# Spectral instability for non-selfadjoint operators*

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**Abstract**

We describe a recent result of M. Hager, stating roughly that for non-selfadjoint ordinary differential operators with a small random perturbation we have a Weyl law for the distribution of eigenvalues with a probability very close to 1.

**Keywords and Phrases:** Non-selfadjoint, eigenvalue, random perturbation
**Mathematics Subject Classification 2000**: 34L20, 47B80

## Contents



## 1 Introduction and some history

In this talk we discuss a recent result by M. Hager [9] which is part of her thesis. Some of the basic ideas in that work have their roots in the general theory of partial

*Dedicated to Professor Takahiro Kawai on the occasion of his sixtieth birthday differential equations. Let us recall that H. Lewy [11] gave an example of an operator in $\mathbf{R}^3$ which is not locally solvable near any point,

$$P = D_{x_1} - x_2 D_{x_3} + i(D_{x_2} + x_1 D_{x_3}). \tag{1.1}$$

Hörmander [10] discovered the role of the Poisson brackets in this context and gave a very general result on non-local solvability of linear partial differential equations saying that a differential operator with smooth coefficients is non locally solvable if the Poisson bracket $\frac{1}{2i}\{p, \overline{p}\}$ is not identically zero on the characteristic set $p = 0$, where $p$ denotes the principal symbol of the operator. The simplest model of a non locally solvable operator is perhaps the the Mizohata operator in two or more dimensions ([12])

$$P = D_{x_2} + i x_2 D_{x_1}. \tag{1.2}$$

It is not locally solvable near any point of the hyperplane $x_2 = 0$. Since then, there have been great developments in the solvability theory, but we shall here follow a slightly different historical line.

We first recall that the proof in [10] is based on what nowadays is called a quasi-mode construction for the adjoint operator: If this adjoint is denoted by $P$ and its principal symbol by $p$, let $(x_0, \xi_0) \in \mathbf{R}^n_x \times (\mathbf{R}^n_\xi \setminus \{0\})$ be a point in the cotangent space where $p = 0$ and $\frac{1}{2i}\{p, \overline{p}\} > 0$. Then there exists a smooth function $\phi \in C^\infty(\mathrm{neigh}\,(x_0))$ with $\phi'(x_0) = \xi_0$, $\mathrm{Im}\,\phi''(x_0) > 0$, $\phi(x_0) = 0$ and a classical symbol

$$a \sim a_0(x) + h a_1(x) + ... \text{ in } C^\infty(\mathrm{neigh}\,(x_0)),\ a_0(x_0) \neq 0,$$

such that

$$P(x, D_x)(a(x; h) e^{\frac{i}{h}\phi(x)}) = \mathcal{O}(h^\infty),\ h \to 0,\ \text{in } C^\infty(\mathrm{neigh}\,(x_0)). \tag{1.3}$$

Here $\mathcal{O}(h^\infty)$ stands for "$\mathcal{O}_N(h^N)$ for every $N \geq 0$". When $P$ has analytic coefficients, we can replace $\mathcal{O}(h^\infty)$ by $\mathcal{O}(e^{-1/(C_0 h)})$ for some constant $C_0 > 0$. The latter fact follows from the work of Sato-Kawai-Kashiwara [14], but the result is at least partially older (L. Boutet de Monvel, P. Kree [1]).

Following work of Yu. Egorov and V. Kondratiev on the oblique derivative problem ([6]), L. Hörmander asked me to make a more complete study of pseudodifferential operators $P$ on a (para-)compact manifold $X$, with

$$\{p, \overline{p}\}(x, \xi) \neq 0 \text{ whenever } p(x, \xi) = 0,\ \xi \neq 0. \tag{1.4}$$

