CHAPTER 10

SPECIALIZATION OF FILTERED $\mathcal{D}$-MODULES

Summary. In this chapter, we take up the notion of specialization and the compatibility property with proper pushforward for filtered $\mathcal{D}_X$-modules. Compared with the approach of Sections 9.3–9.8, we insist in keeping the strictness property, that is, we only work with filtered $\mathcal{D}_X$-modules, not graded modules over the Rees ring $R_F\mathcal{D}_X$. We will compare the two approaches in Section 10.7.

10.1. Introduction

One can introduce the notion of filtered $\mathcal{D}$-module by keeping the data of the $\mathcal{D}$-module and its filtration. The advantage is to keep a hand on the filtration at each step. The main goal of this chapter (Theorem 10.5.4) is to give a proof of the criterion given in Theorem 9.8.8 from this point of view. One should be careful since the category of filtered $\mathcal{D}$-modules is not abelian anymore. As a consequence, dealing with derived categories, as needed when considering pushforward, needs some care, as well as strictness for bi-filtered complexes.

The comparison between the present approach and that of Chapter 9 will be done in Section 10.7. Of particular interest is the property that, for a strict graded $R_F\mathcal{D}_X$-module, strict $\mathbb{R}$-specializability along a smooth divisor H implies a regularity property, which has not been emphasized up to now, but which is essential for the approach in this chapter. In particular, the approach of Section 9.8 does not give as a result the strictness of the pushforward, only its strict specializability. We will show in Section 10.7 how to recover strictness properties from this point of view. On the other hand, the advantage of the approach of Section 9.8 is to allow generalization to cases where the regularity property is not fulfilled (twistor $\mathcal{D}$-modules), since strictness is not used for proving Theorem 9.8.8, only strict specializability is used. Lastly, localization and maximalization also have a natural formalism in the framework of graded $R_F\mathcal{D}_X$-module. We will not take up the corresponding formalism in the setting of filtered $\mathcal{D}_X$-modules.

10.2. Strict and bi-strict complexes

In this section we review the definition and basic properties of strictness for filtered and bi-filtered complexes. We will consider the case of several filtrations in Section 15.1. In particular, when dealing with at least three filtrations, an important role is played by the compatibility condition on filtrations. However, this condition does not arise when dealing with one or two filtrations and the strictness condition on complexes is also very easy to treat directly.

10.2.1. Convention. We work in the abelian category A of sheaves of vector spaces (over some fixed field, that will be the field of complex numbers for our purposes) on some topological space T.[(1)] In such a category, all filtered direct limits exist and are exact. Given an object A in this category, we only consider increasing filtrations $F_\bullet A$ that are indexed by $\mathbb{Z}$ and satisfy $\varinjlim_k F_k A = A$. We write a filtered object in A as (A, F), where $F = (F_k A)_{k\in\mathbb{Z}}$.

Note that if (A, F) is a filtered object, then a subobject B of A carries the induced filtration $(F_k A \cap B)_{k\in\mathbb{Z}}$, while a quotient object A/A' carries the induced filtration $((F_k A + A')/A')_{k\in\mathbb{Z}}$. It is easy to see that the two possible induced filtrations on a subquotient B/A' of A agree.

***10.2.2. Definition* (Strictness of filtered complexes).** Consider a complex $(C^\bullet, F)$ of filtered objects in A. This is a *strict complex* if all morphisms $d\colon C^i \to C^{i+1}$ are strict, in the sense that the isomorphism $\mathrm{Coim}(d) \to \mathrm{Im}(d)$ is an isomorphism of filtered objects, that is, we have

$$d(F_k C^i) = F_k C^{i+1} \cap d(C^i) \quad \text{for all } k, i \in \mathbb{Z}.$$

We will be interested in complexes of bi-filtered objects in A. These are objects of A carrying two filtrations (A, F', F''). We write

$$F'_k F''_\ell A := F'_k A \cap F''_\ell A. \tag{10.2.3}$$

The morphisms in this case are required to be compatible with each of the two filtrations.

10.2.4. Definition. Let $(C^\bullet, F', F'')$ be a complex of bi-filtered objects. We say that the complex is *strict* (or *bi-strict*, if we want to emphasize the fact that we consider two filtrations) if for every i, p, and q, the natural maps in the commutative square

$$\begin{array}{ccc} H^i(F'_k F''_\ell C^\bullet) & \longrightarrow & H^i(F'_k C^\bullet) \\ \big\downarrow & & \big\downarrow \\ H^i(F''_\ell C^\bullet) & \longrightarrow & H^i(C^\bullet) \end{array}$$

[(1)] One can consider more general abelian categories where all filtered direct limits exist and are exact. However, we will not need such a generality. We refer the reader interested in more general categories to **[Sai88**, §1].

are injective, and furthermore, the square is Cartesian, that is, $H^i(F'_kF''_\ell C^\bullet) = H^i(F'_kC^\bullet) \cap H^i(F''_\ell C^\bullet)$.

10.2.5. Remark. It follows from Remark 10.2 that $(C^\bullet, F', F'')$ is strict if and only if all canonical morphisms

$$H^i(F'_kC^\bullet) \longrightarrow F'_kH^i(C^\bullet), \quad H^i(F''_\ell C^\bullet) \longrightarrow F''_\ell H^i(C_\bullet),$$

and

$$H^i(F'_kF''_\ell C^\bullet) \longrightarrow F'_kF''_\ell H^i(C^\bullet)$$

are isomorphisms. (See Exercise 10.6 for the case of a bi-strict morphism.)

10.2.6. Lemma. *If $(C^\bullet, F', F'')$ is a strict complex of bi-filtered objects, then the complexes $(C^\bullet, F')$ and $(F'_kC^\bullet, F'')$ are strict for every $k \in \mathbb{Z}$. In particular, we have a short exact sequence*

$$0 \longrightarrow H^i(F'_kF''_\ell C^\bullet) \longrightarrow H^i(F'_kF''_m C^\bullet) \longrightarrow H^i\left(F'_k(F''_mC^\bullet/F''_\ell C^\bullet)\right) \longrightarrow 0$$

for every $\ell < m$ and every i. Furthermore, for every k, every $\ell < m < n$, and every i, we have short exact sequences

$$0 \to H^i\Big(F'_k(F''_mC^\bullet/F''_\ell C^\bullet)\Big) \longrightarrow H^i\Big(F'_k(C^\bullet/F''_\ell C^\bullet)\Big) \longrightarrow H^i\Big(F'_k(C^\bullet/F''_mC^\bullet)\Big) \to 0$$

and

$$0 \to H^i\Big(F'_k(F''_mC^\bullet/F''_\ell C^\bullet)\Big) \longrightarrow H^i\Big(F'_k(F''_nC^\bullet/F''_\ell C^\bullet)\Big) \\ \longrightarrow H^i\Big(F'_k(F''_nC^\bullet/F''_mC^\bullet)\Big) \to 0.$$

Proof. The first assertion is an immediate consequence of the definition, while the exact sequences follow from the strictness of $(F'_kC^\bullet, F'')$, using Remarks 10.3 and 10.4. $\square$

10.2.7. Lemma. *If $(C^\bullet, F', F'')$ is a strict complex of bi-filtered objects, then for every $k < q$, the complex $(F''_kC^\bullet/F''_\ell C^\bullet, F')$ is strict. In particular, each complex $(\mathrm{gr}_k^{F''}(C^\bullet), F')$ is strict.*

Proof. It follows from Lemma 10.2.6 that for every s and i, in the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^i(F'_mF''_kC^\bullet) \arrow[r] \arrow[d, "u"] & H^i(F'_mF''_\ell C^\bullet) \arrow[r] \arrow[d, "v"] & H^i\Big(F'_m(F''_\ell C^\bullet/F''_kC^\bullet)\Big) \arrow[r] \arrow[d, "w"] & 0 \\
0 \arrow[r] & H^i(F''_kC^\bullet) \arrow[r] & H^i(F''_\ell C^\bullet) \arrow[r] & H^i\left(F''_\ell C^\bullet/F''_kC^\bullet\right) \arrow[r] & 0
\end{tikzcd}$$

the rows are exact. Furthermore, since $(C^\bullet, F', F'')$ is a strict complex, it follows that u and v are injective and the left square is Cartesian (this follows by describing all the objects that appear in that square as subobjects of $H^i(C^\bullet)$). This implies that w is injective, hence $(F''_\ell C^\bullet/F''_kC^\bullet, F')$ is a strict complex. $\square$

10.3. Bi-filtered $\mathcal{D}_X$-modules

In the remaining part of this chapter, we will work with right $\mathcal{D}_X$-modules, since we are mainly interested in the pushforward theorem. Accordingly, we will consider increasing V-filtrations.

We consider the setting of Section 9.2 with a smooth hypersurface $H \subset X$ locally defined by a coordinate t. The ring $\mathcal{D}_X$ is then equipped with the F-filtration, and the V-filtration corresponding to H. We consider correspondingly $\mathcal{D}_X$-modules $\mathcal{M}$ equipped with an F-filtration and a V-filtration. In view of future use, we consider V-filtrations indexed by $A + \mathbb{Z}$ for some finite subset $A \subset (-1, 0]$. We extend in a trivial way the filtration $V_\bullet \mathcal{D}_X$ indexed by $\mathbb{Z}$ to a filtration indexed by $A+\mathbb{Z}$, that is, we set for any $\alpha \in A + \mathbb{Z}$,

$$V_\alpha \mathcal{D}_X = V_{[\alpha]} \mathcal{D}_X.$$

In this section, we mainly consider the interaction of both filtrations, without consideration of coherence or $\mathbb{R}$-specializability.

For a $\mathcal{D}_X$-module $\mathcal{M}$ equipped with filtrations $F_\bullet \mathcal{M}$ ($p \in \mathbb{Z}$) and $V_\bullet \mathcal{M}$ ($\alpha \in A+\mathbb{Z}$), we set $F_p V_\alpha \mathcal{M} := F_p \mathcal{M} \cap V_\alpha \mathcal{M}$ as in (10.2.3), and $F_p \mathrm{gr}_\alpha^V \mathcal{M} := F_p V_\alpha \mathcal{M} / F_p V_{<\alpha} \mathcal{M}$. We notice that $(\mathcal{D}_X, F_\bullet, V_\bullet)$ satisfies the following properties:

(a) Multiplication by t induces an isomorphism $F_p V_\alpha \mathcal{D}_X \simeq F_p V_{\alpha-1} \mathcal{D}_X$ whenever $\alpha \leqslant 0$.

(b) Multiplication by ∂_t induces an isomorphism $F_p \mathrm{gr}_\alpha^V \mathcal{D}_X \simeq F_{p+1} \mathrm{gr}_{\alpha+1}^V \mathcal{D}_X$ whenever $\alpha > -1$.

These will be the basic relations we impose to bi-filtered $\mathcal{D}_X$-modules. We consider then the category $\mathsf{FV}(\mathcal{D}_X)$ consisting of triples $(\mathcal{M}, F, V)$, where $\mathcal{M}$ is a right $\mathcal{D}_X$-module, F is a (usual) filtration and V is a V-filtration indexed by $A+\mathbb{Z}$ on $\mathcal{M}$, for some finite set $A \in (-1, 0]$, such that the following conditions are satisfied (no coherence assumption is made here):

(i) the F and V-filtrations $F_\bullet \mathcal{M}$ and $V_\bullet \mathcal{M}$ are exhaustive,

(ii) $F_p \mathcal{M} = 0$ for $p \ll 0$,

(iii) multiplication by t induces an isomorphism $F_p V_\alpha \mathcal{M} \simeq F_p V_{\alpha-1} \mathcal{M}$ whenever $\alpha < 0$,

(iv) multiplication by ∂_t induces an isomorphism $F_p \mathrm{gr}_\alpha^V \mathcal{M} \simeq F_{p+1} \mathrm{gr}_{\alpha+1}^V \mathcal{M}$ whenever $\alpha > -1$.

The morphisms in $\mathsf{FV}(\mathcal{D}_X)$ are morphisms of right $\mathcal{D}_X$-modules that are compatible with both filtrations. We usually refer to the objects of $\mathsf{FV}(\mathcal{D}_X)$ simply as *bi-filtered $\mathcal{D}_X$-modules*. We note that Condition (iii) implies in particular that, for all $\alpha < 0$, $V_\alpha \mathcal{M}$ has no t-torsion and $V_\alpha \mathcal{M} \cdot t = V_{\alpha-1} \mathcal{M}$ (note that we do not include $\alpha = 0$ for arbitrary bi-filtered $\mathcal{D}_X$-modules). However, we do not assume that $(V_\alpha \mathcal{M})_{\alpha \in \mathbb{R}}$ is a coherent V-filtration with respect to H (more precisely, we do not require any coherence condition or the fact that $t\partial_t - \alpha$ is nilpotent on $\mathrm{gr}_\alpha^V \mathcal{M}$).

10.3.1. Remark. It is not true that a morphism φ in $\mathsf{FV}(\mathcal{D}_X)$ has kernels and cokernels (it is not necessarily true that the induced filtrations on the $\mathcal{D}_X$-modules kernels or cokernels satisfy conditions (iii) and (iv) above). However, this is the case if φ is bi-strict: indeed, since taking F_pV_α and $F_p\mathrm{gr}_\alpha^V$ both commute with taking Ker and Coker, (see Exercise 10.6(3)), the isomorphism condition on t and ∂_t is preserved. If φ is bi-strict, we have an isomorphism $\mathrm{Coim}(\varphi)\simeq\mathrm{Im}(\varphi)$ in $\mathsf{FV}(\mathcal{D}_X)$.

Let $\mathcal{L}$ be an $\mathcal{O}_X$-module. It defines an induced $\mathcal{D}_X$-module $\mathcal{L}\otimes_{\mathcal{O}_X}\mathcal{D}_X$ (see Section 8.5), equipped with filtrations $\mathcal{L}\otimes_{\mathcal{O}_X}(\mathcal{D}_X,F_\bullet)$ and $\mathcal{L}\otimes_{\mathcal{O}_X}(\mathcal{D}_X,,V_\bullet)$. By shifting the filtrations on $\mathcal{D}_X$, we obtain for each $p\in\mathbb{Z}$ and $\alpha\in A+\mathbb{Z}$ an induced $\mathcal{D}_X$-module with filtrations $\mathcal{L}\otimes_{\mathcal{O}_X}(\mathcal{D}_X,F[p]_\bullet)$ and $\mathcal{L}\otimes_{\mathcal{O}_X}(\mathcal{D}_X,V[\alpha]_\bullet)$:

$$\begin{aligned}F_q(\mathcal{L}\otimes\mathcal{D}_X)&=\mathrm{Im}(\mathcal{L}\otimes F_{q-p}\mathcal{D}_X\hookrightarrow\mathcal{L}\otimes\mathcal{D}_X)\\ V_\beta(\mathcal{L}\otimes\mathcal{D}_X)&=\mathrm{Im}(\mathcal{L}\otimes V_{\beta-\alpha}\mathcal{D}_X\hookrightarrow\mathcal{L}\otimes\mathcal{D}_X).\end{aligned}$$

In order to also obtain "induced properties" for F_qV_β, we are led to the following definition.

10.3.2. Definition **(Induced bi-filtered $\mathcal{D}_X$-modules).**

(1) An *elementary induced bi-filtered $\mathcal{D}_X$-module* is a bi-filtered $\mathcal{D}_X$-module of the form

$$\mathcal{L}\otimes_{\mathcal{O}_X}(\mathcal{D}_X,F[p]_\bullet,V[\alpha]_\bullet),\quad p\in\mathbb{Z},\ \alpha\in[-1,0],$$

such that

(a) if $\alpha\in[-1,0)$, $\mathcal{L}$ *has no t-torsion*,

(b) if $\alpha=0$, $\mathcal{L}$ has *t-torsion of order at most one*, that is,

$$\{u\in\mathcal{L}\mid ut^j=0\text{ for some }j\geqslant 1\}=\{u\in\mathcal{L}\mid ut=0\}.$$

(2) An *induced bi-filtered $\mathcal{D}_X$-module* is an object of $\mathsf{FV}(\mathcal{D}_X)$ that is isomorphic to a direct sum of elementary induced bi-filtered $\mathcal{D}_X$-modules

$$\bigoplus_i\big(\mathcal{L}_i\otimes_{\mathcal{O}_X}(\mathcal{D}_X,F[p_i],V[\alpha_i])\big),\quad p_i\in\mathbb{Z},\ \alpha_i\in[-1,0].$$

The full subcategory of $\mathsf{FV}(\mathcal{D}_X)$ consisting of induced objects is denoted $\mathsf{FV}_i(\mathcal{D}_X)$.

We nevertheless need to justify that elementary induced bi-filtered $\mathcal{D}_X$-modules as defined above belong to $\mathsf{FV}(\mathcal{D}_X)$, that is, satisfy Properties (i)–(iv) above. In order to do so, it is convenient to treat separately the case when $\mathcal{L}$ has no t-torsion and when $\mathcal{L}t=0$, the general case following using the existence of an exact sequence

$$0\longrightarrow\mathcal{L}'\longrightarrow\mathcal{L}\longrightarrow\mathcal{L}''\longrightarrow 0,$$

with $\mathcal{L}'t=0$ and $\mathcal{L}''$ without t-torsion.

