R-equivalence on low degree complete intersections

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Abstract

Let k be a function field in one variable over $\mathbb{C}$ or the field $\mathbb{C}((t))$. Let X be a k-rationally simply connected variety defined over k. In this paper we show that R-equivalence on rational points of X is trivial and that the Chow group of zero-cycles of degree zero $A_0(X)$ is zero. In particular, this holds for a smooth complete intersection of r hypersurfaces in $\mathbb{P}^n_k$ of respective degrees $d_1, \ldots, d_r$ with $\sum\limits_{i=1}^{r} d_i^2 \leq n+1$.

1 Introduction

Let X be a projective variety over a field k. Two rational points x_1, x_2 of X are called *directly R-equivalent* if there is a morphism $f : \mathbb{P}^1_k \to X$ such that x_1 and x_2 belong to the image of $\mathbb{P}^1_k(k)$. This generates an equivalence relation called *R-equivalence* [Man]. The set of R-equivalence classes is denoted by $X(k)/R$. If $X(k) = \emptyset$ we set $X(k)/R = 0$.

>From the above definition, the study of R-equivalence on $X(k)$ is closely related to the study of rational curves on X, so that we need many rational curves on X. The following class of varieties sharing this property was introduced in the 1990s by Kollár, Miyaoka and Mori [KMM1], and independently by Campana.

Definition 1.1. Let k be a field of characteristic zero. A projective geometrically integral variety X over k is called *rationally connected* if for any algebraically closed field Ω containing k, two general Ω-points x_1, x_2 of X can be connected by a rational curve: there is a morphism $\mathbb{P}^1_\Omega \to X_\Omega$ such that its image contains x_1 and x_2.

Remark 1.2. One can define rational connectedness by other properties ([Ko96], Section IV.3). For instance, if k is uncountable, one may ask that the condition above is satisfied for *any* two points $x_1, x_2 \in X(\Omega)$.

By a result of Campana [Ca] and Kollár-Miyaoka-Mori [KMM2], smooth Fano varieties are rationally connected. In particular, a smooth complete intersection of

r hypersurfaces in $\mathbb{P}^n_k$ of respective degrees $d_1, \dots, d_r$ with $\sum_{i=1}^{r} d_i \leq n$ is rationally connected. Another important result about rationally connected varieties has been established by Graber, Harris and Starr [GHS]. Let k be a function field in one variable over $\mathbb{C}$, that is, k is the function field of a complex curve. Graber, Harris and Starr prove that any smooth rationally connected variety over k has a rational point.

One can see rationally connected varieties as an analogue of path connected spaces in topology. From this point of view, de Jong and Starr introduce the notion of rationally simply connected varieties as an algebro-geometric analogue of simply connected spaces. For X a projective variety with a fixed ample divisor H we denote by $\overline{M}_{0,2}(X, d)$ the Kontsevich moduli space for all genus zero stable curves over X of degree d with two marked points (see section 2 for more details). In this paper we use the following definition:

Definition 1.3. Let k be a field of characteristic zero. Let X be a projective geometrically integral variety over k. Suppose that $H_2(X, \mathbb{Z})$ has rank one. We say that X is *k-rationally simply connected* if for any sufficiently large integer e there exists a geometrically irreducible component $M_{e,2} \subset \overline{M}_{0,2}(X, e)$ intersecting the open locus of irreducible curves $M_{0,2}(X, e)$ and such that the restriction of the evaluation morphism

$$ev_2 : M_{e,2} \to X \times X$$

is dominant with rationally connected general fiber.

Remark 1.4. Following the work of de Jong and Starr, we restrict ourselves to the case where $H_2(X, \mathbb{Z})$ has rank one.

Note that a k-rationally simply connected variety X over a field k is rationally connected as $X \times X$ is dominated by $M_{e,2} \cap M_{0,2}(X, e)$ from the definition above. This implies that over any algebraically closed field $\Omega \supset k$ two general points of $X(\Omega)$ can be connected by a rational curve.