Under some additional assumptions (in order to get a global result), I obtained in my thesis [15] that a certain operator

$$\mathcal{P} = \begin{pmatrix} P & R_- \\ R_+ & 0 \end{pmatrix} : \mathcal{D}(P) \times \mathcal{H}_- \to L^2(X) \times \mathcal{H}_+ \tag{1.5}$$

has an inverse

$$\mathcal{E} = \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix}$$

modulo smoothing operators, where $E_{-+}$ is smoothing. Here $\mathcal{H}_\pm$ are Sobolev spaces on manifolds $\Gamma_\pm$ with $\dim \Gamma_\pm = \dim (X) - 1$. This result implies that the null-space of

$$P : \mathcal{D}'(X)/C^\infty(X) \to \mathcal{D}'(X)/C^\infty(X)$$

is equal to

$$\{E_+ v_+;\ v_+ \in \mathcal{D}'(\Gamma_+)\}/C^\infty(X),$$

which is a big space, and a similar statement can be made about the cokernel of $P$ in $\mathcal{D}'(X)/C^\infty(X)$.

At about the same time, Sato-Kawai-Kashiwara [14] made a complete study in the analytic category, and they showed among many other things that operators satisfying (1.4) can be reduced to the Mizohata operator.

In September 1972 I met T. Kawai for the first time and we had a very interesting discussion. He pointed out to me that the results of my thesis do not imply that the space of local solutions to the exact equation $Pu = 0$ is large, while in the analytic category the corresponding results ([14]) do so, thanks to the Cauchy-Kowalewski theorem. Indeed, it was showed by L. Nirenberg [13] for perturbations of the H. Lewy operator that there may be a radical difference between the analytic and the $C^\infty$ case. (See also [17], [15] for related results for perturbations of the Mizohata operator.)

This talk deals with closely related problems for eigenvalues and we will establish an "opposite result": By destroying analyticity, the spectral properties "improve" with high probability.

Before describing the precise result in the next section, we end this introduction with an extremely quick review of the notion of pseudospectrum which has been developped by L.N. Trefethen and other specialists in numerical analysis and which then migrated towards PDE thanks to the efforts of E.B. Davies and M. Zworski. See [7, 2, 18]. Let $P : \mathcal{H} \to \mathcal{H}$ be a closed densely defined operator, where $\mathcal{H}$ is a complex Hilbert space. For $\epsilon > 0$, we define the $\epsilon$-pseudospectrum by

$$\sigma_\epsilon(P) = \sigma(P) \cup \{z \in \mathbf{C} \setminus \sigma(P);\ \|(z-P)^{-1}\| > \frac{1}{\epsilon}\}, \tag{1.6}$$

where $\sigma(P)$ denotes the spectrum of $P$. We have

$$\sigma_\epsilon(P) = \bigcup_{\substack{Q \in \mathcal{L}(\mathcal{H}),\\ \|Q\| < \epsilon}} \sigma(P+Q), \tag{1.7}$$

which shows that $\sigma_\epsilon$ is a region of spectral instability.

When $P$ is selfadjoint or more generally normal, we have

$$\sigma_\epsilon(P) = \{z \in \mathbf{C};\ \mathrm{dist}\,(z, \sigma(P)) < \epsilon\},$$

but in general, $\sigma_\epsilon(P)$ is much larger than the right hand side in the above equation.

**Example 1.1** Consider a Jordan block

$$J = \begin{pmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ .. & .. & .. & ... & .. \\ 0 & 0 & 0 & ... & 1 \\ 0 & .. & .. & ... & 0 \end{pmatrix} :\ \mathbf{C}^N \to \mathbf{C}^N,\ N \gg 1.$$

Then $\|(z-J)^{-1}\| \geq |z|^{-N}$, so $\sigma_\epsilon(J) \supset D(0, \epsilon^{1/N})$, while $\sigma(J) = \{0\}$.

