ANSWER TO A QUESTION BY BURR AND ERDŐS ON RESTRICTED ADDITION, AND RELATED RESULTS

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Abstract. We study the gaps in the sequence of sums of h pairwise distinct elements of a given sequence $\mathcal{A}$ in relation with the gaps in the sequence of sums of h not necessarily distinct elements of $\mathcal{A}$. We present several results on this topic. One of them gives a negative answer to a question by Burr and Erdős.

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1. Introduction

In [1], Erdős writes:

> Here is a really recent problem of Burr and myself : An infinite sequence of integers $a_1 < a_2 < \cdots$ is called an asymptotic basis of order k, if every large integer is the sum of k or fewer of the a's. Let now $b_1 < b_2 < \cdots$ be the sequence of integers which is *(sic)* the sum of k or fewer distinct a's. Is it true that
>
> $$\limsup(b_{i+1} - b_i) < \infty.$$
>
> In other words the gaps between the b's are bounded. The bound may of course depend on k and on the sequence $a_1 < a_2 < \cdots$.

For $h \geq 1$, we will use the following notation for addition and restricted addition: $h\mathcal{A}$ will denote the set of sums of h not necessarily distinct elements of $\mathcal{A}$, and $h \times \mathcal{A}$, the set of sums of h pairwise distinct elements of $\mathcal{A}$.

If $\mathcal{A}$ is an increasing sequence of integers $a_1 < a_2 < \cdots$, the largest asymptotic gap in $\mathcal{A}$, that is

$$\limsup_{i\to+\infty}(a_{i+1} - a_i),$$

is denoted by $\Delta(\mathcal{A})$.

We shall write $\mathcal{A} \sim \mathbb{N}$ to denote that a set of integers $\mathcal{A}$ contains all but finitely many positive integers. According to the Erdős-Burr definition, a set of integers $\mathcal{A}$ is an asymptotic basis of order h if h is the smallest integer such that $\bigcup_{j=1}^{h} j\mathcal{A} \sim \mathbb{N}$, or equivalently such that $h(\mathcal{A} \cup \{0\}) \sim \mathbb{N}$.

The lower asymptotic density of a set of integers $\mathcal{A}$ is defined by

$$\underline{\mathrm{d}}\mathcal{A} = \liminf_{x\to+\infty} \frac{|\{a \in \mathcal{A} \text{ such that } 1 \leq a \leq x\}|}{x},$$

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where the notation $|F|$ denotes the cardinality of a finite set F.

The question of Burr and Erdős takes the shorter form: is it true that if $h(\{0\} \cup \mathcal{A}) \sim \mathbb{N}$, then

$$\Delta(\mathcal{A} \cup 2 \times \mathcal{A} \cup \cdots \cup h \times \mathcal{A}) < +\infty\,?$$

We may also ask the following even more natural question: is it true that $\Delta(h\mathcal{A}) < +\infty$ (or at least $h\mathcal{A} \sim \mathbb{N}$) implies $\Delta(h \times \mathcal{A}) < +\infty$? This would imply (and thus give another proof of) the main result in [5] which states that if $\mathcal{A}$ is an asymptotic basis of order h, then $h \times \mathcal{A}$ has a positive lower asymptotic density, as it was conjectured in [2].

We will show that the answer to both questions is no, except if $h = 2$:

Theorem 1. (i) *If* $(\mathcal{A} \cup 2\mathcal{A}) \sim \mathbb{N}$ *then*

$$\Delta(\mathcal{A} \cup 2 \times \mathcal{A}) \leq 2.$$

If $2\mathcal{A} \sim \mathbb{N}$ *then* $\Delta(2 \times \mathcal{A}) \leq 2$.

(ii) *Let* $h \geq 3$. *There exists a set* $\mathcal{A}$ *such that* $h(\{0\} \cup \mathcal{A}) \sim \mathbb{N}$ *and*

$$\Delta(\mathcal{A} \cup 2 \times \mathcal{A} \cup \cdots \cup h \times \mathcal{A}) = +\infty.$$

There exists a set $\mathcal{A}$ *such that* $h\mathcal{A} \sim \mathbb{N}$ *and* $\Delta(h \times \mathcal{A}) = +\infty$.