It is useful to note that since $\mathcal{L}'t=0$, we have locally

$$\begin{aligned}\mathcal{L}'\otimes\mathcal{D}_X&=\mathcal{L}'\otimes_{\mathcal{O}_H}\mathcal{D}_H[\partial_t],\\ F_q(\mathcal{L}'\otimes\mathcal{D}_X)&=\oplus_{j\geqslant0}(\mathcal{L}'\otimes_{\mathcal{O}_H}(F_{q-p-j}\mathcal{D}_H)\partial_t^j)\\ V_\beta(\mathcal{L}'\otimes\mathcal{D}_X)&=\oplus_{j=0}^{\lfloor\beta\rfloor}(\mathcal{L}'\otimes_{\mathcal{O}_H}\mathcal{D}_H\partial_t^j).\end{aligned}\tag{10.3.3}$$

10.3.4. Lemma. *With the above notation, for every q and β, we have*

(i) $F_qV_\beta(\mathcal{L}\otimes\mathcal{D}_X)=\mathrm{Im}(\mathcal{L}\otimes F_{q-p}V_{\beta-\alpha}\mathcal{D}_X\to\mathcal{L}\otimes\mathcal{D}_X)$.

(ii) *There is an exact sequence*

$$0 \longrightarrow F_qV_\beta(\mathcal{L}' \otimes \mathcal{D}_X) \longrightarrow F_qV_\beta(\mathcal{L} \otimes \mathcal{D}_X) \longrightarrow F_qV_\beta(\mathcal{L}'' \otimes \mathcal{D}_X) \longrightarrow 0.$$

Furthermore, we have $\mathcal{L} \otimes (\mathcal{D}_X, F[p], V[\alpha]) \in \mathsf{FV}(\mathcal{D}_X)$.

Proof. The assertion in (i) follows easily when $\mathcal{L}$ has no t-torsion, using the fact that the following maps are injective:

$$\mathcal{L} \otimes V_{\beta-\alpha}\mathcal{D}_X \longrightarrow \mathcal{L} \otimes \mathcal{D}_X, \quad \mathcal{L} \otimes F_{q-p}V_{\beta-\alpha}\mathcal{D}_X \longrightarrow \mathcal{L} \otimes \mathcal{D}_X,$$

and
$$\mathcal{L} \otimes (V_{\beta-\alpha}\mathcal{D}_X / F_{q-p}V_{\beta-\alpha}\mathcal{D}_X) \longrightarrow \mathcal{L} \otimes \mathcal{D}_X / F_{q-p}\mathcal{D}_X.$$

When $\mathcal{L}t = 0$, we deduce (i) from the explicit description in (10.3.3).

We now note that we have exact sequences

$$0 \longrightarrow F_q(\mathcal{L}' \otimes \mathcal{D}_X) \longrightarrow F_q(\mathcal{L} \otimes \mathcal{D}_X) \longrightarrow F_q(\mathcal{L}'' \otimes \mathcal{D}_X) \longrightarrow 0$$

and
$$0 \longrightarrow V_\beta(\mathcal{L}' \otimes \mathcal{D}_X) \longrightarrow V_\beta(\mathcal{L} \otimes \mathcal{D}_X) \longrightarrow V_\beta(\mathcal{L}'' \otimes \mathcal{D}_X) \longrightarrow 0$$

(exactness follows from definition and the fact that the maps

$$\mathcal{L}'' \otimes F_{q-p}\mathcal{D}_X \longrightarrow \mathcal{L}'' \otimes \mathcal{D}_X \quad \text{and} \quad \mathcal{L}'' \otimes V_{\beta-\alpha}\mathcal{D}_X \longrightarrow \mathcal{L}'' \otimes \mathcal{D}_X$$

are injective. Let

$$M = \mathrm{Im}(\mathcal{L} \otimes F_{q-p}V_{\beta-\alpha}\mathcal{D}_X \longrightarrow \mathcal{L} \otimes \mathcal{D}_X)$$

and we similarly define M' and M''. We deduce that we have a commutative diagram with exact rows and injective vertical maps

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] \arrow[d, "j'"] & M \arrow[r] \arrow[d, "j"] & M'' \arrow[r] \arrow[d, "j''"] & 0 \\
0 \arrow[r] & F_qV_\beta(\mathcal{L}' \otimes \mathcal{D}_X) \arrow[r] & F_qV_\beta(\mathcal{L} \otimes \mathcal{D}_X) \arrow[r] & F_qV_\beta(\mathcal{L}'' \otimes \mathcal{D}_X) &
\end{tikzcd}$$

(for the exactness of the top row we use the fact that the map

$$\mathcal{L}'' \otimes F_{q-p}V_{\beta-\alpha}\mathcal{D}_X \longrightarrow \mathcal{L}'' \otimes \mathcal{D}_X$$

is injective; the exactness of the bottom row follows from the above two exact sequences). Since we know that j' and j'' are surjective, it follows that j is also surjective. This completes the proof of both (i) and (ii). The last assertion in the lemma is easy to check for $\mathcal{L}'$ and $\mathcal{L}''$, and we deduce it also for $\mathcal{L}$ using (ii). $\square$

10.3.5. Remark. Given $(\mathcal{M}, F, V) \in \mathsf{FV}(\mathcal{D}_X)$, note that for every $\alpha \in [-1, 0]$ and every $p \in \mathbb{Z}$, we obtain an elementary induced bi-filtered $\mathcal{D}_X$-module as

$$F_pV_\alpha\mathcal{M} \otimes (\mathcal{D}_X, F[p], V[\alpha]).$$

Indeed, we know by Condition (iii) that $F_pV_\alpha\mathcal{M}$ has no t-torsion when $\alpha < 0$. Furthermore, if $u \in F_pV_0\mathcal{M}$ is such that $t^ju = 0$ for some $j \geqslant 2$, then $tu \in F_pV_{-1}\mathcal{M}$ and $t(tu) = 0$, hence $tu = 0$. We have a strict surjective morphism

$$\bigoplus_{\substack{p\in\mathbb{Z}\\ \alpha\in[0,1]}} F_pV_\alpha\mathcal{M} \otimes (\mathcal{D}_X, F[p], V[\alpha]) \longrightarrow (\mathcal{M}, F, V)$$

(in this case strictness simply means that the filtrations on the target are induced by the ones on the source, see Exercise 10.6(1)). Indeed, the surjectivity is a consequence of Conditions (iii) and (iv) in the definition of the category $\mathsf{FV}(\mathcal{D}_X)$. By applying the same argument to the kernel, with the induced filtrations (note that this lies in $\mathsf{FV}(\mathcal{D}_X)$), we obtain a (possibly infinite) resolution of $(\mathcal{M}, F, V)$ by induced objects.

We consider the category of complexes $\mathsf{C}^*(\mathsf{FV}(\mathcal{D}_X))$, where $*$ stands for $+$, $-$, b, or for the empty set. We assume that all complexes $C^\bullet$ in this category satisfy the following assumptions:

(i) For $p \ll 0$, we have $F_pC^\bullet = 0$.
(ii) There exists a finite set $A \subset [-1, 0)$ suitable for each term of $C^\bullet$.

We have a corresponding homotopic category $\mathsf{K}^*(\mathsf{FV}(\mathcal{D}_X))$. A morphism $C_1^\bullet \to C_2^\bullet$ in $\mathsf{K}(\mathsf{FV}(\mathcal{D}_X))$ is a *filtered quasi-isomorphism* if $\mathcal{H}^i(F_pV_\alpha C_1^\bullet) \to \mathcal{H}^i(F_pV_\alpha C_2^\bullet)$ is an isomorphism for all $p \in \mathbb{Z}$ and $\alpha \in \mathbb{R}$. Note that since we work with exhaustive filtrations, every filtered quasi-isomorphism is, in particular, a quasi-isomorphism.

We obtain the filtered derived categories $\mathsf{D}^*(\mathsf{FV}(\mathcal{D}_X))$ by localizing $\mathsf{K}^*(\mathsf{FV}(\mathcal{D}_X))$ at the class of filtered quasi-isomorphisms. As in the case of the derived category of an abelian category, one shows that each $\mathsf{D}^*(\mathsf{FV}(\mathcal{D}_X))$ is a triangulated category. It follows from the universal property of the localization that we get exact functors

$$H^iF_pV_\alpha(-)\colon \mathsf{D}^*(\mathsf{FV}(\mathcal{D}_X)) \longrightarrow \mathsf{D}^*(\mathcal{O}_X),$$

where $\mathsf{D}^*(\mathcal{O}_X)$ is the derived category of $\mathcal{O}_X$-modules, with the suitable boundedness condition.

10.3.6. Remark. Let us assume that X is a product $X \simeq H \times \Delta_t$. Note that for every $\alpha \in \mathbb{R}$, taking $(C^\bullet, F, V)$ to $(\mathrm{gr}_\alpha^V(C^\bullet), F)$ defines an exact functor $\mathsf{D}^*(\mathsf{FV}(\mathcal{D}_X)) \to \mathsf{D}^*(\mathsf{F}(\mathcal{D}_H))$, where $\mathsf{D}^*(\mathsf{F}(\mathcal{D}_H))$ is the filtered derived category of filtered $\mathcal{D}_H$-modules (with suitable boundedness conditions).

Let $\mathsf{K}^*(\mathsf{FV}_i(\mathcal{D}_X))$ be the homotopic category of complexes of induced objects in $\mathsf{FV}(\mathcal{D}_X)$, with suitable boundedness conditions. By localizing this with respect to filtered quasi-isomorphisms, we get $\mathsf{D}^*(\mathsf{FV}_i(\mathcal{D}_X))$.

10.3.7. Lemma. *The exact functor*

$$\mathsf{D}^-(\mathsf{FV}_i(\mathcal{D}_X)) \longrightarrow \mathsf{D}^-(\mathsf{FV}(\mathcal{D}_X))$$

induced by inclusion is an equivalence of categories.

Proof. For every $(\mathcal{M}, F, V) \in \mathsf{FV}(\mathcal{D}_X)$, we construct the resolution $\mathcal{I}^\bullet(\mathcal{M}, F, V)$ by induced bi-filtered $\mathcal{D}_X$-modules as in Remark 10.3.5. It is clear that this is functorial and we extend the construction to a functor $\mathsf{C}^-(\mathsf{FV}(\mathcal{D}_X)) \to \mathsf{C}^-(\mathsf{FV}_i(\mathcal{D}_X))$, by mapping a complex $(\mathcal{M}^\bullet, F, V)$ to the total complex of the double complex $\mathcal{I}^\bullet(\mathcal{M}^\bullet, F, V)$. It is standard to check that this induces a functor between the corresponding filtered derived categories and that this gives an inverse for the functor induced by the inclusion. $\square$

10.3.8. Remark. If $(\mathcal{M}, F, V)$ is a bi-filtered $\mathcal{D}_X$-module, we can choose a finite subset $A \subset [-1, 0]$ such that $\mathrm{gr}^V_\alpha(\mathcal{M}) = 0$ for all $\alpha \in [-1, 0] \smallsetminus A$. As in Remark 10.3.5, we obtain a strict surjective morphism

$$\bigoplus_{\substack{p\in\mathbb{Z}\\ \alpha\in A}} F_p V_\alpha \mathcal{M} \otimes (\mathcal{D}_X, F[p], V[\alpha]) \longrightarrow (\mathcal{M}, F, V),$$

and by iterating this construction, we obtain a resolution $(\mathcal{J}^\bullet, F, V)$ of $(\mathcal{M}, F, V)$ by induced objects such that each $(\mathcal{J}^i, F, V)$ is the direct sum of elementary bi-filtered modules $\mathcal{L} \otimes \mathcal{O}_X (\mathcal{D}_X, F[p_i], V[\alpha_i])$, with the α_i varying over a finite set. In particular, since for every q and β we have

$$F_q V_\beta (\mathcal{L} \otimes \mathcal{O}_X (\mathcal{D}_X, F[p_i], V[\alpha_i])) = 0 \text{ unless } p_i \leqslant q,$$

we conclude that if $F_p V_\alpha \mathcal{M}$ is a coherent $\mathcal{O}_X$-module for every p and α, then $F_p V_\alpha \mathcal{J}^j$ is a coherent $\mathcal{O}_X$-module for every p, α, and j.

10.3.9. Lemma. *Consider two elementary induced bi-filtered $\mathcal{D}_X$-modules*

$$(\mathcal{M}_i, F, V) = \mathcal{L}_i \otimes (\mathcal{D}_X, F[p], V[\alpha]) \quad i = 1, 2,$$

and consider the exact sequences

$$0 \longrightarrow \mathcal{L}'_i \longrightarrow \mathcal{L}_i \longrightarrow \mathcal{L}''_i \longrightarrow 0,$$

where $\mathcal{L}'_i t = 0$ and $\mathcal{L}''_i$ has no t-torsion. If $u \colon \mathcal{L}_1 \to \mathcal{L}_2$ is an injective morphism such that the induced morphism $u'' \colon \mathcal{L}''_1 \to \mathcal{L}''_2$ has the property that $\mathrm{Coker}(u'')$ *has no t-torsion, then the induced morphism $\overline{u} \colon (\mathcal{M}_1, F, V) \to (\mathcal{M}_2, F, V)$ is strict and* $\mathrm{Coker}(\overline{u}) \simeq \mathrm{Coker}(u) \otimes (\mathcal{D}_X, F[p], V[\alpha])$.

Proof. We need to show that if we consider on $\mathrm{Coker}(\overline{u}) \simeq \mathrm{Coker}(u) \otimes \mathcal{D}_X$ the induced filtrations, then for every q and β, the sequence

$$0 \longrightarrow F_q V_\beta (\mathcal{L}_1 \otimes \mathcal{D}_X) \longrightarrow F_q V_\beta (\mathcal{L}_2 \otimes \mathcal{D}_X) \longrightarrow F_q V_\beta (\mathrm{Coker}(u) \otimes \mathcal{D}_X) \longrightarrow 0$$

is exact. This is easy to check when both $\mathcal{L}_i$ have no t-torsion and it follows from the explicit description in (10.3.3) when $\mathcal{L}_i t = 0$ for $i = 1, 2$.

We now consider the general case. Let $u' \colon \mathcal{L}'_1 \to \mathcal{L}'_2$ be the morphism induced by u. Note first that the Snake lemma gives an exact sequence

$$0 \longrightarrow \mathrm{Coker}(u') \longrightarrow \mathrm{Coker}(u) \longrightarrow \mathrm{Coker}(u'') \longrightarrow 0.$$

(since $\mathrm{Ker}(u'')$ has no t-torsion, it has to be zero). This exact sequence is the canonical one associated to $\mathrm{Coker}(u)$ such that the first term is annihilated by t and the third one has no t-torsion.

Consider the commutative diagram

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & F_qV_\beta(\mathcal{L}_1'\otimes\mathcal{D}_X) \arrow[r] \arrow[d] & F_qV_\beta(\mathcal{L}_1\otimes\mathcal{D}_X) \arrow[r] \arrow[d] & F_qV_\beta(\mathcal{L}_1''\otimes\mathcal{D}_X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_qV_\beta(\mathcal{L}_2'\otimes\mathcal{D}_X) \arrow[r] \arrow[d] & F_qV_\beta(\mathcal{L}_2\otimes\mathcal{D}_X) \arrow[r] \arrow[d] & F_qV_\beta(\mathcal{L}_2''\otimes\mathcal{D}_X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_qV_\beta(\mathrm{Coker}(u')\otimes\mathcal{D}_X) \arrow[r] \arrow[d] & F_qV_\beta(\mathrm{Coker}(u)\otimes\mathcal{D}_X) \arrow[r] \arrow[d] & F_qV_\beta(\mathrm{Coker}(u'')\otimes\mathcal{D}_X) \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0. &
\end{tikzcd}$$

The first and the third columns are exact by what we have already discussed. Moreover, the rows are all exact by Lemma 10.3.4. Therefore the middle column is also exact, which is what we had to prove. □

In order to define functors between filtered derived categories, it will be convenient to use the Godement resolution (see Definition 8.7.14), that we now extend to our bi-filtered setting.

For $(\mathcal{M},F,V)\in \mathrm{FV}(\mathcal{D}_X)$, we define $\mathcal{C}^0(\mathcal{M},F,V)$ to be the bi-filtered $\mathcal{D}_X$-module $\mathcal{N}=\bigcup_{p,\alpha}\mathcal{C}^0(F_pV_\alpha\mathcal{M})\subseteq\mathcal{C}^0(\mathcal{M})$, with the filtrations given by $F_p\mathcal{N}=\bigcup_\alpha\mathcal{C}^0(F_pV_\alpha\mathcal{M})$ and $V_\alpha\mathcal{N}=\bigcup_p\mathcal{C}^0(F_pV_\alpha\mathcal{M})$ for $p\in\mathbb{Z}$, $\alpha\in\mathbb{R}$. One checks that

$$\mathcal{C}^0(F_pV_\alpha\mathcal{M})\cap\mathcal{C}^0(F_qV_\beta\mathcal{M})=\mathcal{C}^0(F_{\min(p,q)}V_{\min(\alpha,\beta)}\mathcal{M}).$$

It follows that $F_pV_\alpha\mathcal{N}=\mathcal{C}^0(F_pV_\alpha\mathcal{M})$, hence each $F_pV_\alpha\mathcal{N}$ is flabby. We have a natural strict monomorphism $(\mathcal{M},F,V)\hookrightarrow\mathcal{C}^0(\mathcal{M},F,V)$, whose cokernel is also a bi-filtered $\mathcal{D}_X$-module, and we can proceed inductively as in Definition 8.7.14 to define the complex $\mathrm{God}^\bullet(\mathcal{M},F,V)$ in $\mathsf{C}^+(\mathsf{FV}(\mathcal{D}_X))$ that is filtered quasi-isomorphic to $(\mathcal{M},F,V)$.