E.B. Davies [3] considered a general Schrödinger operator

$$P = -h^2 \frac{d^2}{dx^2} + V(x) \tag{1.8}$$

on $\mathbf{R}$, where the potential $V(x)$ is smooth and complex-valued. Here we are interested in the semi-classical limit $h \to 0$. The associated semi-classical symbol is $p(x, \xi) = \xi^2 + V(x)$. Davies observed that if

$$z = \xi_0^2 + V(x_0), \tag{1.9}$$

with $\xi_0 \neq 0$, $\mathrm{Im}\, V'(x_0) \neq 0$, then there exists a function

$$u(x) = u(x; h) = a(x; h) e^{i\phi(x)/h}$$

with $\phi(x_0) = 0$, $\phi'(x_0) = \pm\xi_0$, $\mathrm{Im}\, \phi''(x_0) > 0$, $a(x; h) \sim h^{-1/4}(a_0(x) + h a_1(x) + ...)$ in $C^\infty$, such that

$$\|u\| = 1,\ \|(P - z)u\| = \mathcal{O}(h^\infty).$$

This implies that

$$\text{either } z \in \sigma(P) \text{ or } \|(z-P)^{-1}\| \geq \frac{1}{\mathcal{O}(h^\infty)}.$$

In many cases when $V(x)$ is analytic, the spectrum of $V$ is confined to a union of curves, much smaller then the set of values in (1.9).

M. Zworski [18] observed that this was a rediscovery of the old Hörmander construction. Then with N. Dencker and Zworski [4], we established

**Proposition 1.2** *Let $P(x, hD_x; h)$ be an h-pseudodifferential operator on $\mathbf{R}^n$ with symbol*

$$P(x, \xi; h) \sim p(x, \xi) + hp_{-1}(x, \xi) + ... \in \text{ a suitable class.}$$

*Assume*

$$z = p(x_0, \xi_0),\ \frac{1}{2i}\{p, \overline{p}\}(x_0, \xi_0) > 0.$$

*Then there exists a function $u = u_h = a(x; h)e^{i\phi(x)/h}$ with the same properties as in and around (1.3) such that $\|u_h\| = 1$, $\|(P - z)u_h\| = \mathcal{O}(h^\infty)$. When $P$ is analytic (in a suitable class) we may replace $\mathcal{O}(h^\infty)$ by $\mathcal{O}(e^{-1/(Ch)})$ for some $C > 0$.*

In the Schrödinger case $p = \xi^2 + V(x)$ we have $\frac{1}{2i}\{p, \overline{p}\}(x, \xi) = -\xi \cdot \operatorname{Im} V'(x)$.

# 2 The result of M. Hager

We work in $L^2(\mathbf{R})$. Let $P = p^w(x, hD_x)$ be the Weyl quantization of $p(x, h\xi)$. Assume that $p(x, \xi)$ is holomorphic in a tubular neighborhood of $\mathbf{R}^2$, and

$$p(x, \xi) = \mathcal{O}(m(\operatorname{Re}(x, \xi))), \tag{2.1}$$

where $1 \leq m$ is an order function on $\mathbf{R}^2$, in the sense that

$$m(\rho) \leq C_0 \langle \rho - \mu \rangle^{N_0} m(\mu),\ \rho, \mu \in \mathbf{R}^2, \tag{2.2}$$

where $\langle \rho - \mu \rangle = \sqrt{1 + |\rho - \mu|^2}$. We may assume without loss of generality that $m$ belongs to its own symbol class: $\partial^\alpha m = \mathcal{O}(m)$ for every $\alpha \in \mathbf{N}^2$. Then for $h > 0$ small enough, $P : L^2(\mathbf{R}) \to L^2(\mathbf{R})$ is a closed densely defined operator with domain $\mathcal{D}(P) = H(m) := (m^w(x, hD))^{-1}(L^2(\mathbf{R}))$.