By a recent result of de Jong and Starr [dJS], a smooth complete intersection X of r hypersurfaces in $\mathbb{P}^n_k$ of respective degrees $d_1, \dots, d_r$ and of dimension at least 3 is k-rationally simply connected if $\sum_{i=1}^{r} d_i^2 \leq n+1$.

Let us now recall the definition of $A_0(X)$. Denote $Z_0(X)$ the free abelian group generated by the closed points of X. *The Chow group of degree* 0 is the quotient of the group $Z_0(X)$ by the subgroup generated by the $\pi_*(\mathrm{div}_C(g))$, where $\pi : C \to X$ is a proper morphism from a normal integral curve C, g is a rational function on C and $\mathrm{div}_C(g)$ is its divisor. It is denoted by $CH_0(X)$. If X is projective, the degree map $Z_0(X) \to \mathbb{Z}$ which sends a closed point $x \in X$ to its degree $[k(x) : k]$ induces a map $\deg : CH_0(X) \to \mathbb{Z}$ and we denote $A_0(X)$ its kernel.

At least in characteristic zero, the set $X(k)/R$ and the group $A_0(X)$ are k-birational invariants of smooth projective k-varieties, and they are reduced to one element if X is a projective space. Thus $X(k)/R = 1$ and $A_0(X) = 0$ if X is a smooth projective k-rational variety.

For k a function field in one variable over $\mathbb{C}$ and for $k = \mathbb{C}((t))$, one wonders whether a similar statement holds for arbitrary smooth rationally connected k-varieties ([CT], 10.11 and 11.3). This has been established for some special classes of varieties ([CT83], [CTSa], [CTSk], [CTSaSD], [Ma]). Most of these results hold if k is a C_1-field or, more generally, if $cd(k) \leq 1$.

In this paper, we prove the following result:

Theorem 1.5. *Let k be either a function field in one variable over $\mathbb{C}$ or the field $\mathbb{C}((t))$. Let X be a k-rationally simply connected variety over k. Then*

(i) $X(k)/R = 1$;

(ii) $A_0(X) = 0$.

Combined with the theorem of de Jong and Starr, this gives:

Corollary 1.6. *Let k be either a function field in one variable over $\mathbb{C}$ or the field $\mathbb{C}((t))$. Let X be a smooth complete intersection of r hypersurfaces in $\mathbb{P}^n_k$ of respective degrees $d_1, \ldots, d_r$. Assume that $\sum_{i=1}^{r} d_i^2 \leq n+1$. Then*

(i) $X(k)/R = 1$;

(ii) $A_0(X) = 0$.

The methods we use in the proof of the theorem apply more generally over a field k of characteristic zero such that any rationally connected variety over k has a rational point. As for the corollary, one can prove it in a simpler way for any C_1 field k in the case $\sum d_i^2 \leq n$ (see section 4).

Note that one knows better results for smooth cubics and smooth intersections of two quadrics. Let k be either a function field in one variable over $\mathbb{C}$ or the field $\mathbb{C}((t))$.

In the case of smooth cubic hypersurfaces in $\mathbb{P}^n_k$ we have $X(k)/R = 1$ if $n \geq 5$ ([Ma], 1.4). It follows that $A_0(X) = 0$ if $n \geq 5$. In fact, we have $A_0(X) = 0$ if $n \geq 3$. One can prove it by reduction to the case $n = 3$. The latter case follows from the result on geometrically rational k-surfaces ([CT83], Thm.A), obtained by K-theoretic methods.

In the case of smooth intersections of two quadrics in $\mathbb{P}^n_k$ we have $X(k)/R = 1$ if $n \geq 5$ ([CTSaSD], 3.27). Hence $A_0(X) = 0$ if $n \geq 5$. In fact, we also have $A_0(X) = 0$ if $n = 4$ as a particular case of [CT83], Thm.A.

It was also known that under the assumption of the theorem there is a bound $N = N(d_1, \ldots, d_r)$ such that $A_0(X) = 0$ if $n \geq N$ ([Pa], 5.4). The bound N here is defined recursively: $N(d_1, \ldots, d_r) = f(d_1, \ldots, d_r; N(d_1 - 1, \ldots, d_r - 1))$ where the function f grows rapidly with the degrees. For example, one can deduce that $N(3) = 12$ and $N(4) = 3 + 3^{12}$.

