IRREDUCIBLE CHARACTERS OF THE METAPLECTIC GROUP II: FUNCTORIALITY

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ABSTRACT. We extend our previous work [RT] and establish a complete duality theory for the (nonalgebraic) real metaplectic group. As a consequence, we obtain an intrinsic local Langlands conjecture for this group and, in particular, develop a geometric theory of endoscopic lifting. We also investigate the behavior of this formalism with theta lifting to equal-size orthogonal groups, and prove that for the kinds of infinitesimal character for which stability is empty, theta lifting preserves Kazhdan-Lusztig character formulas. Finally we interpret a character lifting due to Adams as an instance of functoriality for $Mp(2n,\mathbb{R})$.

1. INTRODUCTION

A fundamental guiding principle in the representation theory of a reductive algebraic group over a local field F is the local Langlands conjecture. The conjecture predicts remarkable relationships ("functorialities") between groups which ostensibly have nothing to do with each other. Perhaps surprisingly, this theory interacts in deep and interesting ways with the representation theory of certain nonalgebraic groups. The most famous example is the metaplectic double cover of the symplectic group, and its role in the construction of theta series. Because of this example and others, it is a natural and important problem to bring nonalgebraic reductive groups into the Langlands formalism. From a purely utilitarian perspective, the functorial relationships that the formalism predicts may offer deeper insights into the representation theory of nonalgebraic groups.

The original conjectures of Langlands offer little insight into how one might approach nonalgebraic groups, and so we turn to the geometric reformulation of Langlands' ideas codified most completely in [V5] (but see the extensive references given there). Very roughly speaking, at least on the level of Grothendieck groups, the category of admissible representations of the F-points of a reductive algebraic F-group is conjecturally dual to a precisely defined category of equivariant sheaves. (This is made more precise in the beginning of Section 6.) The appropriate category of sheaves is especially delicate to define when $F = \mathbb{R}$ [ABV], in the sense that one must abandon the classical Weil-Deligne group. Yet once this is done, something miraculous happens at the real place which, at least literally, does not happen at the non-archimedian places: the geometric category of sheaves itself has an intrinsic representation-theoretic interpretation[1]. As a consequence, the duality between representations and sheaves (on the level of Grothendieck groups) is actually equivalent to a duality between characters of a real group and those of a reductive subgroup of the Langlands dual of its complexification. This is the character duality of [V4], which depends crucially on

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[1] This is slightly misleading: at $\mathbb{R}$, the representation theoretic duality predated the geometry.

the Kazhdan-Lusztig algorithm ([V3]), and hence on the group being algebraic. Roughly speaking, one concludes that if a nonalgebraic group admits a duality theory like [V4], then this can be translated into a local Langlands formalism.

The primary purpose of the present paper is to state and prove an analog of the local Langlands conjecture for $Mp(2n,\mathbb{R})$, the nonalgebraic metaplectic double cover of the symplectic group. This turns out to be closely related to the theory for the indefinite orthogonal group and although precise statements require some preparation, the qualitative features of our main result are very easy to understand. The local Langlands conjecture (in the form of [ABV]) for the indefinite orthogonal group provides a complex algebraic variety $\mathbb{X}$ on which the complex dual group $G^\vee_\mathbb{C}$ acts and a duality between the representation theory of the various inner forms of $SO(p,q)$ and the $G^\vee_\mathbb{C}$ equivariant geometry of $\mathbb{X}$. Our main result can be interpreted as providing a duality between the representation theory of $Mp(2n,\mathbb{R})$ and the $\widetilde{G}^\vee_\mathbb{C}$ equivariant geometry of the *same* space $\mathbb{X}$; here $\widetilde{G}^\vee_\mathbb{C}$ is an algebraic two-fold central extension of $G^\vee_\mathbb{C}$.[2]

Before discussing this result in more detail, we note that once the formalism of the local Langlands conjecture is in place, we can immediately exploit some of the functorialities it predicts. For instance, since $\widetilde{G}^\vee_\mathbb{C}$ is a central extension of $G^\vee_\mathbb{C}$, the $\widetilde{G}^\vee_\mathbb{C}$ orbits and $G^\vee_\mathbb{C}$ orbits on $\mathbb{X}$ coincide. The theory of [ABV] interprets such orbits as Langlands-Weil parameters[3]. This allows us to define L-packets for $Mp(2n,\mathbb{R})$ and since the $G^\vee_\mathbb{C}$ and $\widetilde{G}^\vee_\mathbb{C}$ orbits coincide, we obtain a matching of L-packets for $Mp(2n,\mathbb{R})$ and those for the various inner forms of $SO(p,q)$. (The matching is canonical up to twisting by the outer automorphism of $Mp(2n,\mathbb{R})$, and is explained in more detail in the discussion before Theorem 5.4.) Each L-packet gives rise to a stable virtual representation, and all such representations arise this way. Thus matching of L-packets gives rise to an isomorphism, say T, from the space of stable virtual representation of $Mp(2n,\mathbb{R})$ to those of the various $SO(p,q)$. In Theorem 5.5 we prove that T coincides (up to outer automorphism) with the map studied in [A2]. (In fact, the theorem also treats a microlocalization of the map T obtained from the consideration of micro L-packets.) In this way, we interpret Adams' map as a simple functorial consequence of the local Langlands conjecture for $Mp(2n,\mathbb{R})$.

As another application, we obtain a completely geometric interpretation of endoscopic lifting for $Mp(2n,\mathbb{R})$. An endoscopic datum for $Mp(2n,\mathbb{R})$ consists of a pair $(s,\widetilde{G}_\mathbb{R})$, with s an elliptic element of $Sp(2n,\mathbb{R})$ of order two, and $\widetilde{G}_\mathbb{R}$ the preimage of the centralizer $Z_{G_\mathbb{R}}(s)$ in $Mp(2n,\mathbb{R})$. The geometric Langlands formalism for $Mp(2n,\mathbb{R})$ implies that there is a canonical lifting of genuine virtual representations of $\widetilde{G}_\mathbb{R}$ to those of $Mp(2n,\mathbb{R})$. In Section 6, we prove that this lifting coincides with the transpose of the transfer of orbital integrals for $Mp(2n,\mathbb{R})$ and $\widetilde{G}_\mathbb{R}$ defined in [R2].

We now to turn to a more detailed discussion of the local Langlands formalism for $Mp(2n,\mathbb{R})$ which, by the initial discussion above, is equivalent to a character multiplicity duality theory for this group. For a group $U_\mathbb{R}$ isomorphic to a product of a metaplectic group and a linear real reductive group and a regular infinitesimal character λ, there is no problem in extending the classical Langlands classification to $U_\mathbb{R}$. Hence there exists a finite set $\mathcal{P}^{U_\mathbb{R}}_\lambda$ and for each $\gamma \in \mathcal{P}^{U_\mathbb{R}}_\lambda$, we may speak of the standard module $\mathrm{std}_{U_\mathbb{R}}(\gamma)$ and its

[2]One caveat is in order here. While the geometry of [ABV] treats all infinitesimal characters simultaneously, our results are formulated one infinitesimal character at a time.

[3]That is, as dual group orbits on the space of admissible homomorphisms from the Weil-Deligne groups into an appropriate L-group.

unique irreducible quotient $\mathrm{irr}_{U_{\mathbb{R}}}(\gamma)$. (Here and elsewhere, when we deal with representations of a covering group, we will always implicitly assume everything in sight is a genuine representation.) The following is proved in Theorem 5.2.

Theorem 1.1. *Fix a block of irreducible genuine representation $\mathcal{B}$ for $G_{\mathbb{R}} = Mp(2n, \mathbb{R})$ at regular infinitesimal character λ. Then there exists a group $G'_{\mathbb{R}}$ which is a product of a metaplectic group and a linear real reductive group, an infinitesimal character λ' for $G'_{\mathbb{R}}$, a block of genuine irreducible representations $\mathcal{B}'$, and bijection $\mathcal{B} \to \mathcal{B}'$ (denoted $\gamma \mapsto \gamma^\vee$) such that*

$$[std_{G_{\mathbb{R}}}(\gamma)] = \sum_{\delta \in \mathcal{B}} m_{\gamma\delta}[irr_{G_{\mathbb{R}}}(\delta)] \tag{1.2}$$

if and only if

$$[irr_{G'_{\mathbb{R}}}(\delta^\vee)] = \sum_{\gamma^\vee \in \mathcal{B}'} \epsilon_{\gamma\delta} m_{\gamma\delta}[std_{G'_{\mathbb{R}}}(\gamma^\vee)]; \tag{1.3}$$

here $\epsilon_{\pi\tau} = \pm 1$ and is explicitly computable.

We now describe the dual group $G'_{\mathbb{R}}$ and the dual infinitesimal character λ'. If we identify the long roots in $\mathfrak{g} := \mathfrak{sp}(2n, \mathbb{C})$ with the short roots for $\mathfrak{so}(2n+1, \mathbb{C})$, then we obtain an infinitesimal character λ_{SO} for $\mathfrak{so}$ corresponding to λ. (In coordinates, the assignment $\lambda \mapsto \lambda_{SO}$ is the identity; see Section 2.3 for more details.) Adams and Barbasch proved (see Section 2.4) that the Howe correspondence may be interpreted as a bijection

$$\theta : \mathcal{P}^{Mp}_{\lambda} \longrightarrow \coprod \mathcal{P}^{SO(p,q)}_{\lambda_{SO}},$$

where the disjoint union on the right-hand side is over all $p+q = 2n+1$ with the parity of p fixed. Now fix a block $\mathcal{B}$ as in Theorem 1.1, and choose $\pi \in \mathcal{B}$. Let $\mathcal{B}_{SO}$ denote the block for $SO(p, q)$ at infinitesimal character λ_{SO} containing $\theta(\pi)$. (In general $\mathcal{B}_{SO}$ and even the inner form $SO(p, q)$ will depend on the choice of π, and we comment on this below.) Given $\mathcal{B}_{SO}$, we can apply the construction of [V4] to obtain a dual group $H'_{\mathbb{R}}$, a block of representation $(\mathcal{B}_{SO})'$ with trivial infinitesimal character for $H'_{\mathbb{R}}$, and a bijection $\mathcal{B}_{SO} \to (\mathcal{B}_{SO})'$ so that the duality of (1.2)–(1.3) holds. Here $H'_{\mathbb{R}}$ is a real form of $\mathfrak{g}(\lambda)$, the centralizer in $\mathfrak{sp}(2n, \mathbb{C})$ of $\exp(2\pi i \lambda_{SO})$. More precisely, $H'_{\mathbb{R}}$ is a product of a factor of the form $Sp(2k, \mathbb{R})$, a factor of the form $Sp(a, b)$, and a number of $U(r, s)$ factors. Recall that the block $\mathcal{B}_{SO}$ and (hence apparently $H'_{\mathbb{R}}$) depended on a choice of $\pi \in \mathcal{B}$. Yet it turns out that $H'_{\mathbb{R}}$ does not depend on the choice. The dual group of Theorem 1.1 is given by covering the $Sp(2k, \mathbb{R})$ factor of $H'_{\mathbb{R}}$ with the metaplectic double cover, and leaving the other factors unchanged. Finally λ' is the infinitesimal character of the representation of $G'_{\mathbb{R}}$ obtained by taking the external tensor product of the metaplectic representation on the Mp factor, and the trivial representation on the remaining linear factors.

Thus, roughly speaking, the relevant dual group $G'_{\mathbb{R}}$ for a representation of $Mp(2n, \mathbb{R})$ is obtained by taking a metaplectic cover of the dual group for $\theta(\pi)$, and the dual infinitesimal character λ' is the infinitesimal character of the smallest genuine unitary representation of $G'_{\mathbb{R}}$.

Before proceeding further, we make a few comments on the proof of the Theorem 1.1 which is given in Section 5 below. (It is worth pointing out that the bijection of the theorem is constructed explicitly there.) Ultimately we reduce the theorem to two extreme cases: when all the coordinates of λ are half-integers (i.e. infinitesimal characters for which genuine discrete series exist); and when none of them are. In the former case, unwinding the construction of the dual group shows that the theorem gives a duality of $Mp(2n,\mathbb{R})$ at infinitesimal character λ to $Mp(2n,\mathbb{R})$ at infinitesimal character equal to that of the metaplectic representation. Using a translation principle, we can move back to λ and interpret the theorem as a self-duality. This is the content of the main result of [RT].

It remains to treat the non-half-integral case, i.e. when none of the coordinates of λ are half-integers. (In this case, only the quasisplit form of $SO(2n+1)$ admits any representations at infinitesimal character λ_{SO}.) Here the metaplectic factor of $G'_{\mathbb{R}}$ disappears, so $G'_{\mathbb{R}}$ coincides with $H'_{\mathbb{R}}$ for $SO(n,n+1)$. This suggests that the representation theory (at least on the level of characters) of Mp is closely related to that of SO. In fact, the two theories are identical and the Howe correspondence implements the identification. A more precise version of the following theorem is proved in Theorem 4.1 below.

Theorem 1.4. *Fix non-half-integral infinitesimal character λ, and set $\lambda_{SO} = \theta_{\mathrm{ic}}(\lambda)$ (notation as in Section 2.3). Then there is a bijection between the genuine standard representations of $Mp(2n,\mathbb{R})$ with infinitesimal character λ and those of $SO(n,n+1)$ with infinitesimal character λ_{SO} which preserves composition series. More precisely, the Howe correspondence gives a bijection*

$$\theta \,:\, \mathcal{P}^{Mp}_{\lambda} \longrightarrow \mathcal{P}^{SO(n,n+1)}_{\lambda_{SO}}$$

such that

$$[std_{Mp}(\gamma)] = \sum_{\delta \in \mathcal{P}^{Mp}_{\lambda}} n_\delta [irr_{Mp}(\delta)],$$

if and only if

$$[std_{SO}(\theta(\gamma))] = \sum_{\delta \in \mathcal{P}^{Mp}_{\lambda}} n_\delta [irr_{SO}(\theta(\delta))].$$

Hence, in the non-half-integral case, Theorem 1.1 follows from the duality of [V4] for $SO(n,n+1)$. For general reductive dual pairs, it is worth remarking that the Howe correspondence behaves intractably with respect to composition series of standard modules. Yet the present result seems to extend to arbitrary equal-rank dual pairs, as long as the infinitesimal character is suitably far from that of a discrete series (though still not necessarily generic).

Next recall the matching of L-packets for Mp and SO mentioned above. In Theorem 5.4, we prove that (up to possibly twisting by the outer automorphism of $Mp(2n,\mathbb{R})$), the map θ implements the matching. More precisely, if we fix an orbit Q of $G^\vee_{\mathbb{C}}$ (equivalently $\widetilde{G}^\vee_{\mathbb{C}}$) on $\mathbb{X}$, then the image under θ of the L-packet for SO parameterized by Q is the L-packet for Mp parameterized by Q. Said differently, the Langlands-Weil parameters of a genuine irreducible representation of $Mp(2n,\mathbb{R})$ may simply be taken to be those of $\theta(\pi)$. When no coordinate of the infinitesimal character is a half-integer, this follow easily of Theorem 1.4, and as a consequence it is not difficult to reduce Theorem 5.4 to the half-integral case. In turn, this case follows from a deep and intricate geometric reformulation of the main result of [RT] which we now describe. Given a representation π of $Mp(2n,\mathbb{R})$ with half-integral infinitesimal character λ, we can attach two $GL(n,\mathbb{C})$ orbits on the flag variety X for $\mathfrak{sp}(2n,\mathbb{C})$ as follows. The support of the appropriate $\mathcal{D}$-module localization provides one orbit $Q(\pi)$. To get the other, we pass to $\theta(\pi)$ and then its dual in the sense of [V4]; this is a representation of $Sp(2n,\mathbb{R})$ and hence we can take its support to get the other $GL(n,\mathbb{C})$ orbit $Q'(\pi)$. In Section 3.1, we prove this map is injective and that its image is closed under the operation of swapping (Q,Q') with (Q',Q). In this way we obtain an involution on the set of irreducible representations of $Mp(2n,\mathbb{R})$ with infinitesimal character λ. In Theorem 3.16 we prove that this involution implements the character multiplicity duality of Theorem 1.1. It is this formulation of the duality at half-integral infinitesimal character that allows one to check that θ implements the matching of L-packets.

In the interest of mathematical honesty, we remark that the above discussion belies the importance of a very delicate balancing of certain choices. For instance, there is a choice of oscillator defining θ; a more serious problem is the 2^n choices of the duality for the various inner forms $SO(p,q)$. This latter issue is buried in Definition 3.2 and Lemma 3.6 whose proof may charitably be described as dense. In a sense, it is remarkable that the choices can be juggled successfully and, in the end, they amount (at most) to twisting the representation theory of $Mp(2n,\mathbb{R})$ by an outer automorphism. We overlooked this serious subtlety in an earlier version, and thank the referee for bringing it to our attention.

We conclude by remarking that the results proved here likely extend to a much larger class of nonalgebraic reductive Lie groups. (It may appear that the ubiquitous use of θ in the above discussion severely limits any generalization, but in actuality θ is used only to make certain statements (like Theorem 1.4 and the definition of $G'_{\mathbb{R}}$) cleaner.) For instance, in a future joint paper we develop a local Langlands conjecture for the metaplectic cover of $GL(n,\mathbb{R})$. The picture again is very similar: if the representation theory of $GL(n,\mathbb{R})$ is dual (in the sense of [ABV]) to the $G^\vee_{\mathbb{C}}$ equivariant geometry of some space $\mathbb{X}$, we show that the representation theory of $\widetilde{GL}(n,\mathbb{R})$ is dual to the $\widetilde{G}^\vee_{\mathbb{C}}$ equivariant geometry of $\mathbb{X}$ for some algebraic central extension of $G^\vee_{\mathbb{C}}$. Again the orbits of $G^\vee_{\mathbb{C}}$ and $\widetilde{G}^\vee_{\mathbb{C}}$ on $\mathbb{X}$ coincide, so we obtain an analog of the functoriality T described above. Since stability is empty for $GL(n,\mathbb{R})$, this amounts to a lifting of representations from $GL(n,\mathbb{R})$ to $\widetilde{GL}(n,\mathbb{R})$ which we subsequently prove is simply the lifting defined by Kazhdan and Patterson in [KP]. In this sense the liftings of Kazhdan-Patterson [KP] and Adams [A2], while superficially very different, are really vestiges of the same general kind of functoriality.

2. Notation and Background

2.1. **Generalities.** In this section $G_{\mathbb{R}}$ denotes an arbitrary connected real reductive group in the Harish-Chandra class, $K_{\mathbb{R}}$ denotes a maximal compact subgroup of $G_{\mathbb{R}}$ with corresponding Cartan involution σ, and K denotes the complexification of $K_{\mathbb{R}}$. Furthermore, we assume that $\mathrm{rk}(G_{\mathbb{R}}) = \mathrm{rk}(K_{\mathbb{R}})$, i.e. $G_{\mathbb{R}}$ admits discrete series representations, and that all Cartan subgroups of $G_{\mathbb{R}}$ are abelian. (Of course these hypotheses are too restrictive in general, but all of the groups we consider below satisfy them.) We fix a compact Cartan subgroup $T_{\mathbb{R}}$ of $G_{\mathbb{R}}$ contained in $K_{\mathbb{R}}$ and a positive root system $\Delta^+(\mathfrak{g},\mathfrak{t})$ of $\mathfrak{t}$ in $\mathfrak{g}$.