10.3.10. Lemma. *Given an elementary induced bi-filtered $\mathcal{D}_X$-module*

$$(\mathcal{M},F,V)\simeq\mathcal{L}\otimes_{\mathcal{O}_X}(\mathcal{D}_X,F[p],V[\alpha]),$$

we have

$$\mathcal{C}^0(\mathcal{M},F,V)\simeq\mathcal{C}^0(\mathcal{L})\otimes_{\mathcal{O}_X}(\mathcal{D}_X,F[p],V[\alpha]).$$

Proof. If we consider the exact sequence

$$0\longrightarrow\mathcal{L}'\longrightarrow\mathcal{L}\longrightarrow\mathcal{L}''\longrightarrow 0,$$

where $\mathcal{L}'t=0$ and $\mathcal{L}''$ has no t-torsion, then we have an induced exact sequence

$$0\longrightarrow\mathcal{C}^0(\mathcal{L}')\longrightarrow\mathcal{C}^0(\mathcal{L})\longrightarrow\mathcal{C}^0(\mathcal{L}'')\longrightarrow 0$$

and $\mathcal{C}^0(\mathcal{L}')t=0$, while $\mathcal{C}^0(\mathcal{L}'')$ has no t-torsion. In particular, we see that every t-torsion element in $\mathcal{C}^0(\mathcal{L})$ is annihilated by t. We also deduce from this that it is enough to prove the lemma when either $\mathcal{L}$ has no t-torsion or when $\mathcal{L}t=0$.

Suppose first that $\mathcal{L}$ has no t-torsion. In this case we have

$$F_qV_\beta(\mathcal{C}^0(\mathcal{L})\otimes\mathcal{D}_X)=\mathcal{C}^0(\mathcal{L})\otimes F_{q-p}V_{\beta-\alpha}\mathcal{D}_X\simeq\mathcal{C}^0(\mathcal{L}\otimes F_{q-p}V_{\beta-\alpha}\mathcal{D}_X),$$

since $F_{q-p}V_{\beta-\alpha}\mathcal{D}_X$ is a locally free $\mathcal{O}_X$-module, of finite type (see Exercise 8.49(2)). This implies the isomorphism in the lemma. The case when $\mathcal{L}t=0$ follows similarly, using the explicit description in (10.3.3). $\square$

10.3.11. Corollary. *If $(\mathcal{M},F,V)\in\mathsf{FV}(\mathcal{D}_X)$ is induced, then its filtered resolution*

$$0\longrightarrow(\mathcal{M},F,V)\longrightarrow\mathcal{C}^0(\mathcal{M},F,V)\longrightarrow\mathcal{C}^1(\mathcal{M},F,V)\longrightarrow\cdots$$

consists of induced objects and the morphisms are strict and they correspond to morphisms of $\mathcal{O}_X$-modules.

Proof. This follows by combining Lemmas 10.3.9 and 10.3.10. The only thing to note is that if we have a short exact sequence of $\mathcal{O}_X$-modules

$$0\longrightarrow\mathcal{L}'\longrightarrow\mathcal{L}\longrightarrow\mathcal{L}''\longrightarrow0,$$

with $\mathcal{L}'t=0$ and $\mathcal{L}''$ without t-torsion, then $\operatorname{Coker}(\mathcal{L}''\to\mathcal{C}^0(\mathcal{L}''))$ has no t-torsion. $\square$

10.4. The direct image of bi-filtered $\mathcal{D}_X$-modules

Let $f\colon X\to X'$ be a morphism between complex manifolds. We assume that $X'=H'\times\Delta_t$ and $X=H\times\Delta_t$ such that $f=f|_H\times\mathrm{Id}_t$. We set $X_0=H\times\{0\}$ and $X'_0=H'\times 0$. Our first goal is to define a functor ${}_{\mathrm{D}}f_*\colon\mathsf{D}^-(\mathsf{FV}(\mathcal{D}_X))\to\mathsf{D}^-(\mathsf{FV}(\mathcal{D}_{X'}))$.

In addition to the sheaf $\mathcal{D}_X$, we also have on X the sheaf $f^{-1}(\mathcal{D}_{X'})$. This carries the F-filtration and the V-filtration induced from $\mathcal{D}_{X'}$ (the V-filtration being the one with respect to X'_0). In particular, we may consider the categories $\mathsf{FV}(f^{-1}(\mathcal{D}_{X'}))$ and $\mathsf{FV}_i(f^{-1}(\mathcal{D}_{X'}))$. For example, an object in $\mathsf{FV}_i(f^{-1}(\mathcal{D}_{X'}))$ is one that is isomorphic to a direct sum of objects of the form $\mathcal{L}\otimes_{f^{-1}(\mathcal{O}_{X'})}(f^{-1}(\mathcal{D}_{X'}),F[p],V[\alpha])$, where $\mathcal{L}$ is an $f^{-1}(\mathcal{O}_{X'})$-module that has no t-torsion, unless $\alpha=0$, in which case the all local sections of $\mathcal{L}$ that are annihilated by some power of t are actually annihilated by t. The same construction from before (see Lemma 10.3.7) shows that the inclusion functor determines an equivalence of categories

$$\mathsf{D}^-(\mathsf{FV}_i(f^{-1}(\mathcal{D}_{X'})))\longrightarrow\mathsf{D}^-(\mathsf{FV}(f^{-1}(\mathcal{D}_{X'}))).$$

As in the case of the direct image of non-filtered $\mathcal{D}_X$-modules, the key player in the definition of the direct image for bi-filtered $\mathcal{D}_X$-modules is

$$\mathcal{D}_{X\to X'}:=\mathcal{O}_X\otimes_{f^{-1}(\mathcal{O}_{X'})}f^{-1}(\mathcal{D}_{X'}).$$

This has a structure of left $\mathcal{D}_X$-module and right $f^{-1}(\mathcal{D}_{X'})$-bimodule and carries an F-filtration and a V-filtration induced from $f^{-1}(\mathcal{D}_{X'})$. These are compatible not only with the F and V-filtrations on $f^{-1}(\mathcal{D}_{X'})$, via right multiplication, but also with the F and V-filtrations on $\mathcal{D}_X$, via left multiplication.

10.4.1. Example. The two main examples are when f is smooth and when f is a closed immersion. The typical case for f being smooth is when $f\colon X = X' \times W \to X'$ is the projection onto the first factor. In this case we have a surjection $\mathcal{D}_{X'\times W} \to \mathcal{D}_{X'\times W\to X'}$ such that in local coordinates $w_1, \dots, w_r$ on W, we get an isomorphism

$$\mathcal{D}_{X'\times W\to X'} \simeq \mathcal{D}_{X'\times W}/\mathcal{D}_{X'\times W}\cdot(\partial_{w_1}, \dots, \partial_{w_r}).$$

On the other hand, the typical case when f is a closed immersion is when $f\colon X \hookrightarrow X' = X \times Z$ is given by $f(x) = (x, z_0)$. If we have coordinates $z_1, \dots, z_r$ on Z, then

$$\mathcal{D}_{X\to X\times Z} \simeq \mathcal{D}_X \otimes \mathbb{C}[\partial_{z_1}, \dots, \partial_{z_r}].$$

We first claim that

$$\tag{10.4.2} {}^{\mathrm{p}}\mathrm{DR}_{X/X'}(\mathcal{M}, F, V) = (\mathcal{M}, F, V) \otimes_{\mathcal{D}_X} (\mathcal{D}_{X\to X'}, F, V),$$

with the tensor product of the filtrations from the two factors, defines a functor ${}^{\mathrm{p}}\mathrm{DR}_{X/X'}\colon \mathsf{FV}_i(\mathcal{D}_X) \to \mathsf{FV}_i(f^{-1}(\mathcal{D}_{X'}))$. Indeed, we have

$$\mathcal{L} \otimes_{\mathcal{O}_X} (\mathcal{D}_X, F[p], V[\alpha]) \otimes_{\mathcal{D}_X} (\mathcal{D}_{X\to X'}, F, V) \simeq \mathcal{L} \otimes_{f^{-1}(\mathcal{O}_{X'})} (f^{-1}(\mathcal{D}_{X'}), F[p], V[\alpha]).$$

It is clear on such a formula that this functor is compatible with grading with respect to $V_\bullet$, as defined in Remark 10.3.6.

10.4.3. Lemma. *The functor ${}^{\mathrm{p}}\mathrm{DR}_{X/X'}$ maps a filtered quasi-isomorphism in the category $\mathsf{K}(\mathsf{FV}_i(\mathcal{D}_X))$ to a filtered quasi-isomorphism.*

Proof. We need to prove that if $(C^\bullet, F, V)$ is a complex of bi-filtered $\mathcal{D}_X$-modules such that all complexes $F_pV_\alpha C^\bullet$ are exact, then $F_pV_\alpha\, {}^{\mathrm{p}}\mathrm{DR}_{X/X'}(C^\bullet)$ is exact for all $p \in \mathbb{Z}$ and $\alpha \in \mathbb{R}$. By factoring f as $X \xrightarrow{j} X \times X' \xrightarrow{p} X'$, where p is the second projection and j is the graph of f, we reduce the proof for f to proving the assertion separately for j and p (note that $\mathcal{D}_{X\to X'} \simeq \mathcal{D}_{X\to X\times X'} \otimes_{j^{-1}(\mathcal{D}_{X\times X'})} j^{-1}(\mathcal{D}_{X\times X'\to X'})$).

The assertion for j is trivial since we may assume that we have coordinates $y_1, \dots, y_r$ on X', so that ${}^{\mathrm{p}}\mathrm{DR}_{X/X\times X'}$ can be identified with $\mathbb{C}[\partial_{y_1}, \dots, \partial_{y_r}] \otimes_{\mathbb{C}} (-)$. Let us prove now the assertion for the projection $p\colon X \times X' \to X'$. For every object $(\mathcal{M}, F, V) \in \mathsf{FV}(\mathcal{D}_{X\times X'})$, consider the complex ${}^{\mathrm{p}}\mathrm{DR}_X(\mathcal{M}, F, V)$ consisting of $\mathcal{M} \otimes_{\mathcal{D}_{X\times X'}} \wedge^{-\bullet}\Theta_X$ (given local coordinates $x_1, \dots, x_n$ on X, this complex can be identified to the Koszul-type complex corresponding to $\partial_{x_1}, \dots, \partial_{x_r}$). The filtrations are defined by

$$F_p(\mathcal{M} \otimes \wedge^{-i}\Theta_X) = F_{p+i}\mathcal{M} \otimes \wedge^{-i}\Theta_X, \quad V_\alpha(\mathcal{M} \otimes \wedge^{-i}\Theta_X) = V_\alpha\mathcal{M} \otimes \wedge^{-i}\Theta_X.$$

Note that the morphism $\mathcal{D}_{X\times X'} \otimes_{\mathcal{O}_X} \wedge^{-\bullet}\Theta_X \to \mathcal{D}_{X\to X'}$ induces a morphism

$${}^{\mathrm{p}}\mathrm{DR}_X(\mathcal{M}, F, V) \longrightarrow (\mathcal{M}, F, V) \otimes_{\mathcal{D}_X} (\mathcal{D}_{X\to X'}, F, V)$$

for every bi-filtered $\mathcal{D}_X$-module $(\mathcal{M}, F, V)$. This is a filtered quasi-isomorphism if $(\mathcal{M}, F, V) \simeq \mathcal{L} \otimes (\mathcal{D}_{X\times X'}, F[p], V[\alpha])$, hence for all induced bi-filtered $\mathcal{D}_{X\times X'}$-modules. Indeed, it is enough to check the assertion when either $\mathcal{L}$ has no t-torsion, or when $\mathcal{L}t = 0$; in each case, the verification is straightforward.

On the other hand, it is clear that if all $F_pV_\alpha C^\bullet$ are exact, then also all complexes $F_pV_\alpha(C^\bullet\otimes\wedge^{-i}\Theta_W)$ are exact, hence each $F_pV_\alpha\,{}^{\mathrm{p}}\mathrm{DR}_{X/X'}(C^\bullet)$ is exact by the above discussion. This completes the proof of the lemma. $\square$

As a consequence of the lemma, the functor ${}^{\mathrm{p}}\mathrm{DR}_{X/X'}$ we have defined in (10.4.2) induces an exact functor ${}^{\mathrm{p}}\mathrm{DR}_{X/X'}\colon \mathsf{D}^*(\mathsf{FV}_i(\mathcal{D}_X))\to\mathsf{D}^*(\mathsf{FV}_i(f^{-1}(\mathcal{D}_{X'})))$, where $*$ stands for $+$, $-$, b, or the empty set, and this functor is compatible with V-grading.

We now introduce the topological direct image. We first define it at the level of bi-filtered D-modules. Suppose that $f\colon X\to X'$ is as above. If $(\mathcal{M},F,V)$ is a bi-filtered $f^{-1}(\mathcal{D}_{X'})$-module, we define $f_*(\mathcal{M},F,V)\in\mathsf{FV}(\mathcal{D}_{X'})$ to be given by $(\mathcal{N},F,V)$, where $\mathcal{N}=\bigcup_{p,\alpha}f_*(F_pV_\alpha\mathcal{M})$, with $F_p\mathcal{N}=\bigcup_\alpha f_*(F_pV_\alpha\mathcal{M})$ and $V_\alpha\mathcal{N}=\bigcup_p f_*(F_pV_\alpha\mathcal{M})$. We obtain in this way a functor $f_*\colon\mathsf{FV}(f^{-1}(\mathcal{D}_{X'}))\to\mathsf{FV}(\mathcal{D}_{X'})$. Note that if $\mathcal{L}$ is an $f^{-1}(\mathcal{O}_{X'})$-module, then $f_*(\mathcal{L}\otimes(f^{-1}(\mathcal{D}_{X'}),F[p],V[\alpha]))\simeq f_*(\mathcal{L})\otimes(\mathcal{D}_{X'},F[p],V[\alpha])$ by the projection formula (we use the fact that $\mathcal{D}_{X'}$ is a locally free $\mathcal{O}_{X'}$-module). Therefore we also have a functor $f_*\colon\mathsf{FV}_i(f^{-1}(\mathcal{D}_X))\to\mathsf{FV}_i(\mathcal{D}_{X'})$.

We next define a version of the topological direct image functor at the level of filtered derived categories

$$f_*\colon\mathsf{D}^*(\mathsf{FV}_i(f^{-1}(\mathcal{D}_{X'})))\longrightarrow\mathsf{D}^*(\mathsf{FV}_i(\mathcal{D}_{X'})),$$

as follows. By a variant of Corollary 10.3.11, we associate functorially to every $(\mathcal{M},F,V)\in\mathsf{FV}_i(f^{-1}(\mathcal{D}_{X'}))$ a strict complex $\mathcal{C}^\bullet(\mathcal{M},F,V)$

$$0\longrightarrow\mathcal{C}^0(\mathcal{M},F,V)\longrightarrow\mathcal{C}^1(\mathcal{M},F,V)\longrightarrow\cdots$$

that gives a filtered resolution of $(\mathcal{M},F,V)$ by induced bi-filtered modules. It is convenient to replace this by a bounded complex, hence if $\dim_{\mathbb{R}}(X)=2n$, we consider the complex

$$\widetilde{\mathcal{C}}^\bullet(\mathcal{M},F,V):\{0\to\widetilde{\mathcal{C}}^0(\mathcal{M},F,V)\to\widetilde{\mathcal{C}}^1(\mathcal{M},F,V)\to\cdots\to\widetilde{\mathcal{C}}^{2n}(\mathcal{M},F,V)\to 0\},$$

where

$$\widetilde{\mathcal{C}}^j(\mathcal{M},F,V)=\begin{cases}\mathcal{C}^j(\mathcal{M},F,V), & 0\leq j\leqslant 2n-1;\\[4pt] \mathrm{Coker}(\mathcal{C}^{2n-2}(\mathcal{M},F,V)\to\mathcal{C}^{2n-1}(\mathcal{M},F,V)), & j=2n;\\[4pt] 0, & j\geqslant 2n+1.\end{cases}$$

It follows from the construction that $\widetilde{\mathcal{C}}^\bullet(\mathcal{M},F,V)$ is a strict complex, giving a filtered resolution of $(\mathcal{M},F,V)$ by induced bi-filtered $f^{-1}(\mathcal{D}_{X'})$-modules. Moreover, since we truncated at the dimension of X, we have $R^mf_*(F_pV_\alpha\widetilde{\mathcal{C}}^j(\mathcal{M},F,V))=0$ for every $m\geqslant 1$ and every j, p, and α. Given a complex $(\mathcal{M}^\bullet,F,V)$ in $\mathsf{FV}_i(f^{-1}(\mathcal{D}_{X'}))$, we consider the total complex of the double complex $\widetilde{\mathcal{C}}^\bullet(\mathcal{M}^\bullet,F,V)$. It is now standard to see that this induces exact functors $f_*\colon\mathsf{D}^*(\mathsf{FV}_i(f^{-1}(\mathcal{D}_{X'})))\to\mathsf{D}^*(\mathsf{FV}_i(\mathcal{D}_{X'}))$, where $*$ stands for $+$, $-$, b, or for the empty set. According to Lemma 10.3.10, the above construction $\widetilde{\mathcal{C}}^\bullet$ is compatible with the V-grading functor of Remark 10.3.6 and, by the exactness of $f*$ on the terms of the complexes $\widetilde{\mathcal{C}}^\bullet$, V-grading commutes with f_* as defined by the previous construction.

By composing f_* and ${}^{\mathrm{p}}\mathrm{DR}_{X/X'}$, we obtain an exact functor

$$ {}_{\mathrm{D}}f_*\colon \mathsf{D}^*(\mathsf{FV}_i(\mathcal{D}_X)) \longrightarrow \mathsf{D}^*(\mathsf{FV}_i(\mathcal{D}_{X'})), $$

which in light of Lemma 10.3.7 also gives a functor $\mathsf{D}^-(\mathsf{FV}(\mathcal{D}_X)) \to \mathsf{D}^-(\mathsf{FV}(\mathcal{D}_{X'}))$.

