the virtual character  $Q_0(M, \mathcal{L})$  of  $G$  by

$$Q_0(M, \mathcal{L}) = \sum_{n \in \mathbb{Z}} Q(M_{red}, \mathcal{L}_{red} \otimes \mathcal{T}_{-n}) e^{in\theta}.$$

In particular, the multiplicity of the trivial representation in  $Q_0(M, \mathcal{L})$  is  $Q(M_{red}, \mathcal{L}_{red})$ .

As a corollary of Theorem 17 of this article and of a limit formula à la Witten for integration of equivariant differential forms in a neighbourhood of  $f^{-1}(0)$  proved in [28], we obtain the following decomposition of  $Q(M, \mathcal{L})$  associated to the partition

$$M = M_0 \cup \overline{M_+} \cup \overline{M_-}$$

of  $M$ :

$$Q(M, \mathcal{L}) = Q_0(M, \mathcal{L}) \oplus Q_+(M, \mathcal{L}) \oplus Q_-(M, \mathcal{L}).$$

If a group  $K$  commutes with the action of  $G$ , then this decomposition is a decomposition of representations of  $G \times K$ . As a result the virtual representation  $Q(M, \mathcal{L})^G$  of  $K$  is the virtual representation  $Q(M_{red}, \mathcal{L}_{red})$  of  $K$ .

To extend the results to the action of a torus, as announced in [29], we can apply successively the decomposition above to analyse the decomposition of  $Q(M, \mathcal{L})$  under the action of a torus  $T = (S^1)^n$  in order to obtain  $n(\xi, M, \mathcal{L}) = Q(M_\xi, \mathcal{L}_\xi)$  for a regular value  $\xi$ . This requires an extension of Theorem 17 to an open subset  $V$  of some non compact manifolds  $N$ . The basic example is  $T^*G \times M$  where  $M$  is a compact  $G \times H$  manifold and  $V = T^*G \times U$ . We will not give the details. Instead, we will give in Part II of this article [30] a proof based directly on a deformation of the Dirac operator itself. This proof, although requiring the machinery of transversally elliptic symbols, requires almost no computation and leads directly to the formula  $n(\xi, M, \mathcal{L}) = Q(M_\xi, \mathcal{L}_\xi)$  in the more general case of orbifolds with torus actions. It is parallel to the method we used here. However we feel worthwhile to publish here a detailed and elementary proof for the case of an  $S^1$ -action.

In order to state the results without spin hypothesis, it is necessary to modify as in [23] the notion of Kostant-Souriau line bundle. This is explained in the appendix and is only a technical modification. However, we believe results are more invariantly stated in terms of quantum bundles as in [27]. Indeed if  $\tau$  is a quantum bundle on an even dimensional compact oriented manifold  $M$ , there is a virtual representation  $Q(M, \tau)$  associated to  $\tau$ . Furthermore there is a natural map  $\tau \mapsto \tau_{red}$  from quantum bundles on  $M$  to quantum bundles on the reduction  $M_{red}$ . Our main result on multiplicities is thus:

**Theorem 1** *Let  $G$  be a torus . Let  $K$  be a compact Lie group. Let  $M$  be a  $G \times K$  Hamiltonian manifold. Let  $\mathcal{L}$  be a Kostant-Souriau quantum line bundle for  $G \times K$ . Let  $Q(M, \mathcal{L}) \in R(G \times K)$  be the quantized space. Let  $\mu : M \rightarrow \mathfrak{g}^*$  the moment map for the  $G$ -action. Assume that  $G$  acts freely on  $\mu^{-1}(0)$ . Consider the  $K$ -Hamiltonian manifold  $M_{red} = G \backslash \mu^{-1}(0)$  with Kostant-Souriau quantum line bundle  $\mathcal{L}_{red}$ . Then the virtual representation  $Q(M, \mathcal{L})^G$  is isomorphic to the virtual representation  $Q(M_{red}, \mathcal{L}_{red})$  of  $K$ .*

I am thankful to Michel Duflo for discussion on this problem. The inspiration of this work is our common conjecture ([27]) on universal formula for characters. Indeed the character  $Q(U, \mathcal{E}, \lambda)$  obtained by a limit procedure is the universal character formula for the manifold  $U$  with a cylindrical end attached to it.

## 2 Quantization on compact manifolds

In this section we recall some well known facts due mainly to Atiyah-Bott ([1], [2]) , Atiyah-Segal-Singer ([5], [4], [6]) ) and Berline-Vergne ([9]) on the equivariant index of the Dirac operator.

Let  $G$  be a compact Lie group acting on a compact even dimensional orientable manifold  $M$ . For simplicity, we assume in this article that  $M$  has a  $G$ -invariant spin structure. We will remove this assumption in part II. If  $\mathcal{E}$  is a vector bundle over  $M$ , we denote by  $\Gamma(M, \mathcal{E})$  the space of its smooth sections. Let  $K_G(M)$  be the Grothendieck group of  $G$ -equivariant complex vector bundles over  $M$ .

We denote by  $R(G)$  the ring of virtual finite dimensional representations of  $G$ . If  $V^\pm$  are two finite dimensional representation spaces of  $G$ , then  $[V] = [V^+] - [V^-]$  is an element of  $R(G)$ . We denote  $\dim V = \dim V^+ - \dim V^-$ . If  $G = \{1\}$  we identify  $R(G)$  to  $\mathbb{Z}$  by the function  $\dim V$ .

Let  $o$  be a  $G$ -invariant orientation of  $M$ . There is a well defined quantization map

$$Q^o : K_G(M) \rightarrow R(G).$$

This map can be constructed as follows: Choose a  $G$ -invariant metric on  $M$ . Let  $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$  be the spinor bundle (conventions on gradings are as in [7]). Let  $\mathcal{E}$  be a  $G$ -equivariant complex vector bundle over  $M$ . Let  $\mathcal{S}^\pm \otimes \mathcal{E}$  be the twisted spinor bundle. With the help of a  $G$ -invariant connection  $\mathbb{A}$

on  $\mathcal{E}$ , we can construct a twisted Dirac operator  $D_{\mathcal{E},\mathbb{A}}$ . This gives an elliptic operator

$$D_{\mathcal{E},\mathbb{A}}^+ : \Gamma(M, \mathcal{S}^+ \otimes \mathcal{E}) \rightarrow \Gamma(M, \mathcal{S}^- \otimes \mathcal{E})$$

which commutes with the natural action of  $G$ . The index space of  $D_{\mathcal{E},\mathbb{A}}^+$  is by definition the virtual representation of  $G$  in  $[Ker D_{\mathcal{E},\mathbb{A}}^+] - [Coker D_{\mathcal{E},\mathbb{A}}^+]$ . The virtual representation of  $G$  so obtained is independent of the choice of the metric on  $M$  and of the connection  $\mathbb{A}$  on  $\mathcal{E}$ . It depends only of the bundle  $\mathcal{E}$  and of the orientation  $o$  of  $M$ . We define

$$(1) \quad Q^o(M, \mathcal{E}) = (-1)^{\dim M/2} ([Ker D_{\mathcal{E},\mathbb{A}}^+] - [Coker D_{\mathcal{E},\mathbb{A}}^+])$$

In particular if  $G$  is reduced to the identity and if  $\mathcal{E}$  is a bundle over  $M$ , then  $Q^o(M, \mathcal{E}) \in \mathbb{Z}$  is a number. This number will be called the Riemann-Roch number of the bundle  $\mathcal{E}$  over  $M$ .

Indeed the direct image map  $Q^o$  is the  $C^\infty$ -version of the direct image map in algebraic geometry. If  $M$  is a complex manifold of complex dimension  $d$  and  $\mathcal{E}$  is a holomorphic vector bundle, then the space  $Q^o(M, \mathcal{E})$  coincide up to a sign  $\epsilon(o)$  with the virtual space

$$H(M, \mathcal{E} \otimes \rho^*) = \sum_{k=0}^d (-1)^k [H^k(M, \mathcal{O}(\mathcal{E} \otimes \rho^*))]$$

of cohomology of the sheaf of holomorphic sections of  $\mathcal{E} \otimes \rho^*$ , where  $\rho$  is the square root of the line bundle of  $(n, 0)$ -forms. The line bundle  $\rho$  exists, from our assumption on existence of spin structure. In particular, if  $\mathcal{E}$  is sufficiently positive, the space  $Q^o(M, \mathcal{E})$  is up to sign the space of holomorphic sections of  $\mathcal{E} \otimes \rho^*$ . Our convention on orientations is such that if  $\mathcal{L}$  is a sufficiently positive line bundle and  $o_{\mathcal{L}}$  the orientation of  $M$  induced by the symplectic form determined by the curvature of  $\mathcal{L}$ , then  $Q^{o_{\mathcal{L}}}(M, \mathcal{L}) = H^0(M, \mathcal{L} \otimes \rho^*)$ .

Let  $\mathfrak{g}$  be the Lie algebra of  $G$ . If  $X \in \mathfrak{g}$ , we denote by  $X_M$  the vector field on  $M$  produced by the infinitesimal action of  $\mathfrak{g}$ :

$$(X_M)_x = \frac{d}{d\epsilon} (\exp -\epsilon X) \cdot x|_{\epsilon=0}.$$

A  $G$ -equivariant differential form on  $M$  is a smooth  $G$ -equivariant map, defined on the Lie algebra  $\mathfrak{g}$ , with values in the space  $\mathcal{A}(M)$  of smooth differential forms on  $M$ . We denote the space of  $G$ -equivariant differential

forms by  $\mathcal{A}_G^\infty(\mathfrak{g}, M) = C^\infty(\mathfrak{g}, \mathcal{A}(M))^G$ . Here  $X$  will denote a point on  $\mathfrak{g}$  or the function  $X \mapsto X$ . Thus we may denote a map  $\alpha : \mathfrak{g} \rightarrow \mathcal{A}(M)$  by the notation  $\alpha(X)$ . Similar notations will be common for functions on manifolds, where the notation  $f(x)$  will denote (depending of the context) either the value of the function  $f$  at the point  $x \in M$  or the function  $f$  itself.

We will consider equivariant differential forms  $\alpha(X)$  which are defined only for  $X$  belonging to a  $G$ -invariant open subset  $W \subset \mathfrak{g}$ . We denote by  $\mathcal{A}_G^\infty(W, M) = C^\infty(W, \mathcal{A}(M))^G$  the space of these forms. An element of  $C^\infty(W, \mathcal{A}(M))$  will also be referred to as a differential form on  $M$  depending on  $X \in W$ . The *equivariant coboundary*  $d_{\mathfrak{g}} : \mathcal{A}_G^\infty(W, M) \rightarrow \mathcal{A}_G^\infty(W, M)$  is defined for  $\alpha \in \mathcal{A}_G^\infty(W, M)$  and  $X \in W$  by

$$(d_{\mathfrak{g}}\alpha)(X) = d(\alpha(X)) - \iota(X_M)(\alpha(X)),$$

where  $\iota(X_M)$  is the contraction with the vector field  $X_M$ .

A closed equivariant form is by definition a  $G$ -equivariant differential form  $\alpha$  satisfying  $d_{\mathfrak{g}}\alpha = 0$ .

We also write  $d_X$  for the operator  $d - \iota(X_M)$  acting on forms.

Let  $G$  be a compact Lie group acting on a symplectic manifold  $M$  by a Hamiltonian action. We denote by  $(M, \sigma, \mu)$  such a data, with  $\sigma$  the symplectic form on  $M$  and  $\mu : M \rightarrow \mathfrak{g}^*$  the moment map. For  $X \in \mathfrak{g}$ , let

$$(2) \quad X \mapsto \sigma_{\mathfrak{g}}(X) = \mu(X) + \sigma$$

be the equivariant symplectic form. It is a closed  $G$ -equivariant differential form on  $M$ .

Let  $\mathcal{E}$  be a  $G$ -equivariant vector bundle with  $G$ -invariant connection  $\mathbb{A}$ . Let  $F$  be the curvature of  $\mathbb{A}$ . For  $X \in \mathfrak{g}$ , let  $\mu^{\mathbb{A}}(X) \in \Gamma(M, \text{End}(\mathcal{E}))$  be the moment of  $X$  with respect to the connection  $\mathbb{A}$  ([7]). Let  $X \mapsto F(X) = \mu^{\mathbb{A}}(X) + F$  ( $X \in \mathfrak{g}$ ) be its equivariant curvature. Then  $\text{ch}(\mathcal{E}, \mathbb{A})(X) = \text{Tr}(e^{F(X)})$  is a closed  $G$ -equivariant differential form on  $M$  called the equivariant Chern character. For  $X = 0$ , we denote  $\text{ch}(\mathcal{E}, \mathbb{A})(0) = \text{Tr}(e^F)$  simply by  $\text{ch}(\mathcal{E})$  leaving implicit the choice of connection. The form  $\text{ch}(\mathcal{E})$  is up to normalisations factors of  $2\pi$  the usual Chern character.

**Definition 2** (*Kostant-Souriau*) ([21],[25]). *The Hamiltonian manifold  $(M, \sigma, \mu)$  is said to be quantizable, if there exists a  $G$ -equivariant line bundle  $\mathcal{L} \rightarrow M$  with  $G$ -invariant connection  $\mathbb{A}$  such that the equivariant curvature of  $\mathcal{L}$  is  $i\sigma_{\mathfrak{g}}(X)$ . Such a line bundle will be called a Kostant-Souriau line bundle.*

Thus, if  $(\mathcal{L}, \mathbb{A})$  is a Kostant-Souriau line bundle, we have for  $X \in \mathfrak{g}$ ,

$$(3) \quad \text{ch}(\mathcal{L}, \mathbb{A})(X) = e^{i\sigma_{\mathfrak{g}}(X)}.$$

If  $G$  acts on a set  $E$ , we will denote by  $E^G$  the set of fixed points of  $G$  in  $E$ . If  $s \in G$ , we denote by  $E(s)$  the subset of  $E$  fixed by  $s$ . We denote by  $G(s)$  the centraliser of  $s$  in  $G$ . If  $s \in G$ , the set  $M(s)$  is a submanifold of  $M$ . We denote by  $TM$  the tangent bundle to  $M$ . If  $N$  is a closed submanifold of  $M$ , we denote by  $T_N M = TM|_N/TN$  the normal bundle to  $N$  in  $M$ . If  $S \in \mathfrak{g}$ , we denote by  $M(S) = \{m \in M; (S_M)_m = 0\}$  the manifold of zeroes of the vector field  $S_M$ .