The restricted order of an asymptotic basis $\mathcal{A}$, if it exists, is defined as the smallest integer h such that any large enough integer is the sum of h or fewer pairwise distinct elements of $\mathcal{A}$. We denote it by $\mathrm{ord}_r(\mathcal{A})$. In general, asymptotic bases do not have to possess a (finite) restricted order. However, in the special case of asymptotic bases of order 2, the situation is more simple and can be precisely described (see [7] and [6]): indeed, being given an arbitrary asymptotic basis $\mathcal{A}$ of order 2, its restricted order is known to exist and to satisfy $2 \leq \mathrm{ord}_r(\mathcal{A}) \leq 4$; moreover any integral value in this range can be achieved with asymptotic bases $\mathcal{A}$ such that $2\mathcal{A} = \mathbb{N}$. In particular, there exist asymptotic bases $\mathcal{A}$ containing 0 verifying $\mathrm{ord}_r(\mathcal{A}) > 2$ and for which we consequently have $\Delta(2 \times \mathcal{A}) = \Delta(\mathcal{A} \cup 2 \times \mathcal{A}) \geq 2$. This shows that assertion (i) in Theorem 1 is optimal.

Having Theorem 1 at hand, the next natural question is then: assume that $h\mathcal{A} \sim \mathbb{N}$, that is $h\mathcal{A}$ contains all but finitely many positive integers, is it true that there exists an integer k such that $\Delta(k \times \mathcal{A}) < +\infty$? If so, k could depend on $\mathcal{A}$. But, suppose that such a k exists for all $\mathcal{A}$ satisfying $h\mathcal{A} \sim \mathbb{N}$: is this value of k uniformly (with respect to $\mathcal{A}$) bounded from above (in term of h)? If so, write $k(h)$ for the maximal possible value:

$$k(h) = \max_{h\mathcal{A} \sim \mathbb{N}} \min\{k \in \mathbb{N} \text{ such that } \Delta(k \times \mathcal{A}) \text{ is finite}\}.$$

Theorem 1 implies that $k(2)$ does exist and is equal to 2. No other value of $k(h)$ is known but we believe that the following conjecture is true.

Conjecture 2. *The function* $k(h)$ *is well-defined in the sense that for any integer* $h \geq 1$, $k(h)$ *is finite.*

If this conjecture is true, what is the asymptotic behaviour of $k(h)$? Our proof of Theorem 1 will be based on an explicit counterexample to the Erdős-Burr conjecture. This construction will lead in fact to a lower bound of $k(h)$, which obviously implies Theorem 1 for $h \geq 3$.

Theorem 3. *Let $h \geq 2$. We have*

$$k(h) \geq 2^{h-2} + h - 1.$$

This study is closely related to the following problem: if $\mathcal{A}$ is an asymptotic basis of order h which admits a (finite) restricted order $\text{ord}_r(\mathcal{A})$, is it true that $\text{ord}_r(\mathcal{A})$ is bounded in terms of h? If so, let us define $f(h)$ to be the maximal possible value taken by $\text{ord}_r(\mathcal{A})$, when $\mathcal{A}$ runs over the bases of order h having a finite restricted order. For $h = 2$, the question has been completely solved in [6] where it is shown that $f(2) = 4$. For $h \geq 3$, if we reuse the example leading to the bound of Theorem 3, we obtain an explicit lower bound for $f(h)$.

Theorem 4. *Let $h \geq 3$. One has*

$$f(h) \geq 2^{h-2} + h - 1.$$

In another direction, we can study, for a given set of positive integers $\mathcal{A}$, the asymptotic behaviour of the sequence $\big(\Delta(h \times \mathcal{A})\big)_{h \geq h_0}$. The first observation is that this sequence is well-defined for some h_0 as soon as $\Delta(h_0 \times \mathcal{A})$ is finite. Indeed we have the following proposition.