It is also clear, by applying the arguments separately to f_* and ${}^{\mathrm{p}}\mathrm{DR}_{X/X'}$, that taking the direct image commutes with taking the graded pieces of the V-filtration. More precisely, if $f_0\colon X_0 \to X_0'$ is the restriction of f, then given any $(\mathcal{M}^\bullet, F, V) \in \mathsf{D}^-(\mathsf{FV}(\mathcal{D}_X))$ and any $\alpha \in \mathbb{R}$, we have an isomorphism in $\mathsf{D}^-(\mathsf{F}(\mathcal{D}_{X_0'}))$:

$$ (\mathrm{gr}^V_\alpha\, {}_{\mathrm{D}}f_*(\mathcal{M}^\bullet, F, V), F) \simeq {}_{\mathrm{D}}f_{0*}(\mathrm{gr}^V_\alpha(\mathcal{M}^\bullet, F, V), F). \tag{10.4.4} $$

We consider two conditions on an object $C^\bullet$ of $\mathsf{C}(\mathsf{FV}(\mathcal{D}_X))$:

(a) The action of $t\partial_t - \alpha$ on $H^i(\mathrm{gr}^V_\alpha C^\bullet)$ is nilpotent for all $i \in \mathbb{Z}$, $\alpha \in \mathbb{R}$,

(b) Each $H^i(F_pV_\alpha C^\bullet)$ is a coherent $\mathcal{O}_X$-module.

Let $\mathsf{C}^*_m(\mathsf{FV}(\mathcal{D}_X))$ and $\mathsf{C}^*_{\mathrm{c}}(\mathsf{FV}(\mathcal{D}_X))$ be the full subcategories of $\mathsf{C}^*(\mathsf{FV}(\mathcal{D}_X))$ consisting of those objects that satisfy condition (a), respectively (b), and we similarly define $\mathsf{D}^*_m(\mathsf{FV}(\mathcal{D}_X))$ and $\mathsf{D}^*_{\mathrm{c}}(\mathsf{FV}(\mathcal{D}_X))$ as full subcategories of $\mathsf{D}^*(\mathsf{FV}(\mathcal{D}_X))$

10.4.5. Lemma. *With the above notation, suppose also that f is proper and $(\mathcal{M}, F, V) \in \mathsf{FV}(\mathcal{D}_X)$.*

(i) *If $(\mathcal{M}, F, V) \in \mathsf{C}_{\mathrm{c}}(\mathsf{FV}(\mathcal{D}_X))$, then $f_*(\mathcal{M}, F, V) \in \mathsf{D}^-_{\mathrm{c}}(\mathsf{FV}_{\mathrm{c}}(\mathcal{D}_{X'}))$.*

(ii) *If $(\mathcal{M}, F, V) \in \mathsf{C}(\mathsf{FV}(\mathcal{D}_X))$, then $f_*(\mathcal{M}, F, V) \in \mathsf{D}^-_m(\mathsf{FV}_m(\mathcal{D}_{X'}))$.*

Proof. Let $(C^\bullet, F, V) \to (\mathcal{M}, F, V)$ be a filtered resolution by induced bi-filtered $\mathcal{D}_X$-modules constructed as in Remark 10.3.8. If $F_pV_\alpha\mathcal{M}$ is a coherent $\mathcal{O}_X$-module for every p, α, then $F_pV_\alpha C^k$ is a coherent $\mathcal{O}_X$-module for every p, α, and k. One can then deduce that all $H^k(F_pV_\alpha\, {}^{\mathrm{p}}\mathrm{DR}_{X/X'}(C^\bullet))$ are coherent $f^{-1}(\mathcal{O}_{X'})$-modules, and then that all $H^k(F_pV_\alpha f_*({}^{\mathrm{p}}\mathrm{DR}_{X/X'}(C^\bullet)))$ are coherent $\mathcal{O}_{X'}$-modules.

If the action of $(t\partial_t - \alpha)^m$ on $\mathrm{gr}^V_\alpha(\mathcal{M})$ is zero, then also its action on

$$ f_{0*}(\mathrm{gr}^V_\alpha(\mathcal{M}), F) \simeq \mathrm{gr}^V_\alpha f_*(\mathcal{M}, F, V) $$

is zero, hence the same holds for the action on $H^k(\mathrm{gr}^V_\alpha f_*(\mathcal{M}, F, V))$. $\square$

10.5. Specializability of filtered $\mathcal{D}_X$-modules

We assume that $X = H \times \Delta_t$, where Δ_t is a disc with coordinate t and we set $X_0 = H \times \{0\} \subset X$. We use the notion of a coherent V-filtration for a coherent $\mathcal{D}_X$-module $\mathcal{M}$ as defined in Section 9.3, as well as the notion of $\mathbb{R}$-specializability. Since we are dealing with $\mathcal{D}_X$-modules, the strictness property is not involved.

We now turn to $\mathbb{R}$-specializability for filtered $\mathcal{D}_X$-modules. Suppose that $(\mathcal{M}, F)$ is a coherently F-filtered $\mathcal{D}_X$-module (recall that the *coherence* condition means that the $\mathrm{gr}^F(\mathcal{D}_X)$-module $\mathrm{gr}^F\mathcal{M} := \oplus_m F_m\mathcal{M}/F_{m-1}\mathcal{M}$ is coherent, see Exercise 8.62).

10.5.1. Definition. One says that $(\mathcal{M}, F)$ is $\mathbb{R}$-*specializable along* H if $\mathcal{M}$ is $\mathbb{R}$-specializable along H with V-filtration denoted by $V_\bullet\mathcal{M}$ and if $(\mathcal{M}, F_\bullet, V_\bullet)$ belongs to $\mathsf{FV}(\mathcal{D}_X)$.

In other words, arguing as in Proposition 9.3.25, we have

(a) $(F_pV_\alpha\mathcal{M})\cdot t = F_pV_{\alpha-1}\mathcal{M}$ for all $p\in\mathbb{Z}$ and $\alpha<0$.

(b) $(F_p\mathrm{gr}_\alpha^V\mathcal{M})\cdot\partial_t = F_{p+1}\mathrm{gr}_{\alpha+1}^V\mathcal{M}$ for all $p\in\mathbb{Z}$ and $\alpha>-1$.

Furthermore, we say that $(\mathcal{M},F)$ is *a filtered middle extension along H* $\mathcal{M}$ is a middle extension along H, i.e., the non filtered morphism var resp. can is injective resp. is onto, and moreover if (b) holds also for $\alpha=-1$, that is, the filtered can is onto.

Of course, the inclusions "$\subseteq$" in (a) and (b) always hold for every $\alpha\in\mathbb{R}$. We also note that each $(\mathrm{gr}_\alpha^V\mathcal{M},F)$ is a filtered $\mathcal{D}_H$-module. The first condition can be called a *regularity condition.* Indeed, for a nonzero holonomic $\mathcal{D}_X$-module $\mathcal{M}$ with irregular singularities, we can have $V_\alpha\mathcal{M}=\mathcal{M}$ for every α (e.g. when $\dim X=1$, $\mathcal{M}=\mathcal{D}_X/\mathcal{D}_X(t^2\partial_t+1)$), and the condition $tF_p\mathcal{M}=F_p\mathcal{M}$ cannot be satisfied by a nonzero coherent $\mathcal{O}_X$-module $F_p\mathcal{M}$.

10.5.2. Remark. As in Remark 7.2.30, the conditions (a) and (b) are respectively equivalent to

(a) for $\alpha<0$ and any p, $F_pV_\alpha\mathcal{M}=(j_*j^{-1}F_p\mathcal{M})\cap V_\alpha\mathcal{M}$,

(b) for $\alpha\in(-1,0]$, $k\geqslant 1$ and any p,

$$F_pV_{\alpha+k}\mathcal{M}=\partial_t^kF_pV_\alpha\mathcal{M}+\sum_{j=0}^{k-1}\partial_t^jF_{p-j}V_0\mathcal{M}.$$

Furthermore, if $(\mathcal{M},F_\bullet)$ is a filtered middle extension, (b) is replaced with

(c) for $\alpha\in[-1,0)$, $k\geqslant 1$ and any p,

$$F_pV_{\alpha+k}\mathcal{M}=\partial_t^kF_pV_\alpha\mathcal{M}+\sum_{j=0}^{k-1}\partial_t^jF_{p-j}V_{<0}\mathcal{M}.$$

In particular, $F_p\mathcal{M}=\sum_{j\geqslant 0}\partial_t^jF_{p-j}V_{<0}\mathcal{M}$ and $F_\bullet\mathcal{M}$ is uniquely determined from $j^{-1}F_\bullet\mathcal{M}$.

As above, in the presence of a nonzero $g\in\mathcal{O}(X)$, we consider the graph embedding $\iota_g\colon X\to X\times\mathbb{A}^1_{\mathbb{C}}$. Given a filtered $\mathcal{D}_X$-module $(\mathcal{M},F)$ on X, we say that $(\mathcal{M},F)$ is $\mathbb{R}$-specializable along (g) if $\iota_{g*}(\mathcal{M},F)$ is so along $H\subset X\times\mathbb{A}^1_{\mathbb{C}}$. One can show that if $(g=0)$ is smooth, then this condition holds if and only if $(\mathcal{M},F)$ is $\mathbb{R}$-specializable along (g) (see Exercise 9.22).

10.5.3. Lemma. *Let $(\mathcal{M},F)$ be a coherently F-filtered $\mathcal{D}_X$-module which is $\mathbb{R}$-specializable along H. Then for each $\alpha\in A+\mathbb{Z}$, $F_\bullet V_\alpha\mathcal{M}$ is a coherent $FV_0\mathcal{D}_X$-filtration of $V_\alpha\mathcal{M}$ and $F_\bullet\mathrm{gr}_\alpha^V\mathcal{M}$ is a coherent F-filtration of $\mathrm{gr}_\alpha^V\mathcal{M}$.*

Proof. We first prove that $F_pV_\alpha\mathcal{M}$ is $\mathcal{O}_X$-coherent. This is a local question, and we can then assume that $V_\alpha\mathcal{M}$ is the union of $\mathcal{O}_X$-coherent submodules. The intersection of each such with $F_p\mathcal{M}$ is coherent, according to Corollary 8.8.8, and their union in $F_p\mathcal{M}$ is also coherent, as wanted. Applying a similar reasoning to $R_FV_\alpha\mathcal{M}$ in

$V_\alpha\mathcal{M}[z,z^{-1}]$ gives the coherence of $(V_\alpha\mathcal{M},F_\bullet)$. Since this holds for each α, it follows that $(\mathrm{gr}^V_\alpha\mathcal{M},F_\bullet)$ is coherent as a filtered $\mathrm{gr}^V_0\mathcal{D}_X$-module. Using now $\mathbb{R}$-specializability implies the coherence as a filtered $\mathcal{D}_X$-module. □

We now come to the main result of this chapter.

10.5.4. Theorem. *Let $f\colon X\to X'$ be a morphism as in Section 10.4 and let $f_0\colon X_0\to X'_0$ be the restriction of f. Suppose that f is proper and that $(\mathcal{M},F)$ is a coherently F-filtered $\mathcal{D}_X$-module which is $\mathbb{R}$-specializable, with V-filtration $V_\bullet\mathcal{M}$. If ${}_{\mathrm{D}}f_{0*}(\mathrm{gr}^V_\alpha(\mathcal{M}),F)$ is strict for every $\alpha\in\mathbb{R}$, then ${}_{\mathrm{D}}f_*(\mathcal{M},F,V)$ is strict in a neighborhood of X'_0.*

The strictness assumption means that the natural morphism
$$R^kf_{0*}(F_p\,{}^{\mathrm{p}}\mathrm{DR}_{X_0/X'_0}\,\mathrm{gr}^V_\alpha\mathcal{M})\longrightarrow R^kf_{0*}\,{}^{\mathrm{p}}\mathrm{DR}_{X_0/X'_0}\,\mathrm{gr}^V_\alpha\mathcal{M}={}_{\mathrm{D}}f^{(k)}_{0*}\mathrm{gr}^V_\alpha\mathcal{M}$$
is injective for every k,p,α.

The proof of the theorem will be given at the end of Section 10.6. Let us emphasize some consequences of the theorem.

10.5.5. Consequences. Under all assumptions of Theorem 10.5.4, the following holds.

(1) The complex ${}_{\mathrm{D}}f_*(\mathcal{M},F)$ is strict, i.e., for all k and all $p\in\mathbb{Z}$, the natural morphism $R^kf_*(F_p\,{}^{\mathrm{p}}\mathrm{DR}_{X/X'}\,\mathcal{M})\to R^kf_*\,{}^{\mathrm{p}}\mathrm{DR}_{X/X'}\,\mathcal{M}={}_{\mathrm{D}}f^{(k)}_*\mathcal{M}$ is injective.

(2) For each k, the $\mathcal{D}_{X'}$-module ${}_{\mathrm{D}}f^{(k)}_*\mathcal{M}$ is $\mathbb{R}$-specializable along H' and for each $\alpha\in\mathbb{R}$, $\mathrm{gr}^V_\alpha({}_{\mathrm{D}}f^{(k)}_*\mathcal{M})\simeq{}_{\mathrm{D}}f^{(k)}_{0*}\mathrm{gr}^V_\alpha(\mathcal{M})$.

(3) Moreover, for each $p\in\mathbb{Z}$, $F_p\mathrm{gr}^V_\alpha\,{}_{\mathrm{D}}f^{(k)}_*\mathcal{M}\simeq F_p\,{}_{\mathrm{D}}f^{(k)}_{0*}\mathrm{gr}^V_\alpha\mathcal{M}$.

10.6. A strictness criterion for complexes of filtered $\mathcal{D}$-modules

10.6.a. Setup. Assume that $X=H\times\Delta_t$ and set $X_0=X\times\{0\}$. We consider a *bounded*[2] complex
$$\cdots\longrightarrow\mathcal{M}^{i-1}\xrightarrow{\ d\ }\mathcal{M}^i\xrightarrow{\ d\ }\mathcal{M}^{i+1}\longrightarrow\cdots$$
of $\mathcal{D}_X$-modules. We set $X=H\times\Delta_t$. We make the following assumptions:

(a) Each $\mathcal{M}^i$ has an increasing filtration $F_\bullet\mathcal{M}^i$ by $\mathcal{O}_X$-submodules, exhaustive, locally bounded below, and compatible with the order filtration on $\mathcal{D}_X$.

(b) Each $\mathcal{M}^i$ has an increasing filtration $V_\bullet\mathcal{M}^i$ by $\mathcal{O}_X$-submodules, discretely indexed by $\mathbb{R}$, on which t and ∂_t act in the usual way.

(c) The differentials $d\colon\mathcal{M}^i\to\mathcal{M}^{i+1}$ respect both filtrations $F_\bullet\mathcal{M}^i$ and $V_\bullet\mathcal{M}^i$.

(d) The $\mathcal{O}_X$-modules $H^i(F_pV_\alpha\mathcal{M}^\bullet)$ are coherent for every $i,p\in\mathbb{Z}$ and $\alpha\in\mathbb{R}$.

(e) The morphism $t\colon F_pV_\alpha\mathcal{M}^i\to F_pV_{\alpha-1}\mathcal{M}^i$ is an isomorphism for $i,p\in\mathbb{Z}$ and $\alpha<0$.

(f) The morphism $\partial_t\colon F_p\mathrm{gr}^V_\alpha\mathcal{M}^i\to F_{p+1}\mathrm{gr}^V_{\alpha+1}\mathcal{M}^i$ is an isomorphism for $i,p\in\mathbb{Z}$ and $\alpha>-1$.

[2] We need in fact a weaker condition stated as (2) in Lemma 10.6.7.

(g) For every $\alpha \in \mathbb{R}$, the operator $t\partial_t - \alpha$ acts nilpotently on $H^i(\operatorname{gr}_\alpha^V \mathcal{M}^\bullet)$.

(h) For every $\alpha \in [-1, 0]$, the complex $\operatorname{gr}_\alpha^V \mathcal{M}^\bullet$, with the induced differential and the filtration induced by $F_\bullet \mathcal{M}^\bullet$, is strict.

(i) For every $i \in \mathbb{R}$, the Rees module $\bigoplus_{p\in\mathbb{Z}} H^i(F_p\mathcal{M}^\bullet)z^p$ is coherent over $R_F\mathcal{D}_X$.

Let us denote by $(M^\bullet, d)$ the resulting complex of graded modules over the ring $R = \mathbb{C}[z, v]$; here the z-variable goes with the filtration $F_\bullet\mathcal{M}^i$, and the v-variable with the filtration $V_\bullet\mathcal{M}^i$. Since the latter is indexed by $\mathbb{R}$, this needs a little bit of care. Because we are dealing with a bounded complex, we can choose an increasing sequence of real numbers $\alpha_k \in \mathbb{R}$, indexed by $k \in \mathbb{Z}$, such that all the jumps in the filtrations $V_\bullet\mathcal{M}^b$ happen at some α_k; we then define

$$M^i_{j,k} = F_j V_{\alpha_k}\mathcal{M}^i$$

for $i, j, k \in \mathbb{Z}$. This makes each

$$M^i = \bigoplus_{j,k\in\mathbb{Z}} M^i_{j,k}$$

into a $\mathbb{Z}^2$-graded module over the ring R; since the differentials in the original complex are compatible with both filtrations, they induce morphisms of graded R-modules $d\colon M^i \to M^{i+1}$.

10.6.1. Theorem. *The complex $(\mathcal{M}^\bullet, d)$, equipped with the two filtrations $F_\bullet\mathcal{M}^\bullet$ and $V_\bullet\mathcal{M}^\bullet$, is strict on an open neighborhood of X_0.*

In contrast with the analogous proposition 9.8.10, the proof we give here does not use completions. On the other hand, it makes strong use of the coherence property (d) which does not occur in Proposition 9.8.10. This is related with the strictness assumption of $\widetilde{\mathcal{M}}$ that is implicitly used here since we start from $\widetilde{\mathcal{M}} = R_F\mathcal{M}$ (see corollary 10.7.4).