Recall the localisation formula for  $G$ -equivariant differential forms with compact support on an oriented  $G$ -manifold  $M$ . Let  $S \in \mathfrak{g}$  and consider the zero set  $M(S)$ . We choose a  $G(S)$ -invariant connection on  $T_{M(S)}M$  and we denote by  $R(T_{M(S)}M)$  its  $G(S)$ -equivariant curvature. Choose an orientation  $o$  on  $T_{M(S)}M$ . We denote by  $Eul_o(T_{M(S)}M)$  the  $G(S)$ -equivariant Euler form of  $T_{M(S)}M$ . We have for  $Y \in \mathfrak{g}(S)$

$$Eul_o(T_{M(S)}M)(Y) = (-2\pi)^{-\text{rank}(T_{M(S)}M)/2} \det_o R(T_{M(S)}M)(Y)^{1/2}.$$

Let  $\mathcal{W}$  be a  $G(S)$ -invariant neighbourhood of  $S$  in  $\mathfrak{g}(S)$ . Let  $\alpha \in \mathcal{A}_{G(S)}^\infty(\mathcal{W}, M)$  such that  $\alpha(X) \in \mathcal{A}_{cpt}(M)$  for every  $X \in \mathcal{W}$ . We suppose that  $\alpha$  is a closed  $G(S)$ -equivariant form on  $M$ . The form  $Eul_o(T_{M(S)}M)(Y)_x$  is invertible for  $Y$  sufficiently near  $S$  and  $x$  in the compact support of  $\alpha$ . Then for  $Y \in \mathfrak{g}(S)$  sufficiently close to  $S$ , we have ([8],[10], see also [7] Chapter 7 )

$$(4) \quad \int_M \alpha(Y) = \int_{M(S)} \frac{\alpha(Y)}{Eul_o(T_{M(S)}M)(Y)}.$$

Here the orientations of  $M$ ,  $M(S)$  and the orientation  $o$  on  $T_{M(S)}M$  are chosen in a compatible way.

Assume now that  $\alpha(Y)$  depends holomorphically of  $Y$ , for  $Y$  belonging to an open set  $W$  in  $\mathfrak{g}(S)_{\mathbb{C}}$ . Let  $U$  be the open subset of  $\mathfrak{g}(S)_{\mathbb{C}}$  consisting of those  $Y$  such that  $Eul_o(T_{M(S)}M)(Y)$  is invertible. Let  $W'$  be the connected component of  $W \cap U$  containing  $S$ . By analytic continuation, we then have, for all  $Y \in W'$

$$(5) \quad \int_M \alpha(Y) = \int_{M(S)} \frac{\alpha(Y)}{Eul_o(T_{M(S)}M)(Y)}.$$

For example, if  $\alpha$  is analytic in a small ball  $W \subset \mathfrak{g}(S)_{\mathbb{C}}$ , we can apply Formula 5 to  $Y = zS$ , with  $z$  a non zero complex number of small norm.

A bouquet of equivariant differential forms on  $M$  is a family  $\alpha_s$  where for each  $s \in G$  the form  $\alpha_s$  is a  $G(s)$ -closed equivariant differential form on  $M(s)$  satisfying the following conditions of invariance and compatibility ([13], [11], see also [27]).

1. Invariance:

$$\alpha_{gsg^{-1}} = g \cdot \alpha_s$$

for all  $g \in G$  and  $s \in G$ .

2. Compatibility: Let  $s \in G$ , then for all  $S \in \mathfrak{g}(s)$  and *sufficiently small*

$$\alpha_{se^S}(Y) = \alpha_s(S + Y)|_{M(se^S)}$$

for all  $Y \in \mathfrak{g}(se^S)$ .

**Remark 2.1** *If  $S \in \mathfrak{g}(s)$  is sufficiently small then  $M(se^S) = M(s) \cap M(S)$  and  $\mathfrak{g}(se^S) = \mathfrak{g}(s) \cap \mathfrak{g}(S)$  so that the right hand-side of the equality (2) has a meaning.*

Recall ([13],[11], see also [27]) the definition of the bouquet  $\text{bch}(\mathcal{E}, \mathbb{A})$  of Chern characters of a  $G$ -equivariant vector bundle with  $G$ -invariant connection  $\mathbb{A}$ . By definition  $\text{bch}(\mathcal{E}, \mathbb{A}) = (\text{ch}_s(\mathcal{E}, \mathbb{A}))_{s \in G}$  where

$$(6) \quad \text{ch}_s(\mathcal{E}, \mathbb{A})(X) = \text{Tr}(s^{\mathcal{E}} e^{F(X)}|_{M(s)}) \quad \text{for } X \in \mathfrak{g}(s).$$

If  $\lambda$  is a  $G$ -invariant 1-form, then  $\mathbb{A}(t) = \mathbb{A} - it\lambda I$  is also a  $G$ -invariant connection for all  $t \in \mathbb{R}$ . We have

$$(7) \quad \text{ch}_s(\mathcal{E}, \mathbb{A}(t)) = e^{-itd_{\mathfrak{g}(s)}\lambda|_{M(s)}} \wedge \text{ch}_s(\mathcal{E}, \mathbb{A}).$$

Let us give a formula for  $\text{Tr } Q^o(M, \mathcal{E})$  in the neighbourhood of  $s \in G$  in terms of the equivariant cohomology of  $M(s)$ . As  $M$  is oriented and has a  $G$ -invariant spin structure, the submanifolds  $M(s)$  of  $M$  are orientable (see for example [7] or [27].) To explain the formula for  $\text{Tr } Q^o(M, \mathcal{E})$  in the neighbourhood of  $s \in G$ , we recall the definitions of some other equivariant differential forms:

Let  $s$  be an orthogonal transformation of an Euclidean vector space  $V_1$  such that  $(1 - s)$  is invertible. In particular  $\det_{V_1}(1 - s) > 0$ . Let  $\mathfrak{so}(V_1)(s)$

be the space of orthogonal transformation of  $V_1$  commuting with  $s$ . Define for  $Y \in \mathfrak{so}(V_1)(s)$

$$(8) \quad D_s(V_1)(Y) = \det_{V_1}(1 - se^Y).$$

If  $V_1$  is understood, we write  $D_s(V_1) = D_s$ . The function  $D_s$  has an analytic square-root on  $(\mathfrak{so}(V_1)(s))_{\mathbb{C}}$ . We normalize it by  $D_s^{1/2}(0) > 0$ .

If  $V_1$  is two-dimensional with  $(e_1, e_2)$  as orthonormal basis, and  $s \cdot (e_1 + ie_2) = e^{i\theta}(e_1 + ie_2)$  with  $\theta$  not in  $2\pi\mathbb{Z}$ , then

$$(9) \quad D_s^{1/2}(0) = 2|\sin(\theta/2)|.$$

Let  $\nabla$  be a  $G$ -equivariant Euclidean connection on  $TM$ . Then  $\nabla$  determines Euclidean connections  $\nabla_0$  on  $TM(s)$  and  $\nabla_1$  on  $T_{M(s)}M$ . Let  $R_0(X)$ ,  $R_1(X)$  be the equivariant curvatures of  $\nabla_0$  and  $\nabla_1$ . Then, we define the  $G(s)$ -equivariant form  $J(M(s), \nabla)$  on  $M(s)$  by

$$(10) \quad J(M(s), \nabla)(X) = \det\left(\frac{e^{R_0(X)/2} - e^{-R_0(X)/2}}{R_0(X)}\right)$$

for  $X \in \mathfrak{g}(s)$ . When  $\nabla$  is understood, we write the equivariant differential form  $J(M(s), \nabla)$  simply as  $J(M(s))$ .

For  $X$  in a small neighbourhood of 0 in the complexification of  $\mathfrak{g}(s)$ ,  $J^{1/2}(M(s), \nabla)(X)$  is analytic and invertible.

Let us still denote by  $s$  the transformation of  $T_{M(s)}M$  determined by  $s$ . Then at each point  $x \in M(s)$  the transformation  $s$  is an orthogonal transformation of  $(T_{M(s)}M)_x$  and does not have any eigenvalue equal to 1. We define for  $X \in \mathfrak{g}(s)$ :

$$(11) \quad D_s(T_{M(s)}M, \nabla)(X) = \det(1 - se^{R_1(X)}).$$

When  $\nabla$  is understood, we write the equivariant differential form  $D_s(T_{M(s)}M, \nabla)$  simply as  $D_s(T_{M(s)}M)$ .

We denote by  $L(s, \mathcal{E}, \mathbb{A})$  the  $G(s)$ -equivariantly closed form on  $M(s)$  defined on a sufficiently small neighbourhood of 0 in  $\mathfrak{g}(s)_{\mathbb{C}}$  by

$$(12) \quad L(s, \mathcal{E}, \mathbb{A})(X) = (2\pi)^{-\dim M(s)/2} \text{ch}_s(\mathcal{E}, \mathbb{A})(X) J^{-1/2}(M(s))(X) D_s^{-1/2}(T_{M(s)}M)(X).$$

Let  $o$  be an orientation of  $M$  and  $o'$  an orientation of  $M(s)$ . The action of  $s$  on the spin bundle  $\mathcal{S}$  determines a sign  $\epsilon(s, o, o')$  which is a locally constant function on  $M(s)$ . If  $S$  is small, then  $M(e^S) = M(S)$ . The element  $S$  determines an orientation  $o_S$  of the normal bundle  $T_{M(s)}M$  and  $\epsilon(e^S, o, o') = 1$  in the case where the orientations  $o, o', o_S$  are compatible. The convention for  $o_S$  is as in [13].

**Theorem 3** *Let  $\mathcal{E}$  be a  $G$ -equivariant vector bundle over an even-dimensional compact spin manifold  $M$ . Choose a  $G$ -invariant connection  $\mathbb{A}$  on  $\mathcal{E}$ . Then, for each  $s \in G$ , there exists a neighbourhood  $U_s(0)$  of 0 in  $\mathfrak{g}(s)$  such that we have*

$$\mathrm{Tr} Q^o(M, \mathcal{E})(s \exp X) = i^{-\dim M/2} \int_{M(s), o'} (2\pi)^{-\dim M(s)/2} \frac{\epsilon(s, o, o') \mathrm{ch}_s(\mathcal{E}, \mathbb{A})(X)}{J^{1/2}(M(s))(X) D_s^{1/2}(T_{M(s)}M)(X)}$$

for every  $X \in U_s(0)$ .

We write also this formula

$$(13) \quad \mathrm{Tr} Q^o(M, \mathcal{E})(s \exp X) = i^{-\dim M/2} \int_{M(s), o'} \epsilon(s, o, o') L(s, \mathcal{E}, \mathbb{A})(X)$$

with  $L(s, \mathcal{E}, \mathbb{A})(X)$  given by (12). The formula (13) determines  $\mathrm{Tr} Q^o(M, \mathcal{E})$  in a neighbourhood of  $s \in G$ .

**Remark 2.2** *Let us denote by  $\mathcal{Z}_G(M)$  the space of bouquets of equivariant differential forms. We have defined in ([13], see also [27]) a direct image (or bouquet-integral) map*

$$\int_b^M : \mathcal{Z}_G(M) \rightarrow C^\infty(G)^G.$$

*We can restate the formula of Theorem 3 for the equivariant index of the twisted Dirac operator in function of the bouquet integration as follows*

$$\mathrm{Tr} Q^o(M, \mathcal{E}) = i^{-\dim M/2} \int_b^M \mathrm{bch}(\mathcal{E}, \mathbb{A}).$$

Let  $g = se^S$  with  $S \in \mathfrak{g}(s)$  small. Let  $Y \in \mathfrak{g}(s) \cap \mathfrak{g}(S)$ . Then the two integral expressions for  $\mathrm{Tr} Q^o(M, \mathcal{E})(se^{S+Y}) = \mathrm{Tr} Q^o(M, \mathcal{E})(se^S e^Y)$  either as an integral formula over  $M(s)$  (formula for the  $s$ -part) or over  $M(s) \cap M(S)$

(formula for the  $se^S$ -part) agree, as follows from the bouquet condition on  $\text{ch}_s(\mathcal{E}, \mathbb{A})$  and the localisation formula . For example the two formulas for  $\text{Tr } Q^o(M, \mathcal{E})(e^{S+Y}) = \text{Tr } Q^o(M, \mathcal{E})(e^S e^Y)$  coincide: for  $s = e^S$  with  $S$  small, we have  $M(s) = M(S)$   $G(s) = G(S)$  and the following relation between the  $G(S)$ -equivariant forms over  $M(S)$ :

$$(14) \quad (2\pi)^{-\dim M/2} \frac{\text{ch}(\mathcal{E}, \mathbb{A})(S+Y)|_{M(S)}}{J^{1/2}(M)|_{M(S)}(S+Y) \text{Eul}_{o_s}(T_{M(S)}M)(S+Y)} = \\ (2\pi)^{-\dim M(S)/2} \frac{\text{ch}_s(\mathcal{E}, \mathbb{A})(Y)}{J^{1/2}(M(S))(Y) D_s^{1/2}(T_{M(S)}M)(Y)}$$

for all  $Y \in \mathfrak{g}(S)$  sufficiently small.

Taking  $X = 0$  in Formula (13), we obtain Atiyah-Segal-Singer Formula for the equivariant index of the Dirac operator.

(15)

$$\text{Tr } Q^o(M, \mathcal{E})(s) = i^{-\dim M/2} \int_{M(s), o'} (2\pi)^{-\dim M(s)/2} \frac{\epsilon(s, o, o') \text{ch}_s(\mathcal{E}, \mathbb{A})(0)}{J^{1/2}(M(s))(0) D_s^{1/2}(T_{M(s)}M)(0)}.$$

When  $G = \{1\}$ , we identify the index  $Q^o(M, \mathcal{E})$  with the natural number  $\text{Tr } Q^o(M, \mathcal{E})(1) = \dim Q^o(M, \mathcal{E})$ . We have

$$(16) \quad Q^o(M, \mathcal{E}) = (2i\pi)^{-\dim M/2} \int_M \text{ch}(\mathcal{E}) J^{-1/2}(M).$$

We have denoted  $J^{-1/2}(M)(0)$  simply by  $J^{-1/2}(M)$  . The form  $J^{-1/2}(M)$  is a characteristic form on  $M$  which coincide up to normalisations factors of  $2\pi$  with the  $\hat{A}$ -genus.

When  $(M, \sigma, \mu)$  is a Hamiltonian manifold and  $\mathcal{L}$  a Kostant-Souriau line bundle on  $M$ , the virtual representation  $Q^o(M, \mathcal{L})$  is then the quantized space of the manifold  $(M, \sigma, \mu)$ . Here the orientation  $o$  will always be the symplectic orientation and we will sometimes omit  $o$  in our notation. If the group  $G$  is connected, the bouquet of Chern characters  $\text{bch}(\mathcal{L}, \mathcal{A})$  is entirely determined by  $(\sigma, \mu)$ , thus we could write  $Q^o(M, \mathcal{L}) = Q(M, \sigma, \mu)$ . The (virtual) representation  $Q(M, \sigma, \mu)$  has indeed some deep relations with the original symplectic space  $(M, \sigma)$ . For example, if the  $\hat{A}$ -genus of the manifold  $M$  is equal to 1 ( as is the case when  $M$  is a regular coadjoint orbit of  $G$ ),

then the dimension of  $Q(M, \sigma, \mu)$  is equal to the symplectic volume of  $M$ . In particular, this volume is an integer. One of the aims of this article is to describe the decomposition of  $Q(M, \sigma, \mu)$  in function of the moment map  $\mu$ .

Of course, Atiyah-Singer-Segal pointwise formula (15) for  $\text{Tr } Q^o(M, \mathcal{E})$  determines it. However, as the dependance of  $s$  of the set  $M(s)$  is quite chaotic, it is difficult to employ directly this formula for the geometric study of multiplicities: to compute for example the multiplicity of the trivial representation of  $G$ , we have to compute  $\int_G \text{Tr } Q^o(M, \mathcal{E})(s) ds$ . The equivariant index formula has a better behaviour: the dependance in  $X$  of the integral formula for  $\text{Tr } Q^o(M, \mathcal{E})(se^X)$  is  $C^\infty$  in  $X$ . However, there are still some difficulties, as it seems not possible to give a unique global integral formula valid on  $G$  and with  $C^\infty$  dependance on  $s$  (the formula given above is only valid in a neighbourhood of each point  $s \in G$ ).