Recall that a complex reductive Lie algebra such as $\mathfrak{g}$ is canonically endowed with an "abstract Cartan algebra" $\mathfrak{h}_a$ and a positive root system Δ^+_a (see [ABV], Chapter 16, for details), and that the data of a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a positive root system $\Delta^+(\mathfrak{g},\mathfrak{h})$ give a canonical isomorphism between $\mathfrak{h}$ and $\mathfrak{h}_a$.

We let $\mathcal{HC}(\mathfrak{g},K)$ denote the category of Harish-Chandra modules for $G_{\mathbb{R}}$. For any infinitesimal character $\lambda \in \mathfrak{h}_a^*$, $\mathcal{HC}(\mathfrak{g},K)_\lambda$ is the full subcategory of modules having infinitesimal character λ. The Grothendieck groups of these categories are denoted respectively by $\mathbb{K}\mathcal{HC}(\mathfrak{g},K)$ and $\mathbb{K}\mathcal{HC}(\mathfrak{g},K)_\lambda$, and we write $\mathrm{Irr}(G_\mathbb{R})$ and $\mathrm{Irr}(G_\mathbb{R})_\lambda$ for the irreducible objects in each category. We will write $[\pi]$ for the image of a Harish-Chandra module π in $\mathbb{K}\mathcal{HC}(\mathfrak{g},K)$.

2.1.1. *Langlands classification without L-groups.* Recall that a pseudocharacter of $G_\mathbb{R}$ is a pair $(H_\mathbb{R},\gamma)$ where $H_\mathbb{R}$ is a σ-stable Cartan subgroup of $G_\mathbb{R}$, and $\gamma=(\Gamma,\overline{\gamma})$ consists of an irreducible representation Γ of $H_\mathbb{R}$ and an element $\overline{\gamma}\in\mathfrak{h}^*$, with certain compatibility conditions. (We refer to [Vgr], Chapter 6, for details.) Fix $\lambda\in\mathfrak{h}_a^*$. We say that γ is a λ-pseudocharacter if $\overline{\gamma}$ and λ define the same infinitesimal character. The group $K_\mathbb{R}$ acts on the set of λ-pseudocharacters by conjugation, and we denote the set of orbits by $\mathcal{P}_\lambda$. Given a (K conjugacy class of) λ-pseudocharacter $(H_\mathbb{R},\gamma)$ one can define a standard module $\mathrm{std}(\gamma,H_\mathbb{R})$ ($\mathrm{std}(\gamma)$ for short) which admits a unique irreducible submodule $\mathrm{irr}(\gamma,H_\mathbb{R})$ (again $\mathrm{irr}(\gamma)$ for short), and each element of $\mathrm{Irr}(G_\mathbb{R})_\lambda$ arises (uniquely) in this way. Finally, if $\pi=\mathrm{irr}(\gamma,H_\mathbb{R})$, we say that π is attached to $H_\mathbb{R}$.

Next recall that the following two sets are bases of the Grothendieck group $\mathbb{K}\mathcal{HC}(\mathfrak{g},K)_\lambda$:

$$\{\,[\mathrm{irr}(\gamma)]\,\}_{\gamma\in\mathcal{P}_\lambda}\quad\text{and}\quad\{\,[\mathrm{std}(\gamma)]\,\}_{\gamma\in\mathcal{P}_\lambda}. \tag{2.1}$$

The change of basis matrix defines integers $M(\gamma,\delta)$ and $m(\gamma,\delta)$ as follows

$$[\mathrm{irr}(\delta)]=\sum_{\gamma\in\mathcal{P}_\lambda}M(\gamma,\delta)[\mathrm{std}(\gamma)],\qquad [\mathrm{std}(\delta)]=\sum_{\gamma\in\mathcal{P}_\lambda}m(\gamma,\delta)[\mathrm{irr}(\gamma)]. \tag{2.2}$$

2.1.2. *Hecke modules.* Write $\mathbf{H}(W(\lambda))=\mathbf{H}(\lambda)$ for the Hecke algebra of the integral Weyl group $W(\lambda)$; this is an algebra over $\mathbb{Z}[q,q^{-1}]$, the ring of Laurent polynomials in the indeterminant q. Recall that

$$\mathbf{M}_\lambda:=\mathbb{Z}[q,q^{-1}]\otimes_\mathbb{Z}\mathbb{K}\mathcal{HC}(\mathfrak{g},K)_\lambda$$

is a representation of $\mathbf{H}(\lambda)$. When the infinitesimal character is integral, the definition of this representation is well-known ([V3], Definition 6.4). In the nonintegral case, when simple integral roots may be not be simple in the entire root system, the definition is slightly more intricate: one must cross a series of nonintegral walls (each of which is an equivalence of categories) to move the simple integral root into a position where it is actually simple; then apply the formulas of [V3], Definition 6.4; and finally return to the original category of interest by another series of nonintegral wall crosses. This procedure is especially unpleasant since different sequences of wall crosses are generally required for different nonsimple integral roots. (In the cases we consider in Section 4, this unpleasantness can be avoided.)

The representation of $\mathbf{H}(\lambda)$ on $\mathbf{M}_\lambda$ places a number of explicitly computable constraints on the the matrix $m(\gamma,\delta)$ of Equation (2.2). If $G_\mathbb{R}$ is linear, one of the main consequences of the argument reproduced in [ABV], Chapter 17, is that *the Hecke algebra representation* $\mathbf{M}_\lambda$ *entirely specifies* $m(\gamma,\delta)$. (One can think of the Bruhat $\mathcal{G}$-order and Verdier duality as being axiomatically and uniquely characterized by the Hecke algebra action.) If $G_\mathbb{R}$ is nonlinear, one can expect the Hecke algebra representation to specify $m(\gamma,\delta)$ only in special cases; this will be the case in Section 4 below. In general, however, one needs to keep track of more refined information about the nonintegral wall-crosses. (In the case of the metaplectic groups at half-integral infinitesimal character, for instance, this is captured by extending the action of Hecke algebra to a slightly larger algebra [RT]. This action then completely determines the multiplicity matrix.)

2.1.3. *The support of a Harish-Chandra module.* Fix $\lambda \in \mathfrak{h}_a^*$ dominant and write $\mathcal{D}_\lambda$ for the corresponding twisted sheaf of differential operators on X, the flag variety of $\mathfrak{g}$. Let π be a Harish-Chandra module of infinitesimal character λ and set $\Delta_\lambda(\pi) = \mathcal{D}_\lambda \otimes_{A(\lambda)} \pi$; here $A(\lambda)$ is the quotient of the enveloping algebra of $\mathfrak{g}$ by the minimal primitive ideal with infinitesimal character λ (i.e. $A(\lambda)$ consists of the global sections of $\mathcal{D}_\lambda$). We write $\mathrm{supp}(\pi)$ for the support of $\Delta_\lambda(\pi)$; this is a K-equivariant closed subvariety of X. If π is in fact irreducible then $\mathrm{supp}(\pi)$ is also irreducible, and hence is the closure of a single orbit of K on X. We denote this orbit by $\mathrm{supp}_{\mathrm{o}}(\pi)$. Finally, for $\gamma \in \mathcal{P}_\lambda$, we write $\mathrm{supp}(\gamma)$ for $\mathrm{supp}(\mathrm{irr}(\gamma))$ and likewise for $\mathrm{supp}_{\mathrm{o}}(\gamma)$.

2.2. **The metaplectic group.** Let $Mp(2n,\mathbb{R})$ be the metaplectic group of rank n. It is a nontrivial central extension of order two of of $Sp(2n,\mathbb{R})$, defined by an explicit cocycle (see for example [Tor]).

We denote the projection $Mp(2n,\mathbb{R}) \longrightarrow Sp(2n,\mathbb{R})$ by $\mathbf{pr}$, and write $\mathbf{z}$ for the nontrivial element of $Mp(2n,\mathbb{R})$ in $\mathbf{pr}^{-1}(\{\mathrm{Id}_{2n}\})$, and by $\mathbf{x}$, $\mathbf{y}$ the elements in $\mathbf{pr}^{-1}(\{-\mathrm{Id}_{2n}\})$. The center $Z(Mp(2n,\mathbb{R}))$ of $Mp(2n,\mathbb{R})$ consists in the four elements $\{\mathbf{e},\mathbf{z},\mathbf{x},\mathbf{y}\}$. We will adopt the notational convention that preimages (in $Mp(2n,\mathbb{R})$) of subgroups of $Sp(2n,\mathbb{R})$ will be denoted by adding a tilde. We refer to [RT], Section 2 for structure theory, notations and conventions concerning $Mp(2n,\mathbb{R})$. A number of explicit choices are made there, such as representatives of conjugacy classes of Cartan subgroups, positive root systems, etc. Here we simply recall the notation

$$\Delta_a = \{\pm e_i \pm e_j,\ 1 \leq i < j \leq n\,;\ \pm 2e_i,\ 1 \leq i \leq n\}.$$

This is a root system of type C_n, and $e_1, \dots, e_n$ is a basis of $\mathfrak{h}_a^*$. We also fix a maximal compact subgroup $K_\mathbb{R} \simeq U(n)$ of $Sp(2n,R)$. Its preimage $\widetilde{K}_\mathbb{R}$ is a maximal compact subgroup of $Mp(2n,\mathbb{R})$. We denote the respective complexifications by K and $\widetilde{K}$.

Recall that a Harish-Chandra module for $\widetilde{G}_\mathbb{R}$ is called genuine if $\mathbf{z}$ acts by -1 (so that the module does not factor to the linear group $G_\mathbb{R}$). We denote the full subcategory of $\mathcal{HC}(\mathfrak{sp}(2n,\mathbb{C}),\widetilde{K})$ of genuine modules by $\mathcal{HC}(\mathfrak{sp}(2n,\mathbb{C}),\widetilde{K})^{\mathrm{gen}}$, its Grothendieck group by $\mathbb{K}\mathcal{HC}(\mathfrak{sp}(2n,\mathbb{C}),\widetilde{K})^{\mathrm{gen}}$, and its irreducible objects by $\mathrm{Irr}(Mp(2n,\mathbb{R}))^{\mathrm{gen}}$. Now let $\widetilde{H}_\mathbb{R}$ be a σ-stable Cartan subgroup of $\widetilde{G}_\mathbb{R}$ and let $(\widetilde{H}_\mathbb{R},\gamma) = (\Gamma,\overline{\gamma})$ be a λ-pseudocharacter. We say that γ is genuine if $\Gamma(\mathbf{z}) = -1$. It is essentially obvious that γ is genuine if and only if $\mathrm{std}(\gamma)$ (or $\mathrm{irr}(\gamma)$) is.

Because we will only be interested in genuine representations of the metaplectic group we will let $\mathcal{P}_\lambda^{Mp}$ denote the ($\widetilde{K}_\mathbb{R}$ conjugacy classes of) *genuine* λ-pseudocharacters for $\widetilde{G}_\mathbb{R}$.

2.3. **Notations for $SO(2n+1)$.** For each pair of positive integers p, q such that $p+q = 2n+1$, fix a real $2n+1$ dimensional vector space $V^{p,q}$ equipped with a non-degenerate symmetric bilinear form $(\cdot,\cdot)_{p,q}$ of signature (p,q) and let $V_\mathbb{C}^{p,q}$ be its complexification (endowed with the induced bilinear form). We let $SO(p,q)$ denote the special isometry group of $(\cdot,\cdot)_{p,q}$. Let $SO(2n+1,\mathbb{C})$ be the special isometry group of the usual bilinear form $(X,Y) \mapsto {}^tXY$ on $\mathbb{C}^{2n+1}$, and fix isomorphisms $\iota_{p,q} :\ V_\mathbb{C}^{p,q} \to \mathbb{C}^{2n+1}$ of orthogonal spaces. This induces an embedding

$$\iota_{p,q} :\ SO(p,q) \to SO(2n+1,\mathbb{C})$$

the image being a real form of $SO(2n+1,\mathbb{C})$. All real forms obtained in such a way are inner to the split form $SO(n+1,n)$, and if we fix a parity $\delta = \pm 1$, the image of the various

$V^{p,q}$ with $(-1)^q = \delta$ form a system of representatives of conjugacy classes of real forms of $SO(2n+1,\mathbb{C})$. We identify $SO(p,q)$ with its image in $SO(2n+1,\mathbb{C})$. Notice that since $SO(2n+1,\mathbb{C})$ is adjoint, the notion of real forms and strong real forms ([ABV], Chapter 2) coincide. Thus the definitions and results of [ABV] will apply to $\coprod' SO(p,q)$, where the prime indicates that the union is taken over all p and q with $p+q = 2n+1$ and $(-1)^q = \delta$.

Let $\mathfrak{h}_a^{SO}$ be an abstract Cartan subalgebra of $\mathfrak{so}(2n+1,\mathbb{C})$, let

$$\Delta_a^{SO} = \{e_i' - e_j',\, e_i' \,;\, 1 \leq i < j \leq n\}$$

be the root system of $\mathfrak{h}_a^{SO}$ in $\mathfrak{so}(2n+1,\mathbb{C})$ and let W_a^{SO} be its Weyl group.

Recall the abstract Cartan subalgebra $\mathfrak{h}_a$ of $\mathfrak{sp}(2n,\mathbb{C})$ and the basis $\{e_i\}_{1\leq i\leq n}$ of $\mathfrak{h}_a^*$. Let $\lambda \in \mathfrak{h}_a^*$ and write $(\lambda_1,\dots\lambda_n)$ for its coordinates in the basis $\{e_i\}_{1\leq i\leq n}$. Let $\theta_{\mathrm{ic}}(\lambda) \in (\mathfrak{h}_a^{SO})^*$ be the element with coordinates $(\lambda_1,\dots\lambda_n)$ in the basis $\{e_i'\}_{1\leq i\leq n}$. This defines a map $\theta_{\mathrm{ic}} : \mathfrak{h}_a^* \to (\mathfrak{h}_a^{SO})^*$.

Remark 2.3. Notice that λ is half-integral if and only if $\theta_{\mathrm{ic}}(\lambda)$ is integral. (For instance, if λ is the infinitesimal character of the oscillator for $Mp(2n,\mathbb{R})$, then $\theta_{\mathrm{ic}}(\lambda) = \rho_{SO}$.) Also note that if α is a short root in Δ_a then $\alpha' = \theta_{\mathrm{ic}}(\alpha)$ is a root in Δ_a^{SO} and if α is long then $\alpha' = \frac{1}{2}\theta_{\mathrm{ic}}(\alpha)$ is a root in Δ_a^{SO}. This gives a bijection $\alpha \leftrightarrow \alpha'$ from Δ_a to Δ_a^{SO}. When λ has half-integral coordinates, the integral root systems for λ and λ' are not in bijection.

2.4. **Adams-Barbasch correspondence.** Consider the dual pair

$$(O(p,q), Sp(2n,\mathbb{R})), \qquad p+q = 2n+1,$$

in $Sp(2n(2n+1),\mathbb{R})$, and denote their preimages in $Mp(2n(2n+1),\mathbb{R})$ by $(\widetilde{O}(p,q), Mp(2n,\mathbb{R}))$. Fix a choice of an oscillator representation for $Mp(2n(2n+1),\mathbb{R})$ (there are two non-equivalent such choices). The theta correspondence then gives a bijection between subsets of irreducible genuine representations of $\widetilde{O}(p,q)$ and $Mp(2n,\mathbb{R})$. It is relatively easy to see that every genuine representations of $Mp(2n,\mathbb{R})$ occurs in the correspondence, and that exactly half of those for $\widetilde{O}(p,q)$ do. This can be quantified more precisely as follows. Let π be a representation of $SO(p,q)$ with $p+q = 2n+1$, and let ξ denote a genuine character of $\widetilde{O}(p,q)$ which is trivial on the identity component. (The structure of the covering dictates that exactly two such characters exist.) There are two representations of $O(p,q)$ which restrict to π, and after tensoring with ξ, exactly one of them occurs in the theta correspondence. This gives a map from irreducible representations of the various $SO(p,q)$'s to genuine representations of $Mp(2n,\mathbb{R})$, and one in fact obtains a bijection

$$\theta : \mathrm{Irr}(Mp(2n,\mathbb{R}))_\lambda^{gen} \xrightarrow{\sim} \coprod{}' \mathrm{Irr}(SO(p,q))_{\theta_{\mathrm{ic}}(\lambda)}, \tag{2.4}$$

where the prime in the disjoint union means that we take only pairs of positive integers (p,q) such that $p+q = 2n+1$, $(-1)^q = \delta$ (δ as in Section 2.3), and $\theta_{\mathrm{ic}}(\lambda)$ is defined in Section 2.3. This bijection has been explicitly computed by Adams and Barbasch ([AB1], Theorem 5.1).

Let $\mathcal{P}_{\lambda'}^{SO} = \coprod' \mathcal{P}_{\lambda'}^{SO(p,q)}$ be the set of Langlands parameters for the various $SO(p,q)$'s ($p+q = 2n+1$, $(-1)^q = \delta$) at infinitesimal character $\lambda' = \theta_{\mathrm{ic}}(\lambda)$. We denote again by θ the bijection obtained from 2.4 on the level of Langlands parameters :

$$\theta : \mathcal{P}_\lambda^{Mp} \xrightarrow{\sim} \mathcal{P}_{\theta_{\mathrm{ic}}(\lambda)}^{SO}, \tag{2.5}$$

Notice that there were four choices involved in the definition of θ: two from the choice of oscillator representation and two from the choice of genuine character ξ. The effect of these choices can be extracted from [AB1].

Proposition 2.6. *Retain the notation introduced above, and let ζ denote the nontrivial character of each noncompact $SO(p,q)$ which is trivial on the identity component. (If $pq = 0$ adopt the convention that ζ is the trivial representation.)*

1. *Let θ be defined with respect to a fixed choice of oscillator and character ξ. Let θ' be defined with respect to the other oscillator and (the same) character ξ. Then $\theta^{-1} \circ \theta'$ coincides with twisting by an outer automorphism of $Mp(2n, \mathbb{R})$.*
2. *Let θ be defined with respect to a fixed choice of oscillator and character ξ. Let θ' be defined with respect to the same oscillator and a character $\xi' \neq \xi$ which coincides with ξ on the identity component of $\widetilde{O}(p,q)$. Then $\theta' \circ \theta^{-1}$ coincides with tensoring with ζ.*

For future reference, we need to record some qualitative features of the computation of [AB1]. Informally speaking, the bijection θ is the identity on the level of Langlands parameters. When one considers more refined representation theoretic information, the way in which θ differs from the identity is confined only to the genuine discrete series of $Mp(2n, \mathbb{R})$. If we avoid the kind of infinitesimal character where such discrete series can infiltrate pseudocharacters (by requiring none of the coordinates to be half-integers), these differences disappear. This will be of crucial importance in Section 4.