Proposition 5. *Let $\mathcal{A}$ be a set of positive integers. Assume that $\Delta(h_0 \times \mathcal{A})$ is finite for some integer h_0, then for any $h \geq h_0$, $\Delta(h \times \mathcal{A})$ is finite.*

This result implies that

$$k(h) = 1 + \max_{h\mathcal{A} \sim \mathbb{N}} \max\{k \in \mathbb{N} \text{ such that } \Delta(k \times \mathcal{A}) = +\infty\}.$$

According to what obviously happens in the case of usual addition, it would be of some interest to establish, for any given set of integers $\mathcal{A}$, the monotonicity of the sequence $\big(\Delta(h \times \mathcal{A})\big)_{h \geq 1}$:

Conjecture 6. *Let $\mathcal{A}$ be a set of positive integers, then the sequence $\big(\Delta(h \times \mathcal{A})\big)_{h \geq 1}$ is non-increasing.*

We will observe firstly the following:

Proposition 7. *Let $\mathcal{A}$ be a set of positive integers, then*

$$\Delta(3 \times \mathcal{A}) \leq \Delta(2 \times \mathcal{A}).$$

More interestingly, we will show the following partial result in the direction of Conjecture 6:

Theorem 8. *Let $\mathcal{A}$ be a set of positive integers. Then there exists an increasing sequence of integers $(h_j)_{j \geq 1}$ such that $\big(\Delta(h_j \times \mathcal{A})\big)_{j \geq 1}$ is non-increasing.*

This theorem clearly implies that for a given set of positive integers $\mathcal{A}$, the inequality $\Delta\big((h+1) \times \mathcal{A}\big) \leq \Delta(h \times \mathcal{A})$ holds for infinitely many positive integers h. Theorem 8 is a direct consequence of the following more precise result.

Theorem 9. *Let $\mathcal{A}$ be a set of positive integers and h be the smallest positive integer such that $\Delta(h \times \mathcal{A})$ is finite. Then there exists an increasing sequence of integers $(h_j)_{j \geq 0}$ with $h_0 = h$ such that for any $j \geq 1$, one has $h_j + 2 \leq h_{j+1} \leq h_j + h + 1$ and $\Delta(h_{j+1} \times \mathcal{A}) \leq \Delta(h_j \times \mathcal{A})$.*

This shows that for a given set of positive integers $\mathcal{A}$, the inequality $\Delta\big((h+1)\times\mathcal{A}\big) \le \Delta(h\times\mathcal{A})$ holds for any h belonging to some set of positive integers having a positive lower asymptotic density bounded from below by $1/(h+1)$.

Let $\mathcal{A}$ be a set of integers satisfying the weaker condition $\underline{\mathrm{d}}h\mathcal{A} > 0$ (instead of $h\mathcal{A}\sim\mathbb{N}$). We will establish in Theorem 10 that the validity of Conjecture 2 would imply that $\Delta(k\times\mathcal{A})$ is finite for some integer k under this weaker condition. Clearly this result, if true, could not be uniform in $\mathcal{A}$. Henceforth, we introduce, for $\beta>0$, the quantity

$$k_1(\beta,h) = \max_{\underline{\mathrm{d}}h\mathcal{A}\ge\beta} \min\left\{k\in\mathbb{N} \text{ such that } \Delta(k\times\mathcal{A}) \text{ is finite}\right\}.$$

Our final result shows that k_1 is as well-defined as k, in some sense.

Theorem 10. *Assume that Conjecture 2 holds. Then for any real number β such that $0<\beta\le 1$ and any positive integer h, we have*

$$k_1(\beta,h) \le k\left(\left\lceil\left(1+\frac{1}{h}\right)\frac{1}{\beta}\right\rceil h\right),$$

where $\lceil u\rceil$ is the ceiling of u.

2. The proofs

For any real numbers x and y, $[x,y]$ and $[x,y)$ will denote the sets of all integers n (called intervals of integers) such that $x\le n\le y$ and $x\le n<y$ respectively.

Proof of Theorems 1, 3 and 4. Let us first consider the case $h=2$. Clearly the odd elements in $2\mathcal{A}$ do belong to $2\times\mathcal{A}$. This implies that if $2\mathcal{A}\sim\mathbb{N}$, then $\Delta(2\times\mathcal{A})\le 2$. This also implies that the odd elements in $\mathcal{A}\cup 2\mathcal{A}$ are in $\mathcal{A}\cup(2\times\mathcal{A})$. It follows that $\mathcal{A}\cup 2\mathcal{A}\sim\mathbb{N}$ implies $\Delta(\mathcal{A}\cup(2\times\mathcal{A}))\le 2$.