Consider the particular case of a manifold  $M$  with a trivial action of a torus  $T$ .

Let  $T$  be a torus. Let  $\mathfrak{t}$  be the Lie algebra of  $T$ . Let  $P \subset i\mathfrak{t}^*$  be the set of differentials of unitary characters of  $T$ . We will call an element of  $P$  a weight of  $T$ . If  $\xi \in P$ , we denote by  $e_\xi \in \hat{T}$  the corresponding character of  $T$ . For  $X \in \mathfrak{t}$ , we have  $e_\xi(\exp X) = e^{(\xi, X)}$ . Then  $R(T)$  is the free  $\mathbb{Z}$ -module with basis  $e_\xi$ ,  $\xi \in P$ . Let  $\mathcal{E}$  be a  $T$ -equivariant vector bundle over  $M$ . The vector bundle  $\mathcal{E}$  is a sum of its subbundles  $\mathcal{E}_\xi$  such that  $T$  acts on  $\mathcal{E}_\xi$  by  $e_\xi$ . Let  $\mathbb{A}_\xi$  be the connection induced by the  $T$ -invariant connection  $\mathbb{A}$  on  $\mathcal{E}_\xi$ . Thus the function  $s \mapsto \text{ch}_s(\mathcal{E}, \mathbb{A})(0) = \text{Tr}(se^F) = \sum_\xi e_\xi(s) \text{ch}(\mathcal{E}_\xi, \mathbb{A}_\xi)(0)$  is a regular function on  $T$  with values differential forms on  $M$ . We denote it by  $\text{ch}(\mathcal{E})(s)$  leaving implicit the choice of  $\mathbb{A}$ . The equivariant index  $Q^o(M, \mathcal{E})$  of  $\mathcal{E}$  is the element of  $R(T)$  such that for  $s \in T$

$$(17) \quad \text{Tr } Q^o(M, \mathcal{E})(s) = (2i\pi)^{-\dim M/2} \int_M \text{ch}(\mathcal{E})(s) J^{-1/2}(M).$$

In particular the set of weights appearing in the virtual representation  $Q^o(M, \mathcal{E})$  is contained in the set of weights  $\xi$  such that  $\mathcal{E}_\xi$  is non zero.

Let  $G$  be a compact connected Lie group. Atiyah-Segal-Singer formula (15) gives us a formula similar to Hermann Weyl formula for the character of  $Q^o(M, \mathcal{E})$ . Let  $T$  be the maximal torus of  $G$ .

We have a decomposition

$$\text{Tr } Q^o(M, \mathcal{E})(s) = \sum_{\xi \in P} n(\xi, M, \mathcal{E}) e_\xi(s)$$

for  $s \in T$  where  $n(\xi, M, \mathcal{E}) \in \mathbb{Z}$  and is non zero for a finite number of  $\xi$ . The number  $n(\xi, M, \mathcal{E})$  is called the multiplicity of the weight  $e_\xi$  in  $Q^o(M, \mathcal{E})$ .

Let  $M^T$  be the set of fixed points of the action of  $T$  in  $M$ . Let  $o'$  be an orientation of  $M^T$ . Let  $\mathcal{N}$  be the normal bundle of  $M^T$  in  $M$ . The formula (18)

$$\mathrm{Tr} Q^o(M, \mathcal{E})(s) = i^{-\dim M/2} \int_{M^T, o'} (2\pi)^{-(\dim M^T)/2} \frac{\epsilon(s, o, o') \mathrm{ch}_s(\mathcal{E}, \mathbb{A})(0)}{J^{1/2}(M^T)(0) D_s^{1/2}(\mathcal{N})(0)}$$

is valid for the dense set of  $s \in T$  such that  $M(s) = M^T$ .

Let  $\mathcal{F}$  be the set of connected components of  $M^T$ . For  $a \in \mathcal{F}$ , we denote by  $M_a$  the corresponding connected component of  $M^T$ . We denote by  $\mathcal{E}_a$  the restriction of the vector bundle  $\mathcal{E}$  on  $M_a$ . Let  $\mathcal{N}_a$  be the normal bundle  $T_{M_a}M$ .

**Definition 4** *We say that  $s$  is  $a$ -regular if  $\det_{\mathcal{N}_a}(1 - s) \neq 0$ .*

We denote by  $T_{reg,a}$  the set of  $a$ -regular elements of  $T$ . We fix an orientation  $o_a$  on  $M_a$ . We consider the part of Formula (18) for  $\mathrm{Tr} Q^o(M, \mathcal{E})$  coming from integration on the connected component  $M_a$  of  $M^T$ .

**Definition 5** *Let  $a \in \mathcal{F}$ . Let  $\Theta_a^o(M, \mathcal{E})$  be the function on  $T_{a,reg}$  given by:*

$$\Theta_a^o(M, \mathcal{E})(s) = i^{-\dim M/2} \int_{M_a, o_a} (2\pi)^{-(\dim M_a)/2} \frac{\epsilon(s, o, o_a) \mathrm{ch}_s(\mathcal{E}_a, \mathbb{A})(0)}{J^{1/2}(M_a)(0) D_s^{1/2}(\mathcal{N}_a)(0)}.$$

**Remark 2.3**  $\Theta_a^o(M, \mathcal{E})(s)$  *does not depend of the choice of  $o_a$ .*

**Lemma 6** *The function  $\Theta_a^o(M, \mathcal{E})$  is the restriction to  $T_{reg,a}$  of a rational function on  $T$ .*

**Proof.** We need to analyse the behaviour in  $s$  of the terms of the above integral formula for  $\Theta_a^o(M, \mathcal{E})$ . Recall that we have chosen  $G$ -invariant connections  $\nabla$  on  $TM$  and  $\mathbb{A}$  on  $\mathcal{E}$ . The function  $s \rightarrow \mathrm{ch}_s(\mathcal{E}_a, \mathbb{A})(0) = \mathrm{ch}(\mathcal{E}_a)(s)$  is a regular function on  $T$  with values differential forms on  $M_a$ .

Let us describe the function  $s \mapsto D_s^{1/2}(\mathcal{N}_a)(0)$ . For  $x \in M_a$ , the representation of  $T$  on the vector space  $(\mathcal{N}_a)_{x \otimes \mathbb{R}} \mathbb{C}$  breaks up into non-zero weights. These weights and their multiplicities are independent of the point  $x \in M_a$ . We denote by  $\Delta_a$  the set of weights  $\alpha$  appearing in the action of  $T$  in  $(\mathcal{N}_a)_{x \otimes \mathbb{R}} \mathbb{C}$  for some  $x \in M_a$ . They are all non-zero weights. For  $\alpha \in \Delta_a$

let  $\mathcal{N}_a(\alpha)$  be the subbundle of  $\mathcal{N}_a \otimes_{\mathbb{R}} \mathbb{C}$  where  $T$  acts by a multiple of the weight  $e_\alpha$ . Let  $n_a(\alpha)$  be the rank of  $\mathcal{N}_a(\alpha)$ . The set  $T_{a,reg}$  is the set of elements  $t \in T$  such that  $(1 - e_\alpha(t)) \neq 0$  for all  $\alpha \in \Delta_a$ . Let  $R_{a,\alpha}$  be the curvature of the connection determined by  $\nabla$  on  $\mathcal{N}_a(\alpha)$ .

Weights of  $T$  on  $\mathcal{N}_a$  appear in pairs  $\pm\alpha_k$ . If  $S_a \in \mathfrak{t}$  is such that  $i\alpha(S_a) \neq 0$  for all  $\alpha \in \Delta_a$  then we define

$$(20) \quad \Delta_a^+(S_a) = \{\alpha \in \Delta_a; i\alpha(S_a) < 0\}$$

**Definition 7** A subset  $\Delta_a^+$  of  $\Delta_a$  such that there exists  $S_a \in \mathfrak{t}$  with  $\Delta_a^+ = \Delta_a^+(S_a)$  is called a positive system.

Then  $\Delta_a = \Delta_a^+ \cup (-\Delta_a^+)$ . For a positive system we denote

$$C(\Delta_a^+) = \{S \in \mathfrak{t}; i\alpha(S) < 0, \text{ for all } \alpha \in \Delta_a^+\}.$$

We fix a positive system  $\Delta_a^+$  and define

$$(21) \quad \mathcal{N}_a^+ = \bigoplus_{\alpha \in \Delta_a^+} \mathcal{N}_a(\alpha).$$

We denote by  $R_a^+$  the curvature of the connection determined by  $\nabla$  on  $\mathcal{N}_a^+$ .

The fact that the representation of  $T$  on  $M$  lifts to the spin bundle implies that there exists a  $T$ -equivariant line bundle  $\mathcal{L}_a^+$  over  $M_a$  such that

$$(\mathcal{L}_a^+)^2 = \Lambda^{max} \mathcal{N}_a^+.$$

Let

$$(22) \quad \rho_a^+ = \frac{1}{2} \sum_{\alpha \in \Delta_a^+} n_a(\alpha) \alpha.$$

Then  $\rho_a^+$  is a weight of  $T$ .

Let  $r_a^+ = \frac{1}{2} \sum_{\alpha \in \Delta_a^+} \text{Tr}_{\mathcal{N}_a(\alpha)} R_{a,\alpha}$ . Then  $r_a^+$  is a 2-form on  $M_a$ . We have

$$\text{ch}(\mathcal{L}_a^+)(s) = e_{\rho_a^+}(s) e^{r_a^+}.$$

We have (see Formula( 9))

$$D_s^{1/2}(\mathcal{N}_a)(0) = i^{n_a^+} e_{-\rho_a^+}(s) e^{-r_a^+} \epsilon(s, \Delta_a^+) \det_{\mathcal{N}_a^+}(1 - s e^{R_a^+})$$

where  $\epsilon(s, \Delta_a^+)$  is a sign. The system  $\Delta_a^+$  determines an orientation on  $\mathcal{N}_a$ : we choose as oriented basis  $e_1, e_2, e_3, e_4, \dots$  a basis such that  $e_1 - ie_2, e_3 - ie_4, \dots$  is a basis of  $\mathcal{N}_a^+$ . Define

$$(23) \quad \epsilon(o, o_a, \Delta_a^+) = \pm 1$$

according to the cases where the orientations on  $\mathcal{N}_a$  given by  $\Delta_a^+$  and  $o/o_a$  coincide or not. We have  $\epsilon(s, o, o_a)\epsilon(s, \Delta_a^+) = \epsilon(o, o_a, \Delta_a^+)$  thus

$$\epsilon(s, o, o_a)D_s^{1/2}(\mathcal{N}_a)(0) = \epsilon(o, o_a, \Delta_a^+)i^{n_a^+}e_{-\rho_a^+}(s)e^{-r_a^+} \det_{\mathcal{N}_a^+}(1 - se^{R_a^+})$$

and

$$(24) \quad \Theta_a^o(M, \mathcal{E})(s) = (2i\pi)^{-\dim M_a} \int_{M_a, o_a} \epsilon(o, o_a, \Delta_a^+) \frac{\text{ch}(\mathcal{E}_a \otimes \mathcal{L}_a^+)(s)}{J^{1/2}(M_a)} \frac{-1}{\det_{\mathcal{N}_a^+}(1 - se^{R_a^+})}.$$

Thus we see on this formula that  $\Theta_a^o(M, \mathcal{E})(s)$  is a rational function of  $s$  on  $T_{reg, a}$ . We have used a positive system to give a rational expression. However  $\Theta_a^o(M, \mathcal{E})(s)$  is independent of the choice of positive system. ■

Let us say that  $s \in T$  is  $M$ -regular if  $s$  is  $a$ -regular (Definition 4) for all  $a \in \mathcal{F}$ .

**Lemma 8** *For an arbitrary choice of positive systems  $\Delta_a^+$  of  $\Delta_a$ , we have the equality*

$$\text{Tr } Q^o(M, \mathcal{E})(s) = \sum_{a \in \mathcal{F}} \Theta_a^o(M, \mathcal{E})(s)$$

for any  $M$ -regular element  $s \in T$ .

**Proof.** The character of the finite dimensional virtual representation  $Q^o(M, \mathcal{E})$  is an analytic function on  $T$ . The preceding formula holds for all  $s \in T$  such that  $M(s) = M^T$ . By analyticity, it holds for all  $M$ -regular elements of  $T$ . ■

A choice of positive system  $\Delta_a^+$  determines a natural extension of the function  $\Theta_a^o(M, \mathcal{E})$  defined on  $T_{a, reg}$  as a generalised function on  $T$ .

**Definition 9** We denote by  $R^{-\infty}(T)$  the set of generalised characters of  $T$ . An element  $\theta \in R^{-\infty}(T)$  is a sum  $\sum n_\xi e_\xi$  of characters of  $T$  with coefficients  $n_\xi$  in  $\mathbb{Z}$  and such that the coefficients  $n_\xi$  have at most polynomial growth. The support of  $\theta$  is the set of  $\xi \in P$  such that  $n_\xi \neq 0$ . For  $\theta \in R^{-\infty}(T)$ , we denote by  $\text{Tr } \theta$  the generalised function

$$\text{Tr } \theta(t) = \sum_{\xi} n_\xi e_\xi(t).$$

Let  $\Delta_a^+$  be a choice of positive systems for  $\Delta_a$ . Let  $\mathcal{N}_a^+$  be the  $T$ -equivariant vector bundle given by Formula(21). Let  $S(\mathcal{N}_a^+) = \bigoplus_0^\infty S^m(\mathcal{N}_a^+)$  be the series of complex finite dimensional vector bundles obtained from the symmetric powers of  $\mathcal{N}_a^+$ . If  $\mathcal{H}$  is a  $T$ -equivariant bundle on  $M_a$ , then the equivariant index  $Q^{o_a}(M_a, \mathcal{H})$  is an element of  $R(T)$ .

**Definition 10** Define the series of characters of  $T$ :

$$A_a^o(M, \mathcal{E}, \Delta_a^+) = \epsilon(o, o_a, \Delta_a^+) \sum_{k=0}^{\infty} Q^{o_a}(M_a, \mathcal{E}_a \otimes S^k(\mathcal{N}_a^+) \otimes \mathcal{L}_a^+).$$

It is easy to see that  $A_a^o(M, \mathcal{E}, \Delta_a^+)$  is in  $R^{-\infty}(T)$ .

**Proposition 11** For  $s \in T_{a, \text{reg}}$ , we have

$$\Theta_a^o(M, \mathcal{E})(s) = \text{Tr}(A_a^o(M, \mathcal{E}, \Delta_a^+))(s).$$

**Proof.** Using Formula (24), this formula is a consequence of the formula:  $\det_{\mathcal{N}_a^+}(1 - se^{R_a^+}) \text{Tr}_{S(\mathcal{N}_a^+)}(se^{R_a^+}) = 1$  and of the index formula for a manifold with trivial  $T$ -action (Formula (17)). ■

Let  $T_{\mathbb{C}} = (\mathbb{C}^*)^{\dim T}$  be the complexification of  $T$ . Elements of  $R(T)$  extend on holomorphic functions on  $T_{\mathbb{C}}$ . Let

$$T_{\mathbb{C}}(\Delta_a^+) = \{g = \exp(X + iY); X \in \mathfrak{t}, Y \in C(\Delta_a^+)\}.$$

Remark that for  $g \in T_{\mathbb{C}}(\Delta_a^+)$  then  $\det_{\mathcal{N}_a^+}^{-1}(1 - ge^{R_a^+})$  is not 0 as  $(1 - e_\alpha(g)) \neq 0$  for all  $\alpha \in \Delta_a$ .