10.6.b. Proof of Theorem 10.6.1. Note first that each M^i is a flat R-module. Using the above definition of the complex $(M^\bullet, d)$, we clearly have

$$(M^\bullet/vM^\bullet)_{j,k} = \frac{F_j V_{\alpha_k}\mathcal{M}^\bullet}{F_j V_{\alpha_{k-1}}\mathcal{M}^\bullet} = F_j \operatorname{gr}^V_{\alpha_k}\mathcal{M}^\bullet.$$

The condition in (h) has the following interpretation.

10.6.2. Lemma. *All cohomology modules of the complex $(M^\bullet/vM^\bullet, d)$ are flat over the ring $R/vR = \mathbb{C}[z]$.*

Proof. Together with (e) and (f), the condition in (h) says that the complex $\operatorname{gr}_\alpha^V\mathcal{M}^\bullet$ is strict for every $\alpha \in \mathbb{R}$. In terms of graded modules, this means that multiplication by z is injective on the cohomology of the complex $M^\bullet/vM^\bullet$, which is equivalent to flatness over the ring $\mathbb{C}[z]$. $\square$

The next step in the proof involves a local argument, and so we fix a point $x \in X_0$ and localize everything at x. Although we keep the same notation as above, in the remainder of this section, each $\mathcal{M}^i$ is a $\mathcal{D}_{X,x}$-module, the condition in (d) reads

$H^i(F_pV_\alpha\mathcal{M}^\bullet)$ is a finitely generated $\mathcal{O}_{X,x}$-module, etc. With this convention in place, consider the short exact sequence of complexes

$$0 \longrightarrow M^\bullet \longrightarrow M^\bullet \longrightarrow M^\bullet/vM^\bullet \longrightarrow 0,$$

in which the morphism from $M^\bullet$ to $M^\bullet$ is multiplication by v. (To keep the notation simple, we are leaving out the change in the grading.) The resulting long exact sequence in cohomology looks like this:

$$\cdots \longrightarrow H^i(M^\bullet) \longrightarrow H^i(M^\bullet) \longrightarrow H^i(M^\bullet/vM^\bullet) \longrightarrow H^{i+1}(M^\bullet) \longrightarrow H^{i+1}M^\bullet \longrightarrow \cdots$$

The following result constitutes the heart of the proof.

10.6.3. Proposition. *The connecting homomorphisms $\delta\colon H^i(M^\bullet/vM^\bullet) \to H^{i+1}(M^\bullet)$ in the long exact sequence are trivial.*

Once we have proved the proposition, we will know that the multiplication morphisms $v\colon H^i(M^\bullet) \to H^i(M^\bullet)$ are injective and that

$$\frac{H^i(M^\bullet)}{vH^i(M^\bullet)} \simeq H^i(M^\bullet/vM^\bullet).$$

Together with Lemma 10.6.2, this will tell us that v, z is a regular sequence on $H^i(M^\bullet)$, which is two thirds of what we need to prove that $H^i(M^\bullet)$ is a flat R-module.

In preparation for the proof, let us consider the graded pieces in a fixed bidegree (j,k) in the long exact sequence; to simplify the notation, set $p = j$ and $\alpha = \alpha_k$. We then have the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & H^{i+1}(F_pV_\beta\mathcal{M}^\bullet) \arrow[d] & \\
H^i(F_p\mathrm{gr}^V_\alpha\mathcal{M}^\bullet) \arrow[r, "\delta"] \arrow[dr, "\varepsilon"'] & H^{i+1}(F_pV_{<\alpha}\mathcal{M}^\bullet) \arrow[r] \arrow[d] & H^{i+1}(F_pV_\alpha\mathcal{M}^\bullet) \\
 & H^{i+1}(F_pV_{(\beta,\alpha)}\mathcal{M}^\bullet) &
\end{tikzcd}$$

Here $\beta < \alpha$, and the notation $V_{(\beta,\alpha)}\mathcal{M}^\bullet$ is an abbreviation for $V_{<\alpha}\mathcal{M}^\bullet/V_\beta\mathcal{M}^\bullet$. We observe that the morphism ε is trivial because the source and the target have different "weights" with respect to the action of the operator $t\partial_t$.

10.6.4. Lemma. *With notation as above, the morphism*

$$\varepsilon\colon H^i(F_p\mathrm{gr}^V_\alpha\mathcal{M}^\bullet) \longrightarrow H^{i+1}(F_pV_{(\beta,\alpha)}\mathcal{M}^\bullet)$$

is trivial.

Proof. We have a commutative diagram

$$\begin{tikzcd} H^i(F_p \mathrm{gr}_\alpha^V \mathcal{M}^\bullet) \arrow[r, "\varepsilon"] \arrow[d, hook] & H^{i+1}(F_p V_{(\beta,\alpha)} \mathcal{M}^\bullet) \arrow[d, hook] \\ H^i(\mathrm{gr}_\alpha^V \mathcal{M}^\bullet) \arrow[r] & H^{i+1}(V_{(\beta,\alpha)} \mathcal{M}^\bullet) \end{tikzcd}$$

in which the two vertical morphisms are injective because of (h). Now the operator $t\partial_t$ acts on the $\mathcal{O}_X$-module in the lower left corner with α as its only eigenvalue, and on the $\mathcal{O}_X$-module in the lower right corner with eigenvalues contained in the interval (β, α); this is a consequence of (g). Since the bottom arrow is compatible with the action of $t\partial_t$, it must be zero; but then ε is also zero. □

We conclude from the lemma that the image of

$$\delta \colon H^i(F_p \mathrm{gr}_\alpha^V \mathcal{M}^\bullet) \to H^{i+1}(F_p V_{<\alpha} \mathcal{M}^\bullet)$$

is contained in the intersection

$$\bigcap_{\beta<\alpha} \mathrm{Im}\Big(H^{i+1}(F_p V_\beta \mathcal{M}^\bullet) \to H^{i+1}(F_p V_{<\alpha} \mathcal{M}^\bullet) \Big)$$

We can now use (e) and Krull's intersection theorem to prove that this intersection is trivial (in the local ring $\mathcal{O}_{X,x}$).

10.6.5. Lemma. *We have*

$$\bigcap_{\beta<\alpha} \mathrm{Im}\Big(H^{i}(F_p V_\beta \mathcal{M}^\bullet) \to H^{i}(F_p V_{\alpha} \mathcal{M}^\bullet) \Big) = \{0\}.$$

Proof. Consider the following commutative diagram:

$$\begin{tikzcd} F_p V_\beta \mathcal{M}^{i-1} \arrow[r, "d"] \arrow[d, hook] & F_p V_\beta \mathcal{M}^{i} \arrow[r, "d"] \arrow[d, hook] & F_p V_\beta \mathcal{M}^{i+1} \arrow[d, hook] \\ F_p V_\alpha \mathcal{M}^{i-1} \arrow[r, "d"] & F_p V_\alpha \mathcal{M}^{i} \arrow[r, "d"] & F_p V_\alpha \mathcal{M}^{i+1} \end{tikzcd}$$

Suppose that we have an element $m \in F_p V_\alpha \mathcal{M}^i$ with $dm = 0$ that belongs to the image of $H^i(F_p V_\beta \mathcal{M}^\bullet)$. Then

$$m = dm_0 + m_1$$

for some $m_0 \in F_p V_\alpha \mathcal{M}^{i-1}$ and some $m_1 \in F_p V_\beta \mathcal{M}^i$. Now if $\beta < -1$, then by (e), we have $m_1 = m_2 t$ for a unique $m_2 \in F_p V_{\beta+1} \mathcal{M}^i$. Since multiplication by t is injective on $F_p V_{\beta+1} \mathcal{M}^{i+1}$, the fact that $dm_1 = 0$ implies that $dm_2 = 0$. As long as $\beta + 1 \leqslant \alpha$, we also have

$$m_2 t \in (F_p V_{\beta+1} \mathcal{M}^i) \cdot t \subseteq (F_p V_\alpha \mathcal{M}^i) \cdot t,$$

and therefore $m \in d(F_p V_\alpha \mathcal{M}^{i-1}) + (F_p V_\alpha \mathcal{M}^i) \cdot t$. By this type of argument, one shows more generally that

$$\bigcap_{\beta<\alpha} \mathrm{Im}\Big(H^{i}(F_p V_\beta \mathcal{M}^\bullet) \to H^{i}(F_p V_{\alpha} \mathcal{M}^\bullet) \Big) \subseteq \bigcap_{m\in\mathbb{N}} H^i(F_p V_\alpha \mathcal{M}^\bullet) \cdot t^m.$$

Since $H^i(F_pV_\alpha\mathcal{M}^\bullet)$ is finitely generated as an $\mathcal{O}_{X,x}$-module by (d), Krull's intersection theorem implies that the right-hand side is equal to zero. □

The conclusion is that $\delta = 0$, and hence that v, z form a regular sequence on $H^i(M^\bullet)$. Together with the following result, this proves that $H^i(M^\bullet)$ is flat as an R-module (see Section 15.2.a for details on flatness for graded R-modules).

10.6.6. Lemma. *The morphism $z\colon H^i(M^\bullet) \to H^i(M^\bullet)$ is injective.*

Proof. Since v, z form a regular sequence on $H^i(M^\bullet)$, the corresponding Koszul complex is exact. By the same argument as in the proof of Proposition 15.2.7, every element in the kernel of $z\colon H^i(M^\bullet) \to H^i(M^\bullet)$ can be written as v times another element in the kernel; consequently,

$$\operatorname{Ker}\Big(z\colon H^i(M^\bullet) \to H^i(M^\bullet)\Big) \subseteq \bigcap_{m\geqslant 1} v^m H^i(M^\bullet).$$

Looking at a fixed bidegree (j,k) and setting $p = j$ and $\alpha = \alpha_k$ as above, the right-hand side equals

$$\bigcap_{\beta<\alpha} \operatorname{Im}\Big(H^i(F_pV_\beta\mathcal{M}^\bullet) \to H^i(F_pV_\alpha\mathcal{M}^\bullet)\Big),$$

which is equal to zero by Lemma 10.6.5. □

In summary, we have shown that for every point $x \in X_0$, the localization of the complex $(\mathcal{M}^\bullet, d)$ is strict (as a complex of $\mathcal{D}_{X,x}$-modules with two filtrations). Now it remains to prove that the complex $(\mathcal{M}^\bullet, d)$ is strict on an open neighborhood of X_0, using the coherence condition in (i). This will end the proof of Theorem 10.6.1.

10.6.7. Lemma. *If $(\mathcal{M}^\bullet, F, V)$ is a complex of bi-filtered $\mathcal{D}_X$-modules whose restriction to X_0 is strict and which satisfies the following two conditions:*

(1) *for every j, the $R_F\mathcal{D}_X$-module $\oplus_{p\in\mathbb{Z}} H^j(F_p\mathcal{M}^\bullet)z^p$ is coherent;*
(2) *we have $H^j(F_p\mathcal{M}^\bullet) = 0$ for $|j| \gg 0$ and all p.*

Then $(\mathcal{M}^\bullet, F, V)$ is strict in a neighborhood of X_0.

Proof. Note that over $X \smallsetminus X_0$ we have $V_\alpha\mathcal{D}_X = \mathcal{D}_X$ for every α. Since $\bigcup_\alpha V_\alpha\mathcal{M} = \mathcal{M}$, it is easy to deduce that over this open subset, $V_\alpha\mathcal{M} = \mathcal{M}$ for every α. Therefore $(\mathcal{M}^\bullet, F, V)$ is strict over an open subset $U \subseteq X \smallsetminus X_0$ if and only if $(\mathcal{M}^\bullet, F)$ is strict over U.

By assumption, $(\mathcal{M}^\bullet, F, V)$ is strict at the points $x \in X_0$, hence in order to complete the proof of the lemma, it is enough to show that if $(\mathcal{M}^\bullet, F)$ is strict at a point $x \in X$, then it is strict in an open neighborhood of x. Since the F-filtration on $\mathcal{M}^\bullet$ is exhaustive, it follows from Exercise 10.5 that $(\mathcal{M}^\bullet, F)$ is strict at $x \in X$ if and only if the natural map $H^j(F_p\mathcal{M}^\bullet)_x \to H^j(F_{p+1}\mathcal{M}^\bullet)_x$ is injective for all p and j. For every j, consider the coherent $R_F\mathcal{D}_X$-module $\mathcal{M}_j := \oplus_{p\in\mathbb{Z}} H^j(F_p\mathcal{M}^\bullet)$. We see that $(\mathcal{M}^\bullet, F)$ is strict at $x \in X$ if and only if the map $u_j\colon \mathcal{M}_j \to \mathcal{M}_j$ given by multiplication with z is injective for all j. Furthermore, by (2) we only need to consider finitely many j. Since $\mathcal{M}_j$ is a coherent $R_F\mathcal{D}_X$-module, it follows that $\operatorname{Ker}(u_j)$ is a coherent $R_F\mathcal{D}_X$-module. In the neighborhood of a given point $x \in X$, we have a finite set of generators

$s_1,\dots,s_r$ of $\mathrm{Ker}(u_j)$ over $R_F\mathcal{D}_X$. If all the s_i vanish at x, then they also vanish in an open neighborhood of x and u_j is injective in this neighborhood. Since we can argue in this way simultaneously for finitely many j, this concludes the proof of the lemma. $\square$

10.6.c. Proof of Theorem 10.5.4. We will apply Theorem 10.6.1 to the bounded complex ${}_{\mathrm{D}}f_*(\mathcal{M},F,V)$. We first check that the conditions (a)–(i) of Section 10.6.a are fulfilled.

Since $(\mathcal{M},F,V)$ is an object of $\mathsf{FV}(\mathcal{D}_X)$, an application of Lemma 10.4.5 gives that ${}_{\mathrm{D}}f_*(\mathcal{M},F,V)\in\mathsf{D}^-(\mathsf{FV}(\mathcal{D}_{X'}))$. Moreover, by hypothesis we have that

$$(\mathrm{gr}^V_\alpha\,{}_{\mathrm{D}}f_*(\mathcal{M}^\bullet,F,V),F)\simeq{}_{\mathrm{D}}f_{0*}(\mathrm{gr}^V_\alpha(\mathcal{M}^\bullet,F,V),F)$$

is strict (the isomorphism is given by (10.4.4)). On the other hand, since $(\mathcal{M},F)$ is coherent, $(\mathcal{M},F,V)\in\mathsf{FV}_{\mathrm{c}}(\mathcal{D}_X)$. Therefore another application of Lemma 10.4.5 implies that ${}_{\mathrm{D}}f_*(\mathcal{M},F,V)\in\mathsf{D}^-_{\mathrm{c}}(\mathsf{FV}(\mathcal{D}_{X'}))$. As a consequence, the conditions (a)–(h) are thus fulfilled by ${}_{\mathrm{D}}f_*(\mathcal{M},F,V)$. Lastly, the coherence condition (i) follows from the coherence theorem 8.8.15. Therefore Theorem 10.6.1 implies that ${}_{\mathrm{D}}f_*(\mathcal{M},F,V)$ is strict in a neighborhood of X'_0. $\square$

10.7. Comparison with the results of Chapter 9

In this section we compare the notion of specializability for filtered $\mathcal{D}_X$-modules, as developed in this chapter, and that for a strict $R_F\mathcal{D}_X$-module, as considered in Chapter 9 (see Definition 9.3.18). We also discuss consequences about the regularity of the underlying holonomic $\mathcal{D}_X$-module and its strict holonomicity.

10.7.a. Strictness of strictly $\mathbb{R}$-specializable $R_F\mathcal{D}_X$-modules. Let $\widetilde{\mathcal{M}}$ be a (right) coherent graded $R_F\mathcal{D}_X$-module which is strictly $\mathbb{R}$-specializable along a smooth hypersurface H and let $V_\bullet\widetilde{\mathcal{M}}$ denotes its Kashiwara-Malgrange filtration. Then $\widetilde{\mathcal{M}}$ is strict if and only if $V_\alpha\widetilde{\mathcal{M}}$ is strict for some α, since all $\mathrm{gr}^V_\gamma\widetilde{\mathcal{M}}$ are assumed to be strict. The former property is equivalent to the existence of a coherent $\mathcal{D}_X$-module $\mathcal{M}$ equipped with a coherent F-filtration $F_\bullet\mathcal{M}$ such that $\widetilde{\mathcal{M}}=R_F\mathcal{M}$, while the latter is equivalent to the existence of a coherent $V_0\mathcal{D}_X$-module $V_\alpha\mathcal{M}$ equipped with a coherent F-filtration $F_\bullet V_\alpha\mathcal{M}$ such that $V_\alpha\widetilde{\mathcal{M}}=R_FV_\alpha\mathcal{M}$.

Assume thus that $\widetilde{\mathcal{M}}$ is strict and let $(\mathcal{M},F_\bullet\mathcal{M})$ be the coherently F-filtered $\mathcal{D}_X$-module such that $\widetilde{\mathcal{M}}=R_F\mathcal{M}$. Then $\mathcal{M}$ is $\mathbb{R}$-specializable along H and we have (see Exercise 9.24):

$$V_\alpha\mathcal{M}=V_\alpha\widetilde{\mathcal{M}}/(z-1)V_\alpha\widetilde{\mathcal{M}}\quad\text{and}\quad V_\alpha\widetilde{\mathcal{M}}[z^{-1}]=V_\alpha\mathcal{M}[z,z^{-1}].$$

10.7.1. Lemma. *Let $\widetilde{\mathcal{M}}$ be as above. Then the Kashiwara-Malgrange filtration of $\widetilde{\mathcal{M}}$ satisfies*

$$V_\alpha\widetilde{\mathcal{M}}=\widetilde{\mathcal{M}}\cap(V_\alpha\widetilde{\mathcal{M}}[z^{-1}]),\tag{10.7.1 $*$}$$

where the intersection takes place in $\widetilde{\mathcal{M}}[z^{-1}]$.