Theorem 2.7. *Let π, π_1, and π_2 be irreducible genuine representations of $Mp(2n, \mathbb{R})$. The following statements hold independent of the choices (Proposition 2.6) defining θ.*

1. *θ behaves well with respect to the pseudocharacter parameterization of Section 2.1. More precisely, if π is attached to a Cartan subgroup whose image under* **pr** *(Notation 2.2) is isomorphic to $(\mathbb{R}^*)^m \times (\mathbb{C}^*)^r \times (S^1)^s$, then $\theta(\pi)$ is attached to a Cartan subgroup isomorphic to $(\mathbb{R}^*)^m \times (\mathbb{C}^*)^r \times (S^1)^s$. In particular,*
   (a) *π is a discrete series representation if and only if $\theta(\pi)$ is.*
   (b) *π is an irreducible Langlands quotient of a principal series if and only if $\theta(\pi)$ is.*
2. *θ behaves well with respect to supports. More precisely,*
$$\mathrm{supp}_{\mathrm{o}}(\pi_1) = \mathrm{supp}_{\mathrm{o}}(\pi_2) \iff \mathrm{supp}_{\mathrm{o}}(\theta(\pi_1)) = \mathrm{supp}_{\mathrm{o}}(\theta(\pi_2))$$
3. *θ behaves well with respect to τ-invariants. More precisely, suppose π has infinitesimal character λ, and that α is a simple integral root for λ such that α' is a simple integral root for λ' (see remark 2.3). Then*
$$\alpha \in \tau(\pi) \Longleftrightarrow \alpha' \in \tau(\theta(\pi)).$$
   Moreover, suppose that either λ has no coordinates which are half-integers, or λ is arbitrary but α is short. Then α is a type II noncompact imaginary for π if and only if α' is type II noncompact imaginary for $\theta(\pi)$. The same conclusion holds for type I noncompact imaginary roots, compact imaginary roots, complex roots, and type I and type II roots satisfying (or not satisfying) the parity condition.
4. *Suppose that either no coordinate of λ is a half-integer and α is a simple integral root; or that λ arbitrary and α is short, simple, and integral. Then θ behaves well with respect to Cayley transforms and the cross action in α. More precisely, retain the assumption on λ and α, write α' as in Remark 2.3, and fix $\gamma \in \mathcal{P}_{\lambda}^{Mp}$ (notation as in Section 2.1.1). Write $\theta(irr_{Mp}(\gamma)) = irr_{SO}(\theta(\gamma))$ for some $\theta(\gamma) \in \mathcal{P}^{SO}_{\theta_{\mathrm{ic}}(\lambda)}$. Then*
   (a) *We have*
$$\theta(s_\alpha \times \gamma) = s_{\alpha'} \times \theta(\gamma).$$

(b) *If α is type II noncompact imaginary for γ, write $c_\alpha(\gamma) = \{\gamma_+^\alpha, \gamma_-^\alpha\}$ for the Cayley transform of γ through λ, and likewise for $c_{\alpha'}(\theta(\gamma))$. Then*

$$c_{\alpha'}(\theta(\gamma)) = \{\theta(\gamma_+^\alpha), \theta(\gamma_-^\alpha)\}.$$

The obvious statement holds for type I noncompact imaginary roots, and for real roots and inverse Cayley transforms.

We refer to [Vgr] or [RT] for material concerning Cayley transforms and cross action.

3. L-packets, microlocal L-packets, and stability: the half-integral case

In this section, we fix $\lambda \in (\mathfrak{h}_a^{Sp})^*$ regular, half-integral and dominant. (The reduction of singular case to the regular case is discussed in Remark 3.20 below.) Write X for the flag manifold of $\mathfrak{sp}(2n,\mathbb{C})$, and recall the complexified maximal compact subgroups K and $\widetilde{K}$ of Section 2.2. Note that since the action of $\widetilde{K}$ on $\mathfrak{sp}(2n,\mathbb{C})$ factors through K, the orbits of $\widetilde{K}$ on X coincide with those of K. It is very important to note that $\mathrm{supp}_o(\pi) \in K\backslash X$ for π in *either* $\mathrm{Irr}(Mp(2n,\mathbb{R}))$ or $\mathrm{Irr}(Sp(2n,\mathbb{R}))$.

3.1. **Geometric reformulation of** [RT]**.** In this section, we give a clean geometric definition of the duality of [RT]. (This does not circumvent [RT]: to prove that the definition given below has the crucial representation theoretic properties of Theorem 3.16 requires the full development given in [RT].)

Let $\lambda' = \theta_{\mathrm{ic}}(\lambda) \in \mathfrak{h}_a^{SO}$ (notation as in Section 2.3). The hypothesis on λ above imply that λ' is a regular integral dominant weight. According to [V4] (applied to the case under consideration), there is a bijection:

$$\mathbf{D}_{SO}\ :\ \mathcal{P}_{\lambda'}^{SO} \to \mathcal{P}_\rho^{Sp}, \qquad \gamma \mapsto \gamma^\vee.$$

such that the multiplicity matrices are essentially inverse transposes of each other:

$$[\mathrm{irr}(\delta)] = \sum_{\gamma\in\mathcal{P}_{\lambda'}^{SO}} M(\gamma,\delta)[\mathrm{std}(\gamma)] \quad \text{iff} \quad [\mathrm{std}(\gamma^\vee)] = \sum_{\delta^\vee\in\mathcal{P}_\rho^{Sp}} \epsilon_{\gamma\delta} M(\gamma,\delta)[\mathrm{irr}(\delta^\vee)];$$

here $\epsilon(\gamma,\delta) = (-1)^{d(\gamma,\delta)}$ with

$$d(\gamma,\delta) = \dim \mathrm{Supp}(\mathrm{irr}(\delta)) - \dim \mathrm{Supp}(\mathrm{irr}(\gamma)).$$

As a matter of notation, we will also write $\pi \mapsto \pi^\vee$ for the corresponding map between $\coprod' \mathrm{Irr}(SO(p,q))_{\lambda'}$ and $\mathrm{Irr}(Sp(2n,\mathbb{R}))_\rho$, and refer to $\pi^\vee$ as the dual of π. We write $\mathbf{D}_{p,q}$ for the restriction of $\mathbf{D}_{SO}$ to $\mathcal{P}_{\lambda'}^{SO(p,q)}$.

It will be important for us to pin down the choices involved in defining $\mathbf{D}_{SO}$. Each noncompact $SO(p,q)$ has two connected components and hence two one-dimensional representations with trivial infinitesimal character, the trivial representations and another one we denote by $\zeta_{p,q}$ (or just ζ if the context is clear). For notational convenience fix an outer automorphism of $Sp(2n,\mathbb{R})$, say $\mathbf{A}$, and also write $\mathbf{A}$ for the induced action on $K\backslash X$ and $\mathrm{Irr}(Sp(2n,\mathbb{R}))$.

Lemma 3.1. *There are two possibilities for the restriction of $\mathbf{D}_{SO}$ to each $\mathcal{P}_{\lambda'}^{SO(p,q)}$ with $pq \neq 0$. Denote them by $\mathbf{D}_{p,q}$ and $\mathbf{D}'_{p,q}$. Then $\mathbf{D}'_{SO(p,q)} \circ \mathbf{D}_{SO(p,q)}^{-1} = \mathbf{A}$.*

Proof. This follows easily from [V4]. □

Since there are n noncompact inner forms of SO of a fixed signature, there are 2^n possible choices for $\mathbf{D}_{SO}$. There is a more sophisticated approach to the duality of [V4] which reduces the number of choices of $\mathbf{D}_{SO}$ to just two. This is explained in [ABV], but since it is difficult to extract our particular example from that reference, we recall a few concrete details. Choose a large discrete series, say π_L, of $Sp(2n,\mathbb{R})$; there are two such choices, namely π_L and $\pi'_L := \mathbf{A}(\pi_L)$. They are completely distinguished by their associated varieties. More precisely, there are two $GL(n,\mathbb{C})$ principal orbits (i.e. orbits of maximal dimension) on the nilpotent cone in the complexified Cartan $\mathfrak{p}$ for $Sp(2n,\mathbb{R})$, say $\mathbb{O}$ and $\mathbb{O}'$. We may arrange the labeling so that $\mathrm{AV}(\pi_L) = \overline{\mathbb{O}}$ and $\mathrm{AV}(\pi_L) = \overline{\mathbb{O}'}$.

Fix a choice of $\mathbf{D}_{p,q}$, and let $\mathbf{1}_{p,q}$ denote the trivial representation of $SO(p,q)$. From the general principles of [V4], $\mathbf{1}_{p,q}^{\vee} := \mathbf{D}_{p,q}(\mathbf{1}_{p,q})$ has maximal GK-dimension, and hence its associated variety is either $\overline{\mathbb{O}}$, $\overline{\mathbb{O}'}$, or $\overline{\mathbb{O}}\cup\overline{\mathbb{O}'}$. In fact, this latter possibility occurs if and only if $pq = 0$. This is probably an instance of a general kind of coinduction statement, but here we indicate a direct argument. It is not difficult to see that $\mathbf{1}_{p,q}^{\vee}$ is cohomologically induced (in the good range) from the external tensor product of a large discrete series of a (quasisplit) $U(r,s)$ and the irreducible principal series of a smaller $Sp(2m,\mathbb{R})$; here $r+s+m = n$, and $r+s = \min\{p,q\}$, the real rank of $SO(p,q)$. The irreducible principal series has a reducible associated variety (equal to the union of the closure of the two principal orbits for the smaller symplectic group). By localizing, it is easy to see that associated varieties of cohomologically induced modules are induced in an appropriate sense. In the current context, we conclude that $\mathrm{AV}(\mathbf{1}_{p,q}^{\vee})$ is induced from the product of a principal orbit in a quasisplit $U(r,s)$ and the union of the two principal orbits for $Sp(2m,\mathbb{R})$. Using, for instance, the orbit induction computations of [T2], it is easy to see that $\mathrm{AV}(\mathbf{1}_{p,q}^{\vee})$ is irreducible exactly when $r+s \neq 0$ (equivalently, $pq \neq 0$).

Definition 3.2. Fix a choice of large discrete series π_L as above. We say that a choice of $\mathbf{D}_{p,q}$ is of type π_L if $\mathrm{AV}(\mathbf{D}_{p,q}(\mathbf{1}_{p,q})) = \mathrm{AV}(\pi_L)$. We say that $\mathbf{D}_{SO}$ is pure of type π_L if its restriction to each $SO(p,q)$, $pq \neq 0$, is of type π_L. By the above considerations, there are exactly two pure choices of $\mathbf{D}_{SO}$ corresponding to the two choices of π_L. From Lemma 3.1, it is not difficult to verify that the two choices differ by $\mathbf{A}$.

The pure choices of $\mathbf{D}_{SO}$ have nice properties that we shall exploit below. For now consider the map

$$\Delta \;:\; \mathrm{Irr}(Mp(2n,\mathbb{R}))^{\mathrm{gen}}_{\lambda} \longrightarrow K\backslash X \times K\backslash X \tag{3.3}$$

defined by

$$\pi \mapsto \big(\mathrm{supp}_{\mathrm{o}}(\pi), \mathrm{supp}_{\mathrm{o}}(\theta(\pi)^{\vee})\big).$$

Note that all together there are 2^{n+2} different choices in the definition of Δ : four from the choices defining θ (see Proposition 2.6) and 2^n from the choice of $\mathbf{D}_{SO}$. (One should also be reminded of the choice of δ in Section 2.3. This is fixed once and for all, and is not relevant here.) Write ι for the involution of $K\backslash X \times K\backslash X$ that switches factors: $\iota(Q,Q') = (Q',Q)$.

Proposition 3.4. *The map Δ is injective. Moreover, $\iota(\mathrm{im}(\Delta)) = \mathrm{im}(\Delta)$.*

Sketch. One can prove the proposition by an explicit and more or less straightforward calculation. This is messy of course, so we sketch a slightly more abstract approach.

We first prove injectivity. By Theorem 2.7(2), injectivity is equivalent to the following assertion for $\pi_1, \pi_2 \in \coprod' \operatorname{Irr}(SO(p,q))_{\lambda'}$,

$$(\operatorname{supp}_o(\pi_1), \operatorname{supp}_o(\pi_1^\vee)) = (\operatorname{supp}_o(\pi_2), \operatorname{supp}_o(\pi_2^\vee)) \iff \pi_1 = \pi_2.$$

But $(\operatorname{supp}_o(\pi_1), \operatorname{supp}_o(\pi_1^\vee))$ is simply a geometric reformulation of what Adams-Barbasch-Vogan ([A1], Definition 2.9) call a set of L-data for SO. Hence [A1], Theorem 2.12, implies the indicated injectivity statement.

To prove the final assertion of the proposition, we first observe that ι is a bijection between

$$\operatorname{Im}(\Delta; \mathrm{ds}) := \Delta(\{\pi \mid \pi \text{ is a discrete series}\})$$

and

$$\operatorname{Im}(\Delta; \mathrm{ps}) := \Delta(\{\pi \mid \pi \text{ is a quotient of a principal series}\}).$$

More precisely, if we denote the open orbit of K on X by Q_o, it is easy to check from Theorem 2.7(1a–b), and the fact that $\mathbf{D}_{SO}$ interchanges discrete and principal series representations, that

$$\operatorname{Im}(\Delta; \mathrm{ds}) = \{(Q_o, Q') \mid Q' \text{ is closed}\},$$

and likewise that

$$\operatorname{Im}(\Delta; \mathrm{ps}) = \{(Q', Q_o) \mid Q' \text{ is closed}\}.$$

Hence the observation above follows.

Next suppose $n = 2k$ and define $\operatorname{Im}(\Delta; \mathbb{C}^*)$ to be the image of Δ applied to all representations which attached to a Cartan subgroup whose image under $\mathbf{pr}$ is isomorphic to $(\mathbb{C}^*)^k$. Now θ is a bijection between such representations for $Mp(2n,\mathbb{R})$ and such representations for SO (Theorem 2.7(1)); and $\mathbf{D}_{SO}$ is a bijection between such representations for SO and those for Sp. Hence we conclude that

$$\iota(\operatorname{Im}(\Delta; \mathbb{C}^*)) = \operatorname{Im}(\Delta; \mathbb{C}^*).$$

The previous two paragraphs prove that ι preserves the image of Δ restricted to those representations attached to Cartan subgroups whose image under $\mathbf{pr}$ is either $(\mathbb{R}^*)^n$, $(\mathbb{C}^*)^k$, or $(S^1)^n$. The general case of the final assertion of the proposition follows by combining these three cases. We omit the details. ☐

As a consequence of Proposition 3.4, we obtain an involution

$$\mathbf{D}_{Mp} := \Delta^{-1} \circ \iota \circ \Delta \; : \; \operatorname{Irr}(Mp(2n,\mathbb{R}))_\lambda^{gen} \longrightarrow \operatorname{Irr}(Mp(2n,\mathbb{R}))_\lambda^{gen} \tag{3.5}$$

On the level of pseudocharacters, we also write

$$\mathbf{D}_{Mp} \; : \; \mathcal{P}_\lambda^{Mp} \longrightarrow \mathcal{P}_\lambda^{Mp}.$$

Again there are 2^{n+2} buried in the definition of $\mathbf{D}_{Mp}$, but as the second part of the next lemma indicates, only the pure choices have good representation theoretic applications.

Lemma 3.6. *Fix $\gamma \in \mathcal{P}_\lambda^{Mp}$, and recall that λ is regular and half-integral. Fix a choice of θ and $\mathbf{D}_{SO}$ defining $\mathbf{D}_{Mp}$.*

1. *Let α be a short root in Δ_a^{Sp} (and hence α is integral for λ). Then, independent of the choices defining $\mathbf{D}_{Mp}$,*
   (a) *$\mathbf{D}_{Mp}$ commutes with cross action in α,*

$$s_\alpha \times \mathbf{D}_{Mp}(\gamma) = \mathbf{D}_{Mp}(s_\alpha \times \gamma).$$

(b) *If α is noncompact imaginary for γ, then α is real and satisfies the parity condition for $\mathbf{D}_{Mp}(\gamma)$. Moreover $\mathbf{D}_{Mp}$ intertwines Cayley and inverse Cayley transforms,*

$$\mathbf{D}_{Mp}(\gamma^\alpha) = \left[\mathbf{D}_{Mp}(\gamma)\right]_\alpha.$$

2. *Let α be a long imaginary or real root for γ (hence half-integral), and let T_α denote the (infinitesimal character preserving) nonintegral wall cross in α. Then $\mathbf{D}_{Mp}$ commutes with T_α,*

$$(T_\alpha \circ \mathbf{D}_{Mp})(\gamma) = (\mathbf{D}_{Mp} \circ T_\alpha)(\gamma),$$

if and only if the choice of $\mathbf{D}_{SO}$ defining $\mathbf{D}_{Mp}$ is pure in the sense of Definition 3.2.

Proof. We first consider (1a). Retaining the hypothesis and notation of the current lemma, Proposition 3.4 implies that it is enough to establish

$$\operatorname{supp}_{\mathrm{o}}\big(s_\alpha \times \mathbf{D}_{Mp}(\gamma)\big) = \operatorname{supp}_{\mathrm{o}}\big(\mathbf{D}_{Mp}(s_\alpha \times \gamma)\big); \text{ and} \tag{3.7}$$

$$\operatorname{supp}_{\mathrm{o}}\big(\mathbf{D}_{SO}(\theta(s_\alpha \times \mathbf{D}_{Mp}(\gamma)))\big) = \operatorname{supp}_{\mathrm{o}}\big(\mathbf{D}_{SO}(\theta(\mathbf{D}_{Mp}(s_\alpha \times \gamma)))\big). \tag{3.8}$$

Consider (3.7). We have

$$\begin{aligned} \operatorname{supp}_{\mathrm{o}}\big(\mathbf{D}_{Mp}(s_\alpha \times \gamma)\big) &= \operatorname{supp}_{\mathrm{o}}\big(\mathbf{D}_{SO}(\theta(s_\alpha \times \gamma))\big) && \text{by definition of } \mathbf{D}_{Mp} \\ &= \operatorname{supp}_{\mathrm{o}}\big(\mathbf{D}_{SO}(s_{\alpha'} \times \theta(\gamma))\big) && \text{by Theorem 2.7(4)} \\ &= \operatorname{supp}_{\mathrm{o}}\big(s_\alpha \times \mathbf{D}_{SO}(\theta(\gamma))\big) && \text{by the general properties of [V4].} \end{aligned}$$

Recall that, by definition, $\operatorname{supp}_{\mathrm{o}}(\mathbf{D}_{Mp}(\gamma)) = \operatorname{supp}_{\mathrm{o}}(\mathbf{D}_{SO}(\theta(\gamma)))$. It is a general property of the cross action that if π and π' have the same support, then so do $s_\alpha \times \pi$ and $s_\alpha \times \pi'$; so (3.7) follows from the previous sentence and the previous displayed equation. The proof of (3.8) and the statement in (1b) is entirely similar. We omit the details.

It is important to point out that purity was not needed in the above argument. The reason is that if $\theta(\gamma)$ is a (conjugacy class of) pseudocharacter of $SO(p,q)$, then $\theta(s_\alpha \times \gamma) = s_{\alpha'} \times \theta(\gamma)$ is a pseudocharacter of the *same* inner form, $SO(p,q)$, so there is no issue of the compatibility of the restriction of $\mathbf{D}_{SO}$ to the various inner forms. This is not the case if the cross action is replaced by T_α as in (2), and purity necessarily plays a role. Definition 3.2 is difficult to apply, and we need an alternative characterization of purity in the course of the proof of (2). This requires some auxiliary ideas.

Given an orbit $Q \in K\backslash X$, let Q_α denote the image of Q under the projection of X to the partial flag variety defined by a long root α. Since α defines a subgroup of $Sp(2n,\mathbb{C})$ isomorphic to $Sp(2,\mathbb{C})$, it gives rise to an inclusion ψ_α of $\mathbb{P}^1$ into X. Let Q^α denote the intersection of Q with the image of ψ_α. If α is an imaginary root for orbits Q and Q', then $Q = Q'$ if and only if $Q_\alpha = Q'_\alpha$ and $Q^\alpha = (Q')^\alpha$. The same conclusion holds if α is a real root for both Q and Q'. (In the real case, if $Q_\alpha = Q'_\alpha$, one always has $Q^\alpha = (Q')^\alpha$, so this condition is vacuous.)