Theorem 1.5 is inspired by the work of de Jong and Starr [dJS] and we use their ideas in the proof. In section 2 we recall some notions about the moduli space of curves used in [dJS] and we analyse the case when we can deduce some information about R-equivalence on X from the existence of a rational point on the moduli space. Next, in section 3 we deduce Theorem 1.5. In section 4 we give an application to complete intersections.

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2 Rational points on a moduli space of curves

2.1 The moduli space $\overline{M}_{0,n}(X,d)$

Let X be a projective variety over a field k of characteristic zero with an ample divisor H. Let $\bar{k}$ be an algebraic closure of k. While studying R-equivalence on rational points of X we need to work with a space parametrizing rational curves on X. We fix the degree of the curves we consider in order to have a space of finite type.

The space of rational curves of fixed degree on X is not compact in general. One way to compactify it, due to Kontsevich ([KonMan], [FP]), is to use stable curves.

Definition 2.1. A *stable* curve over X of degree d with n marked points is a datum $(C, p_1, \dots, p_n, f)$ of

(i) a proper geometrically connected reduced k-curve C with only nodal singularities,

(ii) an ordered collection $p_1, \dots, p_n$ of distinct smooth k-rational points of C,

(iii) a k-morphism $f : C \to X$ with $\deg_C f^*H = d$,

such that the stability condition is satisfied:

(iv) C has only finitely many $\bar{k}$-automorphisms fixing the points $p_1, \dots, p_n$ and commuting with f.

We say that two stable curves $(C, p_1, \dots, p_n, f)$ and $(C', p'_1, \dots, p'_n, f')$ are *isomorphic* if there exists an isomorphism $\phi : C \to C'$ such that $\phi(p_i) = p'_i$, $i = 1, \dots, n$ and $f' \circ \phi = f$.

We use the construction of Araujo and Kollár [AK] to parametrize stable curves. They show that there exists a *coarse moduli space* $\overline{M}_{0,n}(X,d)$ for all genus zero stable curves over X of degree d with n marked points, which is a projective k-scheme ([AK], Thm. 50). Over $\mathbb{C}$ the construction was first given in [FP]. The result in [AK] holds over an arbitrary, not necessarily algebraically closed field and, more generally, over a noetherian base.

We denote by $M_{0,n}(X,d)$ the open locus corresponding to irreducible curves and by

$$ev_n : \overline{M}_{0,n}(X,d) \to \underbrace{X \times \ldots \times X}_{n}$$

the evaluation morphism which sends a stable curve to the image of its marked points.

When one says that $\overline{M}_{0,n}(X,d)$ is a coarse moduli space, it means that the following two conditions are satisfied:

(i) there is a bijection of sets:

$$\Phi : \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{genus zero stable curves over } \bar{k} \\ f : C \to X_{\bar{k}} \text{ with } n \text{ marked points,} \\ \deg_C f^*H = d \end{array} \right\} \xrightarrow{\sim} \overline{M}_{0,n}(X,d)(\bar{k});$$

(ii) if $\mathcal{C} \to S$ is a family of genus zero stable curves of degree d with n marked points, parametrized by a k-scheme S, then there exists a unique morphism $M_S : S \to \overline{M}_{0,n}(X,d)$ such that for every $s \in S(\bar{k})$ we have

$$M_S(s) = \Phi(\mathcal{C}_s).$$

In general, over nonclosed fields we do not have a bijection between isomorphism classes of stable curves and rational points of the corresponding moduli space, see [AK] p.31. In particular, a k-point of $\overline{M}_{0,n}(X,d)$ does not in general correspond to a stable curve defined over k.

2.2 Rational points on $\overline{M}_{0,2}(X,d)$

Let P and Q be two k-points of X. Suppose there exists a stable curve $f : C \to X_{\bar{k}}$ over $\bar{k}$ with two marked points mapping to P and Q, such that the corresponding point $\Phi(f)$ is a k-point of $\overline{M}_{0,2}(X,d)$. Even if we are not able to prove that f can be defined over k, using combinatorial arguments we will show that P and Q are R-equivalent over k. Let us state the main result of this section.