**Lemma 12** The generalised function  $\text{Tr } A_a^o(M, \mathcal{E}, \Delta_a^+)$  is the boundary value of an holomorphic function  $\Psi_a^o$  on  $T_{\mathbb{C}}(\Delta_a^+)$ . We have for  $g \in T_{\mathbb{C}}(\Delta_a^+)$ :

$$\Psi_a^o(g) = (2i\pi)^{-\dim M_a} \int_{M_a, o_a} \epsilon(o, o_a, \Delta_a^+) \frac{\text{ch}(\mathcal{E}_a \otimes \mathcal{L}_a^+)(g)}{J^{1/2}(M_a)} \det_{\mathcal{N}_a^+}^{-1}(1 - ge^{R_a^+}).$$

**Proof.** For each  $k$ , the function  $g \mapsto \text{Tr } Q^{o_a}(M_a, \mathcal{E}_a \otimes S^k(\mathcal{N}_a^+) \otimes \mathcal{L}_a^+)(g)$  extends holomorphically to  $T_{\mathbb{C}}$ . Furthermore, the series

$$\sum_{k=0}^{\infty} \text{Tr } Q^{o_a}(M_a, \mathcal{E}_a \otimes S^k(\mathcal{N}_a^+) \otimes \mathcal{L}_a^+)(g)$$

defines an holomorphic function on  $T_{\mathbb{C}}(\Delta_a^+)$ . Indeed, writing  $g = \exp(X+iY)$  with  $Y \in C(\Delta_a^+)$  and decomposing  $\mathcal{N}_a^+$  in line bundles, this follows from the fact that for any  $\alpha \in \Delta_a^+$ ,

$$\sum_{k=0}^{\infty} e^{k\alpha(X+iY)}$$

defines an holomorphic function on  $T_{\mathbb{C}}(\Delta_a^+)$  as  $i\alpha(Y) < 0$  for  $Y \in C(\Delta_a^+)$ .

Consider the case where  $(M, \sigma, \mu)$  is a quantizable symplectic manifold. Then  $\mu : M \rightarrow \mathfrak{t}^*$  takes constant values  $\mu_a$  on each connected component  $M_a$  of  $M^T$ . Furthermore it follows from the definition of the Kostant-Souriau line bundle with connection  $(\mathcal{L}, \mathbb{A})$  that  $T$  acts by  $e_{i\mu_a}$  on  $\mathcal{L}_a$ . In particular  $i\mu_a$  is a weight. By definition of  $A^o(M, \mathcal{L}, \mathbb{A})$  and the remark following Formula 17, we have the following lemma.

**Lemma 13** *The support of  $A^o(M, \mathcal{L}, \mathbb{A})$  is contained in the set*

$$i\mu_a + \rho_a^+ + \left\{ \sum n_{\alpha} \alpha; n_{\alpha} \geq 0, \alpha \in \Delta_a^+ \right\}.$$

Let  $a \mapsto \Delta_a^+$  be an arbitrary choice of positive systems for  $\Delta_a^+$ , when  $a$  varies in  $\mathcal{F}$ . Then, over the open subset of  $M$ -regular elements, we have from Lemma 8 and Proposition 11 the equality

$$(25) \quad \text{Tr } Q^o(M, \mathcal{E})(s) = \sum_a \text{Tr } A_a^o(M, \mathcal{E}, \Delta_a^+)(s)$$

However in general the equality above does not hold over  $T$ .

Let  $S \in \mathfrak{t}$  be an element such  $\alpha(S) \neq 0$  for all  $\alpha \in \cup_{a \in \mathcal{F}} \Delta_a$ . The element  $S$  determines a consistent choice of positive systems  $\Delta_a^+(S)$  of  $\Delta_a$  when  $a$  varies in  $\mathcal{F}$ .

**Proposition 14** *(Guillemin-Lerman-Sternberg, Guillemin-Prato) ([15], [16])*  
*Let  $\mathcal{F}$  be the set of connected components of  $M^T$ . Let  $S \in \mathfrak{t}$  be an element*

such  $\alpha(S) \neq 0$  for all  $\alpha \in \cup_{a \in \mathcal{F}} \Delta_a$ . We have the identity of generalised functions over  $T$ :

$$\mathrm{Tr} Q^o(M, \mathcal{E})(s) = \sum_{a \in \mathcal{F}} \mathrm{Tr} A_a^o(M, \mathcal{E}, \Delta_a^+(S))(s).$$

**Proof.** Let us give a proof using the integral expression (13) for  $\mathrm{Tr} Q^o(M, \mathcal{E})(s \exp X)$  as this proof will generalised easily to a proof of Theorem 17. Consider the function  $s \mapsto \mathrm{Tr} Q^o(M, \mathcal{E})(s)$ . It extends holomorphically to  $T_{\mathbb{C}}$ . We can choose  $S$  such that  $M(S) = M^T$ . Let  $t$  be a small positive number. We have  $\mathrm{Tr} Q^o(M, \mathcal{E})(s) = \lim_{t \rightarrow 0} \mathrm{Tr} Q^o(M, \mathcal{E})(s \exp itS)$ .

The differential form  $L(s, \mathcal{E}, \mathbb{A})(X)$  extends holomorphically in a neighbourhood of 0 in  $\mathfrak{t}_{\mathbb{C}}$ . It is clear that

$$\mathrm{Tr} Q^o(M, \mathcal{E})(s \exp itS) = i^{-\dim M/2} \int_{M(s), o'} \epsilon(s, o, o') L(s, \mathcal{E}, \mathbb{A})(itS)$$

for every  $t$  sufficiently small.

We now use the localisation formula (5)(applied to  $M = M(s)$  and  $Y = itS$ ) and we obtain, for any small positive  $t$ ,

$$i^{-\dim M/2} \int_{M(s), o'} \epsilon(s, o, o') L(s, \mathcal{E}, \mathbb{A})(itS) = i^{-\dim M/2} \sum_{a \in \mathcal{F}} \int_{M_a, o_a} \epsilon(s, o, o') \frac{L(s, \mathcal{E}, \mathbb{A})(itS)}{\mathrm{Eul}_{o'/o_a}(T_{M_a} M(s))(itS)}$$

We can check by calculations similar to those of Lemma 6 that on  $M_a$

$$(26) \quad i^{-\dim M/2} \epsilon(s, o, o') \frac{L(s, \mathcal{E}, \mathbb{A})(itS)|_{M_a}}{\mathrm{Eul}_{o'/o_a}(T_{M_a} M(s))(itS)} = \\ (2i\pi)^{-\dim M_a} \epsilon(o, o_a, \Delta_a^+) \frac{\mathrm{ch}(\mathcal{E}_a \otimes \mathcal{L}_a^+)(s \exp itS)}{J^{1/2}(M_a)} \det_{\mathcal{N}_a^+}^{-1}(1 - s \exp itS e^{R_a^+}).$$

The element  $s \exp itS$  is in  $T_{\mathbb{C}}(\Delta_a^+)$  for any  $t > 0$ . Thus we obtain that for any  $t > 0$ ,

$$\mathrm{Tr} Q^o(M, \mathcal{E})(s \exp itS) = \sum_{a \in \mathcal{F}} \Psi_a^o(s \exp itS)$$

and we conclude by Lemma 12. ■

### 3 Quantization of manifolds with boundaries

Consider a compact  $G$ -manifold  $M$  (oriented and with  $G$ -invariant spin structure) and let  $U$  be a  $G$ -invariant open subset of  $M$ . If  $(\mathcal{E}, \mathbb{A})$  is a  $G$ -equivariant vector bundle over  $M$  with connection  $\mathbb{A}$ , we would like to give a meaning to the quantized space  $Q^o(U, \mathcal{E}, \mathbb{A})$ . Consider the character formula (13) for  $Q^o(M, \mathcal{E})$ . Let  $s \in G$  and  $Y \in U_s(0)$  a small neighbourhood of 0 in  $\mathfrak{g}(s)$ , then

$$\mathrm{Tr} Q^o(M, \mathcal{E}, \mathbb{A})(s \exp Y) = i^{-\dim M/2} \int_{M(s), o'} \epsilon(s, o, o') L(s, \mathcal{E}, \mathbb{A})(Y).$$

It would be naive to try to define a character  $\mathrm{Tr} Q^o(U, \mathcal{E}, \mathbb{A})$  by a truncated formula:

$$\mathrm{Tr} Q^o(U, \mathcal{E}, \mathbb{A})(s \exp Y) = i^{-\dim M/2} \int_{U(s), o'} \epsilon(s, o, o') L(s, \mathcal{E}, \mathbb{A})(Y).$$

This will never define a character on  $G$  nor even a global function  $\Theta$  on  $G$  as the localisation formula (which relies on Stokes' theorem) does not hold for manifolds with boundaries.

Consider a  $G$ -invariant 1-form  $\lambda$  on  $M$ . As suggested by Witten's localisation procedure, we introduce for every  $t \in \mathbb{R}$  the connection  $\mathbb{A}(t) = \mathbb{A} - it\lambda$  on  $\mathcal{E}$ . By Formula (7) we have  $L(s, \mathcal{E}, \mathbb{A}(t))(Y) = e^{-itd_Y\lambda} L(s, \mathcal{E}, \mathbb{A})(Y)$ .

We consider

$$\mathrm{Tr} Q^o(M, \mathcal{E}, \mathbb{A}(t))(s \exp Y) = i^{-\dim M/2} \int_{M(s), o'} \epsilon(s, o, o') e^{-itd_Y\lambda} L(s, \mathcal{E}, \mathbb{A})(Y).$$

The integral is independent of  $t$  as seen from the fact that  $e^{-itd_Y\lambda}$  is congruent to 1 in equivariant cohomology. However for every  $s \in G$  the truncated integral

$$TQ^o(U, \mathcal{E}, \mathbb{A}, \lambda, s, t)(Y) = i^{-\dim M/2} \int_{U(s), o'} \epsilon(s, o, o') e^{-itd_Y\lambda} L(s, \mathcal{E}, \mathbb{A})(Y)$$

is a  $C^\infty$ -function on  $U_s(0)$  depending on  $t$ .

We assume that  $U = \{x \in M; f(x) > 0\}$  where  $f$  is a  $G$ -invariant function from  $M$  to  $\mathbb{R}$  such that 0 is a regular value of  $f$ .

**Lemma 15** Assume that the map  $\mu_\lambda : M \rightarrow \mathfrak{g}^*$  given by  $\mu_\lambda(X) = \lambda(X_M)$  does not vanish at any point of the boundary  $B$  of  $U$ . Then for each  $s \in G$  the limit  $\Theta^o(U, \mathcal{E}, \mathbb{A}, \lambda, s)$  when  $t \mapsto \infty$  of  $TQ^o(U, \mathcal{E}, \mathbb{A}, \lambda, s, t)$  exists in the space of generalised functions on  $U_s(0)$ .

**Proof.** For  $s \in G$ , the boundary of  $U(s)$  is smooth and given by  $B(s)$ . Indeed we have  $B = f^{-1}(0)$ . Then the differential  $df$  vanishes on  $(1-s)T_x M$  for  $x \in M(s)$ . It follows that 0 is a regular value for the restriction of  $f$  to  $M(s)$ .

Let  $\alpha$  be any closed  $G(s)$ -equivariant form on  $M(s)$ . We write  $\mathfrak{g}(s) = \mathfrak{z}$ . For  $Y \in \mathfrak{z}$ , write  $\Theta(s, t)(Y) = \int_{U(s)} e^{-itd_Y \lambda} \alpha(Y)$ . Then, we have

$$\frac{d}{dt} e^{-itd_Y \lambda} \alpha = -id_{\mathfrak{z}}(\lambda e^{-itd_Y \lambda} \alpha).$$

We then obtain

$$e^{-itd_Y \lambda} \alpha = \alpha - id_{\mathfrak{z}} \left( \int_0^t \lambda e^{-iud_Y \lambda} \alpha du \right).$$

Integration over  $U(s)$  and using Stokes' formula leads to

$$\Theta(s, t)(Y) = \int_{U(s)} e^{-itd_Y \lambda} \alpha(Y) = \int_{U(s)} \alpha(Y) - i \int_{B(s)} \left( \int_0^t \lambda e^{-iud_Y \lambda} \alpha(Y) du \right).$$

Let us see that when  $t$  tends to  $\infty$ ,  $\Psi(B, s, t)(Y) = -i \int_{B(s)} \int_0^t \lambda e^{-iud_Y \lambda} \alpha(Y) du$  has a limit in the sense of generalised functions given by

$$\Psi(B, s)(Y) = -i \int_{B(s)} \int_0^\infty \lambda e^{-iud_Y \lambda} \alpha(Y) du.$$

Consider a test function  $\phi$  on  $\mathfrak{z}$ . Define for  $f \in \mathfrak{z}^*$   $(\hat{\alpha}\phi)(f) = \int_{\mathfrak{z}} e^{if(Y)} \alpha(Y) \phi(Y) dY$ .

Then  $(\hat{\alpha}\phi)(f)$  is a differential form on  $M$  depending of  $f \in \mathfrak{z}^*$ . When  $f \mapsto \infty$ , this differential form converges uniformly to 0 on  $M$ . We have  $d_Y \lambda = -\mu_\lambda(Y) + d\lambda$  by definition of  $\mu_\lambda$  so that  $e^{-iud_Y \lambda} = e^{iu\mu_\lambda(Y)} e^{-iud\lambda}$ . The restriction of  $\mu_\lambda : M \rightarrow \mathfrak{g}^*$  to  $M(s)$  is valued in  $\mathfrak{z}^*$ . Thus

$$(27) \quad \int_{\mathfrak{z}} \Psi(B, s, t)(Y) \phi(Y) dY = -i \int_{B(s) \times [0, t]} \lambda e^{-iud\lambda} (\hat{\alpha}\phi)(u\mu_\lambda(m)) du$$

In this integral expression, we see that  $\Psi(B, s, t)$  has a limit. Indeed for  $m \in B(s)$ , the differential form  $(\hat{\alpha}\phi)(u\mu_\lambda(m))$  is rapidly decreasing in  $u$  (as  $\mu_\lambda(m)$  is never 0 on  $B(s)$ ) while  $e^{-iud\lambda}$  is polynomial in  $u$ .