Let T_α denote the involution of $\mathcal{P}_\lambda^{Mp}$ corresponding to the infinitesimal character preserving translation functor that crosses the α wall. We thus obtain an involution $T'_\alpha := \theta \circ T_\alpha \circ \theta^{-1}$ of $\mathcal{P}_{\lambda'}^{SO}$. We claim that the choice of $\mathbf{D}_{SO}$ is pure if and only if for any $\gamma \in \mathcal{P}_{\lambda'}^{SO}$ for which α' is a real (short) root,

$$\operatorname{supp}_{\mathrm{o}}(\mathbf{D}_{SO}(\gamma))_\alpha = \operatorname{supp}_{\mathrm{o}}\big(\mathbf{D}_{SO}(T'_\alpha(\gamma))\big)_\alpha. \tag{3.9}$$

The important point to notice is that γ and $T'_\alpha(\gamma)$ generally live on different real forms.

The proof of (3.9) is sketched below. For now, let us assume it holds and deduce (2) from it. The first step is to establish (2) directly for $Mp(2,\mathbb{R})$ and for the eight possible choices of θ and $\mathbf{D}_{SO}$. (In rank one, every choice of $\mathbf{D}_{SO}$ is pure.) This is lengthy but entirely elementary, and we omit the details. Return to the general case and fix a pure choice of $\mathbf{D}_{SO}$ defining $\mathbf{D}_{Mp}$. Let π_1 be a genuine irreducible representation of $Mp(2n,\mathbb{R})$ for which α is real root. (The argument for α imaginary is identical to this case and we make no further mention of it.) Set $\pi_2 = \mathbf{D}_{Mp}(\pi_1)$. Unwinding the definitions, this means that

$$\text{supp}_{\text{o}}(\pi_1) = \text{supp}_{\text{o}}\big(\mathbf{D}_{SO}(\theta(\pi_2))\big); \text{ and} \tag{3.10}$$

$$\text{supp}_{\text{o}}(\pi_2) = \text{supp}_{\text{o}}\big(\mathbf{D}_{SO}(\theta(\pi_1))\big). \tag{3.11}$$

We are trying to prove

$$\text{supp}_{\text{o}}(T_\alpha(\pi_1)) = \text{supp}_{\text{o}}\big[\mathbf{D}_{SO}\big(\theta(T_\alpha(\pi_2))\big)\big]; \text{ and} \tag{3.12}$$

$$\text{supp}_{\text{o}}(T_\alpha(\pi_2)) = \text{supp}_{\text{o}}\big[\mathbf{D}_{SO}\big(\theta(T_\alpha(\pi_1))\big)\big]. \tag{3.13}$$

Because α is real for π_1, the proof of Proposition 3.4 shows α is imaginary for π_2 and hence the general properties of $\mathbf{D}_{SO}$ together with Theorem 2.7 imply α is real for $\text{supp}_{\text{o}}\big(\mathbf{D}_{SO}(\theta(T_\alpha(\pi_2)))\big)$.

We first treat (3.12). By the above comments, to establish (3.12), it suffices to prove

$$\text{supp}_{\text{o}}(T_\alpha(\pi_1))^\alpha = \text{supp}_{\text{o}}\big[\mathbf{D}_{SO}\big(\theta(T_\alpha(\pi_2))\big)\big]^\alpha; \text{ and} \tag{3.14}$$

$$\text{supp}_{\text{o}}(T_\alpha(\pi_1))_\alpha = \text{supp}_{\text{o}}\big[\mathbf{D}_{SO}\big(\theta(T_\alpha(\pi_2))\big)\big]_\alpha. \tag{3.15}$$

Because of the explicit computation of T_α given in [V4], Corollary 4.8 and Lemma 4.9, (3.14) reduces to the $Mp(2,\mathbb{R})$ case already treated. (Of course one must use (3.10) and (3.11) in the argument.) Consider now the left-hand side of (3.15). Again by the explicit computation of T_α in [V4], it is easy to check that $\text{supp}_{\text{o}}(T_\alpha(\pi_1)))_\alpha = \text{supp}_{\text{o}}(\pi_1)_\alpha$. Turning to the right-hand side of (3.15),

$$\begin{aligned} \text{supp}_{\text{o}}\big[\mathbf{D}_{SO}\big(\theta(T_\alpha(\pi_2))\big)\big]_\alpha &= \text{supp}_{\text{o}}\big[\mathbf{D}_{SO}\big(T'_\alpha(\theta(\pi_2))\big)\big]_\alpha && \text{by definition of } T'_\alpha \\ &= \text{supp}_{\text{o}}\big[\mathbf{D}_{SO}\big(\theta(\pi_2)\big)\big]_\alpha && \text{by (3.9).} \end{aligned}$$

Thus (3.15) is equivalent to

$$\text{supp}_{\text{o}}(\pi_1)_\alpha = \text{supp}_{\text{o}}\big[\mathbf{D}_{SO}\big(\theta(\pi_2)\big)\big]_\alpha.$$

But this is guaranteed by (3.10). This completes the proof of (3.12). The proof of (3.13) is identical.

Thus it remains to establish that (3.9) holds if and only if $\mathbf{D}_{SO}$ is pure. This is extremely complicated, and we give only a rough sketch. Start with a representation π of $SO(p,q)$ for which α is real. It may happen that $\pi' := T'_\alpha(\pi)$ is also a representation of $SO(p,q)$. If this is the case, then the computation of T_α in [V1] mentioned above shows that π' is the inverse Cayley transform (through α) of π and (3.9) follows from the computation of T_α and general properties of [V4]. (No assumption of purity is required here.) So we may suppose that π' is a representation of $SO(p',q') \neq SO(p,q)$. We may find a sequence of cross actions and Cayley transforms that takes π' to $\mathbf{1}_{p',q'}$, the trivial representation of $SO(p',q')$. One may check that it makes sense to apply this same sequence (suitably interpreted) to π to arrive at a new representation π_1 which differs from $\mathbf{1}_{p,q}$ by a Cayley transform. Suppose this sequence takes the root α to the root β which again will necessarily be long. It is enough to check that $\text{supp}_{\text{o}}(\mathbf{D}_{SO}(\pi_1))_\beta = \text{supp}_{\text{o}}(\mathbf{D}_{SO}(\mathbf{1}_{p',q'}))_\beta$ if and only if $\mathbf{D}_{p,q}$ and $\mathbf{D}_{p',q'}$ are of the same type. Suppose the latter is of type π_L. We need to recall the distance between two element Q and Q' in $K\backslash X$. This is the minimal length of a saturated chain between Q

and Q'. (By saturated chain, we mean a sequence of elements in $K\backslash X$ beginning at Q and ending at Q'' such that if Q_1 and Q_2 are successive terms in the sequence, either $Q_1 > Q_2$ or $Q_2 > Q_1$ in the closure order and, moreover, the complex dimensions of Q_1 and Q_2 differ by exactly one.) Because $\mathbf{D}_{p',q'}$ is of type π_L, the distance between $\mathrm{supp}_o(\mathbf{D}_{SO}(\mathbf{1}_{p',q'}))$ and $\mathrm{supp}_o(\pi_L)$ is shorter than the distance between $\mathrm{supp}_o(\mathbf{D}_{SO}(\mathbf{1}_{p',q'}))$ and $\mathrm{supp}_o(\pi'_L)$. Since π_1 differs from $\mathbf{1}_{p,q}$ by only a Cayley transform, we know $\mathrm{supp}_o(\mathbf{D}_{SO}(\pi_1))$ will be closer to $\mathrm{supp}_o(\pi_L)$ than $\mathrm{supp}_o(\pi'_L)$ if and only if $\mathbf{D}_{p,q}$ is of type π_L. It then remains to show that the fact that both $\mathrm{supp}_o(\mathbf{D}_{SO}(\mathbf{1}_{p',q'}))$ and $\mathrm{supp}_o(\mathbf{D}_{SO}(\pi_1))$ are closer to $\mathrm{supp}_o(\pi_L)$ than $\mathrm{supp}_o(\pi'_L)$ implies (3.9). This follows because one can directly establish that *either* (3.9) holds or the equality in (3.9) holds with the right-hand side twisted by $\mathbf{A}$. But if the this latter case holds, it contradicts the fact that both $\mathrm{supp}_o(\mathbf{D}_{SO}(\mathbf{1}_{p',q'}))$ and $\mathrm{supp}_o(\mathbf{D}_{SO}(\pi_1))$ are closer to $\mathrm{supp}_o(\pi_L)$ than $\mathrm{supp}_o(\pi'_L)$. □

We now turn to the representation theoretic significance of $\mathbf{D}_{Mp}$. The deep and miraculous fact is that for pure choices of $\mathbf{D}_{SO}$, $\mathbf{D}_{Mp}$ implements a character multiplicity duality.

Theorem 3.16. *Recall the assumption that the infinitesimal character λ is half-integral and regular. Fix θ and a pure choice of $\mathbf{D}_{SO}$ (Definition 3.2), and define $\mathbf{D}_{Mp}$ with respect to these choices (Equation 3.5). On the level of pseudocharacters write $\gamma^\vee = \mathbf{D}_{Mp}(\gamma)$. Then*

$$(3.17)\qquad [irr(\delta)] = \sum_{\gamma\in\mathcal{P}^{Mp}_\lambda} M(\gamma,\delta)[std(\gamma)] \quad \textit{iff} \quad [std(\gamma^\vee)] = \sum_{\delta^\vee\in\mathcal{P}^{Mp}_\lambda} \epsilon_{\gamma\delta} M(\gamma,\delta)[irr(\delta^\vee)];$$

where $\epsilon(\gamma,\delta) = (-1)^{d(\gamma,\delta)}$ with

$$d(\gamma,\delta) = \dim \mathrm{supp}(irr(\delta)) - \dim \mathrm{supp}(irr(\gamma)).$$

Sketch. We first recall that in [RT], Definition 6.31, we defined an involution

$$\mathbf{D}_{\mathrm{RT}} \,:\, \mathrm{Irr}(Mp(2n,\mathbb{R}))^{gen}_\lambda \longrightarrow \mathrm{Irr}(Mp(2n,\mathbb{R}))^{gen}_\lambda .$$

On the level of pseudocharacters, write

$$\mathbf{D}_{\mathrm{RT}} \,:\, \mathcal{P}^{Mp}_\lambda \longrightarrow \mathcal{P}^{Mp}_\lambda$$

Actually there were two such involutions defined in [RT] and they differed by composing with $\mathbf{A}$ (see [RT], Remark 6.35) and it is impossible to make a distinguished choice among these two. For either choice, the main result of [RT] shows that $\mathbf{D}_{\mathrm{RT}}$ implements a character multiplicity duality in the sense that of Equation (3.17). Fix θ and a pure choice of $\mathbf{D}_{SO}$ defining $\mathbf{D}_{Mp}$. Our task is to show that there is a choice of $\mathbf{D}_{\mathrm{RT}}$ such that $\mathbf{D}_{\mathrm{RT}} = \mathbf{D}_{Mp}$.

First we comment on the definition of $\mathbf{D}_{\mathrm{RT}}$. Let π be any genuine discrete series, let Π_s denote the set of simple short roots in the abstract root system, and let $\tau_s(\pi)$ denote the subset of Π_s consisting of those roots roots in the τ invariant of π. Remark 6.34 of [RT] implies a choice of $\mathbf{D}_{\mathrm{RT}}$ is completely characterized by two conditions: $\mathbf{D}_{\mathrm{RT}}(\pi)$ is a principal series such that $\tau_s(\mathbf{D}_{\mathrm{RT}}(\pi)) = \{\alpha \in \Pi_s \mid \alpha \notin \tau_s(\pi)\}$; and $\mathbf{D}_{\mathrm{RT}}$ satisfies the conclusion of Lemma 3.6.

Let π denote the genuine discrete series of $Mp(2n,\mathbb{R})$ at infinitesimal character λ whose image under θ is the representation with infinitesimal character λ' of the compact inner form of SO with discriminant δ. Since every simple root is in the τ invariant of π', Theorem 2.7(3) implies that every short simple root is in the τ-invariant of π. From Theorem 2.7(1), we know π is a discrete series, and so from the proof of Proposition 3.4 we conclude that $\mathbf{D}_{Mp}(\pi)$ is a principal series. From the definition of $\mathbf{D}_{Mp}$, we know that $\theta(\mathbf{D}_{Mp}(\pi)) = \mathbf{D}_{SO}(\pi')$. Again from Theorem 2.7(3) together with the fact ([V4]) that $\mathbf{D}_{SO}$ inverts τ-invariant, we conclude that no short simple root is in the τ-invariant of $\mathbf{D}_{Mp}(\pi)$. From the characterization of $\mathbf{D}_{\mathrm{RT}}$ in the previous paragraph, Lemma 3.6 implies that $\mathbf{D}_{Mp}$ coincides with a choice of $\mathbf{D}_{\mathrm{RT}}$. □

The reader will note the strong analogy of this theorem with the symmetric reformulation, due to Adams-Vogan (see [A1]), of [V4]. From this perspective, it is clear that the set $K\backslash X$ plays the role of the set of Langlands-Weil parameters for L-packets of $Mp(2n,\mathbb{R})$ with infinitesimal character λ. We pursue this in the next section.

We conclude by examining the effects of the choices defining $\mathbf{D}_{Mp}$. Recall that there were two pure choices of $\mathbf{D}_{SO}$ and four choices of θ. The following propositions indicate that modifying the choices amounts to twisting $\mathbf{D}_{Mp}$ by the outer automorphism $\mathbf{A}$.

Proposition 3.18. *Fix a choice of θ. Let $\mathbf{D}_{Mp}$ be defined with respect to θ and a pure choice of $\mathbf{D}_{SO}$, and let $\mathbf{D}'_{Mp}$ be defined with respect to the other pure choice. Then $\mathbf{D}'_{Mp} = \mathbf{A}\circ\mathbf{D}_{Mp}$.*

Proof. Retain the notation of the proof of Theorem 3.16. From Lemma 3.1, it is easy to check that $\mathbf{D}_{\mathrm{RT}}(\pi) \neq \mathbf{D}'_{\mathrm{RT}}(\pi)$. Yet both $\mathbf{D}_{Mp}$ and $\mathbf{D}'_{Mp}$ coincide with some choice of $\mathbf{D}_{\mathrm{RT}}$. Since these choices differ by $\mathbf{A}$, the proposition follows. □

Unlike the dichotomy of Proposition 2.6, both kinds of choices for θ amount to the outer automorphism of $Mp(2n,\mathbb{R})$.

Proposition 3.19. *Recall that the definition of $\mathbf{D}_{Mp}$ depends on a choice of θ and $\mathbf{D}_{SO}$. For the following statements, fix a pure choice of $\mathbf{D}_{SO}$.*

1. *Retain the notation of Proposition 2.6(1), and let $\mathbf{D}_{Mp}$ and $\mathbf{D}'_{Mp}$ denote the corresponding involutions of $\mathcal{P}^{Mp}_{\lambda}$. Then $\mathbf{D}'_{Mp} = \mathbf{A}\circ\mathbf{D}_{Mp}$.*
2. *Retain the notation of Proposition 2.6(2), and let $\mathbf{D}_{Mp}$ and $\mathbf{D}'_{Mp}$ denote the corresponding involutions of $\mathcal{P}^{Mp}_{\lambda}$. Then $\mathbf{D}'_{Mp} = \mathbf{A}\circ\mathbf{D}_{Mp}$.*

Proof. Consider (1) and retain the notation of the proof of Theorem 3.16. From Proposition 2.6(1), it is easy to check that $\mathbf{D}_{\mathrm{RT}}(\pi) \neq \mathbf{D}'_{\mathrm{RT}}(\pi)$. Now the conclusion of the proposition follows exactly as in the proof of Proposition 3.18. The proof of (2) is identical. □

Remark 3.20. Suppose λ is not regular, but still half-integral. The situation is then analogous to the case of parabolic-singular duality for linear groups. Since this contains some ideas only implicit in [V4], we briefly recall them. Suppose a representation π of a linear group $G_{\mathbb{R}}$ has singular infinitesimal character λ_s. Let $G^\vee(\lambda_s)$ denote the centralizer of $\exp(2\pi i\lambda_s)$ in the complex dual group $G^\vee$, and write $P^\vee(\lambda_s)$ for the parabolic subgroup of $G^\vee$ defined by the roots which are singular for λ_s. Let π_{reg} denote some irreducible representation of $G_{\mathbb{R}}$ with regular infinitesimal character which translates (via a translation functor, say, T) to π, and let $\mathcal{B}_{\mathrm{reg}}$ denote the block containing π_{reg}. Then [V4] supplies a block $\mathcal{B}^\vee_{\mathrm{reg}}$ of representations with trivial infinitesimal character for a real form $G^\vee_{\mathbb{R}}(\lambda_s)$ of $G^\vee(\lambda_s)$ and a bijection $\mathbf{D} : \mathcal{B}_{\mathrm{reg}} \longrightarrow \mathcal{B}^\vee_{\mathrm{reg}}$. Consider the block $\mathcal{B}$ containing π; so $\mathcal{B} = T(\mathcal{B}_{\mathrm{reg}})$. Then the image $\mathcal{B}^\vee := \mathbf{D}(\mathcal{B})$ consists of the irreducibles in the category of representations of $G^\vee_{\mathbb{R}}(\lambda_s)$ with trivial infinitesimal character obtained by localizing on $G^\vee(\lambda)/P^\vee(\lambda_s)$. (It is possible to formulate a character multiplicity duality statement in this context, but since standard modules in the parabolic category are more delicate, we omit a precise statement.) Let $K^\vee$ denote the complexification of the maximal compact in $G^\vee_{\mathbb{R}}(\lambda_s)$. Roughly speaking, in the regular case the approach of [ABV] reinterprets the Langlands-Weil parameters as the set of $K^\vee$ orbits on the flag variety of $G^\vee(\lambda_s)$. At the singular infinitesimal character λ_s, the appropriate set of parameters is the set of K orbits on the partial flag variety $G^\vee(\lambda_s)/P^\vee(\lambda_s)$.

Here the situation here is exactly the same. If λ_s is singular and half-integral, and π is an irreducible genuine representation of $Mp(2n,\mathbb{R})$, the its dual is obtained by localization on the partial flag variety $Sp(2n,\mathbb{C})/P(\lambda_s)$, where $P(\lambda_s)$ is the defined as above. Since this case reduces to the regular case in the same way that the linear case does, we will focus all of our attention on the regular case.

3.2. **L-packets.** Fix $Q \in K\backslash X$ and λ regular and half-integral as above. Following [ABV], we write $\mathbb{L}_{SO}(Q,\lambda')$ for the super L-packet of representations with infinitesimal character λ'; more precisely, fix a pure choice of the duality $\mathbf{D}_{SO}$ of [V4] (Definition 3.2), and define

$$\mathbb{L}_{SO}(Q,\lambda') = \{\ \pi \in \coprod{}' \mathrm{Irr}(SO(p,q))_{\lambda'} \text{ with } \mathrm{supp}_{\mathrm{o}}\big(\mathbf{D}_{SO}(\pi) \ = Q\};$$

By analogy, fix a choice of $\mathbf{D}_{Mp}$ as in Section 3.1 and define

$$\mathbb{L}_{Mp}(Q,\lambda) = \{\pi \in \mathrm{Irr}(Mp(2n,R))^{gen}_{\lambda} \text{ with } \mathrm{supp}_{\mathrm{o}}\big(\mathbf{D}_{Mp}(\pi)\big) \ = Q\}.$$

It is important to note that while the definitions of L-packets depend on the indicated choices of duality, the set of L-packets with fixed infinitesimal character does not.