Let

$$\Sigma = \overline{p(\mathbf{R}^2)},$$

and let $\Sigma_\infty$ denote the set of accumulation points of $p$ at $(x, \xi) = \infty$. Let $\widetilde{\Omega} \subset\subset \mathbf{C} \setminus \Sigma_\infty$ be a connected open set not entirely contained in $\Sigma$. Assume that

$$|p(x, \xi) - z_0| \geq \frac{m(x, \xi)}{C_0},\ (x, \xi) \in \mathbf{R}^2,$$

for some $z_0 \in \widetilde{\Omega} \setminus \Sigma$ and some constant $C_0 > 0$. If $\widetilde{K} \subset \mathbf{C} \setminus \Sigma$ is compact, then $\sigma(P) \cap \widetilde{K} = \emptyset$ for $h$ small enough. Moreover, the spectrum of $P$ in $\widetilde{\Omega}$ is discrete when $h > 0$ is small enough.

Assume that

$$\xi \mapsto p(x,\xi) \text{ is even.} \tag{2.3}$$

Let $\Omega \subset\subset \widetilde{\Omega}$ be open and simply connected. Assume

$$p(x,\xi) \in \overline{\Omega} \Rightarrow \{p,\overline{p}\}(x,\xi) \neq 0. \tag{2.4}$$

Then for $z \in \overline{\Omega}$:

$$\begin{gathered} p^{-1}(z) = \{\rho_j^+(z), \rho_j^-(z);\, j = 1,2,..,n\}, \\ \rho_j^\pm = (x_j, \pm\xi_j),\ \pm\frac{1}{2i}\{p,\overline{p}\}(\rho_j^\pm) > 0. \end{gathered} \tag{2.5}$$

Assume for simplicity that $x_j \neq x_k$ for $j \neq k$.

We now add a small random perturbation $\delta q_\omega(x)$ where $\omega$ denotes the random parameter and $\delta$ is a small parameter satisfying

$$e^{-\frac{1}{D_0 h}} < \delta < \frac{1}{C_0} h^{\frac{3}{2}},\ D_0, C_0 \gg 1. \tag{2.6}$$

$q_\omega$ will be a random linear combination of eigenfunctions of an auxiliary operator

$$\begin{gathered} \widetilde{P} = \widetilde{p}^w,\ \text{where } \partial^\alpha \widetilde{p} = \mathcal{O}(\widetilde{m}),\ \widetilde{p} \geq \frac{\widetilde{m}}{C}, \\ \langle\rho\rangle^{k_0} \leq \widetilde{m}(\rho) \leq C_0 \langle \rho - \mu\rangle^{N_0} \widetilde{m}(\mu),\ k_0 > 0. \end{gathered} \tag{2.7}$$

Let $q_1, q_2, ...$ be an orthonormal basis of eigenfunctions of $\widetilde{P}$ corresponding to the eigenvalues $E_1 \leq E_2 \leq ... \to +\infty$. Let $N = C/h$ for $C \gg 1$ and put

$$q_\omega(x) = \sum_{\ell \leq N} \alpha_\ell(\omega) q_\ell(x), \tag{2.8}$$

where $\alpha_\ell$ are independent identically distributed complex random variables with $\langle \alpha_\ell \rangle = 0$, variance $\sigma = \delta^{2/n}$ and distribution:

$$\frac{1}{\pi\sigma^2} e^{-|\alpha|^2/\sigma^2} d(\mathrm{Re}\,\alpha) d(\mathrm{Im}\,\alpha).$$

**Theorem 2.1** **(M. Hager)** *Let $\Gamma \subset\subset \Omega$ be open with smooth boundary. There exist $C, K, D > 0$ such that for $h > 0$ sufficiently small, we have with probability $\geq 1 - C\delta^{\frac{1}{2n}} h^{-K}$ that*

$$|\#(\sigma(P + \delta q_\omega) \cap \Gamma) - \frac{1}{2\pi h} \mathrm{vol}\,(p^{-1}(\Gamma))| \leq D\Big(\frac{\ln\frac{1}{\delta}}{h}\Big)^{\frac{1}{2}}. \tag{2.9}$$

Hager also has a similar theorem saying that with a probability very close to 1, the Weyl asymptotics (2.9) holds simultaneously for all $\Gamma$ varying in a class of sets that satisfy the above assumptions uniformly.