Applying this calculation to  $TQ^\circ(U, \mathcal{E}, \mathbb{A}, \lambda, s, t)$  we see that  $TQ^\circ(U, \mathcal{E}, \mathbb{A}, \lambda, s, t)$  has limit when  $t \rightarrow \infty$  the generalised function  $\Theta^\circ(U, \mathcal{E}, \mathbb{A}, \lambda, s)$  given for  $Y \in U_s(0)$  by:

$$(28) \quad \Theta^\circ(U, \mathcal{E}, \mathbb{A}, \lambda, s)(Y) = i^{-\dim M/2} \left( \int_{U(s)} \epsilon(s, o, o') L(s, \mathcal{E}, \mathbb{A})(Y) - i \int_{B(s)} \left( \int_0^\infty \lambda e^{-iud_Y \lambda} \epsilon(s, o, o') L(s, \mathcal{E}, \mathbb{A})(Y) du \right) \right).$$

■

*We conjecture:*

Under the same hypothesis as Lemma 15 there exists a virtual trace class representation  $Q^\circ(U, \mathcal{E}, \mathbb{A}, \lambda)$  of  $G$  such that for every  $s \in G$ , we have for  $Y \in U_s(0)$

$$\text{Tr } Q^\circ(U, \mathcal{E}, \mathbb{A}, \lambda)(s \exp Y) = \lim_{t \rightarrow \infty} TQ^\circ(U, \mathcal{E}, \mathbb{A}, \lambda, s, t)(Y) = \Theta^\circ(U, \mathcal{E}, \mathbb{A}, \lambda, s)(Y).$$

Remark that it is not even clear that there exists a  $G$ -invariant generalised function  $\Theta$  on  $G$  such that, for  $Y \in U_s(0)$ ,

$$\Theta(s \exp Y) = \Theta^\circ(U, \mathcal{E}, \mathbb{A}, \lambda, s)(Y).$$

**Remark 3.1** *It is possible to understand Formula (28) for  $\Theta^\circ(U, \mathcal{E}, \mathbb{A}, \lambda, s)$  in the framework of bouquet integrals:*

*Let us consider the manifold  $\tilde{M} = M \times \mathbb{R}$  where  $G$  acts trivially on  $\mathbb{R}$ . We embed  $M$  in  $M \times \mathbb{R}$  by  $m \mapsto (m, 0)$ . We write  $(m, u)$  an element of  $\tilde{M}$ . We consider the differential form  $\tilde{\lambda} = u\lambda$  as a differential form on  $\tilde{M}$ . Let us consider  $C \subset \tilde{M}$  the cylinder with base  $B$ :*

$$C = B \times \mathbb{R}^+.$$

*The boundary of  $C$  in  $M \times \mathbb{R}$  is equal to the boundary of  $U$  both being the manifold  $B$ . If  $R$  is a tubular neighbourhood of  $B$  in  $M$ , we can identify  $C$  to the open subset  $R - \bar{U}$  of  $M$ . This gives an orientation  $o_{out}$  to  $C$ .*

*Define*

$$(29) \quad Z = U \cup (C, o_{out}).$$

*Then  $Z$  is an oriented cycle in  $\tilde{M}$ . It can be also identified to the manifold  $U$  with the cylindrical end  $C$  attached to it. Consider on  $\tilde{M}$  the pull back*

$\tilde{\mathcal{E}}$  of the vector bundle  $\mathcal{E}$  with connection  $\tilde{\mathbb{A}} = \mathbb{A} - i\tilde{\lambda}$ . Then Formula (28) for  $\Theta^o(U, \mathcal{E}, \mathbb{A}, \lambda, s)(Y)$  is nothing but the  $s$ -part of the universal formula for  $i^{-\dim M/2} \int_b^{U \cup C} \text{bch}(\tilde{\mathcal{E}}, \tilde{\mathbb{A}})$ . The conjecture above is then: There exists a virtual trace class representation  $Q^o(U \cup C, \tilde{\mathcal{E}}, \tilde{\mathbb{A}}) = Q^o(U, \mathcal{E}, \mathbb{A}, \lambda)$  such that

$$(30) \quad \text{Tr } Q^o(U, \mathcal{E}, \mathbb{A}, \lambda) = i^{-\dim M/2} \int_b^{U \cup C} \text{bch}(\tilde{\mathcal{E}}, \tilde{\mathbb{A}})$$

and is consistent with the hope (see [27]) than for good non compact manifolds and good bouquets, then the bouquet integration produces global (generalised) functions on  $G$ .

We will prove this conjecture together with an explicit formula for  $Q^o(U, \mathcal{E}, \mathbb{A}, \lambda)$  in a simple case. The following localisation formula for the manifold  $U$  with boundary  $B$  is due to Kalkman ([18]).

**Lemma 16** *Let  $\alpha$  be a closed  $G$ -equivariant differential form on  $M$ . Assume there exists a central element  $S \in \mathfrak{g}$  such that  $(\lambda, S_M)$  does not vanish on  $B$ . Then for every  $Y \in \mathfrak{g}$  sufficiently close to  $S$*

$$\int_U \alpha(Y) - \int_B \frac{\lambda}{(d_{\mathfrak{g}}\lambda)(Y)} \alpha(Y) = \int_{U(S)} \frac{\alpha(Y)}{\text{Eul}_o(T_{M(S)}M)(Y)}.$$

**Proof.** Our hypothesis implies  $M(S) \cap B = \emptyset$ . Recall (see for example [13]) that we may write  $\alpha(Y) = \beta(Y) + d_{\mathfrak{g}}\nu(Y)$  where  $\beta(Y)$  is supported in a small neighbourhood of  $M(S)$ . Using a  $G$ -invariant partition of unity we may write also  $\nu(Y) = \nu_0(Y) + \nu_1(Y)$  where  $\nu_0(Y)$  is supported on a neighbourhood of  $B$  and  $\nu_1(Y)$  is identically 0 on  $B$ . Applying the localisation formula (4) to the compactly supported closed form  $\alpha_1(Y) = \beta(Y) + (d_{\mathfrak{g}}\nu_1)(Y)$  on  $U$ , we obtain

$$\int_U \alpha_1(Y) = \int_{U(S)} \frac{\alpha_1(Y)}{\text{Eul}_o(T_{M(S)}M)(Y)}.$$

As  $\alpha_1$  is in the same cohomology class as  $\alpha$ ,

$$\int_{U(S)} \frac{\alpha_1(Y)}{\text{Eul}_o(T_{M(S)}M)(Y)} = \int_{U(S)} \frac{\alpha(Y)}{\text{Eul}_o(T_{M(S)}M)(Y)}.$$

Let  $\alpha_0(Y) = (d_{\mathfrak{g}}\nu_0)(Y)$ . It remains to show that

$$\int_U \alpha_0(Y) = \int_B \frac{\lambda}{(d_{\mathfrak{g}}\lambda)(Y)} \alpha(Y).$$

We have  $\alpha_0(Y) = d_Y(\lambda(d_Y\lambda)^{-1}\alpha_0(Y))$  as  $\alpha_0(Y)$  is compactly supported near  $B$  and  $d_Y\lambda$  invertible on  $B$ . Let  $n = \dim M$ . This implies that the term of maximal degree of  $\alpha_0(Y)$  is exact and equal to  $d(\lambda(d_Y\lambda)^{-1}\alpha_0(Y))_{[n-1]}$ . By Stokes' theorem, we obtain  $\int_U \alpha_0(Y) = \int_B \lambda(d_Y\lambda)^{-1}\alpha_0(Y)$ . But  $\alpha = \alpha_0$  on  $B$  and we obtain our result. ■

Assume that  $\alpha(Y)$  depends holomorphically of  $Y$  for  $Y$  belonging to an open subset  $W$  of  $\mathfrak{g}_{\mathbb{C}}$ . Let  $U$  be the open subset consisting of the elements  $Y \in \mathfrak{g}_{\mathbb{C}}$  such that  $d_{\mathfrak{g}}\lambda(Y)|_B$  and  $Eul_o(T_{M(S)}M)(Y)$  are invertible. Let  $W'$  be the connected component of  $S$  in  $W \cap U$ . Then for all  $Y \in W'$ , we have

$$(31) \quad \int_U \alpha(Y) - \int_B \frac{\lambda}{(d_{\mathfrak{g}}\lambda)(Y)} \alpha(Y) = \int_{U(S)} \frac{\alpha(Y)}{Eul_o(T_{M(S)}M)(Y)}.$$

We consider now a compact manifold  $M$  with an action of a torus  $T$ . Let  $U$  be a  $T$ -invariant open subset of  $M$  with smooth boundary  $B$ . Assume that there exists  $S \in \mathfrak{t}$  such that  $\mu_{\lambda}(S) > 0$  on  $B$ . The element  $S \in \mathfrak{t}$  can be assumed sufficiently generic so that  $M(S) = M^T$ . We have  $M^T \cap B = \emptyset$ . Let  $\mathcal{F}$  be the set of connected components of  $M^T$ . Let  $\mathcal{F}(U)$  be the subset of connected components of  $M^T$  which are contained in  $U$ . The element  $S \in \mathfrak{t}$  determines positive systems  $\Delta_a^+(S)$  defined by Formula (20). Recall Definition 10 for  $A_a^o(M, \mathcal{E}, \Delta_a^+(S))$  and Formula (28) for  $\Theta^o(U, \mathcal{E}, \mathbb{A}, \lambda, s)(X)$ . The main theorem of this section is

**Theorem 17** *Let  $T$  be a torus. Let  $U$  be a  $T$ -invariant open subset of  $M$  with boundary  $B$ . Let  $\lambda$  be a  $T$ -invariant 1-form on  $M$  such that there exists  $S \in \mathfrak{t}$  with  $(\lambda, S_M) > 0$  on  $B$ . Define*

$$Q^o(U, \mathcal{E}, \mathbb{A}, \lambda) = \sum_{a \in \mathcal{F}(U)} A_a^o(M, \mathcal{E}, \Delta_a^+(S)).$$

*Then for each  $s \in T$ , there exists small neighbourhood  $U_s(0)$  of  $0 \in \mathfrak{t}$ , such that in  $C^{-\infty}(U_s(0))$ :*

$$\text{Tr } Q^o(U, \mathcal{E}, \mathbb{A}, \lambda)(s \exp X) = \Theta^o(U, \mathcal{E}, \mathbb{A}, \lambda, s)(X).$$

**Remark 3.2** *If  $M = U$ , then  $B$  is empty, and we obtain the formula of Guillemin-Lerman-Sternberg, Guillemin-Prato (Proposition 14)*

**Proof.** Let  $s \in T$ . Let  $\Theta(s) = \Theta^o(U, \mathcal{E}, \mathbb{A}, \lambda, s)$ . Then by Formula (28)

$$\Theta(s)(X) = i^{-\dim M/2}(F(U, s)(X) + \Psi(B, s)(X))$$

with

$$F(U, s)(X) = \int_{U(s)} \epsilon(s, o, o')L(s, \mathcal{E}, \mathbb{A})(X)$$

and

$$\Psi(B, s)(X) = -i \int_{B(s)} \left( \int_0^\infty \lambda e^{-iudx\lambda} \epsilon(s, o, o')L(s, \mathcal{E}, \mathbb{A})(X) du \right).$$

The form  $L(s, \mathcal{E}, \mathbb{A})(X)$  is analytic on a neighbourhood of 0 in  $\mathfrak{t}_{\mathbb{C}}$ . Let  $t \geq 0$  be a small positive number. Define

$$F_t(U, s)(X) = \int_{U(s)} \epsilon(s, o, o')L(s, \mathcal{E}, \mathbb{A})(X + itS).$$

Then

$$\lim_{t \rightarrow 0} F_t(U, s)(X) = F(U, s)(X).$$

Define  $d_t\lambda(X + itS) = -\mu_\lambda(X + itS) + d\lambda$ . Define

$$\Psi_t(B, s)(X) = -i \int_{B(s)} \int_0^\infty \lambda e^{-iu(d_t\lambda)(X+itS)} \epsilon(s, o, o')L(s, \mathcal{E}, \mathbb{A})(X + itS) du.$$

We have

$$\lim_{t \rightarrow 0} \Psi_t(B, s)(X) = \Psi(B, s)(X)$$

in the space of generalised functions.

Let  $t > 0$ . As  $(\mu_\lambda, S) > 0$  on  $B$ , the function  $u \mapsto e^{-ut(\mu_\lambda, S)}$  is rapidly decreasing on  $B(s) \times \mathbb{R}^+$ . Now

$$\begin{aligned} \Psi_t(B, s)(X) &= -i \int_{B(s)} \int_0^\infty \lambda e^{-iu(d_t\lambda)(X+itS)} \epsilon(s, o, o')L(s, \mathcal{E}, \mathbb{A})(X + itS) du \\ &= -i \int_{B(s)} \int_0^\infty \lambda e^{-ut(\mu_\lambda, S)} e^{i\mu_\lambda(X)} e^{-iud\lambda} \epsilon(s, o, o')L(s, \mathcal{E}, \mathbb{A})(X + itS) du \end{aligned}$$

is an analytic function of  $X \in U_s(0)$ . We have

$$\Psi_t(B, s)(X) = - \int_{B(s)} \frac{\lambda}{(d_t\lambda)(X + itS)} \epsilon(s, o, o')L(s, \mathcal{E}, \mathbb{A})(X + itS).$$

We write  $\Delta_a^+(S) = \Delta_a^+$ . As  $S \in C(\Delta_a^+)$  the element  $g = s \exp(X + itS)$  is in  $T_{\mathbb{C}}(\Delta_a^+)$ . We can define also (see Lemma 12)

$$\mathrm{Tr} A_a^o(M, \mathcal{E}, \Delta_a^+)(s \exp(X + itS))$$

and

$$\lim_{t \rightarrow 0} \mathrm{Tr} A_a^o(M, \mathcal{E}, \Delta_a^+)(s \exp(X + itS)) = \mathrm{Tr} A_a^o(M, \mathcal{E}, \Delta_a^+)(s \exp X)$$

in the space of generalised functions.

To prove the theorem it remains to see that if  $P_t(X) = F_t(U, s)(X) + \Psi_t(B, s)(X)$

$$i^{-\dim M/2} P_t(X) = \sum_{a \in \mathcal{F}(U)} \mathrm{Tr} A_a^o(M, \mathcal{E}, \Delta_a^+)(s \exp(X + itS))$$

for any  $t > 0$ .