As a matter of notation, it will be also convenient to define

$$\mathcal{P}^{Mp}_{\lambda}(Q) = \{\gamma \in \mathcal{P}^{Mp}_{\lambda} \,|\, \mathrm{irr}(\gamma) \in \mathbb{L}_{Mp}(Q,\lambda)\}, \tag{3.21}$$

and similarly

$$\mathcal{P}^{SO}_{\lambda'}(Q) = \{\gamma \in \mathcal{P}^{SO}_{\lambda'} \,|\, \mathrm{irr}(\gamma) \in \mathbb{L}_{Mp}(Q,\lambda')\}. \tag{3.22}$$

Previously [AB1] suggested defining L-packets for $Mp(2n,\mathbb{R})$ as the theta lifts of super-packets for SO. We now show that this definition coincides with our intrinsic definition of L-packets for Mp.

Proposition 3.23. *Fix $Q \in K\backslash X$, fix regular half-integral infinitesimal character λ, and Set $\lambda' = \theta_{\mathrm{ic}}(\lambda)$ (Notation 2.3). Fix a pure choice of $\mathbf{D}_{SO}$ (Definition 3.2), fix a choice of θ, and let $\mathbf{D}_{Mp}$ denote the corresponding choice of duality for Mp (Equation (3.5)). Define L-packets for Mp and SO with respect to these choices. Then theta lifting preserves L-packets:*

$$\theta(\mathbb{L}_{Mp}(Q,\lambda)) = \mathbb{L}_{SO}(Q,\lambda').$$

Proof. Let π be any genuine irreducible representation π of $Mp(2n,\mathbb{R})$ with infinitesimal character λ. Then it is a trivial consequence of the definition of $\mathbf{D}_{Mp}$ given above that

$$\mathrm{supp}_{\mathrm{o}}\big(\mathbf{D}_{Mp}(\pi)\big) = \mathrm{supp}_{\mathrm{o}}\big(\mathbf{D}_{SO}(\theta(\pi))\big).$$

The theorem now follows immediately from the definition of $\mathbb{L}_{Mp}(Q,\lambda)$ and $\mathbb{L}_{SO}(Q,\lambda')$. ☐

3.3. **L-packets and stability.** We now turn our attention to stable virtual representations supported on L-packets (or strongly stable virtual representations supported on super L-packets)[4]. For $Mp(2n,\mathbb{R})$, stability is defined in [A2] (see also [R1], [R2]). The following theorem gathers some results of [ABV], [AB1] and [A2].

Theorem 3.24. *Fix $Q \in K\backslash X$, and λ regular and half-integral. Set $\lambda' = \theta_{\mathrm{ic}}(\lambda)$ (Notation 2.3), fix a pure choice of $\mathbf{D}_{SO}$ and a choice of $\mathbf{D}_{Mp}$, and recall the notation of Equations (3.21)–(3.22).*

[4]Since we have assumed the infinitesimal character is regular, there is no difference between stable virtual representations and stable eigendistributions: the isomorphism is obtained by simply taking characters.

1. *The sum of the standard representations in the super L-packet for* SO *defined by Q and λ' is strongly stable. More precisely,*

$$\eta^{\text{loc}}(\mathbb{L}_{SO}(Q,\lambda')) := \sum_{\gamma\in\mathcal{P}^{SO}_{\lambda'}(Q)} [\text{std}(\gamma)]$$

is a strongly stable virtual representation for $\coprod' SO(p,q)$. Furthermore, the set

$$\{\eta^{\text{loc}}(\mathbb{L}_{SO}(Q,\lambda')) \mid Q \in K\backslash X\}$$

is a basis of $\mathbb{K}\mathcal{HC}(SO(2n+1,\mathbb{C}))^{st}_{\lambda'}$, the space of strongly stable virtual representations for $\coprod' SO(p,q)$ with infinitesimal character λ'.

2. *Similarly,*

$$\eta^{\text{loc}}(\mathbb{L}_{Mp}(Q,\lambda)) := \sum_{\gamma\in\mathcal{P}^{Mp}_{\lambda}(Q)} [\text{std}(\gamma)]$$

is a stable virtual representation of $Mp(2n,\mathbb{R})$. Furthermore, the set

$$\{\eta^{\text{loc}}(\mathbb{L}_{Mp}(Q,\lambda)) \mid Q \in K\backslash X\}$$

is a basis of $\mathbb{K}\mathcal{HC}(Mp(2n,\mathbb{R}))^{st}_{\lambda}$, the space of stable virtual representations of $Mp(2n,\mathbb{R})$ with infinitesimal character λ.

Proof. The fact that the indicated statements are independent of the choices of $\mathbf{D}_{SO}$ and $\mathbf{D}_{Mp}$ follows from the observation (mentioned above) that the set of L-packets with a fixed infinitesimal character is independent of such choices. All the assertion in (1) about $SO(2n+1)$ are in Theorem 18.14 of [ABV], which is a reformulation of [Sh], Lemmas 5.2 and 5.3. By carefully examining these references, one sees that the same arguments carry over to prove (2).

For readers not satisfied with this approach, we give an extrinsic argument that deduces (2) from (1) as follows. Since everything is independent of the choices of $\mathbf{D}_{Mp}$ and $\mathbf{D}_{SO}$, we may make take a fixed pure choice of $\mathbf{D}_{SO}$, fix a choice of θ, and fix $\mathbf{D}_{Mp}$ defined with respect to these choices. In [A2], given a choice of oscillator Adams gives an isomorphism from the space of stable virtual representations of SO at λ' to those of $Mp(2n,\mathbb{R})$ at λ. Part (4) of the theorem in the introduction to [A2] together with Theorem 8.8 of [AB1] implies that for the choice of oscillator defining θ, the isomorphism takes $\eta^{\text{loc}}(\mathbb{L}_{SO}(Q,\lambda'))$ to the sum of the standard representations corresponding to elements of $\theta^{-1}(\mathbb{L}_{SO}(Q,\lambda'))$. Now (2) follows from (1) and Proposition 3.23. □

3.4. **Micro L-packets.** We now define microlocal L-packets ("micro L-packets" for short) for SO and Mp when λ is regular and half-integral. For linear groups, in particular the case of SO, the definition is given in [ABV], Chapter 19, where they are simple called micro-packets. The definition in the case of the metaplectic group is new.

Given a Harish-Chandra module π for either Mp or Sp, we write

$$CC(\pi) = \sum_{Q\in K\backslash X} m_Q(\pi)\overline{T^*_Q X}$$

for its characteristic cycle; here $m_Q(\pi)$ are positive integers (which can be zero, since we are summing over all K orbits on X). Note that it is an easy consequence of the definition of the characteristic cycle that

$$m_{\mathrm{supp}_o(\pi)}(\pi) = 1. \tag{3.25}$$

Fix Q in $K\backslash X$, fix a pure choice of $\mathbf{D}_{SO}$, and define the micro L-packet for SO as

$$\mathbb{L}_{SO}^{\mathrm{mic}}(Q,\lambda') := \{\pi \in \coprod{}' \mathrm{Irr}(SO(p,q))_{\lambda'} \mid m_Q(\mathbf{D}_{SO}(\pi)) \neq 0\},$$

i.e. $\mathbb{L}_{SO}^{\mathrm{mic}}(Q,\lambda')$ consists of all representations in $\coprod{}'\mathrm{Irr}(SO(p,q))_{\lambda'}$ whose dual representations in $\mathrm{Irr}(Sp(2n,\mathbb{R}))_\rho$ have characteristic cycles containing $\overline{T_Q^*X}$ with non-zero multiplicity. Note that Equation (3.25) immediately implies

$$\mathbb{L}_{SO}(Q,\lambda') \subset \mathbb{L}_{SO}^{\mathrm{mic}}(Q,\lambda'),$$

but the containment is generally proper. (For examples of this latter phenomenon, see the $n=1$ example in Section 3.5.)

Similarly we fix a choice of $\mathbf{D}_{Mp}$ and define the micro L-packet

$$\mathbb{L}_{Mp}^{\mathrm{mic}}(Q,\lambda) := \{\pi \in \mathrm{Irr}(Mp(2n,\mathbb{R}))_\lambda^{gen} \mid m_Q(\mathbf{D}_{Mp}(\pi)) \neq 0\},$$

the set of representations in $\mathrm{Irr}(Mp(2n,\mathbb{R}))_\lambda^{gen}$ whose dual representations (which are also in $\mathrm{Irr}(Mp(2n,\mathbb{R}))_\lambda^{gen}$) have characteristic cycles containing $\overline{T_Q^*X}$ with non-zero multiplicity. Again we have

$$\mathbb{L}_{Mp}(Q,\lambda) \subset \mathbb{L}_{Mp}^{\mathrm{mic}}(Q,\lambda).$$

3.5. **Behavior of micro L-packets under theta lifting.** By analogy with Proposition 3.23, it is tempting to ask whether micro L-packets lift to micro L-packet under theta. This is easily seen to be false, already for $n=1$. The dual of the trivial representation of $SO(3)$ is the irreducible (nonunitary) principal series of $Sp(2,\mathbb{R}) \simeq SL(2,\mathbb{R})$. The characteristic cycle of the nonunitary principal series is easily seen to consist of the sum of the conormal bundles to each of the three orbits of $K \simeq \mathbb{C}^*$ on $X \simeq \mathbb{P}^1$ (each with multiplicity one). Hence we conclude that the trivial representation of $SO(3)$ belongs to each of the three micro L-packets for SO with infinitesimal character $\lambda' = \rho_{SO} = \theta_{\mathrm{ic}}((1/2)\rho_{Sp})$. It is easy to see that there is no genuine irreducible representation of $Mp(2,\mathbb{R})$ with infinitesimal character $\lambda = (1/2)\rho_{Sp}$ whose characteristic cycle is supported on all three conormal bundles. Hence we conclude that the intersection of the three micro L-packets of genuine representations of $Mp(2,\mathbb{R})$ with infinitesimal character λ is empty. Since θ is a bijection, it cannot take micro L-packets to micro L-packets.

3.6. **Micro L-packets and stability.** We turn to the application of micro L-packets to stable virtual representations.

Theorem 3.26. *Retain the notation of Theorem 3.24, and define micro L-packets as above.*

1. *The sum of the irreducible representations in a micro L-packet for* SO*, weighted by coefficients from the characteristic cycles of their duals, is strongly stable. More precisely*

$$\eta^{\mathrm{mic}}(\mathbb{L}_{SO}^{\mathrm{mic}}(Q,\lambda')) := \sum_{\pi \in \mathbb{L}_{SO}^{\mathrm{mic}}(Q,\lambda')} e(\pi)(-1)^{\dim(\mathrm{supp}(\pi^\vee)) - \dim(Q)} m_Q(\pi^\vee)[\pi]$$

is a strongly stable virtual representation for $\coprod' SO(p,q)$; here $e(\pi)$ is the Kottwitz sign attached to π (i.e. $e(\pi) = (-1)^{\frac{p-q}{2}}$ if $\pi \in Irr(SO(p,q))_{\lambda'}$). Furthermore, the set

$$\{\eta^{\mathrm{mic}}(\mathbb{L}_{SO}^{\mathrm{mic}}(Q,\lambda')) \mid Q \in K\backslash X\}$$

is a basis of $\mathbb{K}\mathcal{HC}(SO(2n+1,\mathbb{C}))^{st}_{\lambda'}$ *(with notation as in Theorem 3.24(1)).*

2. *Similarly,*

$$\eta^{\mathrm{mic}}(\mathbb{L}^{\mathrm{mic}}_{Mp}(Q,\lambda)) := \sum_{\pi\in\mathbb{L}^{\mathrm{mic}}_{Mp}(Q,\lambda)} (-1)^{\dim(\mathrm{supp}(\pi^\vee))-\dim(Q)} m_Q(\pi^\vee)[\pi]$$

is a stable virtual representation of $Mp(2n,\mathbb{R})$. *Furthermore, the set*

$$\{\eta^{\mathrm{mic}}(\mathbb{L}^{\mathrm{mic}}_{Mp}(Q,\lambda)) \mid Q \in K\backslash X\}$$

is a basis of $\mathbb{K}\mathcal{HC}(Mp(2n,\mathbb{R}))^{st}_{\lambda}$ *(with notation as in Theorem 3.24(2)).*

Proof. Part (1) is a special case of [ABV], Corollary 1.32. It follows from part (1) of Theorem 3.24 and [ABV], Theorem 1.31). These results also hold in the relevant setting for Mp. So we deduce (2) of the present theorem from (2) of Theorem 3.24 in exactly the same way. ☐

Remark 3.27. The proof given in [ABV] of their Theorem 1.31 uses the relatively deep index theorem of Kashiwara. We sketch a very short and elementary proof of the [ABV] result that makes no use of Kashiwara's theorem, and which makes the proof of our Theorem 3.26 entirely self-contained. Equation (3.33) below proves the required statement in the context of the metaplectic group, and its derivation (which depends only on some linear algebra and the easy Lemma 3.30) immediately generalizes to the statement given in [ABV], Theorem 1.31.

Remark 3.28. For later use it will be useful to rewrite the stable virtual representations appearing in the theorem. For instance in (2), we can write

$$\eta^{\mathrm{mic}}(\mathbb{L}^{\mathrm{mic}}_{Mp}(Q,\lambda)) := \sum_{\gamma\in\mathcal{P}^{Mp}_\lambda} (-1)^{\dim(\mathrm{supp}(\gamma^\vee))-\dim(Q)} m_Q(\mathrm{irr}(\gamma^\vee))[\mathrm{irr}(\gamma)],$$

since, by definition, $m_Q(\pi^\vee)\neq 0$ if and only if $\pi\in\mathbb{L}^{\mathrm{mic}}(Q,\lambda)$; here $\pi^\vee = \mathbf{D}_{Mp}(\pi)$ where the choice of $\mathbf{D}_{Mp}$ is the one used to define micro L-packets. We can use the linearity of the characteristic cycle to express the above sum in terms of standard (rather than irreducible) representations

$$\eta^{\mathrm{mic}}(\mathbb{L}^{\mathrm{mic}}_{Mp}(Q,\lambda)) = \sum_{\gamma\in\mathcal{P}^{Mp}_\lambda} \epsilon^{Mp}_\gamma m_Q(\mathrm{std}(\gamma^\vee))[\mathrm{std}(\gamma)],$$

where $\epsilon^{Mp}_\gamma = (-1)^{\dim(\mathrm{supp}(\gamma^\vee))-\dim(Q)}$ for $\gamma\in\mathcal{P}^{Mp}_\lambda$. This should be compared with Theorem 3.24(2). Similarly we can apply the same considerations to Theorem 3.26(1) to write

$$\eta^{\mathrm{mic}}(\mathbb{L}^{\mathrm{mic}}_{SO}(Q,\lambda')) = \sum_{\gamma\in\mathcal{P}^{SO}_{\lambda'}} \epsilon^{SO}_\gamma m_Q(\mathrm{std}(\gamma^\vee))[\mathrm{std}(\gamma)],$$

where $\epsilon^{SO}_\gamma = (-1)^{\dim(\mathrm{supp}(\gamma^\vee)-\dim(Q)}$ for $\gamma\in\mathcal{P}^{SO}_{\lambda'}$.

We now define now maps between spaces of stable virtual representations, and investigate their relation with the one defined in [A2]. Fix, as always, a regular half-integral infinitesimal character λ. Fix a pure choice of $\mathbf{D}_{SO}$ and a choice of $\mathbf{D}_{Mp}$. Retain the notation of Theorems 3.24 and 3.26, and recall that those theorems imply that the assignments

$$T^{\mathrm{loc}} \;:\; \eta^{\mathrm{loc}}(\mathbb{L}_{SO}(Q,\lambda')) \mapsto \eta^{\mathrm{loc}}(\mathbb{L}_{Mp}(Q,\lambda)),$$

$$T^{\text{mic}} \;:\; \eta^{\text{mic}}(\mathbb{L}^{\text{mic}}_{SO}(Q,\lambda')) \mapsto \eta^{\text{mic}}(\mathbb{L}^{\text{mic}}_{Mp}(Q,\lambda)),$$

for each $Q \in K\backslash X$, extend to linear isomorphisms

$$T^{\text{loc}},\, T^{\text{mic}} :\; \mathbb{K}\mathcal{HC}(SO(2n+1,\mathbb{C}))^{st}_{\lambda'} \longrightarrow \mathbb{K}\mathcal{HC}(Mp(2n,\mathbb{R}))^{st}_{\lambda}.$$

The maps T^{loc} and T^{mic} both depend on the choice of $\mathbf{D}_{SO}$ and $\mathbf{D}_{Mp}$. From Lemma 3.1 and the fact that the two choices of $\mathbf{D}_{Mp}$ differ by $\mathbf{A}$ (see Propositions 3.18 and 3.19), making an alternate choice in the definition of T^{loc} or T^{mic} amounts to twisting their image by $\mathbf{A}$.

Theorem 3.29. *Fix regular half-integral infinitesimal character λ. With the notation of the previous paragraph, and for a fixed choices of $\mathbf{D}_{Mp}$ and $\mathbf{D}_{SO}$, $T^{\text{loc}} = T^{\text{mic}}$. Furthermore these maps coincide with Adams' transfer of stable distributions up to outer automorphism. More precisely, Adams' map depends on a choice of oscillator, and there is a choice of oscillator such that Adams' map coincides with $T^{\text{loc}} = T^{\text{mic}}$.*

Proof. Let us first show that Adams' map coincide with T^{loc} up to outer automorphism. For a packet of discrete series, this is already in [A2]. The general case follows from this and the induction principle. (See §12 of [A2] and the proof of Theorem 3.24).

We now turn to the proof that $T^{\text{loc}} = T^{\text{mic}}$. This is more subtle and requires the well-known (and easy) fact that the characteristic cycle of a standard module depends only on its support. (Proposition 2.6.2 in [Ch] is a convenient reference.) We isolate the very special cases that we will need.