Proposition 2.2. *Let X be a projective variety over a field k of characteristic zero. Let P and Q be k-points of X. Let $f : C \to X_{\bar{k}}$ be a stable curve over $\bar{k}$ of genus zero with two marked points mapping to P and Q. Let H be a fixed ample divisor on X and let $d = \deg_C f^*H$. If the corresponding point $\Phi(f) \in \overline{M}_{0,2}(X,d)$ is a k-point of $\overline{M}_{0,2}(X,d)$, then the points P and Q are R-equivalent over k.*

Let us first fix some notation. Let k be a field of characteristic zero. Let us fix an algebraic closure $\bar{k}$ of k. Let $L \overset{i}{\hookrightarrow} \bar{k}$ be a finite Galois extension of k, and let $G = \mathrm{Aut}_k(L)$. For any $\sigma \in G$ we denote $\sigma^* : \mathrm{Spec}\, L \to \mathrm{Spec}\, L$ the induced morphism. If Y is an L-variety, denote ${}^{\sigma}Y$ the base change of Y by σ^* and ${}^{\sigma}Y_{\bar{k}}$ the base change by $(i \circ \sigma)^*$. We denote the projection ${}^{\sigma}Y \to Y$ by σ^* too. If $f : Z \to Y$ is an L-morphism of L-varieties, then we denote ${}^{\sigma}f : {}^{\sigma}Z \to {}^{\sigma}Y$ and ${}^{\sigma}f_{\bar{k}} : {}^{\sigma}Z_{\bar{k}} \to {}^{\sigma}Y_{\bar{k}}$ the induced morphisms.

Note that if $Y \subset \mathbb{P}^n_L$ is a projective variety, then ${}^{\sigma}Y$ can be obtained by applying σ to each coefficient in the equations defining Y. Thus, if Y is defined over k, then the subvarieties Y, ${}^{\sigma}Y$ of $\mathbb{P}^n_L$ are given by the same embedding for all $\sigma \in G$. In this case the collection of morphisms $\{\sigma^* : Y \to Y\}_{\sigma \in G}$ defines a right action of G on Y. By Galois descent ([BLR], 6.2), if a subvariety $Z \subset Y$ is stable under this action of G, then Z also is defined over k.

For lack of a suitable reference, let us next give a proof of the following lemma:

Lemma 2.3. *Let C be a projective geometrically connected curve of arithmetic genus $p_a(C) = h^1(C, O_C) = 0$ over a perfect field k. Assume C has only nodal singularities. Then any two smooth k-points a, b of C are R-equivalent.*

Proof. Since the arithmetic genus of C is zero, its geometric components are smooth rational curves over $\bar{k}$ intersecting transversally and, moreover, there exists a unique (minimal) chain of $\bar{k}$-components joining a and b. We may assume that all the components of the chain, as well as their intersection points, are defined over some finite Galois extension L of k. By unicity we see that every component of the chain is stable under the action of $\mathrm{Aut}_k(L)$ on C_L, hence it is defined over k. By the same argument, the intersection points of the components of the chain are k-points. We conclude that the points a and b are R-equivalent over k. □

Let us now give the proof of Proposition 2.2.

We call a and b the marked points of C. We may assume that C, f, a and b are defined over a finite Galois extension $L \overset{i}{\hookrightarrow} \bar{k}$ of k. Let us denote $T = \mathrm{Spec}\, L$ and $G = \mathrm{Aut}_k(L)$. Note that $T \times_k \bar{k} = \coprod_G \mathrm{Spec}\, \bar{k}$ where the morphism $\mathrm{Spec}\, \bar{k} \to T$ is given by $(i \circ \sigma)^*$ on the corresponding component. We view L as a k-scheme and $f : C \to X \times_k T$ as a family of stable curves parametrized by T. Thus we have a moduli map $M_T : T = \mathrm{Spec}\, L \to \overline{M}_{0,2}(X, d)$ defined over k and such that for every $t \in T(\bar{k})$, corresponding to $\sigma \in G$, we have

$$M_T(t) = \Phi({}^{\sigma}C_{\bar{k}})$$

where ${}^{\sigma}f_{\bar{k}} : {}^{\sigma}C_{\bar{k}} \to X_{\bar{k}}$ and the marked points of ${}^{\sigma}C_{\bar{k}}$ are $\sigma(a)$ and $\sigma(b)$.