Define  $\alpha(Z) = \epsilon(s, o, o') L(s, \mathcal{E}, \mathbb{A})(Z)$  for  $Z$  in a small neighbourhood of 0 in  $\mathfrak{t}_{\mathbb{C}}$ . Then  $\alpha(Z)$  depends holomorphically of  $Z$ . For  $t > 0$ ,

$$P_t(X) = \int_{U(s)} \alpha(X + itS) - \int_{B(s)} \frac{\lambda}{(d_t \lambda)(X + itS)} \alpha(X + itS).$$

Recall from our assumption on  $S$  that  $U(s)(S) = U^T = \cup_{a \in \mathcal{F}(U)} M_a$ . Consider the localisation formula (31) (with  $U$  replaced by  $U(s)$ ). We can apply it to  $Y = X + itS$ . As  $X$  is of small norm,  $W'$  contains  $Y = X + zS$  with  $|z| = 1$ . Furthermore as  $X$  is real, it contains elements  $Y = X + itS$  with  $t \neq 0$ . We obtain

$$P_t(X) = \sum_{a \in \mathcal{F}(U)} \int_{M_a, o_a} \frac{\epsilon(s, o, o') L(s, \mathcal{E}, \mathbb{A})(X + itS)}{\mathrm{Eul}_{o'/o_a}(T_{M_a} M(s))(X + itS)}.$$

Comparing with the formula in Lemma 12 for  $\mathrm{Tr} A_a^o(M, \mathcal{E}, \Delta_a^+)(g)$ , with

$$g = s \exp(X + itS),$$

it remains to see that on  $M_a$

$$\begin{aligned} & i^{-\dim M/2} \frac{\epsilon(s, o, o') L(s, \mathcal{E}, \mathbb{A})|_{M_a}(X + itS)}{\mathrm{Eul}_{o'/o_a}(T_{M_a} M(s))(X + itS)} \\ &= (2i\pi)^{-\dim M_a} \epsilon(o, o_a, \Delta_a^+) \frac{\mathrm{ch}(\mathcal{E}_a \otimes \mathcal{L}_a^+)(g)}{J^{1/2}(M_a)} \frac{-1}{\mathcal{N}_a^+} \det(1 - ge^{R_a^+}). \end{aligned}$$

■

## 4 Geometry of the moment map and decomposition of the quantized representation for a $S^1$ -action

Let  $G = \{e^{i\theta}; \theta \in \mathbb{R}\}$ . Let  $\mathfrak{g} = Lie(S^1) = \mathbb{R}E$  with  $E \in \mathfrak{g}$  such that  $\exp \theta E = e^{i\theta}$ . Let  $(M, \sigma, \mu)$  be a  $G$ -Hamiltonian manifold with symplectic form  $\sigma$  and moment map  $\mu$ . Let  $f : M \rightarrow \mathbb{R}$  be the  $G$ -invariant map

$$f(m) = \mu(E)(m).$$

We assume that  $(M, \sigma, \mu)$  is prequantized and let  $\mathcal{L}$  be a Kostant-Souriau line bundle (Definition 2) with its connection  $\mathbb{A}$ . As before we assume for simplicity that  $M$  carries a  $G$ -invariant spin structure. We fix the orientation  $o$  given by the symplectic structure and write  $Q(M, \mathcal{L})$  instead of  $Q^o(M, \mathcal{L})$ . Our aim is to understand the decomposition of  $Q(M, \mathcal{L})$  in irreducible representations of  $G$  in function of the geometry of the map  $f$ . Let  $\xi$  be a regular value of  $f$ . Let  $P_\xi = f^{-1}(\xi)$ . We assume that  $G$  acts freely on  $f^{-1}(\xi)$ . We can then consider the manifold  $M_{red}(\xi) = G \backslash P_\xi$ . It is a symplectic manifold with symplectic form  $\sigma_\xi$ . Then as we will show in Lemma 26 the manifold  $M_{red}(\xi)$  carries also a spin structure. Consider the line bundle  $\mathcal{L}$ . Then  $\mathcal{L}_{red}(\xi) = (G \backslash \mathcal{L}|_{P_\xi})$  is a line bundle on  $M_{red}(\xi)$  which is a Kostant-Souriau line bundle for  $\sigma_\xi$ . In this section, we show:

**Theorem 18** *Let  $G = S^1$ . Let  $\xi \in \mathfrak{g}^*$  such that  $i\xi$  is a weight of  $G$ . Assume that  $G$  acts freely on  $f^{-1}(\xi)$ . Then the multiplicity  $n(i\xi, M, \mathcal{L})$  of  $e_{i\xi}$  in  $Q(M, \mathcal{L})$  is equal to  $Q(M_{red}(\xi), \mathcal{L}_{red}(\xi))$ .*

By changing the moment map  $\mu$  to  $\mu - \xi$ , we can suppose that  $\xi = 0$ . We will then denote  $M_{red}(0)$  simply by  $M_{red}$  and  $\mathcal{L}_{red}(0)$  simply by  $\mathcal{L}_{red}$ .

To study the multiplicity of the trivial representation in  $Q(M, \mathcal{L})$  we will decompose the virtual character  $Q(M, \mathcal{L})$  as the sum of *three infinite dimensional virtual characters*  $\Theta_0$ ,  $\Theta_+$  and  $\Theta_-$ .

We denote  $f^{-1}(0)$  by  $P$ . Consider the principal fibration  $q : P \rightarrow M_{red} = G \backslash P$  with structure group  $G = S^1$ . Let  $n \in \mathbb{Z}$ . Consider the character  $\chi_n(\exp \theta E) = e^{in\theta}$  of  $G$  and let  $\mathcal{T}_n$  be the associated line bundle on  $G \backslash P = M_{red}$ . Then  $Q(M_{red}, \mathcal{L}_{red} \otimes \mathcal{T}_n)$  is a relative integer.

**Definition 19** Define the virtual character  $Q_0(M, \mathcal{L})$  of  $G$  by

$$Q_0(M, \mathcal{L}) = \sum_{n \in \mathbb{Z}} Q(M_{red}, \mathcal{L}_{red} \otimes \mathcal{T}_{-n}) e^{in\theta}.$$

The multiplicity of the trivial representation in  $Q_0(M, \mathcal{L})$  is  $Q(M_{red}, \mathcal{L}_{red})$ .

It follows from Atiyah-Singer formula (16) that the function  $n \mapsto Q(M_{red}, \mathcal{L}_{red} \otimes \mathcal{T}_{-n})$  is polynomial in  $n$  so that  $Q_0(M, \mathcal{L})$  is indeed a trace-class virtual representation. Furthermore,  $\text{Tr } Q_0(M, \mathcal{L})$  defines a generalised function on  $G$  supported at the identity of  $G$ .

Let  $\mathcal{F}$  be the set of connected components of  $M^T$ . Let  $a \in \mathcal{F}$  and let  $M_a$  be a connected component of  $M^T$ . Then  $f$  is constant on  $M_a$ . Let  $\mathcal{F}^+$  (resp.  $\mathcal{F}^-$ ) be the set of  $a \in \mathcal{F}$  such that  $f(M_a) > 0$  (resp.  $f(M_a) < 0$ ). Then  $\mathcal{F} = \mathcal{F}^+ \cup \mathcal{F}^-$ . We then choose for each component  $a \in \mathcal{F}$  the following order:

$$\Delta_a^{out} = \{\alpha \in \Delta_a, i\alpha(E)f(M_a) < 0\}.$$

More precisely for  $a \in \mathcal{F}^+$  we choose:  $\Delta_a^+ = \Delta_a^+(E) = \{\alpha \in \Delta_a, i\alpha(E) < 0\}$ , while we choose for each component  $a \in \mathcal{F}^-$  the order  $\Delta_a^+(-E)$ .

Recall the definition (Definition 10) of the element  $A_a(M, \mathcal{L}, \Delta_a^+)$  in  $R^{-\infty}(G)$ .

**Definition 20** Define  $Q_{\pm}(M, \mathcal{L}) \in R^{-\infty}(G)$  by

$$Q_+(M, \mathcal{L}) = \sum_{a \in \mathcal{F}^+} A_a(M, \mathcal{L}, \Delta_a^{out}).$$

$$Q_-(M, \mathcal{L}) = \sum_{a \in \mathcal{F}^-} A_a(M, \mathcal{L}, \Delta_a^{out}).$$

By definition of  $\mathcal{F}_+$ , the constant value  $\mu_a$  of  $\mu(E)$  on  $M_a$  is a positive integer. By definition of  $\Delta_a^+(E)$  the number  $p_a = -i\rho_a^+(E)$  is a positive integer. The following lemma follows from Lemma 13.

**Lemma 21** Let  $m > 0$  be the minimum of all numbers  $\mu_a + p_a$  for  $a \in \mathcal{F}^+$ . The virtual character  $Q_+(M, \mathcal{L})$  is of the form  $Q_+(M, \mathcal{L}) = \sum a_n e^{in\theta}$  with  $n \geq m$ . In particular  $Q_+(M, \mathcal{L})$  does not contain the trivial representation of  $G = S^1$ .

Similarly the virtual character  $Q_-(M, \mathcal{L})$  is of the form  $Q_-(M, \mathcal{L}) = \sum_{n < 0} a_n e^{in\theta}$  and  $Q_-(M, \mathcal{L})$  does not contain the trivial representation of  $S^1 = G$ .

The main theorem of this section is the following

**Theorem 22** *We have the decomposition*

$$Q(M, \mathcal{L}) = Q_0(M, \mathcal{L}) \oplus Q_+(M, \mathcal{L}) \oplus Q_-(M, \mathcal{L}).$$

It is clear that this theorem implies Theorem 18 for the case of an action of  $S^1$ . Furthermore this theorem should allow us to compare multiplicities when we cross a singular value of  $f$ .

**Remark 4.1** *This writing of  $Q(M, \mathcal{L})$  is in agreement with Formula (25) for  $Q(M, \mathcal{L})$ . Indeed we know from Lemma 8 that  $\text{Tr}(Q(M, \mathcal{L}) - \text{Tr}(Q_+(M, \mathcal{L}) \oplus Q_-(M, \mathcal{L})))$  is a generalised function supported on singular elements. Theorem 22 is an explicit description of this generalised function in function of the fiber  $f^{-1}(0)$ .*

**Proof.** Choosing a basis  $E$  of  $\mathfrak{g}$ , we consider the map  $f = \mu(E)$  from  $M \rightarrow \mathbb{R}$ . Let  $r$  be a small positive number. Let

$$M_0 = \{x; |f(x)| < r\}, \quad M_+ = \{x; f(x) > r\}, \quad M_- = \{x; f(x) < -r\}.$$

We can choose  $r$  such that  $S^1$  acts freely on  $\overline{M_0}$ . Thus the vector field  $E_M$  does not vanish on  $\overline{M_0}$ . As in Witten, consider the function  $w = \frac{1}{2}f^2$ . Then  $w$  gives rise to the Hamiltonian vector field  $H_w = \mu(E)E_M = fE_M$ . Let  $(\cdot, \cdot)$  be a  $G$ -invariant metric on  $M$ . Let

$$(32) \quad \lambda(\cdot) = (H_w, \cdot) = f(E_M, \cdot)$$

be the  $G$ -invariant 1-form determined by  $H_w$  and the choice of  $G$ -invariant metric  $(\cdot, \cdot)$  on  $M$ .

Remark that  $\lambda(E_M) = \mu(E)\|E_M\|^2$  is strictly positive on the boundary of  $M_+$  and strictly negative on the boundary of  $M_-$ .

We associated to  $M_0$ ,  $M_+$  and  $M_-$  three generalised functions on  $G$  with the help of  $\lambda$  by truncating formulas for the  $s$ -part of the character of  $Q(M, \mathcal{L}, \mathbb{A}(t))$  with  $\mathbb{A}(t) = \mathbb{A} - it\lambda$  on  $\mathcal{L}$ . Clearly as  $Q(M, \mathcal{L}, \mathbb{A}(t))$  is independent of  $t$  for all  $s \in G$  and all  $t \in \mathbb{R}$ , and  $X \in \mathfrak{g}$  small we have

$$\text{Tr } Q(M, \mathcal{L})(s \exp X) =$$

$$TQ(M_0, \mathcal{L}, \mathbb{A}, \lambda, s, t)(X) + TQ(M_+, \mathcal{L}, \mathbb{A}, \lambda, s, t)(X) + TQ(M_-, \mathcal{L}, \mathbb{A}, \lambda, s, t)(X).$$

Furthermore Lemma 15 and Theorem 17 implies that

$$\lim_{t \rightarrow \infty} TQ(M_+, \mathcal{L}, \mathbb{A}, \lambda, s, t)(X) = \text{Tr } Q_+(M, \mathcal{L}, \mathbb{A})(s \exp X)$$

and

$$\lim_{t \rightarrow \infty} TQ(M_-, \mathcal{L}, \mathbb{A}, \lambda, s, t)(X) = \text{Tr } Q_-(M, \mathcal{L}, \mathbb{A})(s \exp X).$$

Thus it remains to see

**Lemma 23** *For each  $s \in G$  and  $X \in U_s(0)$  a small neighbourhood of 0 in  $\mathfrak{g}$ , we have in  $C^{-\infty}(U_s(0))$ :*

$$\text{Tr } Q_0(M, \mathcal{L})(s \exp X) = \lim_{t \rightarrow \infty} TQ(M_0, \mathcal{L}, \mathbb{A}, \lambda, s, t)(X)$$

**Proof.**

By our hypothesis on the free action,  $TQ(M_0, \mathcal{L}, \mathbb{A}, \lambda, s, t)(X)$  is equal to 0 for  $s \neq 1$  as  $M_0(s) = \emptyset$ . The generalised function  $\text{Tr } Q_0(M, \mathcal{L})$  is supported at 1, thus  $\text{Tr } Q_0(M, \mathcal{L})(s \exp X) = 0$  for  $s \neq 1$  and  $X$  small. We need only to verify the formula of Lemma 23 for  $s = 1$ . Let  $s = 1$  and let us study the limit in the space of generalised functions on  $U_1(0)$  of the truncated integral

$$TQ(M_0, t, 1)(X) = (2i\pi)^{-\dim M/2} \int_{M_0} \frac{\text{ch}(\mathcal{L}, \mathbb{A}(t))(X)}{J^{1/2}(M)(X)}.$$

As  $\mathbb{A}(t) = \mathbb{A} - it\lambda$ , we have  $\text{ch}(\mathcal{L}, \mathbb{A}(t))(X) = e^{-id_X \lambda} \text{ch}(\mathcal{L}, \mathbb{A})(X)$ .

Let  $\phi$  be a test function on  $\mathfrak{g}$  with support in  $U_1(0)$  so that  $J^{1/2}(M)(X)$  is invertible on the support of  $\phi$ . We have

$$(33) \quad \int_{\mathfrak{g}} TQ(M_0, t, 1)(X) \phi(X) dX = \int_{M_0} \left( \int_{\mathfrak{g}} e^{-itd_X \lambda} L(X) \phi(X) dX \right)$$

with

$$L(X) = (2i\pi)^{-\dim M/2} \text{ch}(\mathcal{L}, \mathbb{A})(X) J^{-1/2}(M)(X)$$

a closed  $G$ -equivariant differential form on  $M$ .

Let us recall the results of [28]. As  $G$  acts freely on  $P$ , there exists an isomorphism  $W : H_G^\infty(\mathfrak{g}, P) \rightarrow H^*(M_{red})$ . If  $\omega$  is a connection form with curvature  $\Omega$  for the fibration  $q : P \rightarrow M_{red}$ , then  $W$  coincides on  $S(\mathfrak{g}^*)$  with the Chern-Weil homomorphism  $\phi \mapsto \phi(\Omega)$ . The inverse of  $W$  is simply given by  $q^*$ .

Let  $\alpha \in \mathcal{A}_G^\infty(\mathfrak{g}, M)$  be a  $G$ -equivariant closed form on  $M$ . The restriction  $\alpha|_P$  of  $\alpha$  to  $P$  is a  $G$ -equivariant closed form on  $P$ . We denote by  $\alpha_{red}$  the element  $W(\alpha|_P)$ . Then  $\alpha_{red}$  is a De Rham cohomology class on  $M_{red}$ .

We have the following proposition ([28], Theorem 19 and Remark)

**Proposition 24** *Let  $\alpha \in \mathcal{A}_G^\infty(\mathfrak{g}, M)$  be a  $G$ -equivariant closed form on  $M$ . Let  $\phi$  be a test function on  $\mathfrak{g}$ . Then*

$$\lim_{t \rightarrow \infty} \int_{M_0} \int_{\mathfrak{g}} e^{-itd_X \lambda} \alpha(X) \phi(X) dX = i(2\pi)^2 \int_{M_{red}} \alpha_{red} \phi(\Omega).$$

**Remark 4.2** *The form  $\phi(\Omega)$  is defined via the Taylor series of  $\phi$  at 0. As  $\Omega$  is nilpotent, the form  $\phi(\Omega)$  involves only finitely many derivatives of  $\phi$  at 0. Thus the map  $\phi \mapsto \int_{M_{red}} \alpha_{red} \phi(\Omega)$  is a distribution of support 0.*

We then obtained that the limit when  $t$  tends to  $\infty$  of the generalised function  $TQ(M_0, t, 1)(X)$  exists. We denote it by  $\Theta_0$ . We have by Proposition 24, for  $\phi$  a test function on  $\mathfrak{g}$ :

$$\int_{\mathfrak{g}} \Theta_0(X) \phi(X) dX = i(2\pi)^2 \int_{M_{red}} L_{red} \phi(\Omega)$$

where  $L(X) = (2i\pi)^{-\dim M/2} \text{ch}(\mathcal{L}, \mathbb{A})(X) J^{-1/2}(M)(X)$ .