## 3 Quick outline of the proof

For more details, see [9]. Let $e_j^+, e_j^-$ be normalized Davies quasimodes associated to $(P-z, \rho_j^+)$ and $((P-z)^*, \rho_j^-)$ respectively with exponentially small remainders as in the last part of Proposition 1.2. Consider

$$
\begin{aligned}
&R_+ : H(m) \to \mathbf{C}^n,\ (R_+u)(j) = (u|e_j^+), \\
&R_- : \mathbf{C}^n \to L^2(\mathbf{R}),\ R_-u_- = \sum_1^n u_-(j)e_j^-.
\end{aligned}
\tag{3.1}
$$

With high probability we have $\|q_\omega\|_{L^\infty} \le 1$, and then

$$
\mathcal{P}^\delta(z) = \begin{pmatrix} P+\delta q_\omega - z & R_-(z) \\ R_+(z) & 0 \end{pmatrix} = H(m)\times \mathbf{C}^n \to L^2 \times \mathbf{C}^n \tag{3.2}
$$

is invertible with inverse

$$
\mathcal{E}^\delta(z) = \begin{pmatrix} E^\delta & E_+^\delta \\ E_-^\delta & E_{-+}^\delta \end{pmatrix}, \tag{3.3}
$$

where

$$
\begin{aligned}
E_{-+}^\delta &= E_{-+}^0 + \delta E_{-+}^{(1)} + \mathcal{O}(\frac{\delta^2}{\sqrt{h}}) \\
&= \delta E_{-+}^{(1)} + \mathcal{O}(\frac{\delta^2}{\sqrt{h}}),
\end{aligned}
\tag{3.4}
$$

$$
(E_{-+}^{(1)})_{j,k} = -(q_\omega e_+^k | e_-^j) + \mathcal{O}(e^{-\frac{1}{Ch}}). \tag{3.5}
$$

Now,

$$
z \in \sigma(P+\delta q_\omega) \Leftrightarrow \det E_{-+}^\delta = 0.
$$

We can show that there exists a function

$$
\ell^\delta(z) = \ell^0(z) + \mathcal{O}(\frac{\delta}{\sqrt{h}}),\ \ell^0 \in C^\infty(\Omega),
$$

such that

$$
F^\delta(z) = e^{\frac{\ell^\delta(z)}{h}} \det E_{-+}^\delta(z) \tag{3.6}
$$

is holomorphic. Moreover,

$$
(\Delta \mathrm{Re}\, \ell_0(z) + \mathcal{O}(h)) d(\mathrm{Re}\, z) d(\mathrm{Im}\, z) = \sum_j (d\xi_j^-(z)\wedge dx_j^-(z) - d\xi_j^+(z)\wedge dx_j^+(z)) \tag{3.7}
$$

so

$$\iint_\Gamma \Delta(\mathrm{Re}\,\ell_0) d(\mathrm{Re}\,z) d(\mathrm{Im}\,z) = \mathrm{vol}\,(p^{-1}(\Gamma)) + \mathcal{O}(h). \tag{3.8}$$

Now,

$$|F^\delta(z)| \le e^{\frac{\mathrm{Re}\,\ell_0(z)}{h}}, \tag{3.9}$$

and for every $z \in \Omega$ we have with a high probability that

$$|F^\delta(z)| \ge e^{\frac{\mathrm{Re}\,\ell_0(z)}{h} - \frac{\epsilon}{h}}. \tag{3.10}$$

Here $\epsilon \ll 1$ should be suitably chosen, possibly depending on $h$. It then suffices to apply

**Proposition 3.1** *Let $\Gamma$ and $\Omega$ be as above, $\phi \in C^\infty(\Omega; \mathbf{R})$. Let $f$ be holomorphic in $\Omega$ with*

$$|f(z;h)| \le e^{\phi(z)/h}, \; z \in \Omega.$$

*Assume there exist $\epsilon \ll 1$, $z_k \in \Omega$, $k \in J$, such that*

$$\partial\Gamma \subset \bigcup_{k\in J} D(z_k, \sqrt{\epsilon}), \; \#J = \mathcal{O}(\frac{1}{\sqrt{\epsilon}}),$$
$$|f(z_k;h)| \ge e^{\frac{1}{h}(\phi(z_k)-\epsilon)}, \; k \in J.$$