Since the curve $f_{\bar{k}} : C_{\bar{k}} \to X_{\bar{k}}$ corresponds to a k-point of $\overline{M}_{0,2}(X, d)$, we can factor M_T as

$$T = \mathrm{Spec}\, L \to \mathrm{Spec}\, k \overset{\Phi(f_{\bar{k}})}{\to} \overline{M}_{0,2}(X, d).$$

We thus see that for every $t \in T(\bar{k})$ the point $M_T(t)$ is the same point $\Phi(f_{\bar{k}})$ of $\overline{M}_{0,2}(X,d)$. Hence for every $\sigma \in G$ the curves ${}^\sigma C_{\bar{k}}$ and $C_{\bar{k}}$ are isomorphic as stable curves. This means that there exists a $\bar{k}$-morphism $\phi_\sigma : C_{\bar{k}} \to {}^\sigma C_{\bar{k}}$, such that $\phi_\sigma(a) = \sigma(a)$, $\phi_\sigma(b) = \sigma(b)$ and ${}^\sigma f_{\bar{k}} \circ \phi_\sigma = f_{\bar{k}}$.

As a consequence, the proposition results from the following lemma.

Lemma 2.4. *Let X be a projective variety over a perfect field k. Let L be a finite Galois extension of k. Denote $G = \mathrm{Aut}_k(L)$. Let P and Q be k-points of X. Suppose we can find an L-stable curve of genus zero $f : C \to X_L$ with two marked points $a, b \in C(L)$, satisfying the following conditions:*

(i) $f(a) = P$, $f(b) = Q$;

(ii) for every $\sigma \in G$ there exists a $\bar{k}$-morphism $\phi_\sigma : C_{\bar{k}} \to {}^\sigma C_{\bar{k}}$ such that

$$\phi_\sigma(a) = \sigma(a),\ \phi_\sigma(b) = \sigma(b) \text{ and } {}^\sigma f_{\bar{k}} \circ \phi_\sigma = f_{\bar{k}}.$$

Then the points P and Q are R-equivalent over k.

Proof. By lemma 2.3, we have a unique (minimal) chain $\{C_1, \dots, C_m\}$ of geometrically irreducible L-components of C, joining $a \in C_1(L)$ and $b \in C_m(L)$.

Let us take $\sigma \in G$. We have two chains of $\bar{k}$-components of ${}^\sigma C$ joining $\sigma(a)$ and $\sigma(b)$: $\{{}^\sigma C_{1,\bar{k}}, \dots, {}^\sigma C_{m,\bar{k}}\}$ and $\{\phi_\sigma(C_{1,\bar{k}}), \dots, \phi_\sigma(C_{m,\bar{k}})\}$. Since the arithmetic genus of ${}^\sigma C_{\bar{k}}$ is zero, we thus have

$$\phi_\sigma(C_{i,\bar{k}}) = {}^\sigma C_{i,\bar{k}},\ i = 1, \dots, m.$$

Let us fix $1 \le i \le m$. Denote the image $f(C_i)$ of C_i in X_L by Z_i. From the commutative diagram

$$\begin{array}{ccc} {}^\sigma C_i & \xrightarrow{{}^\sigma f} & {}^\sigma X \\ \downarrow & & \downarrow \\ C_i & \xrightarrow{f} & X \end{array}$$