In the case $h\ge 3$, it is enough to construct an explicit example. We first introduce the sequence defined by $x_0=h$ and $x_{n+1}=(3\cdot 2^{h-2}-1)x_n^2+hx_n$ for $n\ge 0$, and let

$$\mathcal{A}_n = [0,x_n^2)\cup\left\{2^j x_n^2 \,:\, j=0,1,2,\dots,h-2\right\}.$$

Finally we define

$$\mathcal{A} = \{0\}\cup\bigcup_{n\ge 0}(x_n+\mathcal{A}_n).$$

Since any positive integer less than or equal to $2^{h-1}-2$ can be written as a sum of at most $h-2$ (distinct) powers of 2 taken from $\{2^j \,:\, j=0,1,\dots,h-2\}$, any integer in $[0,(2^{h-1}-1)x_n^2)$ can be written as a sum of $h-1$ elements of $\mathcal{A}_n$. Thus it follows

$$[0,(3\cdot 2^{h-2}-1)x_n^2)\subset\{0,2^{h-2}x_n^2\}+[0,(2^{h-1}-1)x_n^2)\subset\{0,2^{h-2}x_n^2\}+(h-1)\mathcal{A}_n\subset h\mathcal{A}_n.$$

We therefore infer that $[hx_n,x_{n+1})\subset h(x_n+\mathcal{A}_n)$. Moreover, since $hx_n\le x_n^2$, we have $[x_n,hx_n]\subset[x_n,x_n^2]\subset x_n+\mathcal{A}_n$. It follows that, for any $n\ge 0$, we have

$$[x_n,x_{n+1})\subset h\big((x_n+\mathcal{A}_n)\cup\{0\}\big)\subset h\mathcal{A}.$$

Consequently $h\mathcal{A}\sim\mathbb{N}$.

On the other hand, $(h-1)\mathcal{A}\not\sim\mathbb{N}$. Indeed, this assertion follows from the more precise fact that, for any $n\ge 0$, no integer in the range $[(2^{h-1}-1)x_n^2+(h-1)x_n+1, 2^{h-1}x_n^2-1]$ (an interval of integers with a length tending to infinity with n) can be written as a sum of $h-1$ elements of $\mathcal{A}$. Let us prove this fact by contradiction and assume the existence of an integer

$$u \in [(2^{h-1}-1)x_n^2 + (h-1)x_n + 1, 2^{h-1}x_n^2 - 1] \cap (h-1)\mathcal{A}.$$

Since we have (using $h \geq 3$)

$$u \leq 2^{h-1}x_n^2 - 1 < x_{n+1},$$

we deduce that

$$\begin{aligned} u &\in (h-1)\left(\{0\} \cup \bigcup_{i=0}^{n} (x_i + \mathcal{A}_i)\right) \\ &\subset (h-1)\big([0, x_n + x_n^2] \cup \big\{2^j x_n^2 + x_n \,:\, j = 1, 2, \ldots, h-2\big\}\,\big). \end{aligned}$$

In other words, we can express u as a sum of the form

$$\begin{aligned} u &= \alpha_{h-2}\left(2^{h-2}x_n^2 + x_n\right) + \cdots + \alpha_1\left(2x_n^2 + x_n\right) + \rho\left(x_n + x_n^2\right) \\ &= \left(2^{h-2}\alpha_{h-2} + \cdots + 2\alpha_1 + \rho\right)x_n^2 + \left(\alpha_{h-2} + \cdots + \alpha_1 + \rho\right)x_n, \end{aligned}$$

with $\alpha_1, \ldots, \alpha_{h-2} \in \mathbb{N}$, ρ a positive real number and

$$\alpha_{h-2} + \cdots + \alpha_1 + \rho \leq h-1.$$

If we denote by $[\rho]$ the integral part of ρ, this implies that

$$\left(2^{h-2}\alpha_{h-2} + \cdots + 2\alpha_1 + [\rho]\right)x_n^2 \leq u \leq \left(2^{h-2}\alpha_{h-2} + \cdots + 2\alpha_1 + \rho\right)x_n^2 + (h-1)x_n$$

and in view of $u \in [(2^{h-1}-1)x_n^2 + (h-1)x_n + 1, 2^{h-1}x_n^2 - 1]$, we deduce that