Proof. For $\gamma > \alpha$, we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & V_\alpha\widetilde{\mathcal{M}} \arrow[r, hook] \arrow[d] & V_\gamma\widetilde{\mathcal{M}} \arrow[r] \arrow[d] & V_\gamma\widetilde{\mathcal{M}}/V_\alpha\widetilde{\mathcal{M}} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & V_\alpha\widetilde{\mathcal{M}}[z^{-1}] \arrow[r, hook] & V_\gamma\widetilde{\mathcal{M}}[z^{-1}] \arrow[r] & (V_\gamma\widetilde{\mathcal{M}}/V_\alpha\widetilde{\mathcal{M}})[z^{-1}] \arrow[r] & 0 \end{tikzcd}$$

The upper horizontal line is clearly exact, and the lower one is so because $\mathbb{C}[z, z^{-1}]$ is flat over $\mathbb{C}[z]$. The first two vertical maps are injective since $\widetilde{\mathcal{M}}$ is strict. The third vertical map is injective since $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable. It follows that $V_\alpha\widetilde{\mathcal{M}} = V_\gamma\widetilde{\mathcal{M}} \cap V_\alpha\widetilde{\mathcal{M}}[z^{-1}]$ in $\widetilde{\mathcal{M}}[z^{-1}]$. Taking the limit for $\gamma \to \infty$ gives the assertion. $\square$

Consider on $\mathcal{M}$ the bi-filtration $F_pV_\alpha\mathcal{M} := F_p\mathcal{M} \cap V_\alpha\mathcal{M}$. Then (10.7.1 $*$) means that the filtration $U_\bullet\widetilde{\mathcal{M}}$ defined by $U_\alpha\widetilde{\mathcal{M}} := \bigoplus_p(F_pV_\alpha\mathcal{M})z^p$ satisfies the properties of the Kashiwara-Malgrange filtration of a strictly $\mathbb{R}$-specializable $R_F\mathcal{D}_X$-module. In particular we get, according to 9.3.25(a) and (d):

(a) $\forall\, p$ and $\forall\, \alpha < 0$, $t : F_pV_\alpha\mathcal{M} \to F_pV_{\alpha-1}\mathcal{M}$ is an isomorphism,

(b) $\forall\, p$ and $\forall\, \alpha > -1$, $\partial_t : F_p\mathrm{gr}^V_\alpha\mathcal{M} \to F_{p+1}\mathrm{gr}^V_{\alpha+1}\mathcal{M}$ is an isomorphism.

In other words, $(\mathcal{M}, F_\bullet, V_\bullet)$ is an object of $\mathsf{FV}(\mathcal{D}_X)$ (see Definition 10.5.1).

10.7.2. Remark. Due to the coherence of each $F_p\mathcal{M}$, the property (a) is equivalent to

(a′) $\forall\, p$ and $\forall\, \alpha < 0$, $F_pV_\alpha\mathcal{M} = (j_*j^{-1}F_p\mathcal{M}) \cap V_\alpha\mathcal{M}$, where $j : X \smallsetminus H \hookrightarrow X$ denotes the open inclusion.

Indeed, let us check the nontrivial implication (a) $\Rightarrow$ (a′). The inclusion $\subset$ is clear and it is enough to check the inclusion $\supset$. For a local section m of $\mathcal{M}$, there exists q such that m is a local section of $F_q\mathcal{M}$. If the restriction of m to $X \smallsetminus H$ is a local section of $F_p\mathcal{M}$ for some $p < q$, there exists $k \geqslant 1$ such that $m \cdot t^k$ is a local section of $F_p\mathcal{M}$. Therefore, if m is a local section of $(j_*j^{-1}F_p\mathcal{M}) \cap V_\alpha\mathcal{M}$, $m \cdot t^k$ is a local section of $F_p\mathcal{M} \cap V_{\alpha-k}\mathcal{M} = (F_p\mathcal{M} \cap V_\alpha\mathcal{M}) \cdot t^k$. Since $t^k : V_\alpha\mathcal{M} \to V_{\alpha-k}\mathcal{M}$ is bijective for $\alpha < 0$, the conclusion follows.

10.7.3. Proposition. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module which is $\mathbb{R}$-specializable along H, equipped with a coherent F-filtration. The following properties are equivalent:*

(1) *$R_F\mathcal{M}$ is strictly $\mathbb{R}$-specializable along H,*

(2) *$(\mathcal{M}, F_\bullet, V_\bullet)$ is an object of $\mathsf{FV}(\mathcal{D}_X)$.*

Moreover, when these conditions are fulfilled, the filtration $F_\bullet\mathcal{M}$ induces in some neighbourhood of H on each $V_\alpha\mathcal{M}$ a coherent $F_\bullet\mathcal{D}_{X/\mathbb{C}}$-filtration with respect to any local reduced equation $t : X \to \mathbb{C}$ of H, i.e., each $V_\alpha R_F\mathcal{M} = R_FV_\alpha\mathcal{M}$ is $R_F\mathcal{D}_{X/\mathbb{C}}$-coherent in some neighbourhood of H.

Proof. We have already seen that (1) implies (2). Conversely, let us assume (2) and let us set

$$U_\alpha R_F\mathcal{M} = \bigoplus_p (F_pV_\alpha\mathcal{M})z^p.$$

For a local section mz^p of $(F_pV_\alpha\mathcal{M})z^p$, we have $m(t\partial_t-\alpha z)^{\nu_m}z^p\in(F_{p+\nu_m}V_{<\alpha}\mathcal{M})z^{p+\nu_m}$, showing the $\mathbb{R}$-specializability of $R_F\mathcal{M}$ and the fact that $U_\alpha R_F\mathcal{M}\subset V_\alpha R_F\mathcal{M}$. It is enough to show that $U_\bullet\mathcal{M}$ is a coherent filtration indexed by $A+\mathbb{Z}$, since we obviously have $\mathrm{gr}^U_\alpha R_F\mathcal{M}=R_F\mathrm{gr}^V_\alpha\mathcal{M}$, hence the strictness. According to 10.5.1(a) and (b), it is enough to show the $V_0R_F\mathcal{D}_X$-coherence of $U_\alpha R_F\mathcal{M}$ for $\alpha\in[-1,0)$. For a local reduced equation $t:X\to\mathbb{C}$ of H, we will more precisely show the $R_F\mathcal{D}_{X/\mathbb{C}}$-coherence of $U_\alpha R_F\mathcal{M}$ in some neighbourhood of H, showing both the reverse implication (2) $\Rightarrow$ (1) and the last part of the proposition.

We have already seen (see Lemma 10.5.3) that each $F_pV_\alpha\mathcal{M}$ is $\mathcal{O}_X$-coherent. It is thus enough to show that, locally on X, there exists p_o such that $(F_{p_o}V_\alpha\mathcal{M})\cdot F_p\mathcal{D}_{X/\mathbb{C}}=F_{p+p_o}V_\alpha\mathcal{M}$ for all $p\geqslant 0$. Since $\mathrm{E}-\alpha$ is nilpotent on $\mathrm{gr}^V_\alpha\mathcal{M}$, the filtration $F_\bullet\mathrm{gr}^V_\alpha\mathcal{M}$, being $F_\bullet\mathrm{gr}^V_0\mathcal{D}_X$-coherent for every α, is also $F_\bullet\mathcal{D}_H$-coherent. The same argument applies to the induced filtration $(F_\bullet V_\alpha\mathcal{M})/(F_\bullet V_{\alpha-1}\mathcal{M})$ and therefore there exists locally p_o such that

$$\big[(F_{p_o}V_\alpha\mathcal{M})/(F_{p_o}V_{\alpha-1}\mathcal{M})\big]\cdot F_p\mathcal{D}_H=(F_{p+p_o}V_\alpha\mathcal{M})/(F_{p+p_o}V_{\alpha-1}\mathcal{M}).$$

Let us set $U_{\alpha,p}=(F_{p_o}V_\alpha\mathcal{M})\cdot F_p\mathcal{D}_{X/\mathbb{C}}$. By 10.5.1(a) and since α has been chosen in $[-1,0)$, the left-hand term above can be written as $U_{\alpha,p}/U_{\alpha,p}t$, while the right-hand term is

$$(F_{p+p_o}V_\alpha\mathcal{M})/(F_{p+p_o}V_\alpha\mathcal{M})t,$$

so Nakayama's lemma implies finally $(F_{p_o}V_\alpha\mathcal{M})\cdot F_p\mathcal{D}_{X/\mathbb{C}}=F_{p+p_o}V_\alpha\mathcal{M}$ in some neighbourhood of H, as wanted. $\square$

One can be more precise concerning the behaviour of each term of the F-filtration.

10.7.4. Corollary. *Let $\widetilde{\mathcal{M}}$ be a coherent graded $R_F\mathcal{D}_X$-module which is strictly $\mathbb{R}$-specializable along H. Then $\widetilde{\mathcal{M}}$ strict in some neighbourhood of H if and only if, for some $\alpha<0$ and all p, the p-th graded component $(V_\alpha\widetilde{\mathcal{M}})_p$ is $\mathcal{O}_X$-coherent. In such a case, the properties of Proposition* 10.7.3 *hold true and in particular, $\widetilde{\mathcal{M}}=R_F\mathcal{M}$ and $(V_\alpha\widetilde{\mathcal{M}})_p=F_p\mathcal{M}\cap V_\alpha\mathcal{M}$ for every α,p, where $\mathcal{M}:=\widetilde{\mathcal{M}}/(z-1)\widetilde{\mathcal{M}}$ is a coherent $\mathcal{D}_X$-module which is $\mathbb{R}$-specializable along H and $F_\bullet\mathcal{M}$ is a coherent F-filtration of $\mathcal{M}$.*

Proof. If $\widetilde{\mathcal{M}}$ is strict, we can write $\widetilde{\mathcal{M}}=R_F\mathcal{M}$ for some coherent F-filtration on $\mathcal{M}:=\widetilde{\mathcal{M}}/(z-1)\widetilde{\mathcal{M}}$, and we have, according to Proposition 10.7.3, $(V_\alpha\widetilde{\mathcal{M}})_p=F_p\mathcal{M}\cap V_\alpha\mathcal{M}$, which is $\mathcal{O}_X$-coherent as we have seen in the proof of Proposition 10.7.3.

Conversely, since $\widetilde{\mathcal{M}}$ is assumed to be strictly $\mathbb{R}$-specializable, each $\mathrm{gr}^V_\gamma\widetilde{\mathcal{M}}$ is strict, and it is enough to prove that $V_\alpha\widetilde{\mathcal{M}}$ is strict for some $\alpha<0$. For such an α, $V_\alpha\widetilde{\mathcal{M}}/V_\alpha\widetilde{\mathcal{M}}t^j$ is also strict for every $j\geqslant 1$. By left exactness of $\varprojlim_j$, we deduce that $\varprojlim_j(V_\alpha\widetilde{\mathcal{M}}/V_\alpha\widetilde{\mathcal{M}}t^j)$ is also strict. It is thus enough to show that the natural morphism $V_\alpha\widetilde{\mathcal{M}}\to\varprojlim_j(V_\alpha\widetilde{\mathcal{M}}/V_\alpha\widetilde{\mathcal{M}}t^j)$ is injective.

We choose $\alpha<0$ as given by the assumption of the proposition, and we have $(V_\alpha\widetilde{\mathcal{M}})_pt^j=(V_{\alpha-j}\widetilde{\mathcal{M}})_p$ for $j\geqslant 0$ and any p, due to 9.3.25(a). Then

$$\big(V_\alpha\widetilde{\mathcal{M}}/V_\alpha\widetilde{\mathcal{M}}t^j\big)_p=(V_\alpha\widetilde{\mathcal{M}})_p/(V_\alpha\widetilde{\mathcal{M}})_pt^j$$

for every $j \geqslant 0$ and p, and therefore

$$\big(\varprojlim_j (V_\alpha \widetilde{\mathcal{M}}/V_\alpha \widetilde{\mathcal{M}} t^j)\big)_p = \varprojlim_j \big((V_\alpha \widetilde{\mathcal{M}})_p/(V_\alpha \widetilde{\mathcal{M}})_p t^j\big).$$

Since $(V_\alpha \widetilde{\mathcal{M}})_p$ is $\mathcal{O}_X$-coherent, $\varprojlim_j (V_\alpha \widetilde{\mathcal{M}})_p/(V_\alpha \widetilde{\mathcal{M}})_p t^j = \mathcal{O}_{\widehat{X|H}} \otimes_{\mathcal{O}_{X|H}} (V_\alpha \widetilde{\mathcal{M}})_p$ and the natural morphism $(V_\alpha \widetilde{\mathcal{M}})_{p|H} \to \varprojlim_j (V_\alpha \widetilde{\mathcal{M}})_p/(V_\alpha \widetilde{\mathcal{M}})_p t^j$ is injective. It follows that

$$(V_\alpha \widetilde{\mathcal{M}})_{p|H} \longrightarrow \big(\varprojlim_j (V_\alpha \widetilde{\mathcal{M}}/V_\alpha \widetilde{\mathcal{M}} t^j)\big)_p$$

is injective for every p, and thus so is $(V_\alpha \widetilde{\mathcal{M}})|_H \to \big(\varprojlim_j (V_\alpha \widetilde{\mathcal{M}}/V_\alpha \widetilde{\mathcal{M}} t^j)\big)$, as wanted. $\square$

A useful consequence of regularity along a divisor is provided by the following corollary.

***10.7.5. Corollary* (A vanishing criterion).** *Let $g : X \to \mathbb{C}$ be a holomorphic function and let $\widetilde{\mathcal{M}}$ be strict and strictly $\mathbb{R}$-specializable along g. Then $\operatorname{Supp} \widetilde{\mathcal{M}} \subset g^{-1}(0)$ if and only if $\psi_{g,\lambda} \widetilde{\mathcal{M}} = 0$ for all $\lambda \in \mathbb{S}^1$. If moreover $\phi_{g,1} \widetilde{\mathcal{M}} = 0$, then $\widetilde{\mathcal{M}} = 0$.*

***Proof*.** Let $\iota_g : X \hookrightarrow X \times \mathbb{C}_t$ be the inclusion of the graph of g. The properties hold for $\widetilde{\mathcal{M}}$ and g if and only if they hold for $\widetilde{\mathcal{M}}_g$ and t. We can thus assume that we are in the setting of Corollary 10.7.4. The direction $\Rightarrow$ is clear, since the assumption implies that each local section of $\widetilde{\mathcal{M}}$ is annihilated by some power of t, and strict $\mathbb{R}$-specializability implies then that $V_{<0} \widetilde{\mathcal{M}} = 0$. For the direction $\Leftarrow$, we note that the assumption implies that the filtration $V_\alpha \widetilde{\mathcal{M}}$ is constant for $\alpha < 0$, hence so is the filtration $F_p \mathcal{M} \cap V_\alpha \mathcal{M}$ (p being fixed). Since this is a coherent $\mathcal{O}_X$-module and t induces an isomorphism on it by strict $\mathbb{R}$-specializability, we conclude by Nakayama that it is zero.

The remaining assertion on $\phi_{g,1} \widetilde{\mathcal{M}}$ is then clear. $\square$

***10.7.6. Corollary* (Complement to Corollary 9.3.31).** *Assume that $\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}}$ are strict and strictly $\mathbb{R}$-specializable along (g). If $\varphi : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{N}}$ is strictly $\mathbb{R}$-specializable along (g), then φ is strict in some neighbourhood of $g^{-1}(0)$.*

***Proof*.** It is a matter of proving strictness of $\operatorname{Coker} \varphi$. As in the corollary above, we can assume that we are in the setting of Corollary 10.7.4. By Corollary 9.3.31, we have $(V_\alpha(\operatorname{Coker} \varphi))_p = \operatorname{Coker} \varphi_{|(V_\alpha \widetilde{\mathcal{M}})_p}$, hence is $\mathcal{O}_X$-coherent. $\square$

We can now add the strictness property as a fourth item in Theorem 9.8.8, obtaining thus a complete analogue of Theorem 10.5.4.

***10.7.7. Corollary*.** *With the notation and assumptions of Theorem 9.8.8,*

(4) *if $\widetilde{\mathcal{M}}$ is strict in the neighbourhood of H, then ${}_{\mathrm{D}}f_*^{(i)} \widetilde{\mathcal{M}}$ is strict in the neighbourhood of H'.*

Proof. We replace X' by a suitable neighbourhood of H' and X by the pullback of this neighbourhood, so that $\widetilde{\mathcal{M}}$ is strict on X. By Corollary 10.7.4 it is enough to show the $\mathcal{O}_X$-coherence of $U_\alpha({}_{\mathrm{D}}f_*^{(i)}\widetilde{\mathcal{M}})_p = ({}_{\mathrm{D}}f_*^{(i)}V_\alpha\widetilde{\mathcal{M}})_p$ for some $\alpha < 0$ and each p, i, where the equality holds according to 9.8.8(1).