we see that ${}^\sigma f({}^\sigma C_i) = {}^\sigma Z_i$ and that ${}^\sigma f_{\bar{k}}({}^\sigma C_{i,\bar{k}}) = {}^\sigma Z_{i,\bar{k}}$ (using base change by $i : L \to \bar{k}$ in the first line of the diagram above). On the other hand, since $\phi_\sigma(C_{i,\bar{k}}) = {}^\sigma C_{i,\bar{k}}$ and ${}^\sigma f_{\bar{k}} \circ \phi_\sigma = f_{\bar{k}}$, we have ${}^\sigma Z_{i,\bar{k}} = {}^\sigma f_{\bar{k}}({}^\sigma C_{i,\bar{k}}) = {}^\sigma f_{\bar{k}}(\phi_\sigma(C_{i,\bar{k}})) = f_{\bar{k}}(C_{i,\bar{k}}) = Z_{i,\bar{k}}$. Since ${}^\sigma Z_i$ and Z_i are L-subvarieties of X_L, we deduce that ${}^\sigma Z_i = Z_i$ for all $\sigma \in G$. By Galois descent, there exists a k-curve $D_i \subset X$ such that $Z_i = D_i \times_k L$.

Let $\tilde{D}_i \to D_i$ be the normalisation morphism. Since C_i is smooth, the morphism $f|_{C_i} : C_i \to D_i \times_k L$ extends to a morphism $f_i : C_i \to \tilde{D}_i \times_k L$:

$$\begin{array}{ccc} & & \tilde{D}_i \times_k L \\ & \overset{f_i}{\nearrow} & \downarrow \\ C_i & \xrightarrow{f|_{C_i}} & D_i \times_k L. \end{array}$$

We have $\phi_\sigma(C_i \cap C_{i+1}) = {}^\sigma C_i \cap {}^\sigma C_{i+1}$ and ${}^\sigma f_{i,\bar{k}} \circ \phi_\sigma = f_{i,\bar{k}}$, as this is true over a Zariski open subset of D_i. Using the same argument as above, we deduce that the point $f_i(C_i \cap C_{i+1})$ is a k-point of $\tilde{D}_i$. This implies that $\tilde{D}_i$ is a k-rational curve as it is L-rational and has a k-point. Moreover, the point $f(C_i \cap C_{i+1})$ is a k-point of X as the image of $f_i(C_i \cap C_{i+1})$. We deduce that P and Q are R-equivalent by the chain $\tilde{D}_i \to X$, $i = 1, \dots m$. □

Remark 2.5. If the cohomological dimension of the field k is at most 1, one can use more general arguments to prove Proposition 2.2: see [DDE], cor. 1.3, applied to the fibre of the morphism $\overline{\mathcal{M}}_{0,2}(X,d) \to \overline{M}_{0,2}(X,d)$ over the point $\Phi(f)$, where $\overline{\mathcal{M}}_{0,2}(X,d)$ is the stack of all genus zero stable curves over X of degree d with two marked points.

3 Proof of the theorem

In this section we use the previous arguments to prove Theorem 1.5. Let k be a function field in one variable over $\mathbb{C}$ or the field $\mathbb{C}((t))$. Let X be a k-rationally simply connected variety. In particular, X is rationally connected. Note that by the theorem of Graber, Harris and Starr [GHS], a smooth rationally connected variety over a function field in one variable over $\mathbb{C}$ has a rational point, and the same result is also known over $k = \mathbb{C}((t))$ (cf. [CT] 7.5). As any smooth projective variety equipped with a birational morphism to X is still rationally connected, it has a rational point. This implies that $X(k) \neq \emptyset$.

Let us fix a sufficiently large integer e and an irreducible component $M_{e,2} \subset \overline{M}_{0,2}(X,e)$ such that the restriction of the evaluation morphism $ev_2 : M_{e,2} \to X \times X$ is dominant with rationally connected general fibre.

Let P and Q be two k-points of X. A strategy is the following. We would like to apply [GHS] and to deduce that there is a rational point in a fibre over (P,Q). Then, by Proposition 2.2, we deduce that P and Q are R-equivalent. But we only know that a general fibre of ev_2 is rationally connected. If $k = \mathbb{C}((t))$, this is sufficient as R-equivalence classes are Zariski dense in this case by [Ko99]. If k is a function field in one variable over $\mathbb{C}$, our strategy will also work by a specialization argument of the lemma below (see also [Sta] p.25 and [Lie] 4.5). So we obtain $X(k)/R = 1$ as $X(k) \neq \emptyset$.