$$2^{h-2}\alpha_{h-2} + \cdots + 2\alpha_1 + [\rho] \leq 2^{h-1} - 1$$

and

$$2^{h-2}\alpha_{h-2} + \cdots + 2\alpha_1 + \rho \geq 2^{h-1} - 1.$$

We therefore obtain $2^{h-2}\alpha_{h-2} + \cdots + 2\alpha_1 + [\rho] = 2^{h-1} - 1$. We conclude by the facts that $\alpha_{h-2} + \cdots + \alpha_1 + [\rho] \leq h-1$ and that the only decomposition of $2^{h-1}-1$ as a sum of at most $h-1$ powers of 2 is $2^{h-1} - 1 = 1 + 2 + 2^2 + \cdots + 2^{h-2}$ that $\alpha_1 = \cdots = \alpha_{h-2} = [\rho] = 1$. From this, we deduce that $\rho \leq h - 1 - \alpha_1 - \cdots - \alpha_{h-2} = 1$ and finally $\rho = 1$ which gives $u = (2^{h-1}-1)x_n^2 + (h-1)x_n$, a contradiction. Since $h\mathcal{A} \sim \mathbb{N}$, we deduce that $\mathcal{A}$ is an asymptotic basis of order h.

Concerning restricted addition, we see that for $l \geq h-2$, we have

$$\max(l \times \mathcal{A}_n) \leq (2^{h-1}-2)x_n^2 + (l-h+2)x_n^2 = (2^{h-1}+l-h)x_n^2.$$

Hence

$$x_{n+1} - \max\left(l \times (x_n + \mathcal{A}_n)\,\right) \geq (2^{h-2} - l + h - 1)x_n^2 + (h-l)x_n.$$

If $l \leq 2^{h-2} + h - 2$, then $x_{n+1} - \max\left(l \times (x_n + \mathcal{A}_n)\right) \geq x_n^2 - (2^{h-2}-2)x_n$ which tends to infinity as n tends to infinity. It follows that $k(h) \geq 2^{h-2} + h - 1$, as asserted in Theorem 3.

We now complete the proof of Theorem 4. It is clear from the preceding computations that if the basis $\mathcal{A}$ defined above has a (finite) restricted order $\mathrm{ord}_r(\mathcal{A})$ then it must satisfy $\mathrm{ord}_r(\mathcal{A}) \geq 2^{h-2} + h - 1$. Our goal is to prove that $\mathrm{ord}_r(\mathcal{A})$ exists. We will show more precisely that $\mathrm{ord}_r(\mathcal{A}) = 2^{h-2} + h - 1$. For this purpose, it is enough to prove that any sufficiently large integer is a sum of at most $2^{h-2} + h - 1$ distinct elements of $\mathcal{A}$.

It is readily seen that if n is large enough, any integer in $[x_n, 2^{h-2}x_n^2 + x_n)$ is a sum of at most 2^{h-2} integers of $[x_n, x_n^2 + x_n) \subset x_n + \mathcal{A}_n$. Moreover for any integer m in $[0, 2^{h-1} - 1]$, there exists some integer $t(m)$ verifying $0 \le t(m) \le h - 1$ such that

$$z_m = mx_n^2 + t(m)x_n$$

can be written as a sum of at most $h-1$ distinct elements of $\{x_n + 2^j x_n^2 \,:\, j = 0, 1, 2, \dots, h-2\} \subset x_n + \mathcal{A}_n$. In particular, we observe that $t(0) = 0$ and $t(2^{h-1} - 1) = h - 1$. If we assume that n is large enough, then for any arbitrary integer m the difference $z_{m+1} - z_m$ which satisfies $0 \le z_{m+1} - z_m \le x_n^2 + (h-1)x_n$ is less than the length of the interval $[x_n, 2^{h-2}x_n^2 + x_n)$ by our assumption $h \ge 3$. Thus we infer that any integer in the sumset

$$[x_n, 2^{h-2}x_n^2 + x_n) + \{z_m \,:\, 0 \le m \le 2^{h-1} - 1\} = [x_n, 2^{h-2}x_n^2 + z_{2^{h-1}-1} + x_n)$$

is a sum of at most $2^{h-2} + h - 1$ distinct elements of $x_n + \mathcal{A}_n$. Since $z_{2^{h-1}-1} = (2^{h-1} - 1)x_n^2 + (h-1)x_n$, we deduce that any integer in $[x_n, x_{n+1})$ is a sum of at most $2^{h-2} + h - 1$ distinct elements of $x_n + \mathcal{A}_n$. This being true for any large enough integer n, it follows that the basis $\mathcal{A}$, which is of order h, has a restricted order equal to $2^{h-2} + h - 1$.