If $f : X = X' \times Z \to X'$ is a projection, we have, in a way similar to Exercise 8.49(6), ${}_{\mathrm{D}}f_*V_\alpha\widetilde{\mathcal{M}} = \boldsymbol{R}f_*(V_\alpha\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \wedge^{-\bullet}\widetilde{\Theta}_{X/X'})$, and $({}_{\mathrm{D}}f_*^{(i)}V_\alpha\widetilde{\mathcal{M}})_p$ is the i-th cohomology of the relative Spencer complex ($m = \dim X/X'$)

$$\boldsymbol{R}f_*\Big(0 \to (V_\alpha\widetilde{\mathcal{M}})_{p-m} \otimes \wedge^m\Theta_{X/X'} \to \cdots \to (V_\alpha\widetilde{\mathcal{M}})_{p-1} \otimes \Theta_{X/X'} \to (V_\alpha\widetilde{\mathcal{M}})_p \to 0\Big)$$

whose differentials are $\mathcal{O}_{X'}$-linear. Since each term of the complex is $\mathcal{O}_X$-coherent by our assumption of strictness of $\widetilde{\mathcal{M}}$ and since f is proper, Grauert's coherence theorem together with a standard spectral sequence argument in the category of $\mathcal{O}_{X'}$-complexes show that $({}_{\mathrm{D}}f_*^{(i)}V_\alpha\widetilde{\mathcal{M}})_p$ is $\mathcal{O}_{X'}$-coherent.

If $f : X \hookrightarrow X'$ is a closed immersion, it is locally of the form $(t, x_2, \ldots, x_n) \mapsto (t, x_2, \ldots, x_n, 0, \ldots, 0)$. Then

$${}_{\mathrm{D}}f_*V_\alpha\widetilde{\mathcal{M}} = {}_{\mathrm{D}}f_*^{(0)}V_\alpha\widetilde{\mathcal{M}} = f_*V_\alpha\widetilde{\mathcal{M}}[\widetilde{\partial}_{x'_1}, \ldots, \widetilde{\partial}_{x'_m}]$$

and

$$({}_{\mathrm{D}}f_*^{(0)}V_\alpha\widetilde{\mathcal{M}})_p = \sum_{|a|\leqslant p} f_*(V_\alpha\widetilde{\mathcal{M}})_{p-|a|}\widetilde{\partial}_{x'}^a,$$

which is $\widetilde{\mathcal{O}}_{X'}$-coherent since $(V_\alpha\widetilde{\mathcal{M}})_q = 0$ for $q \ll 0$ locally (use that $(V_\alpha\widetilde{\mathcal{M}})_q = F_q(\widetilde{\mathcal{M}}/(z-1)\widetilde{\mathcal{M}}) \cap V_\alpha(\widetilde{\mathcal{M}}/(z-1)\widetilde{\mathcal{M}})$ according to Corollary 10.7.4, and apply Exercise 8.63(3)). ☐

10.7.8. Corollary. *With the notation and assumptions of Corollary* 9.8.9*, if* $\widetilde{\mathcal{M}}$ *is strict in the neighbourhood of* $g^{-1}(0)$*, so is* ${}_{\mathrm{D}}f_*^{(i)}\widetilde{\mathcal{M}}$ *in the neighbourhood of* $g'^{-1}(0)$*.* ☐

10.7.b. Holonomicity, regularity and specialization. We start with a weak notion of regularity.

10.7.9. Definition **(Regularity along a smooth hypersurface)**

Let H be a a smooth hypersurface defined as the zero set of a function $t : X \to \mathbb{C}$ and let $\mathcal{D}_{X/\mathbb{C}}$ be the corresponding sheaf of relative differential operators. We say that a $\mathcal{D}_X$-module $\mathcal{M}$ which is $\mathbb{R}$-specializable along H is *regular along* H if some (equivalently, any) term $V_\alpha(\mathcal{M})$ of its V-filtration along H is $\mathcal{D}_{X/\mathbb{C}}$-coherent.

By setting $z = 1$ in the second part of Proposition 10.7.3, we find:

10.7.10. Corollary. *Let* $\widetilde{\mathcal{M}}$ *be a strict* $\widetilde{\mathcal{D}}_X$*-module which is strictly* $\mathbb{R}$*-specializable along* H*. Then the underlying* $\mathcal{D}_X$*-module* $\mathcal{M}$ *is regular along* H*.* ☐

Let us now consider the notion of regular holonomicity. If $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module, it is known ([**Kas78**]) that $\mathcal{M}$ is specializable (but possibly not $\mathbb{R}$-specializable in the sense that the roots of the Bernstein polynomials need not be real) along each hypersurface and that nearby and vanishing cycles of $\mathcal{M}$ with respect to any holomorphic function g are holonomic.

Let $\widetilde{\mathcal{M}}$ be a coherent $\widetilde{\mathcal{D}}_X$-module. Recall that $\widetilde{\mathcal{M}}$ is holonomic iff $\mathcal{M}$ is holonomic (see Remark 8.8.24). The following result is thus straightforward.

10.7.11. Corollary. *Assume that $\widetilde{\mathcal{M}}$ is holonomic and strictly $\mathbb{R}$-specializable along (g). Then, $\psi_{g,\lambda}\widetilde{\mathcal{M}}$ ($\lambda \in \mathbb{S}^1$) and $\phi_{g,1}\widetilde{\mathcal{M}}$ are holonomic and strict.* $\square$

As a preparation for the definition of a polarizable Hodge module (with the project of proving regular holonomicity of these), we propose a definition of regular holonomicity, that we will show to be equivalent to any of the standard definitions. This definition is by induction on the dimension of the support.

***10.7.12. Definition* (Regularity).** Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module with support Z of dimension d. We say that $\mathcal{M}$ is *regular* when one of the following conditions is satisfied.

(Reg_0): $d = 0$.

(Reg_d): $d \geqslant 1$ and for any germ $g : (X, x_o) \to (\mathbb{C}, 0)$ of holomorphic function on X,

(1) the $\mathcal{D}_{X\times\mathbb{C}}$-module ${}_{\mathrm{D}}\iota_{g,*}\mathcal{M}$ is regular along $H = X \times \{0\}$ in the sense of Definition 10.7.9,

(2) if $\dim(g^{-1}(0) \cap Z) \leqslant d-1$, the holonomic $\mathcal{D}_X$-modules $\psi_g\mathcal{M}$ and $\phi_{g,1}\mathcal{M}$ satisfy (Reg_{d-1}).

We do not assume $\mathbb{R}$-specializability, and in fact one can argue only with the V-filtrations indexed by $\mathbb{Z}$ so that we do not need to fix a total order on $\mathbb{C}$.

10.7.13. Proposition. *A holonomic $\mathcal{D}_X$-module $\mathcal{M}$ is regular in the sense of Definition* 10.7.12 *if and only if it is regular in any of the usual senses.*

Various definitions of regularity of holonomic $\mathcal{D}_X$-modules can be found in [**Bjö93, Kas03, Meb04**] for example, and are known to be equivalent.

Proof. We denote by ($\mathrm{Reg}_d^{\mathrm{st}}$) any of the standard regularity conditions for $\mathcal{M}$ with support of dimension d. We prove by induction on d that (Reg_d) is equivalent to ($\mathrm{Reg}_d^{\mathrm{st}}$), the case $d = 0$ being clear.

(Reg_d) $\Rightarrow$ ($\mathrm{Reg}_d^{\mathrm{st}}$), *assuming this holds for $d' \leqslant d-1$.* We choose g as in (Reg_d)(2) and and we consider the graph inclusion $\iota_g : X \hookrightarrow X \times \mathbb{C}_t$. By the induction hypothesis, both $\psi_g\mathcal{M}$ and $\phi_{g,1}\mathcal{M}$ satisfy ($\mathrm{Reg}_{d-1}^{\mathrm{st}}$), hence so does the restriction ${}_{\mathrm{D}}\iota_g^*\mathcal{M}$, that we regard as the complex with differential can_t. By stability of ($\mathrm{Reg}_{d-1}^{\mathrm{st}}$) by pullback (see e.g. [**Meb04**, Th. 6.1-1]), it follows that, for any morphism δ from a complex disc to X, the pullback complex ${}_{\mathrm{D}}\delta^*\mathcal{M}$ is regular. This property is also a characterization of ($\mathrm{Reg}_d^{\mathrm{st}}$) (see e.g. [**Meb04**, Th. 6.2-5]), concluding the proof of this implication.

($\mathrm{Reg}_d^{\mathrm{st}}$) $\Rightarrow$ (Reg_d), *assuming this holds for $d' \leqslant d-1$.* It is enough to prove that, under the assumption ($\mathrm{Reg}_d^{\mathrm{st}}$) on $\mathcal{M}$, $\psi_g\mathcal{M}$ satisfies ($\mathrm{Reg}_{d-1}^{\mathrm{st}}$) for any g as in (Reg_d)(2), since ${}_{\mathrm{D}}\iota_g^*\mathcal{M}$ is known to satisfy ($\mathrm{Reg}_{d-1}^{\mathrm{st}}$) (see e.g. [**Meb04**, Th. 6.1-1]), so that $\phi_{g,1}\mathcal{M}$ will also satisfy ($\mathrm{Reg}_{d-1}^{\mathrm{st}}$), and thus both $\psi_g\mathcal{M}, \phi_{g,1}\mathcal{M}$ will satisfy (Reg_{d-1}). The idea is thus to realize $\psi_g\mathcal{M}$ as the pullback to $t=0$ of a finite direct sum of $\mathcal{D}_{X\times\mathbb{C}_t}$-modules

$$\mathcal{H}^{-1}{}_{\mathrm{D}}\iota_t^*({}_{\mathrm{D}}\iota_{g,*}\mathcal{M})_{\alpha,k},\quad \alpha\in\mathbb{C},\ k\in\mathbb{N},$$

where, considering ${}_{\mathrm{D}}\iota_{g,*}\mathcal{M}$ as a left $\mathcal{D}_{X\times\mathbb{C}}$-module, i.e., an $\mathcal{O}_{X\times\mathbb{C}}$-module with flat connection ∇, we define

$$({}_{\mathrm{D}}\iota_{g,*}\mathcal{M})_{\alpha,k}=\Big(({}_{\mathrm{D}}\iota_{g,*}\mathcal{M})\otimes_{\mathcal{O}_X}\mathcal{O}_{X\times\mathbb{C}}(*t)^k,\nabla\otimes\mathrm{Id}+\mathrm{Id}\otimes(\alpha\,\mathrm{Id}+\mathcal{J}_k)\frac{\mathrm{d}t}{t}\Big),$$

with $\mathcal{J}_k$ being the Jordan block (lower, say) of size k. That such an identification exists for k large enough and a finite set of complex numbers α can be proved as Proposition 11.6.10(2) in the next chapter. As $(\mathcal{O}_{X\times\mathbb{C}}(*t)^k,(\alpha\,\mathrm{Id}+\mathcal{J}_k)\mathrm{d}t/t)$ is regular holonomic (being a the pullback of a regular holonomic $\mathcal{D}$-module on the disc with coordinate t, and as the tensor product of two holonomic $\mathcal{D}$-modules satisfying $(\mathrm{Reg}_d^{\mathrm{st}})$ also satisfies $(\mathrm{Reg}_d^{\mathrm{st}})$ (see e.g. [**Meb04**, Cor. 6.2-4]), then so does ${}_{\mathrm{D}}\iota_{g,*}\mathcal{M})_{\alpha,k}$, as well as its restriction $\mathcal{H}^{-1}{}_{\mathrm{D}}\iota_t^*({}_{\mathrm{D}}\iota_{g,*}\mathcal{M})_{\alpha,k}$, hence also $\psi_g\mathcal{M}$. $\square$

10.7.c. A criterion for strict holonomicity. The good behaviour of the duality functor on holonomic $\widetilde{\mathcal{D}}_X$-modules is important in the theory of mixed Hodge module, and for that purpose one has to prove that the underlying $\widetilde{\mathcal{D}}$-module of such an object is strictly holonomic in the sense of Definition 8.8.30. We give here a criterion ensuring that this property holds. As for the regularity property, it is of an inductive nature with respect to the support.

10.7.14. Theorem. *Let $\widetilde{\mathcal{M}}$ be a holonomic $\widetilde{\mathcal{D}}_X$-module and let $g:X\to\mathbb{C}$ be a holomorphic function. Assume that*

(1) *$\widetilde{\mathcal{M}}$ is strict and S-decomposable along (g);*
(2) *for each $\ell\in\mathbb{Z}$, $\mathrm{gr}_\ell^{\mathrm{M}}\psi_{g,\lambda}\widetilde{\mathcal{M}}$ ($\lambda\in S^1$) and $\mathrm{gr}_\ell^{\mathrm{M}}\phi_{g,1}\widetilde{\mathcal{M}}$ are strictly holonomic.*

Then $\widetilde{\mathcal{M}}$ is strictly holonomic in some neighborhood of $g^{-1}(0)$ and $\boldsymbol{D}\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along H.

Summary of the proof. By Proposition 8.8.33, we can assume that $X=H\times\mathbb{C}_t$ and g is the projection $(x,t)\mapsto t$. We consider the right setting. We realize the dual complex $\boldsymbol{D}\widetilde{\mathcal{M}}$ as a V-filtered complex $(\widetilde{\mathcal{N}}^\bullet,(U_k\widetilde{\mathcal{N}})_{k\in\mathbb{Z}})$ and will will show that it satisfies the conditions in Proposition 9.8.10 with (4) replaced by (4') and (5') in Remark 9.8.14. Together with Assumption (2), we conclude that each $\mathcal{E}xt^i_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}},\widetilde{\mathcal{D}}_X)$ is strictly $\mathbb{R}$-specializable along H and the filtration induced by $U_k\widetilde{\mathcal{N}}^\bullet$ is the Kashiwara-Malgrange filtration indexed by $\mathbb{Z}$. By the construction of $U_k\widetilde{\mathcal{N}}^\bullet$, each p-th graded component of $U_k\widetilde{\mathcal{N}}^i$ is $\mathcal{O}_X$-coherent (because it is so for any $V_\ell\widetilde{\mathcal{D}}_X$), and thus the same property holds for $V_kH^i(\widetilde{\mathcal{N}}^\bullet)$. It follows then from Corollary 10.7.4 that each $\mathcal{E}xt^i_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}},\widetilde{\mathcal{D}}_X)$ is strict in some neighborhood of H. Since $\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M},\mathcal{D}_X)=0$ for $i\neq n$ by holonomicity of $\mathcal{M}$, we deduce from Corollary 10.7.5 that $\mathcal{E}xt^i_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}},\widetilde{\mathcal{D}}_X)=0$ for $i\neq n$. In other words, $\widetilde{\mathcal{M}}$ is strictly holonomic in some neighborhood of H. $\square$

The graded case. We set $\widetilde{D}_X = \mathrm{gr}^V \widetilde{\mathcal{D}}_X$ and $\widetilde{D}_t = \widetilde{\mathbb{C}}[t]\langle\widetilde{\partial}_t\rangle$ for simplifying the notation. Let us first consider the graded $\widetilde{D}_X$-module $\widetilde{M} := \mathrm{gr}^V \widetilde{\mathcal{M}} = \bigoplus_{\alpha\in A} \widetilde{M}_\alpha$ with $A \subset (-1, 0]$ finite and $\widetilde{M}_\alpha = \mathrm{gr}^V_{\alpha+\mathbb{Z}}\widetilde{\mathcal{M}} := \bigoplus_{k\in\mathbb{Z}} \mathrm{gr}^V_{\alpha+k}\widetilde{\mathcal{M}}$. We claim that the graded $\widetilde{D}_X$-module $\mathcal{E}xt^i_{\widetilde{D}_X}(\widetilde{M}, \widetilde{D}_X)$ vanishes for $i \neq n$ and $\mathcal{E}xt^n$ is strict. Recall that $\widetilde{M}$ is equipped with a nilpotent operator N and thus with a finite monodromy filtration $\mathrm{M}_\bullet$. By using the long exact sequence of Ext's associated to the short exact sequence

$$0 \longrightarrow \mathrm{M}_{\ell-1}(\widetilde{M}) \longrightarrow \mathrm{M}_\ell(\widetilde{M}) \longrightarrow \mathrm{gr}^{\mathrm{M}}_\ell(\widetilde{M}) \longrightarrow 0,$$

and an obvious induction, we are reduced to proving the same properties for each $\mathrm{gr}^{\mathrm{M}}_\ell(\widetilde{M})$. As we assume that $\widetilde{\mathcal{M}}$ is S-decomposable along H (Assumption (1)), Section 9.7.18 provides the structure of each $\mathrm{gr}^{\mathrm{M}}_\ell(\widetilde{M})$ by means of Formulas (9.7.18 $*$) and (9.7.18 $**$). Lemma 8.8.34 reduces the proof to that for each term occurring in these formulas. For the terms $\mathrm{gr}^{\mathrm{M}}_\ell \mathrm{gr}^V_\alpha \widetilde{\mathcal{M}}$, this is provided by Assumption (2). For the terms $\widetilde{\mathrm{d}}t \otimes \widetilde{\mathbb{C}}[t]$, $\widetilde{\mathrm{d}}t \otimes \widetilde{\mathbb{C}}[\widetilde{\partial}_t]$ and $\widetilde{\mathrm{d}}t \otimes \widetilde{D}_t/(t\widetilde{\partial}_t - \alpha z)\widetilde{D}_t$, a direct computation shows that there $\mathcal{E}xt^0$ vanish and

- $\mathcal{E}xt^1_{\widetilde{D}_t}\big(\widetilde{\mathrm{d}}t \otimes \widetilde{\mathbb{C}}[t]), \widetilde{D}_t\big) \simeq \widetilde{\mathbb{C}}[t]$,
- $\mathcal{E}xt^1_{\widetilde{D}_t}\big(\widetilde{\mathrm{d}}t \otimes \widetilde{\mathbb{C}}[\widetilde{\partial}_t]), \widetilde{D}_t\big) \simeq \widetilde{\mathbb{C}}[\widetilde{\partial}_t]$,
- $\mathcal{E}xt^1_{\widetilde{D}_t}\big(\widetilde{D}_t/(t\widetilde{\partial}_t - \alpha z)\widetilde{D}_t, \widetilde{D}_t\big) \simeq \widetilde{D}_t/\widetilde{D}_t(t\widetilde{\partial}_t - \alpha z)$ as a left $\widetilde{D}_t$-module, with associated right module

$$\widetilde{D}_t/(t\widetilde{\partial}_t + (\alpha+1)z)\widetilde{D}_t.$$

All this justifies the claim. Furthermore, let us set $B = (-A + \mathbb{Z}) \cap (-1, 0]$. The latter computation shows that the graded $\widetilde{D}_X$-module $\widetilde{M}^\vee := \mathcal{E}xt^n_{\widetilde{D}_X}(\widetilde{M}, \widetilde{D}_X)^{\mathrm{right}}$ decomposes as

$$\widetilde{M}^\vee = \bigoplus_{\beta\in B} (\widetilde{M}^\vee)_\beta \quad \text{with} \quad (\widetilde{M}^\vee)_\beta = \begin{cases} \mathcal{E}xt^n_{\widetilde{D}_X}(\widetilde{M}_\beta, \widetilde{D}_X)^{\mathrm{right}} & \text{if } \beta = 0, \\ \mathcal{E}xt^n_{\widetilde{D}_X}(\widetilde{M}_{-\beta+1}, \widetilde{D}_X)^{\mathrm{right}} & \text{if } \beta \neq 0, 1, \end{cases}$$

and $t\widetilde{\partial}_t - (\beta + \ell)$ nilpotent on the component of degree ℓ of $(\widetilde{M}^\vee)_\beta$. Furthermore, as $\widetilde{D}_t/(t\widetilde{\partial}_t + (\alpha+1)z)\widetilde{D}_t$ is strictly $\mathbb{R}$-specializable at the origin, we deduce that $\widetilde{M}^\vee$ is strictly $\mathbb{R}$-specializable along H. $\square$

End of the proof. Since $\widetilde{\mathcal{M}}$ is strict, we can write it as $R_F\mathcal{M}$ with $\mathcal{M}$ holonomic, and since $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable, Proposition 10.7.3 implies that each $F_pV_\alpha\mathcal{M}$ is $\mathcal{O}_X$-coherent. For the sake of simplicity, we now consider the V-filtration as indexed by $\mathbb{Z}$.