*Then,*

$$\#(f^{-1}(0) \cap \Gamma) = \frac{1}{2\pi h} \iint_\Gamma (\Delta\phi) d(\mathrm{Re}\,z) d(\mathrm{Im}\,z) + \mathcal{O}(\frac{\sqrt{\epsilon}}{h}).$$

# 4 Prospects

- Extension to higher dimensions. This works ([Hager-Sj] in preparation).
- More general random perturbations. In higher dimensions we run into questions about random matrices.
- Weyl asymptotics for large eigenvalues in the non-semiclassical case. Here one would like to have results with probabality 1. This is under investigation.
- It would be interesting to see wether one could get similar results about resonances.

# References


[1] L. Boutet de Monvel, P. Kree, *Pseudo-differential operators and Gevrey classes,* Ann. Inst. Fourier, 17(1)(1967), 295–323.

[2] E.B. Davies, *Semi-classical analysis and pseudospectra,* J. Diff. Eq. 216(1)(2005), 153–187.

[3] E.B. Davies, *Semi-classical states for non-self-adjoint Schrödinger operators,* Comm. Math. Phys., 200(1999), 35–41.

[4] N. Dencker, J. Sjöstrand, M. Zworski, *Pseudospectra of (pseudo)differential operators,* Comm. Pure Appl. Math., 57(2004), 384–415.

[5] M. Dimassi, J. Sjöstrand, *Spectral asymptotics in the semi-classical limit,* London Math. Soc. Lecture Note Series, 268(1999).

[6] Yu.V. Egorov, V.A. Kondratiev, *The oblique derivative problem,* Mat. sb. (N.S.) 78(120)(1969), 148–176.

[7] M. Embree, L.N. Trefethen, *Spectra and pseudospectra, the behaviour of nonnormal matrices and operators,* Princeton Univ. Press, Princeton NJ (2005).

[8] M. Hager, *Instabilité spectrale semiclassique pour des opérateurs non-autoadjoints I: un modèle,* préprint 2004, http://hal.ccsd.cnrs.fr/ccsd-00001594, Annales de la Faculté des Sciences de Toulouse, to appear,

[9] M. Hager, *Instabilité spectrale semiclassique d'opérateurs non-autoadjoints II:* http://hal.ccsd.cnrs.fr/ccsd-00004677.

[10] L. Hörmander, *Differential equations without solutions,* Math. Ann. 140(1960), 169-173.

[11] H. Lewy, *An example of a smooth linear partial differential equation without solution,* Ann. of Math., 66(2)(1957), 155–158.

[12] S. Mizohata, *Solutions nulles et solutions non analytiques,* J. Math. Kyoto Univ. 1(1961/1962), 271–302.

[13] L. Nirenberg, *Lectures on linear partial differential equations,* CBMS Regional Conference Series in Mathematics, No. 17. American Mathematical Society, Providence, R.I., 1973.

[14] M. Sato, T. Kawai, M. Kashiwara, *Microfunctions and pseudo-differential equations.* Hyperfunctions and pseudo-differential equations. Lecture Notes in Math., Vol. 287, 265–529 Springer, Berlin, 1973.

[15] J. Sjöstrand, *Operators of principal type with interior boundary conditions,* Acta Math., 130(1973), 1–51.

[16] J. Sjöstrand, *Note on a paper of F. Treves concerning Mizohata type operators,* Duke Math. J., 47(3)(1980), 601-608.

[17] F. Treves, *Remarks about certain first-order linear PDE in two variables,* Comm. PDE 5(4)(1980), 381–425.

[18] M. Zworski, *A remark on a paper of E. B Davies: "Semi-classical states for non-self-adjoint Schrödinger operators",* Comm. Math. Phys. 200(1)(1999), 35–41.