**Lemma 25** *We have*

$$L_{red} = (2i\pi)^{-\dim M/2} e^{i\sigma_{red}} J^{-1/2}(M_{red}).$$

**Proof.** Recall that  $\sigma|_P$  is the pull-back of the symplectic form  $\sigma_{red}$  on  $M_{red}$ . Thus as  $\mu|_P = 0$ , we have  $\text{ch}(\mathcal{L}, \mathbb{A})|_P = e^{i\sigma|_P}$  and  $\text{ch}(\mathcal{L}, \mathbb{A})|_P$  is already in the form  $q^*(e^{i\sigma_{red}})$ .

Let  $J(M_{red})$  the  $J$ -genus of the tangent bundle to  $M_{red}$ . Let us see that

$$J(M)(X)|_P \cong q^* J(M_{red})$$

in cohomology. Indeed  $TM|_P = TP \oplus P \times \mathfrak{g}^*$  and  $TP = q^* TM_{red} \oplus V$  where  $q^* TM_{red}$  is identified to the horizontal tangent bundle and  $V$  to the vertical tangent bundle via the connection form  $\omega$ . Thus  $TM|_P = q^* TM_{red} \oplus P \times (\mathfrak{g} \oplus \mathfrak{g}^*)$ . The equivariant  $J$ -genus of the trivial bundle  $P \times (\mathfrak{g} \oplus \mathfrak{g}^*)$  is identically 1. Thus

$$J(M)|_P \cong q^*(J(M_{red}))$$

and we obtain our lemma. ■

**Lemma 26** *The manifold  $M_{red}$  has a spin structure.*

**Proof.** Let  $\mathcal{S}$  be the spin bundle of  $M$ . Consider the horizontal vector  $H = \partial_f$  on a neighbourhood of  $P$  in  $M$  of the form  $P \times \{f \in ]u, v[ \}$ . Let  $V = E_M$  be the vertical vector field generated by  $E$ . Let  $c_0$  be the endomorphism of  $\mathcal{S}|_P$  obtained by the Clifford action of  $H + iV$  on  $\mathcal{S}$ . Then  $\mathcal{S}_0 = \text{Ker } c_0 \subset \mathcal{S}|_P$  has typical fiber over  $x$  the spinor space of  $T_{q(x)}M_{red}$ . It is a  $G$ -equivariant subbundle of  $\mathcal{S}|_P$ . The bundle  $\mathcal{S}_0/G$  over  $M_{red}$  is a spinor bundle for  $M_{red}$ . ■

As  $\dim M_{red} = \dim M - 2$ , we obtain the following expression of  $\Theta_0$

$$\int_{\mathfrak{g}} \Theta_0(X) \phi(X) dX = (2\pi)(2i\pi)^{-\dim M_{red}/2} \int_{M_{red}} \phi(\Omega) e^{i\sigma_{red}} J^{-1/2}(M_{red}).$$

As  $\phi(\Omega) = \sum_k \frac{\Omega^k}{k!} ((\frac{d}{dX})^k \phi)|_{X=0}$ , we see that  $\Theta_0$  is a distribution on  $\mathfrak{g}$  with support 0. Let us compute its Fourier expansion. Let  $\chi_n(\theta) = e^{in\theta}$ . We have  $\Theta_0(\exp \theta E) = \sum_n a_n e^{in\theta}$  with

$$a_n = (2\pi)^{-1} \int_{\mathbb{R}} \Theta_0(\exp \theta E) \chi_{-n}(\theta) d\theta = (2i\pi)^{-\dim M_{red}/2} \int_{M_{red}} e^{i\sigma_{red}} J^{-1/2}(M_{red}) e^{-in\Omega}.$$

The associated line bundle  $\mathcal{T}_{-n}$  to  $\chi_{-n}$  has Chern character is  $e^{-in\Omega}$ . Thus the index  $Q(M_{red}, \mathcal{L}_{red} \otimes \mathcal{T}_{-n})$  of the twisted Dirac operator  $D_{\mathcal{L}_{red} \otimes \mathcal{T}_{-n}}^+$  on the spin manifold  $M_{red}$  is in  $\mathbb{Z}$ . It is given by the integral formula (16)

$$Q(M_{red}, \mathcal{L}_{red} \otimes \mathcal{T}_{-n}) = (2i\pi)^{-\dim M_{red}/2} \int_{M_{red}} e^{i\sigma_{red}} e^{-in\Omega} J^{-1/2}(M_{red}).$$

We thus obtain Lemma 23 and hence Theorem 22. ■

Assume that a compact group  $K$  acts on  $M$  commuting with the action of  $S^1$  and such that  $(\mathcal{L}, \mathbb{A})$  is a  $K$ -equivariant vector bundle and that the action of  $K$  preserves  $\mathbb{A}$ . Then  $f$  is an  $K$ -invariant function. The manifold  $f^{-1}(0)$  carries an  $K$ -action and  $M_{red}$  is a  $K$ -Hamiltonian manifold with Kostant-Souriau line bundle  $\mathcal{L}_{red}$ . All terms of Theorem 22 are virtual representations of  $G \times K$  and it is clear that the same theorem holds as (virtual) representations of  $G \times K$ .

**Theorem 27** *We have the decomposition*

$$Q(M, \mathcal{L}) = Q_0(M, \mathcal{L}) \oplus Q_+(M, \mathcal{L}) \oplus Q_-(M, \mathcal{L})$$

*as virtual representations of  $G \times K$ .*

The space  $Q(M, \mathcal{L})^G$  is a virtual representation of  $K$ . We have

**Theorem 28** *Let  $G = S^1$ . Let  $K$  be a compact Lie group. Let  $M$  be a  $G \times K$  Hamiltonian manifold with a  $(G \times K)$ -invariant spin structure. Let  $\mathcal{L}$  be a Kostant-Souriau line bundle for  $G \times K$ . Let  $Q(M, \mathcal{L}) \in R(G \times K)$  be the quantized space. Let  $f : M \rightarrow \mathfrak{g}^*$  the moment map for the  $G$ -action. Assume that  $G$  acts freely on  $f^{-1}(0)$ . Consider the  $K$ -Hamiltonian manifold  $M_{red} = G \backslash f^{-1}(0)$  with Kostant-Souriau line bundle  $\mathcal{L}_{red}$ . Then the virtual representation  $Q(M, \mathcal{L})^G$  is isomorphic to the virtual representation  $Q(M_{red}, \mathcal{L}_{red})$  of  $K$ .*

Theorem 27 should be interpreted in the non-compact manifold  $\tilde{M} = M \times \mathbb{R}$ . We embed  $M$  in  $M \times \mathbb{R}$  by  $x \mapsto (x, 0)$ . Let us consider

$$C_{\pm} = B_{\pm} \times \mathbb{R}^+$$

where  $B_{\pm}$  are the boundary of  $M_{\pm}$ . We define orientations  $o_{out}$  on  $C_{\pm}$  as explained in 29. Define

$$Z_{\pm} = M_{\pm} \cup (C_{\pm}, o_{out})$$

$$Z_0 = (C_-, -o_{out}) \cup M_0 \cup (C_+, -o_{out})$$

Thus  $Z_0, Z_+, Z_-$  are  $G$ -invariant oriented cycles in  $\tilde{M}$ . Clearly, as a sum of oriented cycles, we have:

$$M = [Z_-] + [Z_0] + [Z_+].$$

As shown in the preceding section (Formula (30)), the decomposition  $Q(M, \mathcal{L}) = Q_0(M, \mathcal{L}) \oplus Q_+(M, \mathcal{L}) \oplus Q_-(M, \mathcal{L})$  as a sum of 3 virtual characters of  $G$  corresponds to the decomposition  $M = [Z_-] + [Z_0] + [Z_+]$  in  $\tilde{M}$ . Indeed we have

$$\mathrm{Tr} Q_{\pm}(M, \mathcal{L}) = i^{-\dim Z/2} \int_{Z_{\pm}}^b \mathrm{bch}(Z_{\pm}, \tilde{\mathcal{L}}, \mathbb{A}^{\lambda}).$$

$$\mathrm{Tr} Q_0(M, \mathcal{L}) = i^{-\dim Z/2} \int_{Z_0}^b \mathrm{bch}(Z_0, \tilde{\mathcal{L}}, \mathbb{A}^{\lambda}).$$

## 5 Appendix

### 5.1 The universal character formula for quantum bundles

In this appendix, we state the universal character formula without the restrictive assumption of spin structure on  $M$ .

Let  $G$  be a compact Lie group acting on a smooth oriented manifold  $M$  and preserving the orientation of  $M$ . Let us recall the definition of a  $G$ -equivariant quantum bundle over  $M$ .

If  $V$  is a real vector space, we denote by  $GL^+(V)$  the group of linear transformations of  $V$  which are of positive determinant. We denote by  $j : ML^+(V) \rightarrow GL^+(V)$  the two-fold connected cover of  $GL^+(V)$  and by  $\epsilon \in ML^+(V)$  the non trivial element above  $1 \in GL^+(V)$ .

Let  $W$  be a Hermitian space. Let  $U(W)$  be the group of unitary transformations of  $W$ . We denote by  $-I$  the transformation  $w \rightarrow -w$  of  $W$ . We embed  $\mathbb{Z}/2\mathbb{Z}$  as a central subgroup  $Z$  in  $ML^+(V) \times U(W)$  obtained by sending  $(-1) \in \mathbb{Z}/2\mathbb{Z}$  to  $(\epsilon, -I) \in ML^+(V) \times U(W)$ . Let

$$ML^+(V)^W = (ML^+(V) \times U(W))/Z.$$

We denote by  $[g, u]$  the class of the element  $(g, u) \in ML^+(V) \times U(W)$  in  $ML^+(V)^W$ . We call the group  $ML^+(V)^W$  the metilinear group with coefficients in  $W$ . There are canonical morphisms

$$f : ML^+(V)^W \rightarrow GL^+(V)$$

and

$$u : ML^+(V)^W \rightarrow U(W)/\pm I.$$

If  $V_1 \oplus V_2$  is a direct sum decomposition of  $V$  the subgroup  $\{g \in ML^+(V_1 \oplus V_2)^W; f(g) \in GL^+(V_1)\}$  is isomorphic to  $ML^+(V_1)^W$ . We thus imbed  $ML^+(V_1)^{W_1}$  and  $ML^+(V_2)^{W_2}$  as two commuting subgroups of  $ML^+(V)^{W_1 \otimes W_2}$ .

If  $P \rightarrow M$  is a principal space with group  $ML^+(V)^W$ , we denote by  $P^f$  the vector bundle over  $M$  associated to the representation  $f$  of  $ML^+(V)^W$  in  $V$ .

Recall the definition:

**Definition 29** *Let  $M$  be a manifold of dimension  $n$ . Let  $V$  be a real vector space of dimension  $n$ . Let  $W$  be an Hermitian space. A quantum bundle*

$\tau : P \rightarrow M$  over  $M$  with fiber  $W$  is a principal bundle over  $M$  with structure group  $ML^+(V)^W$  such that the associated bundle  $P^f$  is the tangent bundle  $TM$ .

In this definition it is necessary that the manifold  $M$  is orientable for quantum bundles to exist. The space  $W$  is called the fiber of the quantum bundle  $P$ , although there is no true vector bundle  $\mathcal{W}$  with fiber  $W$  associated to  $P$ . However there is an associated bundle  $P^u$  to  $P$  with structure group  $U(W)/\pm I$ , i.e. a pseudo vector bundle with fiber  $W$  in the terminology of [27]. In particular as the adjoint action of  $U(W)$  on  $End(W)$  factors through  $U(W)/\pm I$ , the bundle  $End(\mathcal{W})$  is well defined.

It is clear that we can add two quantum bundles of fibers  $W_1$  and  $W_2$  and obtain a quantum bundle of fiber  $W_1 \oplus W_2$ . We can tensor a quantum bundle with fiber  $W$  with a Hermitian vector bundle with fiber  $E$  and obtain a quantum bundle with fiber  $W \otimes E$ .

Let  $G$  be a Lie group acting on  $M$  and preserving the orientation of  $M$ . A  $G$ -equivariant quantum bundle is a quantum bundle with a left action of  $G$  such that the associated action of  $G$  on the associated bundle  $P^f$  is the natural action of  $G$  on the tangent bundle  $TM$ .

We denote by  $F_G^t(M)$  the set of  $G$ -equivariant quantum bundles on  $M$  (up to isomorphism). We denote by  $K_G^t(M)$  the associated Grothendieck group. If  $G = \{1\}$ , we denote  $F_G^t(M)$  simply by  $F^t(M)$  and  $K_G^t(M)$  by  $K^t(M)$ .

A  $G$ -equivariant quantum bundle with Hermitian connection is a couple  $(\tau, \mathbb{A})$  consisting of a  $G$ -equivariant quantum bundle  $\tau$  and of a  $G$ -invariant connection  $\mathbb{A}$  on the associated bundle  $\tau^u$  with group  $U(W)/\pm I$ .

Let  $\tau$  be a  $G$ -equivariant quantum bundle on  $M$ . If  $\mathbb{A}$  is a  $G$ -equivariant connection on  $P^u$ , the equivariant curvature  $F(X)$  of  $\mathbb{A}$  is a differential form on  $M$  with values in the bundle  $End(\mathcal{W})$ . Thus we can define the Chern character  $ch(\tau, \mathbb{A})(X)$  as in the case of vector bundles by the formula  $ch(\tau, \mathbb{A})(X) = \text{Tr}(e^{F(X)})$ . It is a  $G$ -equivariant closed form on the manifold  $M$ .

Let  $(M, \sigma, \mu)$  be a symplectic manifold with Hamiltonian action of  $G$ . We slightly modify the notion of the Kostant-Souriau line bundle.