Lemma 3.30. *Fix $\gamma_{Mp}, \gamma'_{Mp} \in \mathcal{P}^{Mp}_{\lambda}$, for λ half-integral and regular, and fix $\delta_{Sp}, \in \mathcal{P}^{Sp}_{\lambda'}$, for λ' regular and integral. Then the characteristic cycles of the corresponding standard modules only depends on their support. More precisely,*

1. *If* $\text{supp}(irr_{Mp}(\gamma_{Mp})) = \text{supp}(irr_{Sp}(\delta_{Sp}))$, *then*
$$CC(std_{Mp}(\gamma_{Mp})) = CC(std_{Sp}(\delta_{Sp})).$$
2. *If* $\text{supp}(irr_{Mp}(\gamma_{Mp})) = \text{supp}(irr_{Mp}(\gamma'_{Mp}))$, *then*
$$CC(std_{Mp}(\gamma_{Mp})) = CC(std_{Mp}(\gamma'_{Mp})).$$

(By (1), the corresponding statement holds for standard modules for Sp.*)* □

To prove that $T^{\text{loc}} = T^{\text{mic}}$, we will show that the change of basis matrix between the bases in Theorems 3.24(1) and 3.26(1) coincides with the corresponding matrix for the bases in Theorems 3.24(2) and 3.26(2). We first rewrite this latter matrix as follows, starting with the expression for $\eta^{\text{mic}}(\mathbb{L}^{\text{mic}}_{Mp}(Q,\lambda))$ in Remark 3.28,

$$\eta^{\text{mic}}(\mathbb{L}^{\text{mic}}_{Mp}(Q,\lambda)) = \sum_{\gamma\in\mathcal{P}^{Mp}_{\lambda}} \epsilon^{Mp}_{\gamma} m_Q(\text{std}_{Mp}(\gamma^{\vee}))[\text{std}_{Mp}(\gamma)]. \tag{3.31}$$

Recall the notation of Equation (3.21), and note that it is obvious from the definitions that $\epsilon^{Mp}_{\gamma} = (-1)^{\dim(Q')-\dim(Q)}$ for all $\gamma \in \mathcal{P}^{Mp}_{\lambda}(Q')$. So Equation (3.31) becomes

$$\eta^{\text{mic}}(\mathbb{L}^{\text{mic}}_{Mp}(Q,\lambda)) = \sum_{Q'} \sum_{\gamma\in\mathcal{P}^{Mp}_{\lambda}(Q')} (-1)^{\dim(Q')-\dim(Q)} m_Q(\text{std}_{Mp}(\gamma^{\vee}))[\text{std}_{Mp}(\gamma)]. \tag{3.32}$$

According to Lemma 3.30(2), $(-1)^{\dim(Q')-\dim(Q)}m_Q(\mathrm{std}_{Mp}(\gamma^\vee))$ is a constant for all $\gamma \in \mathcal{P}_\lambda^{Mp}(Q')$. If we denote this constant $c_{QQ'}^{Mp}$, we can rewrite Equation (3.32) as

$$\eta^{\mathrm{mic}}(\mathbb{L}_{Mp}^{\mathrm{mic}}(Q,\lambda)) = \sum_{Q'} c_{QQ'}^{Mp}\left(\sum_{\gamma\in\mathcal{P}_\lambda^{Mp}(Q')}[\mathrm{std}_{Mp}(\gamma)]\right);$$

or, using the definition of η^{loc} in Theorem 3.24(2),

$$\eta^{\mathrm{mic}}(\mathbb{L}_{Mp}^{\mathrm{mic}}(Q,\lambda)) = \sum_{Q'} c_{QQ'}^{Mp}\left(\eta^{\mathrm{loc}}(\mathbb{L}_{Mp}(Q',\lambda))\right). \tag{3.33}$$

(As noted in Remark 3.27, this equation establishes the stability of $\eta^{\mathrm{mic}}(\mathbb{L}_{Mp}^{\mathrm{mic}}(Q,\lambda))$ and, in fact, [ABV], Theorem 1.31, in this context.) So $c_{QQ'}^{Mp}$ is the change of basis matrix between the bases in Theorems 3.24(2) and 3.26(2).

Now we can simply retrace the chain of equalities in the previous paragraph on the SO side to conclude that

$$\eta^{\mathrm{mic}}(\mathbb{L}_{SO}^{\mathrm{mic}}(Q,\lambda')) = \sum_{Q'} c_{QQ'}^{Sp}\left(\eta^{\mathrm{loc}}(\mathbb{L}_{SO}(Q',\lambda'))\right),$$

where $c_{QQ''}^{Sp}$ is $(-1)^{\dim(Q')-\dim(Q)}$ times the multiplicity of the conormal bundle T_Q^*X in the characteristic cycle of any standard representation of $Sp(2n,\mathbb{R})$ whose irreducible quotient has support $\overline{Q'}$. (Lemma 3.30(2) implies that this number does not depend on the standard representation in question.) So $c_{QQ'}^{Sp}$ is the change of basis matrix between the bases in Theorems 3.24(1) and 3.26(1). But Lemma 3.30(1) implies $c_{QQ'}^{Sp} = c_{QQ'}^{Mp}$. So indeed the two change of basis matrices coincide and the theorem is proved. ☐

4. Non-half-integral infinitesimal character

In this section, we examine the opposite case of the one considered in Section 3, and assume throughout that *none of the coordinates of the infinitesimal character λ are half-integers.* There are no representations of $SO(p,q)$ with infinitesimal character $\lambda' = \theta_{ic}(\lambda)$ unless $|p-q| = 1$. If we fix δ appropriately (in the notation of Section 2.3), we need only consider representations of $SO(n,n+1)$.

If $\theta(\mathrm{irr}_{Mp}(\gamma)) = \mathrm{irr}_{SO}(\gamma')$, we will write $\theta(\gamma) = \gamma'$. As explained in the introduction, the notation is justified by the following more precise version of Theorem 1.4. (For the discussion of how Theorem 1.4 follows from Theorem 4.1, see Section 2.1.2.)

Theorem 4.1. *Fix a choice of θ, recall the notation of Section 2.1.2 and retain the setting of Theorem 1.4. Then the $\mathbb{Z}[q,q^{-1}]$-linear map*

$$\begin{aligned}\theta\ :\ \mathbf{M}_\lambda^{Mp(2n,\mathbb{R})} &\longrightarrow \mathbf{M}_{\theta_{ic}(\lambda)}^{SO(n,n+1)}\\ [std_{Mp}(\gamma)] &\longrightarrow [std_{SO}(\theta(\lambda))]\end{aligned}$$

is an isomorphism of $\mathbf{H}(W_{Mp}(\lambda)) = \mathbf{H}(W_{SO}(\theta_{\mathrm{ic}}(\lambda)))$ modules.

Proof. Fix λ, and recall that none of its coordinates are half-integers. Fix a genuine representation π for $Mp(2n,\mathbb{R})$ with infinitesimal character λ. Because each noncompact simple root for π is necessarily type II (by the infinitesimal character restriction), one can easily check that there are no nonintegral simple roots α which are noncompact imaginary.

By the same token, there can be no nonintegral simple real roots for π which satisfy the Speh-Vogan parity condition. (Crossing such a wall would lead to a nonintegral noncompact imaginary root by [V1], Lemma 4.9.) So the only kinds of nonintegral wall crossings that arise for infinitesimal character λ are the ones treated by [V1], Corollary 4.8, and according to that corollary, these are very simple to describe in the Langlands classification. From this computation and [AB1], Theorem 5.1, it is easy to see that θ commutes with these nonintegral wall crossings. Since such crossings are equivalences of categories, by successively applying them we can reduce the general case of the Theorem 4.1 to the case that every simple integral root for λ is actually simple. (Here we are using that $W(\lambda)$ is the Weyl group of a Levi subalgebra of $\mathfrak{g}$. By contrast, in the half-integral case, the integral Weyl group is of type D, and it is impossible to apply a sequence of nonintegral wall crosses every simple integral root is in fact simple.)

Fix a simple integral root α for λ. By the previous paragraph, we can assume that α is actually simple. Note that α' (notation as in Remark 2.3) is a simple integral root for $\theta(\lambda)$ which is also simple. Write T_α and $T_{\alpha'}$ for the corresponding Hecke operators in $\mathbf{H}(W_{Mp}(\lambda)) \simeq \mathbf{H}(W_{SO}(\theta_{\mathrm{ic}}(\lambda)))$. We are trying to show

$$T_\alpha \cdot [\mathrm{std}_{Mp}(\gamma)] = T_{\alpha'} \cdot [\mathrm{std}_{SO}(\theta(\gamma))]. \tag{4.2}$$

Because α is assumed to be simple, the action of T_α and $T_{\alpha'}$ is described explicitly in [V3], Definition 6.4. The only structural information that appears in that definition is that of the cross-action and Cayley (and inverse Cayley) transforms. Unwinding the definitions immediately implies that Equation (4.2) follows from Theorem 2.7(4). The proof is complete. □

Definition 4.3. Fix non-half-integral infinitesimal character λ and let $\mathbf{D}_{SO}$ denote a choice of the duality of [V4] for $SO(n, n+1)$ at infinitesimal character λ. Fix a choice of θ and set $\mathbf{D}_{Mp} = \mathbf{D}_{SO} \circ \theta$. The dual of a genuine representation $\pi = \overline{X}(\gamma)$ is defined to be $\mathbf{D}_{Mp}(\pi)$ and is denoted by $\pi^\vee$ or, on the level of the pseudocharacter classification, by $\overline{X}(\gamma^\vee)$. This is a representation with trivial infinitesimal character of a real form of $Sp(\lambda')$, the centralizer of $e^{2\pi i\lambda'}$ in $Sp(2n, \mathbb{C})$, where $\lambda' = \theta_{\mathrm{ic}}(\lambda)$.

Given the definition, the following theorem is a direct corollary of Theorem 4.1.

Theorem 4.4. *The duality map of Definition 4.3 is a character multiplicity duality in the sense of Theorem 3.16. In more detail, if λ is regular (see Remark 3.20 for comments on the singular case),*

$$[irr(\delta)] = \sum_{\gamma \in \mathcal{P}_\lambda^{Mp}} M(\gamma, \delta)[std(\gamma)]$$

if and only if

$$[std(\gamma^\vee)] = \sum_{\delta^\vee \in \mathcal{P}_\rho^{Sp(\lambda')}} \epsilon_{\gamma\delta} M(\gamma, \delta)[irr(\delta^\vee)];$$

where $\epsilon(\gamma, \delta) = (-1)^{d(\gamma,\delta)}$ with

$$d(\gamma, \delta) = \dim \mathrm{supp}(irr(\delta)) - \dim \mathrm{supp}(irr(\gamma)).$$

Some special cases of the theorem are especially interesting.

Corollary 4.5. *Suppose the coordinates of λ are of the form,*

$$(\lambda_1, \ldots, \lambda_n) = (l+\lambda'_1, \ldots, l+\lambda'_n),$$

where $\lambda'_i \in \mathbb{Z}$ and $\lambda'_i > \cdots > \lambda'_n > 0$.

1. *Suppose that l is a positive real number that is neither an integer nor a half-integer. Write $\mathcal{P}_\rho^{GL(n,\mathbb{R})}$ for the pseudocharacters for $GL(n,\mathbb{R})$ with infinitesimal character $\rho = ((n-1)/2, \ldots, -(n-1)/2)$ in standard coordinates. Then the following three sets in pairwise bijection*

$$\mathcal{P}_\rho^{GL(n,\mathbb{R})}, \mathcal{P}_\lambda^{Mp(2n,\mathbb{R})}, \mathcal{P}_{\theta_{ic}(\lambda)}^{SO(n,n+1)}$$

and, moreover, the corresponding Hecke modules for $\mathbf{H}(S_n)$ are isomorphic. Hence the character formulas for $Mp(2n,\mathbb{R})$ and $SO(n,n+1)$ reduce to those for $GL(n,\mathbb{R})$. In particular, each set has $n+1$ blocks, dual to the unique blocks in $U(n,0), U(n-1,1), \ldots, U(0,n)$ at infinitesimal character ρ.

2. *Suppose l is a positive integer. Then $\mathcal{P}_\lambda^{Mp(2n,\mathbb{R})} \simeq \mathcal{P}_{\theta_{ic}(\lambda)}^{SO(2n+1)}$ (Theorem 4.1) has $n+1$ blocks which are dual to the unique blocks in $Sp(n,0), \ldots, Sp(0,n)$ of infinitesimal character λ.*

Sketch. Part (2) is simply Theorem 4.1, together with the duality of [V4] worked out explicitly for $SO(n,n+1)$ at this kind of infinitesimal character. The final statement of part (1) is again just the explicit duality for $GL(n,\mathbb{R})$ at infinitesimal character ρ — see [T1], Section 6, for details.

It remains to establish the $\mathbf{H}(S_n)$ isomorphism $\mathbf{M}_\rho^{GL(n)} \simeq \mathbf{M}_{\theta_{\rm ic}(\lambda)}^{SO(n,n+1)}$. (Applying Theorem 4.1 then gives the other pairs of $\mathbf{H}(S_n)$ isomorphisms.) Let π' be an irreducible representation of $GL(n,\mathbb{R})$ with trivial infinitesimal character. Let P be a parabolic for $SO(n,n+1)$ with Levi factor $GL(n,\mathbb{R})$, and let π denote the parabolically induced representation $Ind_P^G(\pi)$. It is easy to see that π is irreducible, and that all irreducibles with the same infinitesimal character as π arise in this way. In fact, using induction in stages it is easy to see that the induction has a very simple description on the level of Langlands parameters. This description shows that the induction intertwines the $\mathbf{H}(S_n)$ action. Since the infinitesimal character of π differs from that of $\theta_{\rm ic}(\lambda)$ by an integral weight, the current result follows from the translation principle. □

Given Definition 4.3 and Theorem 4.4, we can then define L-packets and micro L-packets exactly as in Section 3. In more detail, fix λ, $\mathbf{D}_{SO}$, θ and set $\mathbf{D}_{Mp} = \mathbf{D}_{SO} \circ \theta$ as in Definition 4.3. Fix a representation π with infinitesimal character λ. Consider $\mathrm{supp}_{\mathrm{o}}(\mathbf{D}_{Mp}(\pi))$; this is an orbit of a reductive group K on a (possibly partial) flag variety X for the group $Sp(\lambda')$ of Definition 4.3. Given an orbit $Q \in K\backslash X$, we define

$$\mathbb{L}_{Mp}(Q,\lambda) = \{\pi \in \mathrm{Irr}(Mp(2n,R))_\lambda^{gen} \text{ with } \mathrm{supp}_{\mathrm{o}}(\mathbf{D}_{Mp}(\pi)) = Q\}.$$

$$\mathbb{L}_{Mp}^{\mathrm{mic}}(Q,\lambda) = \{\pi \in \mathrm{Irr}(Mp(2n,\mathbb{R}))_\lambda^{gen} \mid m_Q(\mathbf{D}_{Mp}(\pi)) \neq 0\}.$$

Using $\mathbf{D}_{SO}$ instead of Definition 4.3 allows us to define the L-packet $\mathbb{L}_{SO}(Q,\lambda')$ and micro L-packet $\mathbb{L}_{SO}^{mic}(Q,\lambda')$ of representations of SO with infinitesimal character $\lambda' = \theta_{\rm ic}(\lambda)$ in exactly the same way. The following analog of Proposition 3.23 is obvious from the definitions.

Proposition 4.6. *Fix non-half-integral infinitesimal character λ and $Q \in K\backslash X$ as in the discussion of the previous paragraph. Set $\lambda' = \theta_{\rm ic}(\lambda)$ (Notation 2.3). Fix $\mathbf{D}_{SO}$ and θ and* *define L-packets and micro L-packets as above. Then theta lifting preserves both L-packets and micro L-packets:*

$$\theta(\mathbb{L}_{Mp}(Q,\lambda)) = \mathbb{L}_{SO}(Q,\lambda');$$
$$\theta(\mathbb{L}^{\text{mic}}_{Mp}(Q,\lambda)) = \mathbb{L}^{\text{mic}}_{SO}(Q,\lambda');$$

The relationship between L-packets and stability is even simpler than we have indicated: for the kind of infinitesimal character under consideration, it is easy to see that all L-packets for SO are singletons[5], and so every genuine virtual representation (with non half-integral infinitesimal character) is stable. One can easily check the corresponding statement holds for Mp. As in Section 3.6 we obtain two isomorphisms

$$T^{\text{loc}},\ T^{\text{mic}}:\ \mathbb{K}\mathcal{HC}(SO(2n+1,\mathbb{C}))^{st}_{\theta_{\text{ic}}(\lambda)} \longrightarrow \mathbb{K}\mathcal{HC}(Mp(2n,\mathbb{R}))^{st}_{\lambda}.$$

Explicitly, for instance, we have

$$T^{\text{loc}}([\text{std}_{SO}(\gamma)]) = [\text{std}_{Mp}(\theta(\gamma))].$$

It is a formal consequence of the definitions that T^{loc} and T^{mic} coincide, and clearly the above displayed equation indicates that $T^{\text{loc}} = T^{\text{mic}}$ depends on a choice of θ. It only remains to check their relationship with Adams' lifting of characters ([A2]).

Theorem 4.7. *Fix a choice of θ defining T^{loc} and T^{mic} at regular non-half-integral infinitesimal character as above. Then there is a choice of oscillator defining Adams' lifting of characters (say T) such that $T = T^{\text{loc}} = T^{\text{mic}}$. That is, there is a choice of Adams map T such that*

$$T([std_{SO}(\gamma)]) = [std_{Mp}(\theta(\gamma))].$$

Using Theorem 4.1, we can rewrite this as

$$T([irr_{SO}(\gamma)]) = [irr_{Mp}(\theta(\gamma))].$$

Proof. Once one understands how Adams' lifting behaves under induction ([A2], Theorem 12.16), the first assertion is relatively straightforward. (The point is that the infinitesimal character hypothesis implies that the Levi factor of parabolic subgroup in the inducing data for the standard modules appearing above contains only $GL(1)$ and $GL(2)$ factors.) As indicated, Theorem 1.4 gives the second statement. ☐

5. General infinitesimal character

We now apply the results of the previous two sections to obtain a general duality theory. So let π be an irreducible genuine representation of $Mp(2n,\mathbb{R})$ with arbitrary infinitesimal character λ. Let λ^1 denote the vector consisting of the coordinates of λ which are half-integers, and let λ^2 to be the vector of the coordinates which are not half-integers. Let n_i denote the length of λ_i; so $n = n_1+n_2$. We now define an element s of the diagonal (split) Cartan in $Sp(2n,\mathbb{R})$. In the diagonal position corresponding to the ith coordinate λ_i of λ, place a $+1$ if λ_i is a half-integer and a -1 if λ_i is not a half-integer. This defines the element s.

The centralizer, $M_{\mathbb{R}}$ of s in $Sp(2n,\mathbb{R})$ is isomorphic to $Sp(2n_1,\mathbb{R}) \times Sp(2n_2,\mathbb{R})$. Define $\widetilde{M}_{\mathbb{R}}$ to be the preimage of $M_{\mathbb{R}}$ in $Mp(2n,\mathbb{R})$. Clearly $Mp(2n_1,\mathbb{R}) \times Mp(2n_2,\mathbb{R})$ surjects onto $\widetilde{M}_{\mathbb{R}}$. A little checking (see Equation (5.3) and the surrounding discussion in [R2]) of the

[5]It is important to note that micro L-packets will still be complicated, however.

definitions shows that if π_i are irreducible genuine representation of $Mp(2n_i,\mathbb{R})$, then the external tensor product $\pi_1 \otimes \pi_2$ descends to a genuine representation of $\widetilde{M_\mathbb{R}}$ (which we will denote $\pi_1 \overline{\otimes} \pi_2$), and any such representation is obtained this way.

Let π be the genuine irreducible representation of $Mp(2n,\mathbb{R})$ parameterized by the λ-pseudocharacter $\gamma = (\Gamma, \bar{\gamma})$ for a Cartan subgroup $H_\mathbb{R}$. Now $H_\mathbb{R}$ is conjugate to a Cartan subgroup of $\widetilde{M_\mathbb{R}}$ (see [R2], Section 4, for instance) and one may verify that the pair $(\Gamma, \bar{\gamma})$ satisfies the compatibility conditions in the definition of a pseudocharacter for $\widetilde{M_\mathbb{R}}$; to make the context clear, we will denote this pseudocharacter by γ'. By construction, γ' parameterizes a genuine irreducible representation π' of $\widetilde{M_\mathbb{R}}$ which, by the above remarks, must be of the form $\pi_1 \overline{\otimes} \pi_2$ for genuine irreducible representations π_i of $Mp(2n_i,\mathbb{R})$. It is clear from this construction that the infinitesimal character of π_i is λ^i. Section 3 defines the dual $\pi_1^\vee := \mathbf{D}_{Mp(2n_1)}(\pi_1)$, a genuine irreducible representation of $Mp(2n_1,\mathbb{R})$; this depends on a choice of $\mathbf{D}_{Mp(2n_1)}$ at λ^1. Section 4 defines the dual $\pi_2^\vee := \mathbf{D}_{SO(2n_2+1)}(\theta(\pi_2))$, which is an irreducible representation of some real form of the centralizer of $e^{2\pi i \theta_{ic}(\lambda^2)}$ in $Sp(2n_2,\mathbb{C})$; this depends on a choice of $\mathbf{D}_{SO(2n_2+1)}$ at λ_2' and a choice of θ at rank n_2.