This ends the proof of Theorem 4. □

Proof of Proposition 5. We denote by $a_1 < a_2 < \cdots$ the (increasing sequence of) elements of $\mathcal{A}$ and by $b_1 < b_2 < \cdots$ the elements of $h \times \mathcal{A}$. We assume that $\Delta(h \times \mathcal{A}) = \limsup_{i \to +\infty}(b_{i+1} - b_i)$ is finite.

We define i_0 to be the smallest integer such that $b_{i_0} > a_1 + a_2 + \cdots + a_h$. Hence, for any $i \ge i_0$, there exists an element of $\mathcal{A}$, $\alpha(i) \in \{a_1, a_2, \dots, a_h\}$ such that $b_i \in h \times (\mathcal{A} \smallsetminus \{\alpha(i)\})$; in particular this gives $c_i = \alpha(i) + b_i \in (h+1) \times \mathcal{A}$ for $i \ge i_0$.

If $i \ge i_0$ is large enough, then $(b_{i+1} - b_i) \le \Delta(h \times \mathcal{A})$. Let j be the smallest integer greater than i such that $c_j > c_i$. We have

$$0 < c_j - c_i \le c_j - c_{j-1} = (b_j - b_{j-1}) + (\alpha(j) - \alpha(j-1)) \le \Delta(h \times \mathcal{A}) + (a_h - a_1).$$

This shows that for any large enough $c_i \in (h+1) \times \mathcal{A}$, there exists $c_j \in (h+1) \times \mathcal{A}$ such that $1 \le c_j - c_i \le \Delta(h \times \mathcal{A}) + (a_h - a_1)$. From this, it clearly follows that

$$\Delta\big((h+1) \times \mathcal{A}\big) \le \Delta(h \times \mathcal{A}) + (a_h - a_1),$$

thus in particular $\Delta\big((h+1) \times \mathcal{A}\big)$ is finite.

Proposition 5 follows by an easy induction. □

Proof of Proposition 7. Let $X = \{x_1 < x_2 < \cdots < x_i < \cdots\}$ be a set of positive integers. We denote $D(X) = \max_{i \ge 1}(x_{i+1} - x_i)$ and recall that $\Delta(X) = \limsup_{i \to +\infty}(x_{i+1} - x_i)$.

Let $d > 0$. We shall say that X d-covers an interval of integers I if the union of the balls centered on the elements of X with radius $d/2$ contains I. In other words:

$$\text{for all } r \in I, \text{ there exists } x \in X \text{ such that } |x - r| \le d/2.$$

Let $\mathcal{A} = \{a_1 < a_2 < \cdots < a_i < \cdots\}$. Assume $d = \Delta(2 \times \mathcal{A}) < +\infty$. There exists an x_0 such that $[x_0, +\infty)$ is d-covered by $2 \times \mathcal{A}$. We shall see that for any $a_i \in \mathcal{A}$ large enough, the interval $[a_i + x_0, a_{i+1} + x_0)$ is d-covered by $3 \times \mathcal{A}$. This will imply $\Delta(3 \times \mathcal{A}) \leq d = \Delta(2 \times \mathcal{A})$.

First case : if $a_{i+1} \leq 2a_i - x_0 - d/2$, then $a_i + \big((2 \times \mathcal{A}) \cap [0, a_i)\big)$ is contained in $3 \times \mathcal{A}$ and d-covers $[a_i + x_0, 2a_i - d/2)$ which contains $[a_i + x_0, a_{i+1} + x_0)$ by assumption.

Second case : if $a_{i+1} > 2a_i - x_0 - d/2$, then

$$(2 \times \mathcal{A}) \cap \Big[\frac{3a_i}{2}, a_{i+1}\Big) \subset 2 \times \Big(\mathcal{A} \cap \Big[\frac{a_i}{2}, a_i\Big]\Big) .$$

Indeed, if a and b are two distinct elements of $\mathcal{A}$ such that $3a_i/2 \leq a + b < a_{i+1}$, then $a \leq a_i$ and $b \leq a_i$; consequently we must have $a \geq a_i/2$ and $b \geq a_i/2$.