**Definition 30** *We say that  $(M, \sigma, \mu)$  is prequantizable if there exists a  $G$ -equivariant quantum line bundle  $(\tau, \mathbb{A})$  over  $M$  with equivariant curvature  $F(X) = i\sigma_g(X)$ .*

If the manifold  $M$  is a compact even-dimensional manifold and has a

$G$ -invariant orientation  $o$ , there is a well defined quantization map

$$Q^o : K_G^t(M) \rightarrow R(G).$$

This map can be constructed as follows: Let  $V$  be an oriented even-dimensional Euclidean space. Let  $S = S^+ \oplus S^-$  be the spinor space of  $V$ . Given a quantum bundle  $\tau$  over  $M$  with fiber  $W$ , we can construct with the help of a  $G$ -invariant metric on  $M$ , a graded  $G$ -equivariant Clifford bundle  $\mathcal{S}_W^\pm$  on  $M$  with typical fiber  $S^\pm \otimes W$ . We denote by  $\Gamma(M, \mathcal{S}_W^\pm)$  the space of its smooth sections. With the help of a  $G$ -invariant connection  $\mathbb{A}$  on  $P^u$ , we can construct a twisted Dirac operator  $D_{W, \mathbb{A}}^+$ . This is an elliptic operator

$$D_{W, \mathbb{A}}^+ : \Gamma(M, \mathcal{S}_W^+) \rightarrow \Gamma(M, \mathcal{S}_W^-)$$

which commutes with the natural action of  $G$ . We define

$$Q^o(M, \tau) = (-1)^{\dim M/2} ([Ker D_{W, \mathbb{A}}^+] - [Coker D_{W, \mathbb{A}}^+]).$$

The virtual representation  $Q^o(M, \tau)$  depends only of the quantum bundle  $\tau$  and of the orientation  $o$  of  $M$ .

The character formula for  $\text{Tr } Q^o(M, \tau)$  is a slight modification of the character formula 13 when  $M$  is a spin manifold. The crucial difference is that if we do not assume the existence of  $G$ -invariant spin structure, submanifolds  $M(s)$  are not necessarily oriented, thus we need to produce densities on  $M(s)$  rather than differential forms. The Chern character of equivariant quantum bundles produces such families as we now recall.

Recall that if  $s$  is an elliptic transformation of an oriented vector space  $V$  and if  $\tilde{s}$  is an element of  $ML^+(V)$  above  $s$ , then  $\tilde{s}$  determine an orientation  $o_{\tilde{s}}$  of  $V/V(s)$ . The convention is as in [27].

If  $N$  is a manifold, we introduce the two-fold cover  $N_{or} = \{(m, \xi)\}$  of  $N$ , where  $m \in N$  and  $\xi$  is an orientation of  $T_m M$ . We say that  $\alpha$  is a *folded differential form* on  $N$  if  $\alpha$  is a differential form on  $N_{or}$  such that  $\alpha_\xi = -\alpha_{-\xi}$ . The term of maximum exterior degree of a folded differential form  $\alpha$  is a density on  $N$ . We can then define  $\int_N \alpha$ .

Let  $\tau : P \rightarrow M$  be a  $G$ -equivariant quantum bundle over  $M$  with fiber  $W$  and  $G$ -invariant Hermitian connection  $\mathbb{A}$ . Then we can define the equivariant curvature  $F(X)$  of  $\mathbb{A}$ . For  $X \in \mathfrak{g}$ , it is a differential form on  $M$  with values in  $End(W)$ . If  $s \in G$ ,  $m \in M(s)$  and  $p$  is an element of  $P$  above  $m$ , we denote by  $g(p, s)$  the element  $g(p, s) \in ML^+(V)^W$  such that  $sp = pg(p, s)$ .

Let  $(\tilde{s}, s^W) \in \text{ML}^+(V) \times U(W)$  such that  $[\tilde{s}, s^W] = g(p, s)$ . Let  $o'$  be a local orientation of  $M(s)$ . We write  $\text{sign}(\tilde{s}, o, o') = \pm 1$  depending on whether the orientations  $o_{\tilde{s}}, o, o'$  are compatible or not.

**Definition 31** *The Chern character  $\text{bch}(\tau, \mathbb{A}) = (\text{ch}_s(\tau, \mathbb{A}))_{s \in G}$  of the equivariant quantum bundle  $\tau$  with  $G$ -invariant connection  $\mathbb{A}$  is the family of folded equivariant differential forms*

$$\text{ch}_{s,o'}(\tau, \mathbb{A})(X) = \text{sign}(\tilde{s}, o, o') \text{Tr}(s^W e^{F(X)|_{M(s)}}).$$

The Chern character  $\text{bch}(\tau, \mathbb{A})$  of a  $G$ -equivariant quantum bundle with  $G$ -invariant connection  $\mathbb{A}$  is an admissible bouquet in the sense of [13]. The integration  $\int_b^M$  on the space of admissible bouquets is defined in [13].

**Theorem 32** *Let  $\tau$  be a  $G$ -equivariant quantum bundle over an even-dimensional compact manifold  $M$ . Choose a  $G$ -invariant connection  $\mathbb{A}$  on  $\tau$ . Then*

$$\text{Tr } Q^o(M, \tau, \mathbb{A}) = i^{-\dim M/2} \int_b^M \text{bch}(\tau, \mathbb{A}).$$

*The meaning of this formula is as follows. Let  $s \in G$ . There exists a neighbourhood  $U_s(0)$  of 0 in  $\mathfrak{g}(s)$  such that we have*

$$(34) \quad \text{Tr } Q^o(M, \tau)(s \exp X) = i^{-\dim M/2} \int_{M(s)} (2\pi)^{-\dim M(s)/2} \frac{\text{ch}_s(\tau, \mathbb{A})(X)}{J^{1/2}(M(s))(X) D_s^{1/2}(T_{M(s)}M)(X)}$$

*for  $X \in U_s(0)$ .*

Remark that the term of maximal exterior degree under the integral sign is a density on  $M(s)$ , so that it can be integrated. If  $(M, \sigma, \mu)$  is a prequantizable compact symplectic manifold with Kostant-Souriau quantum line bundle  $\tau$ , the virtual representation  $Q(M, \tau, \mathbb{A})$  is the quantized space of the Hamiltonian space  $(M, \sigma, \mu)$ .

Let  $(M, \sigma, \mu)$  a symplectic manifold with an Hamiltonian action of a compact Lie group  $G$ . Let  $\mu : M \rightarrow \mathfrak{g}^*$  be the moment map. Assume that  $G$  acts freely on  $\mu^{-1}(0)$ . We can then form the reduced manifold  $M_{red}$ . In the next subsection, we show that there is a canonical map  $\tau \mapsto \tau_{red}$  from  $G$ -equivariant bundles on  $M$  to quantum bundles on  $M_{red}$ . This is the generalisation of the Lemma 26 and was already observed in the symplectic context in [24].

The main result (Theorem 1) on multiplicities (see part II [30] for the proof) can thus be stated without spin hypothesis on  $M$ .

Denote  $\mu^{-1}(0)$  by  $P$ . A neighbourhood of  $P$  in  $M$  is diffeomorphic to  $P \times \mathfrak{g}^*$ . In the next subsection, we show that there is an isomorphism between  $G$ -equivariant quantum bundles on  $P \times \mathfrak{g}^*$  and quantum bundles on  $M_{red} = G \backslash P$ .

## 5.2 Reduction of Quantum bundles

Let  $G$  be a compact Lie group with Lie algebra  $\mathfrak{g}$ . Let  $P$  be a compact manifold which is a principal space for the action of  $G$ . Let  $N$  be the manifold

$$N = P \times \mathfrak{g}^*$$

with diagonal action of  $G$ . Let  $N_{red} = G \backslash P$  be the quotient space. We assume that the manifold  $N_{red}$  is oriented. We denote by  $q : P \rightarrow N_{red}$  the quotient map.

We denote an element of  $N$  by  $(x, \xi)$  where  $x \in P$  and  $\xi \in \mathfrak{g}^*$ . We denote by  $q^t : P \times \mathfrak{g}^* \rightarrow N_{red}$  the map  $q^t(x, \xi) = q(x)$ . The fiber of  $q^t$  is  $G \times \mathfrak{g}^*$ . In particular the manifold  $N$  is covered by open subsets of the form  $U \times G \times \mathfrak{g}^*$ , so that it looks locally as the product of  $N_{red}$  by a cotangent space.

If  $\mathcal{W}$  is a vector bundle over  $N_{red}$ , then  $(q^t)^*\mathcal{W}$  is a vector bundle over  $N$ . It is provided with an action of  $G$  given by  $g \cdot ((x, \xi), v) = ((gx, g \cdot \xi), v)$  for  $g \in G, x \in P, \xi \in \mathfrak{g}^*, v \in \mathcal{W}_{q(x)}$ . Every  $G$ -equivariant vector bundle over  $P \times \mathfrak{g}^*$  is isomorphic to the pull-back of a  $G$ -equivariant vector bundle over  $P$ . As the action of  $G$  on  $P$  is free, every  $G$ -equivariant vector bundle over  $P$  is isomorphic to a bundle  $q^*(\mathcal{W})$ . Thus every  $G$ -equivariant vector bundle over  $N$  is isomorphic to the pull-back  $(q^t)^*\mathcal{W}$  of a bundle over  $N_{red}$ . We show that the same result is true for quantum bundles over  $N = P \times \mathfrak{g}^*$ .

To state the correspondance between quantum bundles over  $N_{red}$  and  $N$ , we need some lemmas on the groups  $ML^+(V)$ .

Assume  $\dim V = 2d$  even. Let  $o$  be an orientation of  $V$ . Let  $f_1, f_2, \dots, f_{2d-1}, f_{2d}$  be an oriented basis of  $V$ . Let

$$S^o f_{2j-1} = f_{2j}, \quad S^o f_{2j} = -f_{2j-1}.$$

The element  $r^o = \exp \pi S^o$  of  $ML^+(V)$  covers the element  $-1$  of  $SL(V)$  and depends only of the orientation  $o$  of  $V$ . We have  $(r^o)^2 = \epsilon$  and  $r^o$  is central in  $ML^+(V)$ .

Let  $V_1$  be a real vector space. Let  $V = V_1 \oplus V_1^*$ . On  $V$  consider the orientation  $o = o_{V_1 \oplus V_1^*}$  given as follows. If  $E_i$  is a basis of  $V_1$  with dual basis  $E^i$ , then an oriented basis is  $E_1, E^1, E_2, E^2, \dots$ . Consider the homomorphism  $d(g) = (g, {}^t g^{-1})$  of  $GL(V_1)$  into  $SL(V)$ . Let  $ML^+(V)^\mathbb{C}$  be the metilinear group with coefficients in  $\mathbb{C}$ .

**Lemma 33** *Let  $V_1$  be a real vector space. Let  $V = V_1 \oplus V_1^*$ . There exists a homomorphism  $\tilde{d} : GL(V_1) \rightarrow ML^+(V)^\mathbb{C}$  such that  $f\tilde{d}(s) = d(s)$ .*

*We normalise this isomorphism when  $\dim V_1$  is odd such that*

$$\tilde{d}(-1) = [r^o, i^{\dim V_1}].$$

*If  $\dim V_1$  is even, we embed  $V_1$  in  $V_1 \oplus \mathbb{R}$  and normalise  $\tilde{d}$  such that by*

$$\tilde{d}(g) = \tilde{d}(g, I).$$

**Remark 5.1** *If  $\dim V_1$  is odd, we have  $[r^o, i^{\dim V_1}]^2 = [\epsilon, -1] = 1$  in  $ML^+(V)^\mathbb{C}$  so that the image of  $(-1)$  is of order 2 as it should be.*

Let  $n_0 = \dim N_{red}$  and let  $d = \dim G$ . The dimension  $n$  of  $N$  is equal to  $n_0 + 2d$ . By the choice of a connection on  $P$  the tangent bundle to the fiber of  $q^t : N \rightarrow N_{red}$  is isomorphic to  $N \times (\mathfrak{g} \oplus \mathfrak{g}^*)$ . If  $(N_{red}, o_{red})$  is oriented, we choose as orientation of  $N$  the orientation

$$(35) \quad o^N = o_{red} \wedge o_{\mathfrak{g} \oplus \mathfrak{g}^*}$$

Let  $V_0$  be a real vector space of dimension  $n_0$ . Let  $\tau_0 : R_0 \rightarrow N_{red}$  be a quantum bundle on  $N_{red}$  with Hermitian fiber  $W$ . It is a principal bundle over  $N_{red}$  with structure group  $ML^+(V_0)^W$ . Let  $V = V_0 \oplus \mathfrak{g} \oplus \mathfrak{g}^*$ . Then  $V$  is of dimension  $n = n_0 + 2d$ . Consider the natural homomorphism  $h_0 : ML^+(V_0)^W \rightarrow ML^+(V)^W$  coming from the direct sum decomposition  $V = V_0 \oplus \mathfrak{g} \oplus \mathfrak{g}^*$ . Let  $R(\tau_0) = R_0 \times_{ML^+(V_0)^W} ML^+(V)^W$  be the associated bundle to the homomorphism  $h_0$ . The group  $ML^+(V)^W$  acts on the right on  $R(\tau_0)$ . Thus  $R(\tau_0)$  is a principal bundle over  $N_{red}$  with structure group  $ML^+(V)^W$ . We denote by  $[r, a]$  the class of the element  $(r, a)$   $r \in \tau_0, a \in ML^+(V)^W$  in  $R(\tau_0)$ . Consider the adjoint action  $G \rightarrow GL(\mathfrak{g})$ . Consider the homomorphism  $d : G \rightarrow ML^+(\mathfrak{g} \otimes \mathfrak{g}^*)^\mathbb{C}$  given in Lemma 33. As the subgroups  $ML^+(V_0)^W$  and  $ML^+(\mathfrak{g} \oplus \mathfrak{g}^*)^\mathbb{C}$  commute, the space  $R(\tau_0)$  is provided with an action of  $G$  given by  $g \times [r, a] = [r, \tilde{d}(g)a]$ . The bundle  $(q^t)^* R(\tau_0)$  is thus a  $G$ -equivariant principal bundle over  $N$  with structure group  $DL^+(V)^W$ .

**Lemma 34** *If  $\tau_0$  is a quantum bundle on  $N_{red}$ , the bundle  $(q^t)^*R(\tau_0)$  is a  $G$ -equivariant quantum bundle over  $N$ .*

**Proof.** Let  $\tau = (q^t)^*R(\tau_0)$ . We have to prove that the associated bundle  $\tau^f$  with fiber  $V = V_0 \oplus \mathfrak{g} \oplus \mathfrak{g}^*$  is isomorphic to the tangent bundle to  $N$ . It is clear that the associated bundle  $\tau^f$  is isomorphic to  $(q^t)^*TN_{red} \oplus N \times (\mathfrak{g} \oplus \mathfrak{g}^*)$ . Let us choose a connection form for the principal fibration  $P \rightarrow N_{red}$ . This provides an isomorphism between  $TN$  and  $(q^t)^*TN_{red} \oplus N \times (\mathfrak{g} \oplus \mathfrak{g}^*)$ . and identifies  $\tau^f$  to  $TN$  as  $G$ -equivariant bundles.

**Proposition 35** *Every  $G$ -equivariant quantum bundle  $\tau$  on  $N$  is of the form  $\tau = (p_0^t)^*R(\tau_0)$  for a unique quantum bundle  $\tau_{red}$  on  $N_{red}$ .*

The quantum bundle  $\tau_{red}$  over the reduced manifold  $N_{red}$  such that  $\tau \sim (q^t)^*\tau_{red}$  will be called the reduction of  $\tau$ . The associated bundle  $\tau^u$  is isomorphic to  $(q^t)^*\tau_{red}^u$ .

It is also natural to consider the situation where a product of compact Lie groups  $K \times G$  acts on  $P$ , and  $G$  acts freely on  $P$ . Then there is an action of  $K$  on  $N_{red}$  and we can similarly prove:

**Proposition 36** *The map  $\tau_0 \rightarrow (q^t)^*(R(\tau_0))$  is an isomorphism of  $F_H^t(N_{red})$  and  $F_{G \times H}^t(N)$ .*

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