By (adapting) Remark 10.3.8, we find a resolution of $(\mathcal{M}, F, V)$ by elementary bifiltered $\mathcal{D}_X$-modules which are finite direct sums of terms $\mathcal{L}_{k,\varepsilon} \otimes (\mathcal{D}_X, F[k], V[\varepsilon])$, with $\varepsilon \in \{-1, 0\}$ and $\mathcal{L}_{k,\varepsilon}$ is $\mathcal{O}_X$-coherent. Since the question is local, we can replace $\mathcal{L}_{k,\varepsilon}$ by a finite complex of free $\mathcal{O}_X$-modules of finite rank (as $\widetilde{\mathcal{O}}_{X,x}$ is a regular local ring for any $x \in X$). We will thus assume that $\mathcal{L}_{k,\varepsilon}$ is $\mathcal{O}_X$-free of finite rank. Then we have found locally a resolution (in non-positive degrees, possibly unbounded from below) $R_U\widetilde{\mathcal{M}}^\bullet$ of $R_V\widetilde{\mathcal{M}}$ where each term is a finite direct sum of terms $z^k v^\varepsilon R_V\widetilde{\mathcal{D}}_X$ ($\varepsilon \in \{-1, 0\}$, $k \in \mathbb{Z}$).

Taking duals, that is, $\mathcal{H}om_{R_V\widetilde{\mathcal{D}}_X}(\bullet, R_V\widetilde{\mathcal{D}}_X)^{\mathrm{right}}$, leads to a complex which is in non-negative degrees, whose terms are finite direct sums of terms $z^{-k}v^{-\varepsilon}R_V\widetilde{\mathcal{D}}_X$, that are thus identified with the Rees modules $R_U\widetilde{\mathcal{N}}^\bullet$ of $\widetilde{\mathcal{D}}_X$-modules $\widetilde{\mathcal{N}}^\bullet$ associated to a V-filtration indexed by $\mathbb{Z}$. The complex $R_U\widetilde{\mathcal{N}}^\bullet$ satisfies the following properties:

(a) restricting to $v=1$ induces a resolution $\boldsymbol{D}\widetilde{\mathcal{M}} \xrightarrow{\sim} \widetilde{\mathcal{N}}^\bullet$;

(b) restricting to $v=0$ induces a resolution

$$\boldsymbol{D}\mathrm{gr}^V\widetilde{\mathcal{M}} \xrightarrow{\sim} \mathrm{gr}^U\widetilde{\mathcal{N}}^\bullet. \tag{10.7.15}$$

By the first part of the proof, on each term $(\boldsymbol{D}\mathrm{gr}^V\widetilde{\mathcal{M}})_k$ of the gr^V-grading $\boldsymbol{D}(\mathrm{gr}^V\widetilde{\mathcal{M}}) = \bigoplus_{k\in\mathbb{Z}}(\boldsymbol{D}\mathrm{gr}^V\widetilde{\mathcal{M}})_k$, the operator $\prod_{\alpha\in A}(t\widetilde{\partial}_t + (\alpha+1+k)z)$ is nilpotent on $(\boldsymbol{D}\mathrm{gr}^V\widetilde{\mathcal{M}})_k$.

10.7.16. Lemma. *The complex $R_U\widetilde{\mathcal{N}}^\bullet$, when regarded as a V-filtered complex*

$$(\widetilde{\mathcal{N}}^\bullet, (U_k\widetilde{\mathcal{N}}^\bullet)_{k\in\mathbb{Z}}),$$

satisfies the properties of Proposition 9.8.10*, with* (4) *replaced with* (4') *and* (5') *of Remark* 9.8.14*.*

Proof. The items below refer to those of Proposition 9.8.10 and Remark 9.8.14. By construction, the complex $\widetilde{\mathcal{N}}^\bullet$ is in non-negative degrees, so we can take $-N-1=0$. The conditions concerning $\mathrm{gr}_k^U\widetilde{\mathcal{N}}^\bullet$, that is, (1), (2) and part of (4'), are satisfied according to the first part of the proof. To complete (4'), we have to check the vanishing of $H^j(\widetilde{\mathcal{N}}^\bullet)$ for $j\gg 0$. Since $H^j(\widetilde{\mathcal{N}}^\bullet)\simeq H^j(\boldsymbol{D}\widetilde{\mathcal{M}})$, Lemma 8.8.25 implies the desired vanishing for $j\geqslant 2n+2$.

(3) The bijectivity of $t: U_k\widetilde{\mathcal{N}}^i \to U_{k-1}\widetilde{\mathcal{N}}^i$ for any $k\leqslant -1$ holds for any i, since it holds for $t: V_k\widetilde{\mathcal{D}}_X \to V_{k-1}\widetilde{\mathcal{D}}_X$ for $k\leqslant 0$, according to the structure of $R_U\widetilde{\mathcal{N}}^i$.

(5) The $V_0\widetilde{\mathcal{D}}_X$-coherence of $H^i(U_k\widetilde{\mathcal{N}}^\bullet)$ follows from that of each $U_k\widetilde{\mathcal{N}}^i$, which in turn is a consequence of that of $V_0\widetilde{\mathcal{D}}_X$.

(5') The $\mathcal{O}_X$-coherence of $H^i((U_k\widetilde{\mathcal{N}}^\bullet)_p)$ folllows from that of $(U_k\widetilde{\mathcal{N}}^i)_p$, which in turn is a consequence of that of $(V_0\widetilde{\mathcal{D}}_X)_p = F_pV_0\mathcal{D}_X$. □

For any $i\geqslant 0$, set

$$U_kH^i(\widetilde{\mathcal{N}}^\bullet) = \mathrm{image}[H^i(U_k\widetilde{\mathcal{N}}^\bullet) \hookrightarrow H^i(\widetilde{\mathcal{N}}^\bullet)]$$

(the inclusion follows from Proposition 9.8.10). By Proposition 9.8.10(3), the morphism $t: U_kH^i(\widetilde{\mathcal{N}}^\bullet) \to U_{k-1}H^i(\widetilde{\mathcal{N}}^\bullet)$ is bijective for any $k\leqslant -1$. From (b) above and the conclusion of the theorem in the graded case, we deduce that, for any i, $H^i(\widetilde{\mathcal{N}}^\bullet)$ is strictly $\mathbb{R}$-specializable along H and $U_\bullet H^i(\widetilde{\mathcal{N}}^\bullet)$ is its V-filtration indexed by $\mathbb{Z}$. By Corollary 10.7.4 (that we can apply according to (5')), we conclude that $H^i(\widetilde{\mathcal{N}}^\bullet)$ is strict for any i. But by (a) above, we have $H^i(\mathcal{N}^\bullet) = H^i(\boldsymbol{D}\mathcal{M}) = 0$ if $i\neq 0$ since $\mathcal{M}$ is holonomic. By strictness, we deduce that $H^i(\widetilde{\mathcal{N}}^\bullet)=0$ for $i\neq 0$. This concludes the proof of Theorem 10.7.14. □

10.7.17. Remark. The proof can be adapted to the V-filtration indexed by $\alpha+\mathbb{Z}$ for each $\alpha\in A$, and gives an isomorphism, when $X=H\times\Delta_t$:

$$\boldsymbol{D}\mathrm{gr}^V_{\alpha+\mathbb{Z}}\widetilde{\mathcal{M}}\simeq\mathrm{gr}^V_{\beta+\mathbb{Z}}\boldsymbol{D}\widetilde{\mathcal{M}},$$

with α,β related by (10.7.15). However, it does not relate the action of N on both sides, as the first part of the proof only deals with $\mathrm{gr}^{\mathrm{M}}_{\bullet}(\bullet)$, on which this action has been killed.

On the other hand, when considering non-filtered $\mathcal{D}_X$-modules in the product setting $X=H\times\Delta_t$, one can be more precise (see e.g. [**Sab87a**, §3.2], [**MS89**, §4.6], : for any holonomic $\mathcal{D}_X$-modules, there exist functorial isomorphisms of $\mathcal{D}_H$-modules

$$\begin{aligned}\delta_\alpha:\mathrm{gr}^V_\beta(\boldsymbol{D}\mathcal{M})&\xrightarrow{\ \sim\ }\boldsymbol{D}(\mathrm{gr}^V_\alpha\mathcal{M}),\quad \beta=-\alpha+1\in(-1,0),\\ \delta_{-1}:\mathrm{gr}^V_{-1}(\boldsymbol{D}\mathcal{M})&\xrightarrow{\ \sim\ }\boldsymbol{D}(\mathrm{gr}^V_{-1}\mathcal{M}),\\ \delta_0:\mathrm{gr}^V_0(\boldsymbol{D}\mathcal{M})&\xrightarrow{\ \sim\ }\boldsymbol{D}(\mathrm{gr}^V_0\mathcal{M}),\end{aligned}$$

which satisfy $(\boldsymbol{D}\mathrm{N})\circ\delta_\alpha=-\delta_\alpha\circ\mathrm{N}$ for any $\alpha\in[-1,0]$. Furthermore, $\boldsymbol{D}\,\mathrm{can}_t$ (resp. $\boldsymbol{D}\,\mathrm{var}_t$) on $\mathcal{M}$ corresponds to var_t (resp. $-\mathrm{can}_t$) on $\boldsymbol{D}\mathcal{M}$. It follows that, if $\mathcal{M}$ is holonomic, there exist *local* functorial isomorphisms

$$\begin{aligned}\mathrm{gr}^{\mathrm{M}}_\ell\psi_{g,\lambda}\boldsymbol{D}\mathcal{M}&\simeq\boldsymbol{D}(\mathrm{gr}^{\mathrm{M}}_{-\ell}\psi_{g,\overline{\lambda}}\mathcal{M}),\quad \ell\in\mathbb{Z},\ \lambda\in S^1,\\ \mathrm{gr}^{\mathrm{M}}_\ell\phi_{g,1}\boldsymbol{D}\mathcal{M}&\simeq\boldsymbol{D}(\mathrm{gr}^{\mathrm{M}}_{-\ell}\phi_{g,1}\mathcal{M}),\quad \ell\in\mathbb{Z}.\end{aligned}$$

10.8. Exercises

Exercise 10.1. Show that a complex $(C^\bullet,F)$ which is *bounded from above* is strict if and only if the associated Rees complex $R_FC^\bullet$ is strict in the sense of Definition 5.1.6.

Exercise 10.2. Show that $(C^\bullet,F)$ is strict if and only if the canonical morphism $H^i(F_kC^\bullet)\to H^i(C^\bullet)$ is a monomorphism for all $k,i\in\mathbb{Z}$.

Exercise 10.3. By considering the long exact sequence in cohomology for the exact sequence

$$0\longrightarrow F_kC^\bullet\longrightarrow C^\bullet\longrightarrow C^\bullet/F_kC^\bullet\longrightarrow 0,$$

show that if $(C^\bullet,F)$ is strict, then for every i and k we have a short exact sequence

$$0\longrightarrow H^i(F_kC^\bullet)\longrightarrow H^i(C^\bullet)\longrightarrow H^i(C^\bullet/F_kC^\bullet)\longrightarrow 0.$$

Furthermore, show also that the map $H^i(F_kC^\bullet)\to H^i(F_\ell C^\bullet)$ is a monomorphism for every $k<\ell$, by considering the long exact sequence in cohomology corresponding to

$$0\longrightarrow F_kC^\bullet\longrightarrow F_\ell C^\bullet\longrightarrow F_\ell C^\bullet/F_kC^\bullet\longrightarrow 0,$$

and obtain a short exact sequence

$$0\longrightarrow H^i(F_kC^\bullet)\longrightarrow H^i(F_\ell C^\bullet)\longrightarrow H^i(F_\ell C^\bullet/F_kC^\bullet)\longrightarrow 0$$

for every $i\in\mathbb{Z}$.

Exercise 10.4. Show that if $(C^\bullet, F)$ is a strict complex, then for every $k \in \mathbb{Z}$, the complexes $(F_k C^\bullet, F)$ and $(C^\bullet/F_k C^\bullet, F)$, with the induced filtrations, are strict. In particular, using the second complex and Exercise 10.3, deduce that for every $k < \ell < m$ and every i, we have short exact sequences

$$0 \longrightarrow H^i\left(F_\ell C^\bullet/F_k C^\bullet\right) \longrightarrow H^i\left(C^\bullet/F_k C^\bullet\right) \longrightarrow H^i\left(C^\bullet/F_\ell C^\bullet\right) \longrightarrow 0$$

and

$$0 \longrightarrow H^i\left(F_\ell C^\bullet/F_k C^\bullet\right) \longrightarrow H^i\left(F_m C^\bullet/F_k C^\bullet\right) \longrightarrow H^i\left(F_m C^\bullet/F_\ell C^\bullet\right) \longrightarrow 0.$$

Exercise 10.5. Show that the complex $(C^\bullet, F)$ is strict if and only if all canonical morphisms $H^i(F_k C^\bullet) \to H^i(F_{k+1} C^\bullet)$ are monomorphisms. [*Hint*: It is clear that this condition is necessary; prove sufficiency by showing that the condition implies that $H^i(F_k C^\bullet) \to H^i(F_\ell C^\bullet)$ is a monomorphism for every $k < \ell$; use the exhaustivity of the filtration and the exactness of filtering direct limits to prove that $H^i(C^\bullet) \simeq \varinjlim_\ell H^i(F_\ell C^\bullet)$.]

Exercise 10.6. Let $\varphi : (C^0, F'_\bullet, F''_\bullet) \to (C^1, F'_\bullet, F''_\bullet)$ be a bi-filtered morphism, that we consider as defining a complex with two terms.

(1) Assume that φ is onto. Show that φ is bi-strict if and only if $F'_k F''_\ell C^1 = \varphi(F'_k F''_\ell C^0)$ for all $k, \ell \in \mathbb{Z}$.

(2) In general, show that φ is bi-strict if and only if it is strict with respect to each filtration and moreover

$$\begin{aligned}(F'_k C^1 + \operatorname{Im}\varphi) \cap (F''_\ell C^1 + \operatorname{Im}\varphi) &= F'_k F''_\ell C^1 + \operatorname{Im}\varphi,\\ F'_k F''_\ell C^1 \cap \operatorname{Im}\varphi &= \varphi(F'_k F''_\ell C^0).\end{aligned}$$

(3) Show that, if φ is bi-strict, then taking Ker and Coker commutes with grading with respect to $F'_\bullet$, $F''_\bullet$, or both in any order.

10.9. Comments

The aim of this chapter, which covers part of the content of [**Sai88**, §1 & 3] and whose first sketch has been written by Mircea Mustaţă, is to give a proof of Theorem 10.5.4 which closely follows the original proof of Saito [**Sai88**, Prop. 3.3.17], from which is extracted the formalism of bi-filtered derived categories (see also Section 8.9 which is inspired form [**Sai89a**]). However, the original argument using formal completions, which has been reproduced in the proof of Proposition 9.8.10, has been replaced here (Section 10.6.b) by an argument, due to Christian Schnell, using his interpretation of compatibility of a finite family of filtrations in terms of flatness, which somewhat clarifies [**Sai88**, §1.1]. This interpretation is explained with details in Section 15.2.a. The conclusion of Proposition 10.7.3 is an adaptation of [**Sai88**, Cor. 3.4.7], and is inspired from [**ESY17**, Prop. 2.2.4]. The criterion fo strict holonomicity is taken from [**Sai88**, Lem. 5.1.13].