Definition 5.1. Retain the above notation. The dual of π is defined to be $\pi_1^\vee \otimes \pi_2^\vee$. This is a representation of the direct product of $Mp(2n_1,\mathbb{R})$ and a real form of a Levi subgroup of $Sp(2n_2,\mathbb{C})$, and depends on a choice of $\mathbf{D}_{Mp(2n_1)}$ at λ_1 (or, by Section 3.1, a pure choice of $\mathbf{D}_{SO(2n_2+1)}$ at $(\lambda^1)'$ and a choice of θ at rank n_1), a choice of $\mathbf{D}_{SO(2n_2+1)}$ at $(\lambda^2)'$, and a choice of θ at rank n_2. (See Remark 3.20 for further comments on the singular case.)

Theorem 5.2. *Fix regular infinitesimal character and choices of $\mathbf{D}_{Mp(2n_1)}$, $\mathbf{D}_{SO(2n_2+1)}$, and θ at rank n_2 as in Definition 5.1. Then Definition 5.1 defines a character multiplicity duality in the sense of Theorem 3.16.*

Proof. Let $\mathbf{M}$ (resp. $\mathbf{M}_i$) denote the Grothendieck group of the block containing π (resp. π_i) with scalars extended from $\mathbb{Z}$ to $\mathbb{Z}[q,q^{-1}]$. Similarly write $\mathbf{M}^\vee$ (resp. $\mathbf{M}_i^\vee$) for the analogous groups for $\pi^\vee$ (resp. $\pi_i^\vee$). For instance, as $\mathbb{Z}[q,q^{-1}]$ modules, $\mathbf{M}^\vee = \mathbf{M}_1^\vee \otimes \mathbf{M}_2^\vee$ by definition.

Let W denote the integral Weyl group of λ. Clearly $W = W_1 \times W_2$ where W_i is the integral Weyl group of λ^i. (So, in particular, W_1 is isomorphic to the Weyl group of type D.) Write $\mathbf{H}_i$ for the Hecke algebra of W_i. Since W_1 is of type D, the construction of [RT], Proposition 7.5, produces an extended Hecke algebra $\widetilde{\mathbf{H}}_1$ containing $\mathbf{H}_1$. Set $\mathbf{H} = \widetilde{\mathbf{H}}_1 \otimes \mathbf{H}_2$.

From the formulas given in (and the ones of [V3]) it is clear that: $\mathbf{H} := \widetilde{\mathbf{H}}_1 \otimes \mathbf{H}_2$ acts on $\mathbf{M}$; $\widetilde{\mathbf{H}}_1$ acts on $\mathbf{M}_1$ and $\mathbf{M}_1^\vee$; and $\mathbf{H}_2$ acts on $\mathbf{M}_2$ and $\mathbf{M}_2^\vee$. It is obvious that $\mathbf{M}^\vee = \mathbf{M}_1^\vee \otimes \mathbf{M}_2^\vee$ as $\mathbf{H}$ modules. The key observation here is that $\mathbf{M} = \mathbf{M}_1 \otimes \mathbf{M}_2$ as $\widetilde{\mathbf{H}}_1 \otimes \mathbf{H}_2$ modules. This follows again from the formulas mentioned above: those for $\widetilde{\mathbf{H}}_1$ and those for $\mathbf{H}_2$ never mix terms.

Given a module M for a noncommutative R-algebra A, one needs to specify an antiautomorphism of A to define a module structure on the R-linear dual of M. A particular choice of antiautomorphism for $\widetilde{\mathbf{H}}_1$ and $\mathbf{H}_2$ is given in Section 8 of [RT]. Hence we may speak of the modules $\mathbf{M}^*$, $\mathbf{M}_1^*$, and $\mathbf{M}_2^*$. Using [RT] and [V4] respectively, the duality theorems of Sections 3 and 4 are equivalent to the statements

$$\mathbf{M}_1^\vee \simeq \mathbf{M}_1^* \text{ as } \widetilde{\mathbf{H}}_1 \text{ modules; and}$$
$$\mathbf{M}_2^\vee \simeq \mathbf{M}_2^* \text{ as } \mathbf{H}_2 \text{ modules;}$$

where the isomorphism in each case is obtained by sending a basis element $[\pi_i^\vee]$ to the linear map μ_{π_i} which map the basis element $[\tau]$ to the Kronecker delta $\delta_{\pi_i \tau}$. Since

$$\mathbf{M}^* = (\mathbf{M}_1 \otimes \mathbf{M}_2)^* = (\mathbf{M}_1^* \otimes \mathbf{M}_2^*),$$

we thus conclude that

$$\mathbf{M}^* \simeq (\mathbf{M}_1^\vee \otimes \mathbf{M}_2^\vee) = \mathbf{M}^\vee$$

as $\widetilde{\mathbf{H}}$ modules, with the isomorphism mapping μ_π to $[\pi^\vee]$. But in the formalism of [V4], Section 13, and [RT], Section 8, this is equivalent to the assertion of the current theorem. □

The theorem now allows us to define L-packets and micro L-packets in general. Fix a representation π with infinitesimal character λ, and let $\pi^\vee$ denote its dual as in Definition 5.1. Consider $\mathrm{supp}_{\mathrm{o}}(\pi^\vee)$; this is an orbit of some reductive group J on a (possibly partial) flag variety $\mathcal{X}$. L-packets of genuine representations of $Mp(2n,\mathbb{R})$ are parameterized by $J\backslash\mathcal{X}$: given $Q \in J\backslash\mathcal{X}$, we define

$$\mathbb{L}_{Mp}(Q,\lambda) = \{\pi \in \mathrm{Irr}(Mp(2n,R))_\lambda^{gen} \text{ with } \mathrm{supp}_{\mathrm{o}}(\pi^\vee) = Q\}.$$

Micro L-packets of such representation are also parameterized by $K\backslash X$, and are defined by

$$\mathbb{L}_{Mp}^{\mathrm{mic}}(Q,\lambda) = \{\pi \in \mathrm{Irr}(Mp(2n,\mathbb{R}))_\lambda^{gen} \mid m_Q(\pi^\vee) \neq 0\}.$$

The same definitions allow us to define the super L-packet $\mathbb{L}_{SO}(Q,\lambda')$ and super micro L-packet $\mathbb{L}_{SO}^{\mathrm{mic}}(Q,\lambda')$ of representation of SO at infinitesimal character $\lambda' = \theta_{\mathrm{ic}}(\lambda)$. As usual the parametrizations of these sets depend on choices of the dualities involved, and we need to make them compatibly in the following sense.

Lemma 5.3. *Fix regular infinitesimal character λ and a choice of θ. There is a choice of $\mathbf{D}_{Mp}$ at λ (Definition 5.1) and $\mathbf{D}_{SO}$ at λ' such that*

$$\mathrm{supp}_{\mathrm{o}}\big(\mathbf{D}_{Mp}(\pi)\big) = \mathrm{supp}_{\mathrm{o}}\big(\mathbf{D}_{SO}(\theta(\pi))\big)$$

for all $\pi \in Irr(Mp(2n,\mathbb{R}))_\lambda^{\mathrm{gen}}$.

Proof. We construct the choices. The fixed choice of θ also fixes choices of θ at rank n_1 and θ at rank n_2. Fix a choice of $\mathbf{D}_{Mp}$ as in Definition 5.1. This amounts to a pure choice of $\mathbf{D}_{SO(2n_1+1)}$ at λ_1', a choice of θ at rank n_1, a choice of θ at rank n_2, and a choice of $\mathbf{D}_{SO(2n_2+1)}$ at λ_2'. We assume that the choices of θ at rank n_1 and n_2 are compatible with the fixed choice of θ at rank n. As in the construction of Definition 5.1, the choice of duality for $SO(2n+1)$ at infinitesimal character λ' amounts to a choice of dualities for $SO(2n_1+1)$ and $SO(2n_2+1)$ at respective infinitesimal character λ_1' and λ_2'. The choice of $\mathbf{D}_{Mp}$ fixes this data, and we define $\mathbf{D}_{SO}$ accordingly. By Propositions 3.23 and 4.6 (especially the proof of the former which relies on Theorem 3.16), these choices satisfy the condition of the lemma. □

Theorem 5.4. *Fix choices of $\mathbf{D}_{SO(2n+1)}$ and $\mathbf{D}_{Mp(2n)}$ as in Lemma 5.3, and recall the definition of L-packets given before Lemma 5.3. Then θ preserves L-packets*

$$\theta(\mathbb{L}_{Mp}(Q,\lambda)) = \mathbb{L}_{SO}(Q,\lambda').$$

□

Proof. This is an immediate corollary of Lemma 5.3 and the definitions.

Fix choices as in Lemma 5.3. As in Theorems 3.24 and 3.26 we may form the virtual characters

$$\eta^{\mathrm{loc}}(\mathbb{L}_{Mp}(Q,\lambda)) := \sum_{\gamma\in\mathcal{P}_\lambda^{Mp}(Q)} [\mathrm{std}(\gamma)]$$

and

$$\eta^{\mathrm{mic}}(\mathbb{L}_{Mp}^{\mathrm{mic}}(Q,\lambda)) := \sum_{\pi\in\mathbb{L}_{Mp}^{\mathrm{mic}}(Q,\lambda)} (-1)^{\dim(\mathrm{supp}(\pi^\vee))-\dim(Q)} m_Q(\pi^\vee)[\pi].$$

Arguing as in the proof of Theorems 3.24 and 3.26, we obtain two bases of stable virtual representations of $Mp(2n,\mathbb{R})$ at infinitesimal character λ:

$$\{\eta^{\mathrm{loc}}(\mathbb{L}_{Mp}(Q,\lambda)) \mid Q \in K\backslash X\}$$
$$\{\eta^{\mathrm{mic}}(\mathbb{L}_{Mp}(Q,\lambda)) \mid Q \in K\backslash X\}.$$

In the same way there are the two basis consisting respectively of $\{\eta^{\mathrm{loc}}(\mathbb{L}_{SO}(Q,\lambda'))$ and $\{\eta^{\mathrm{mic}}(\mathbb{L}_{SP}(Q,\lambda'))$, and as in Sections 3 and 4 we obtain two isomorphisms by matching the bases,

$$T^{\mathrm{loc}},\ T^{\mathrm{mic}} :\ \mathbb{K}\mathcal{HC}(SO(2n+1,\mathbb{C}))^{st}_{\theta_{\mathrm{ic}}(\lambda)} \longrightarrow \mathbb{K}\mathcal{HC}(Mp(2n,\mathbb{R}))^{st}_{\lambda}.$$

The second part of the proof of Theorem 3.29 is completely general and again applies here to show that $T^{\mathrm{loc}} = T^{\mathrm{mic}}$. Combining Theorems 3.29 and Theorems 4.7 we obtain the following result at regular infinitesimal character λ.

Theorem 5.5. *Fix infinitesimal character λ and define T^{loc} and T^{mic} as above. Then $T^{\mathrm{loc}} = T^{\mathrm{mic}}$. Moreover there is is a choice of oscillator defining Adams' map T such that $T = T^{\mathrm{loc}} = T^{\mathrm{mic}}$.*

Proof. As remarked above, the only point that requires comment is the case of singular infinitesimal character. Strictly speaking we have only defined T^{loc} and T^{mic} when λ is regular, but the comments in Remark 3.20 show how the translation principal leads to a general definition. More precisely, suppose λ_s is weakly dominant and singular, and λ is a dominant weight translate of λ_s. Let T denote the translation functor from λ to λ_s. Write T' for the obvious analog from λ' to λ'_s. Then T^{loc} at λ_s is defined by the requirement $T_{\mathrm{loc}} \circ T' = T \circ T_{\mathrm{loc}}$, where the latter T^{loc} is defined at the regular λ. A similar comment applies to the definition of T^{mic} in the singular case. It follows easily that $T^{\mathrm{loc}} = T^{\mathrm{mic}}$. Similarly it easy to check from the definition of Adams' map that T behaves in the same way under translation as our T^{loc}. So the result follows. □

6. Endoscopy

In this section, following the philosophy of [ABV], we give a geometric interpretation of endoscopic lifting for $Mp(2n,\mathbb{R})$. To elucidate the main ideas, we begin with a general discussion. Let $H_{\mathbb{R}}$ be a real reductive group, and fix a maximal compact subgroup $U_{\mathbb{R}}$ of $H_{\mathbb{R}}$, and an infinitesimal character λ. A geometric parameter space for $H_{\mathbb{R}}$ at infinitesimal character λ is a complex variety $X(\lambda)$, endowed with an action of an algebraic group ${}^\vee H$ which has a finite number of orbits on $X(\lambda)$, satisfying the following properties. (The notation here is to indicate that ${}^\vee H$ is supposed to be something like a Langlands dual, but we don't want to be that precise for now.) Let $\mathcal{A}(X(\lambda), {}^\vee H)$ be the category of ${}^\vee H$-equivariant perverse sheaves on $X(\lambda)$, and write $\mathcal{C}(X(\lambda), {}^\vee H)$ for the category of equivariant constructible sheaves. Their respective Grothendieck groups are identified through the Euler characteristic map, and will be denoted by $\mathbb{K}(X(\lambda), {}^\vee H)$. Irreducible objects for these two categories are parameterized by the same set Ξ, and for $\xi \in \Xi$, we denote by $P(\xi)$ and $\mu(\xi)$ the corresponding irreducible perverse and constructible sheaves on $X(\lambda)$. As the name indicates, a geometric parameter space for $H_{\mathbb{R}}$ at infinitesimal character λ also parameterizes irreducible Harish-Chandra modules in $\mathcal{HC}(\mathfrak{h}, U)_\lambda$, in the sense that we require that there exist a bijection $\phi : \Xi \to \mathcal{P}_\lambda$. For $\xi \in \Xi$, let us denote by $\mathrm{std}(\xi)$ and $\mathrm{irr}(\xi)$ the corresponding standard and irreducible Harish-Chandra modules. The most crucial requirement is the existence of a duality between the representation theory and the geometry. This is encoded in a perfect pairing

$$\langle \cdot, \cdot \rangle : \ \mathbb{K}\mathcal{HC}(\mathfrak{h}, U)_\lambda \times \mathbb{K}(X(\lambda), {}^\vee H) \longrightarrow \mathbb{Z}$$

between Grothendieck groups, such that $\langle \mathrm{std}(\xi), \mu(\zeta) \rangle = \delta_{\xi,\zeta}$ and $\langle \mathrm{irr}(\xi), P(\zeta) \rangle = \pm\delta_{\xi,\zeta}$. This says that the change of basis matrix $m(\xi, \zeta)$ between the bases $\{[\mathrm{irr}(\xi)]\}_{\xi\in\Xi}$ and $\{[\mathrm{std}(\xi)]\}_{\xi\in\Xi}$ of $\mathbb{K}\mathcal{HC}(\mathfrak{h}, U)_\lambda$ is (up to sign) the transpose of the change of basis matrix $m_g(\xi, \zeta)$ between the bases $\{P(\xi)\}_{\xi\in\Xi}$ and $\{\mu(\xi)\}_{\xi\in\Xi}$ of $\mathbb{K}(X(\lambda), {}^\vee H)$.

In an endoscopic setting, the idea is to obtain endoscopic lifting as the transpose of a map between sheaves on geometric parameter spaces. For algebraic groups, the existence and the construction of such geometric parameter spaces is the main topic of [ABV]. In our setting, existence and construction of geometric parameter spaces for $Mp(2n, \mathbb{R})$ and $\widetilde{G}_{\mathbb{R}}$ (notation as in the introduction) at half-integral infinitesimal character follows from the results of Section 3. It is a simple consequence that the geometric parameter space for $\widetilde{G}_{\mathbb{R}}$ includes into the one for $Mp(2n, \mathbb{R})$, and so we obtain an endoscopic lifting as the transpose of pullback of sheaves. When no coordinate of the infinitesimal character is a half-integer results of Section 4 show that the geometric parameter spaces for the real forms of the linear group $SO(2n+1, \mathbb{C})$ are also geometric parameter spaces for $Mp(2n, \mathbb{R})$. The general case follows by conbining these two as in Section 5.

6.1. **Perverse sheaves.** We begin by interpreting Theorem 3.16 along the lines of [ABV], Theorem 1.24. We are thus working with regular half-integral infinitesimal character, as in Section 3, and cannot apply the Riemann-Hilbert correspondence directly to the category $\mathcal{D}_\lambda(X, \widetilde{K})$ of $\widetilde{K}$ equivariant $\mathcal{D}_\lambda$ modules on X. Following [ABV], Chapter 17, set $\mu = 2(\lambda - \rho_{Mp})$, and let $\mathcal{L}$ the affine line bundle it defines on X. The group $H = \mathbb{C}^* \times \widetilde{K}$ acts on $\mathcal{L}^\times$ with $\mathbb{C}^*$ acting the fibers of $\mathcal{L}^\times \to X$ by

$$z \cdot \xi = z^2 \xi \qquad (z \in \mathbb{C}^*, \xi \in \mathcal{L}^\times).$$

This action of $\mathbb{C}^*$ on $\mathcal{L}^\times$ allows to define "genuine" $\mathbb{C}^*$-equivariant object on $\mathcal{L}^\times$, i.e. objects with the required monodromy. Notice that we have two notions of "genuine", one with respect to this action of $\mathbb{C}^*$, and one with respect to the action of the central element $\mathbf{z}$. There is (cf. [ABV], Proposition 17.5) an equivalence of category between $\mathcal{D}_\lambda(X, \widetilde{K})^{gen}$ and $\mathcal{D}(\mathcal{L}^\times, H)^{gen}$ of H-equivariant genuine (both with respect to $\mathbb{C}^*$ and $\mathbf{z}$) $\mathcal{D}_{\mathcal{L}^\times}$-modules on $\mathcal{L}^\times$. Notice that $\mathcal{D}_{\mathcal{L}^\times}$ is the sheaf of differential operators on $\mathcal{O}_{\mathcal{L}^\times}$, so we can apply the Riemann-Hilbert functor $RHom_{\mathcal{D}_{\mathcal{L}^\times}}(\ . \ , \mathcal{O}_{\mathcal{L}^\times})$ from $\mathcal{D}(\mathcal{L}^\times, H)^{gen}$ to $\mathcal{A}(\mathcal{L}^\times, H)$, the category of H-equivariant genuine perverse sheaves on $\mathcal{L}^\times$. To summarize, we have obtained an equivalence of categories between $\mathcal{HC}(\mathfrak{sp}(2n, \mathbb{C}), \widetilde{K})_\lambda$ and $\mathcal{A}(\mathcal{L}^\times, H)$. Recall that irreducible objects in $\mathcal{HC}(\mathfrak{sp}(2n, \mathbb{C}), \widetilde{K})_\lambda$ are parameterized by the set $\mathcal{P}_\lambda^{Mp}$, and that this set can naturally be viewed as the set of irreducible H-equivariant genuine local systems on $\mathcal{L}^\times$ (see [ABV], Lemma 17.9, for instance). In turn, this latter set can be viewed as the set $\Xi(\mathcal{L}^\times, H)$ of pairs (Q,τ) consisting of an H-orbit Q in $\mathcal{L}^\times$ and an irreducible genuine representation of the component group of the stabilizer of an element $x \in Q$. (Up to isomorphism, this group does not depend on the choice of $x \in Q$, and we will be denote by $A(Q)$.) For future reference, we write

$$\phi : \Xi(\mathcal{L}^\times, H) \to \mathcal{P}_\lambda$$

for the bijection just described.