Let $a \in \mathcal{A}$ such that $d/2 + x_0 < a < a_i/2 - d$ (we may always find such an a if a_i is large enough). Then

$$a + \left((2 \times \mathcal{A}) \cap \Big[\frac{3a_i}{2}, a_{i+1}\Big)\right) \subset 3 \times \mathcal{A}.$$

Since $[3a_i/2, a_{i+1})$ is d-covered by $2 \times \mathcal{A}$, the interval $[3a_i/2 + d/2 + a, a + a_{i+1} - d/2)$ is d-covered by $3 \times \mathcal{A}$. Since, in view of the choice made for a, $3a_i/2 + d/2 + a \leq 2a_i - d/2$ and $a + a_{i+1} - d/2 \geq a_{i+1} + x_0$, we infer that $[2a_i - d/2, a_{i+1} + x_0)$ is d-covered by $3 \times \mathcal{A}$. Moreover, the interval of integers $[a_i + x_0, 2a_i - d/2)$ is d-covered by $a_i + \big((2 \times \mathcal{A}) \cap [0, a_i)\big)$. Therefore we conclude that $[a_i + x_0, a_{i+1} + x_0)$ is d-covered by $3 \times \mathcal{A}$. □

Proof of Theorem 9. Let $\mathcal{A}$ be such that $d = \Delta(h \times \mathcal{A}) < +\infty$. This implies that for any sufficiently large x,

$$A(x) = |\mathcal{A} \cap [1, x]| \geq C x^{1/h},$$

for some positive constant C depending only on d. Now, the number of subsets of $\mathcal{A} \cap [1, x]$ with cardinality $h + 1$ is equal to the binomial coefficient $\binom{A(x)}{h+1} \gg x^{1+1/h}$ where the implied constant depends on both $\mathcal{A}$ and h. Choose an x such that $\binom{A(x)}{h+1} \geq (h + 2)!\, h^{h+2} x$. It thus exists an integer n less than $(h + 1)x$ such that

$$n = a_1^{(i)} + \cdots + a_{h+1}^{(i)}, \qquad \text{for } i = 1, \ldots, (h + 1)!\, h^{h+2},$$

where the $(h+1)!\, h^{h+2}$ sets $E_i = \{a_1^{(i)}, \ldots, a_{h+1}^{(i)}\}$ of $h+1$ pairwise distinct elements of $\mathcal{A}$ are distinct. We now make use of the following intersection theorem for systems of sets due to Erdős and Rado (cf. Theorem III of [3]):

Lemma (Erdős-Rado)**.** *Let m, q, r be positive integers and E_i, $1 \leq i \leq m$, be sets of cardinality at most r. If $m \geq r!\, q^{r+1}$, then there exist an increasing sequence $i_1 < i_2 < \cdots < i_{q+1}$ and a set F such that $E_{i_j} \cap E_{i_k} = F$ as soon as $1 \leq j < k \leq q + 1$.*

By applying this result with $q = h$ and $r = h + 1$, we obtain that there are $h + 1$ sets E_{i_j}, $j = 1, \ldots, h + 1$, and a set F, with $0 \leq |F| \leq h - 1$, such that $E_{i_j} \cap E_{i_k} = F$ if $1 \leq j \neq k \leq h + 1$. Observe that we must have $0 \leq |F| \leq h - 1$ since the E_i's are distinct and the sum of all elements of E_i is equal to n for any i. We obtain that the integer

$$n' = n - \sum_{a \in F} a$$

can be written as a sum of $h+1-|F|$ pairwise distinct elements of $\mathcal{A}$ in at least $h+1$ ways, such that all summands occurring in any of these representations of n' in $(h+1-|F|)\times\mathcal{A}$ are pairwise distinct (equivalently, this means that the set $\cup_{j=1}^{h+1} E_{i_j} \smallsetminus F$ has exactly $(h+1)(h+1-|F|)$ distinct elements). This shows that

$$n' + (h\times\mathcal{A}) \subset (2h+1-|F|)\times\mathcal{A},$$

and finally $\Delta(h\times\mathcal{A}) = \Delta\big(n' + (h\times\mathcal{A})\big) \geq \Delta(h_1\times\mathcal{A})$, where $h_1 = 2h+1-|F|$.