In addition to the perverse category $\mathcal{A}(\mathcal{L}^\times, H)$, we need also to introduce the category $\mathcal{C}(\mathcal{L}^\times, H)$ of H-equivariant genuine constructible sheaves on $\mathcal{L}^\times$. The set $\Xi(\mathcal{L}^\times, H)$ parameterized the irreducible objects in both categories and for $\xi \in \Xi(\mathcal{L}^\times, H)$, we will denote the respective perverse and constructible sheaves by $P(\xi)$ and $\mu(\xi)$. Since the cohomology sheaves of a perverse sheaf are constructible, taking Euler characteristics defines a map from the Grothendieck group of $\mathcal{A}(\mathcal{L}^\times, H)$ to that of $\mathcal{C}(\mathcal{L}^\times, H)$. It is well-known that this is an isomorphism. We will identify the two Grothendieck groups and write $\mathbb{K}(\mathcal{L}^\times, H)$ for the common vector space which is endowed with two canonical bases: $\{P(\xi) |\ \xi \in \Xi(\mathcal{L}^\times, H)\}$ and $\{\mu(\xi) |\ \xi \in \Xi(\mathcal{L}^\times, H)\}$. (Here we caution the reader that $P(\xi)$ denotes both a perverse sheaf and the image of its Euler characteristic in $\mathbb{K}(\mathcal{L}^\times, H)$.) We define the change-of-basis matrix as follows:

$$\mu(\xi) = (-1)^{d(\xi)} \sum_{\xi \in \Xi(\mathcal{L}^\times, H)} m_g(\zeta, \xi)\ P(\zeta);$$

here $m_g(\zeta, \xi)$ an integer.

For all $\xi \in \Xi(\mathcal{L}^\times, H)$, we will write

$$\operatorname{irr}(\xi) := \operatorname{irr}(\phi(\xi)^\vee), \qquad \operatorname{std}(\xi) := \operatorname{std}(\phi(\xi)^\vee).$$

Notice the crucial appearance of the duality in this definition, which allows the following reformulation of Theorem 3.16.

Theorem 6.1. *There is a natural perfect pairing :*

$$\langle\ ,\ \rangle\ : \mathbb{K}\mathcal{HC}(\mathfrak{sp}(2n, \mathbb{C}), \widetilde{K})_\lambda \times \mathbb{K}(\mathcal{L}^\times, H) \longrightarrow \mathbb{Z}$$

between the Grothendieck group of finite length genuine Harish-Chandra modules with half-integral infinitesimal character λ, and that of H-equivariant genuine (perverse or constructible) sheaves on $\mathcal{L}^\times$. This pairing is defined on the level of basis vectors by

$$\langle std(\xi), \mu(\zeta)\rangle = \delta_{\xi,\zeta}.$$

In terms of the other bases of $\mathbb{K}\mathcal{HC}(\mathfrak{sp}(2n, \mathbb{C}), \widetilde{K})_\lambda$ and $\mathbb{K}(\mathcal{L}^\times, H)$ we have

$$\langle irr(\xi), P(\zeta)\rangle = (-1)^{d(\xi)} \delta_{\xi,\zeta}.$$

The content of the theorem is the equivalence between the two possible definitions of the pairing.

6.2. **Endoscopy: half-integral case.** Let $\widetilde{G}_\mathbb{R}$ be one of the endoscopic groups for $Mp(2n, \mathbb{R})$ defined in [R2]. It is obtained in the following way. Let $s = \operatorname{Diag}(a_1, \dots, a_n, a_1, \dots, a_n)$, with $a_i = \pm 1$ and let $G_\mathbb{R} = \operatorname{Cent}(s, Sp(2n, \mathbb{C}))$. Thus $G_\mathbb{R}$ is isomorphic to $Sp(2n_1, \mathbb{R}) \times Sp(2n_2, \mathbb{R})$, where n_1 (resp. n_2) is the number of 1 (resp. -1) in $\{a_1, \dots, a_n\}$. As usual $\widetilde{G}_\mathbb{R}$ denotes the preimage of $G_\mathbb{R}$ in $Mp(2n, \mathbb{R})$. Since $\widetilde{G}_\mathbb{R}$ is a quotient of $Mp(2n_1, \mathbb{R}) \times Mp(2n_2, \mathbb{R})$ by a two elements subgroup it is easy to extend the relevant definitions and results about stability, duality, and L-packets to $\widetilde{G}_{\mathbb{R}}$. In fact, genuine representations for $\widetilde{G}_{\mathbb{R}}$ are given by pairs of genuine representations for $Mp(2n_1,\mathbb{R})$ and $Mp(2n_2,\mathbb{R})$ (see Section 5.3 of [R2]).

Let X_i $(i=1,2)$ be the flag manifold of $\mathfrak{sp}(2n_i,\mathbb{C})$. The choices of base points according to our conventions define an embedding :

$$\epsilon\ :\ X_1\times X_2\longrightarrow X$$

Fix half-integral dominant regular infinitesimal character λ for $Mp(2n,\mathbb{R})$ as above and let $\lambda_G=\lambda-\rho_{Mp}+\rho_G$. A short computation shows that λ_G is still half-integral. Form the bundle $\mathcal{L}^\times$ over X as in the previous paragraph and let $\mathcal{L}_G^\times$ denote the restriction $\epsilon^*\mathcal{L}^\times$. Clearly $\mathcal{L}_G^\times$ is the bundle on $X_1\times X_2$ formed from the integral weight $\mu_G=2(\lambda_G-\rho_G)$. We again write

$$\epsilon:\ \mathcal{L}_G^\times\to\mathcal{L}^\times$$

for the embedding of bundles.

Let $(\widetilde{K}_G)_{\mathbb{R}}$ be the maximal compact subgroup of $\widetilde{G}_{\mathbb{R}}$ chosen according to our conventions, and let $\widetilde{K}_G$ be its complexification. If S_G is a K_G-orbit on the flag manifold $X_1\times X_2$, then the image of S_G under ϵ is contained in a single K-orbit on X. Accordingly, the image of an orbit Q_G of $H_G:=\widetilde{K}_G\times\mathbb{C}^*$ in $\mathcal{L}_G^*$ is contained in a single H-orbit on $\mathcal{L}^\times$. We will denote it by $\epsilon(Q_G)$. Furthermore, the inclusion $\widetilde{K}_G\hookrightarrow\widetilde{K}$ induces a map between component groups

$$A(\epsilon):\ A(Q_G)\to A(Q)$$

where $Q=\epsilon(Q_G)$. It is not difficult to compute explicitly these component groups (they are essentially the groups $F^{m,rs}$ of [RT], Section 2.4), and see that $A(\epsilon)$ is in fact an isomorphism.

Pull-back of sheaves defines a map

$$\epsilon^\bullet:\ \mathbb{K}(\mathcal{L}^\times,H)\to\mathbb{K}(\mathcal{L}_G^\times,H_G).$$

Under the pairing of Theorem 6.1, the transpose of $\epsilon^\bullet$ is a map

$$\epsilon_\bullet:\ \mathbb{K}\mathcal{HC}(\mathfrak{g},\widetilde{K}_G)_{\lambda_G}\to\mathbb{K}\mathcal{HC}(\mathfrak{sp}(2n,\mathbb{C}),\widetilde{K})_\lambda.$$

This map is explicitly computed in [ABV], Proposition 26.4 (with the obvious modifications due to our slightly different context). The endoscopic lifting of characters will be constructed from the map $\epsilon_\bullet$. We start from the endoscopic data $(\widetilde{G}_{\mathbb{R}},s)$. Fix an element $\tilde{s}\in Mp(2n,\mathbb{R})$ in the preimages of s. Then $\tilde{s}\in(\widetilde{K}_G)_{\mathbb{R}}$ and since it is central, it defines for all H_G-orbit Q_G in $\mathcal{L}_G^\times$ a class σ in $A(Q_G)$. Let S_G be the K_G-orbit on $X_1\times X_2$ which is the projection of Q_G. We form the following virtual characters

$$\eta^{loc}(\sigma)(\mathbb{L}(S_G,\lambda_G))=\sum_{\xi=(Q_G,\tau)\in\Xi(\mathcal{L}_G^\times,H_G)}\mathrm{tr}\tau(\sigma)[\mathrm{std}(\xi)]$$

In our context, the relevance of this virtual character is that it is obtained by translation by $\tilde{s}$ in the character formula for $\eta^{loc}(\mathcal{L}(S_G,\lambda_G))$ (the stable virtual representation defined in Section 3.3). More precisely, the map

$$\mathbb{K}\mathcal{HC}(\mathfrak{g},\widetilde{K}_G))_{\lambda_G}^{st,gen}\to\mathbb{K}\mathcal{HC}(\mathfrak{g},\widetilde{K}_G))_{\lambda_G}^{gen}$$
$$\eta^{loc}(\mathbb{L}(S_G,\lambda_G))\mapsto\eta^{loc}(\sigma)(\mathbb{L}(S_G,\lambda_G))$$

is the restriction to virtual characters of the map

$$\tau_{\tilde{s}}: \mathbb{K}\mathcal{HC}(\widetilde{G}_{\mathbb{R}})^{gen,st}_{\lambda_G} \to \mathbb{K}\mathcal{HC}(\widetilde{G}_{\mathbb{R}})^{gen,\kappa}_{\lambda_G}$$

defined in [R2], Section 6.

Define Lift : $\mathbb{K}\mathcal{HC}(\mathfrak{g}, \widetilde{K}_G))^{st,gen}_{\lambda_G} \to \mathbb{K}\mathcal{HC}(\mathfrak{sp}(2n,\mathbb{C}), \widetilde{K}))^{gen}_{\lambda}$ to be the composition $\epsilon_\bullet \circ \tau_{\tilde{s}}$. It is essentially an exercise in keeping track of the definitions to show that Lift is the restriction on virtual genuine stable characters of the map

$$\mathrm{Trans}^*: \mathbb{K}\mathcal{HC}(\widetilde{G}_{\mathbb{R}})^{gen,st}_{\lambda_G} \to \mathbb{K}\mathcal{HC}(Mp(2n,\mathbb{R}))^{gen}_{\lambda}$$

defined in [R2].

6.3. **Endoscopy: non half-integral case.** Assume now that the infinitesimal character λ has no half-integral coordinates. The results of Section 4 imply that we can use the L-group of $\coprod{}' SO(p,q)$, which is

$$^L SO_n := Sp(2n,\mathbb{C}) \times \mathrm{Gal}(\mathbb{C}/\mathbb{R})$$

to construct geometric parameter spaces for genuine representations of $Mp(2n,\mathbb{R})$ with infinitesimal character λ. We just use the construction of [ABV], applied to $\coprod{}' SO(p,q)$ at infinitesimal character $\lambda' = \theta_{ic}(\lambda)$ and the θ-correspondence. Then Theorem 4.1 insures that these geometric parameter spaces will have the required formal properties, i.e that an analog of Theorem 6.1 holds.

Consider now the endoscopic data $(\widetilde{G}_{\mathbb{R}}, s)$ of the previous section. View s as an element in the trivial connected component of $^L SO_n$. Thus

$$\mathrm{Cent}(^L SO_n, s) \simeq Sp(2n_1,\mathbb{C}) \times Sp(2n_2,\mathbb{C}) \times \mathrm{Gal}(\mathbb{C}/\mathbb{R}).$$

The considerations above about representation theory of the group $\widetilde{G}_{\mathbb{R}}$ at infinitesimal character $\lambda_G = \lambda - \rho + \rho_G$ enable us to see this latter group as the L-group for $\widetilde{G}_{\mathbb{R}}$. Furthermore, we have an obvious embedding

$$\epsilon: Sp(2n_1,\mathbb{C}) \times Sp(2n_2,\mathbb{C}) \times \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to {}^L SO_n.$$

We are thus reduced to the usual setting of endoscopy for linear groups. From [ABV], Chapter 26, and we obtain a map:

$$\mathrm{Lift}: \mathbb{K}\mathcal{HC}(\mathfrak{g}, \widetilde{K}_G)^{gen,st}_{\lambda_G} \to \mathbb{K}\mathcal{HC}(\mathfrak{sp}(2n,\mathbb{C}), \widetilde{K})^{gen}_{\lambda}$$

as the transpose of a map between Grothendieck groups of category of equivariant sheaves on the relevant geometric parameter spaces. Again, it is elementary to check that Lift is the restriction to stable genuine virtual characters with infinitesimal character λ_G of the map defined in [R2].

As noted in Section 4, it is important to notice that for this kind of infinitesimal character, all virtual characters are stable, i.e.

$$\mathbb{K}\mathcal{HC}(\mathfrak{g}, \widetilde{K}_G)^{gen,st}_{\lambda_G} = \mathbb{K}\mathcal{HC}(\mathfrak{g}, \widetilde{K}_G)^{gen}_{\lambda_G} \quad \mathbb{K}\mathcal{HC}(\mathfrak{sp}(2n,\mathbb{C}), \widetilde{K})^{gen,st}_{\lambda} = \mathbb{K}\mathcal{HC}(\mathfrak{sp}(2n,\mathbb{C}), \widetilde{K})^{gen}_{\lambda}.$$

As a consequence, we could have defined Lift using the endoscopic lifting for groups $\coprod{}' SO(p,q)$ at infinitesimal character λ' given by ϵ (viewed as a map between L-groups on the SO side), and Adams map ([A1]) to make the connection with genuine virtual characters of $\widetilde{G}_{\mathbb{R}}$ and $Mp(2n,\mathbb{R})$. Of course the theory of Section 4 insures that the two approaches are completely equivalent.

6.4. **General case and further results.** Fix dominant regular infinitesimal character $\lambda = (\lambda_1, \dots, \lambda_2)$ for $Mp(2n, \mathbb{R})$, given in usual coordinates for the split (diagonal) Cartan. As in section 5, the splitting between half-integral and non half-integral coordinates of λ also defines a subgroup $M_{\mathbb{R}}$ of $Sp(2n, \mathbb{R})$ which is isomorphic to a product $Sp(2m_1, \mathbb{R}) \times Sp(2m_2, \mathbb{R})$, where m_1 is the number of halfintegral coordinates and m_2 the number of non halfintegral coordinates (thus $m_1 + m_2 = n$). Let $\widetilde{M_{\mathbb{R}}}$ be the inverse image of $M_{\mathbb{R}}$ in $Mp(2n, \mathbb{R})$. Again, it is isomorphic to a quotient of $Mp(2m_1, \mathbb{R}) \times Mp(2m_2, \mathbb{R})$ by a two element subgroups and genuine representations for $\widetilde{M_{\mathbb{R}}}$ are given by pair of genuine representations for $Mp(2m_1, \mathbb{R})$ and $Mp(2m_2, \mathbb{R})$.

Let λ^1 denotes the ordered subset of half-integral coordinates in $\lambda = (\lambda_1, \dots, \lambda_2)$ and λ^2 the ordered subset of non half-integral coordinates. It was shown in Section 5 how the duality theory of genuine representations of $Mp(2n, \mathbb{R})$ at infinitesimal character λ is reduced to the duality theory of $\widetilde{M_{\mathbb{R}}}$ (also at infinitesimal character λ), and that in turn is given by the duality theory of genuine representations of $Mp(2m_1, \mathbb{R})$ at infinitesimal character λ^1 and representations of $SO(p, q)$'s ($p + q = 2m_2 + 1$) at infinitesimal character $\theta_{ic}(\lambda^2)$. Thus, a geometric parameter space for $Mp(2n, \mathbb{R})$ at infinitesimal character λ is given by the product of geometric parameter space for $Mp(2m_1, \mathbb{R})$ at infinitesimal character λ^1 constructed in 6.2 and the geometric parameter space for $Mp(2m_2, \mathbb{R})$ at infinitesimal character λ^2 constructed in 6.3. Let us write this as $\mathcal{X} = \mathcal{X}^1 \times \mathcal{X}^2$ endowed with the action of an algebraic group $J = J^1 \times J^2$.

Let us now go back to the endoscopic setting. The choice of the element s defining $\widetilde{G}_{\mathbb{R}}$ defines a partition of the coordinates of λ in two pieces, corresponding to the isomorphism $G_{\mathbb{R}} \simeq Sp(2n_1, \mathbb{R}) \times Sp(2n_2, \mathbb{R})$. We can now refine the splitting between the half-integral and non half-integral coordinates of λ to each of these factors, and construct the geometric parameter space for $\widetilde{G}_{\mathbb{R}}$ at infinitesimal character λ_G accordingly as above. Thus the geometric parameter space for $\widetilde{G}_{\mathbb{R}}$ at infinitesimal character λ_G can be written as $\mathcal{X}_G = \mathcal{X}_{n_1} \times \mathcal{X}_{n_2}$ endowed with the action of an algebraic group $J_G = J_{n_1} \times J_{n_2}$, and in turn $\mathcal{X}_{n_1} = \mathcal{X}^1_{n_1} \times \mathcal{X}^2_{n_1}$, $J_{n_1} = J^1_{n_1} \times J^2_{n_1}$ and $\mathcal{X}_{n_2} = \mathcal{X}^1_{n_2} \times \mathcal{X}^2_{n_2}$, $J_{n_2} = J^1_{n_2} \times J^2_{n_2}$.

From the discussion in 6.2, we get a map

$$\epsilon_1 : \mathcal{X}^1_{n_1} \times \mathcal{X}^1_{n_1} \to \mathcal{X}^1$$

and from Section 6.3, we get a map

$$\epsilon_2 : \mathcal{X}^2_{n_1} \times \mathcal{X}^2_{n_2} \to \mathcal{X}^2$$

The product gives a map

$$\epsilon = \epsilon_1 \times \epsilon_2 : \mathcal{X}_{n_1} \times \mathcal{X}_{n_2} \to \mathcal{X},$$

and pull-back of sheaves gives a map

$$\epsilon^\bullet : \mathbb{K}(\mathcal{X}, J) \to \mathbb{K}(\mathcal{X}_G, J_G)$$

with transpose

$$\epsilon_\bullet : \mathbb{K}\mathcal{HC}(\mathfrak{g}, \widetilde{K}_G)_{\lambda_G} \to \mathbb{K}\mathcal{HC}(\mathfrak{sp}(2n, \mathbb{C}), \widetilde{K})_\lambda.$$

Combining Section 6.2 and 6.3, this gives a map

$$\text{Lift} : \mathbb{K}\mathcal{HC}(\mathfrak{g}, \widetilde{K}_G))^{st,gen}_{\lambda_G} \to \mathbb{K}\mathcal{HC}(\mathfrak{sp}(2n, \mathbb{C}), \widetilde{K}))^{gen}_\lambda$$

which is the restriction on genuine virtual stable characters of the map Trans* defined in [R2]

One could also develop the relevant microlocal analysis to obtain an analog of [ABV], Theorem 26.4, computing the lift of a stable genuine virtual character on $\widetilde{G}_{\mathbb{R}}$ coming from a micro L-packet. This too can be done without essential difficulties.

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