Iterating this process, we get an increasing sequence $(h_j)_{j\geq 0}$, with $h_0 = h$, such that

$$\Delta(h_j\times\mathcal{A}) = \Delta\big(n' + (h_j\times\mathcal{A})\big) \geq \Delta(h_{j+1}\times\mathcal{A}),$$

where h_{j+1} is of the form $h_j + h + 1 - |F_j|$ for some set F_j satisfying $0 \leq |F_j| \leq h-1$. We conclude that $h_j + 2 \leq h_{j+1} \leq h_j + h + 1$, as stated. □

Proof of Theorem 10. Let h be a positive integer and $\mathcal{A}$ be a sequence of integers. We put $\mathcal{B} = h\mathcal{A}$ and assume that $\underline{\mathrm{d}}\mathcal{B} \geq \beta > 0$. Define

$$j = \left\lceil \left(1+\frac{1}{h}\right)\frac{1}{\beta} \right\rceil.$$

We thus have

$$j\underline{\mathrm{d}}\mathcal{B} \geq 1 + \frac{1}{h} > 1 \geq \underline{\mathrm{d}}j\mathcal{B}.$$

By Kneser's theorem on addition of sequences of integers (cf. [9, 10], [4] or [12]), we obtain that there exist an integer $g \geq 1$ and a sequence $\mathcal{B}_1$ of integers such that

$$\mathcal{B} \subset \mathcal{B}_1, \quad g + \mathcal{B}_1 \subset \mathcal{B}_1, \quad j\mathcal{B}_1 \smallsetminus j\mathcal{B} \text{ is finite},$$

and

$$\underline{\mathrm{d}}j\mathcal{B}_1 \geq j\underline{\mathrm{d}}\mathcal{B}_1 - \frac{j-1}{g}.$$

We may assume that g is the smallest integer satisfying these conditions.

Since $\underline{\mathrm{d}}\mathcal{B}_1 \geq \underline{\mathrm{d}}\mathcal{B} = \beta$, we deduce from the previous inequality that

$$g \leq \frac{j-1}{j\beta - 1}.$$

Hence $g \leq (j-1)h \leq jh$.

We denote by $\overline{\mathcal{A}} \subset \mathbb{Z}/g\mathbb{Z}$ the image of $\mathcal{A}$ by the canonical homomorphism of $\mathbb{Z}$ onto $\mathbb{Z}/g\mathbb{Z}$, the group of residue classes modulo g. Let H be the period of $g\overline{\mathcal{A}}$, that is the subgroup of $\mathbb{Z}/g\mathbb{Z}$ formed by the elements c such that $c + g\overline{\mathcal{A}} = g\overline{\mathcal{A}}$. Since $g \leq jh$, the sumset $jh\overline{\mathcal{A}} = j\overline{\mathcal{B}} = j\overline{\mathcal{B}_1}$ satisfies

$$j\overline{\mathcal{B}_1} + H = j\overline{\mathcal{B}_1}.$$

It therefore follows from the minimality of g that $H = \{0\}$. Thus, from a repeated application of Kneser's theorem on addition of sets in an abelian group (see [9, 10], [8] or [11]), we deduce

$$g \geq |g\overline{\mathcal{A}}| \geq g(|\overline{\mathcal{A}}| - 1) + 1,$$

which implies $|\overline{\mathcal{A}}| = 1$. Therefore there exists an integer a_0 such that any element of $\mathcal{A}$ can be written in the form $a_0 + gx$ for some integer x. We define $\mathcal{A}_1 = \{(a - a_0)/g \,:\, a \in \mathcal{A}\} \subset \mathbb{N}$.

Since $jh\mathcal{A} = j\mathcal{B} \sim j\mathcal{B}_1$, we get $jh\mathcal{A}_1 \sim \mathbb{N}$. Assuming the validity of Conjecture 2, we obtain that $\Delta\big(k(jh) \times \mathcal{A}_1\big)$ is finite, and accordingly $\Delta\big(k(jh) \times \mathcal{A}\big) < +\infty$. $\square$

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