Let us now prove that the group $A_0(X)$ is trivial. Pick $x_0 \in X(k)$. It is sufficient to prove that for every closed point $x \in X$ of degree d we have that $x - dx_0$ is zero in $CH_0(X)$. Let us take a rational point $x' \in X_{k(x)}$ over a point x. By the first part of the theorem, applied to $X_{k(x)}$, x' is R-equivalent to x_0 over $k(x)$. Hence $x' - x_0$ is zero in $CH_0(X_{k(x)})$. Applying the push-forward by the morphism $p : X_{k(x)} \to X$, we deduce that $x - dx_0$ is zero in $CH_0(X)$. This completes the proof.

□

Lemma 3.1. *Let $k = \mathbb{C}(C)$ be the function field of a (smooth) complex curve C. Let Z and T be projective k-varieties, with T smooth. Let $f : Z \to T$ be a* *morphism with rationally connected general fibre. Then for every $t \in T(k)$ there exists a rational point in the fibre Z_t.*

Proof. One can choose proper models $\mathcal{T} \to C$ and $F : \mathcal{Z} \to \mathcal{T}$ of T and Z respectively with $\mathcal{T}$ smooth. We know that any fibre of F over some open set $U \subset \mathcal{T}$ is rationally connected.

The point $t \in T(k)$ corresponds to a section $s : C \to \mathcal{T}$. What we want is to find a section $C \to \mathcal{Z} \times_{\mathcal{T}} C$. One can view the image $s(C)$ in $\mathcal{T}$ as a component of a complete intersection C' of hyperplane sections of $\mathcal{T}$ for some projective embedding. In fact, it is sufficient to take $\dim \mathcal{T} - 1$ functions in the ideal of $s(C)$ in $\mathcal{T}$ generating this ideal over some open subset of $s(C)$. Moreover, one may assume that C' is a special fibre of a family $\mathcal{C}$ of hyperplane sections with general fibre a smooth curve intersecting U. After localization, we may also assume that $\mathcal{C}$ is parametrized by $\mathbb{C}[[t]]$. Let A be any affine open subset in $\mathcal{C}$ containing the generic point ξ of $s(C)$. We have the following diagram:

$$\begin{array}{ccccccc}
 & & & & \mathcal{Z} \times_{\mathcal{T}} \operatorname{Spec} A & \longrightarrow & \mathcal{Z} \\
 & & & & \downarrow{\scriptstyle F_A} & & \downarrow{\scriptstyle F} \\
\xi & \longrightarrow & \operatorname{Spec} A \otimes_{\mathbb{C}[[t]]} \mathbb{C} & \longrightarrow & \operatorname{Spec} A & \longrightarrow & \mathcal{T} \\
 & & \downarrow & & \downarrow & & \\
 & & Spec\, \mathbb{C} & \longrightarrow & \operatorname{Spec} \mathbb{C}[[t]]. & &
\end{array}$$

Let $K = \mathbb{C}((t))$ and let $\bar{K}$ be an algebraic closure of K. By construction, the generic fibre of $F_{\bar{K}} : \mathcal{Z} \times_{\mathcal{T}} \bar{K} \to \operatorname{Spec} A \otimes_{\mathbb{C}[[t]]} \bar{K}$ is rationally connected. By [GHS] we obtain a rational section of $F_{\bar{K}}$. As $\bar{K}$ is the union of the extensions $\mathbb{C}((t^{1/N}))$ for $N \in \mathbb{N}$, we have a rational section for the morphism $\mathcal{Z} \times_{\mathcal{T}} \mathbb{C}[[t^{1/N}]] \to \operatorname{Spec} A \otimes_{\mathbb{C}[[t]]} \mathbb{C}[[t^{1/N}]]$ for some N. By properness, this section extends to all codimension 1 points of $\operatorname{Spec} A \otimes_{\mathbb{C}[[t]]} \mathbb{C}[[t^{1/N}]]$, in particular, to the point ξ on the special fiber. This extends again to give a section $C \to \mathcal{Z} \times_{\mathcal{T}} C$ as desired. □

4 Proof of the corollary

The following result is essentially contained in [dJS]. We include the proof here as we need the precise statement over a field which is not algebraically closed.

Proposition 4.1. *Let k be a field of characteristic zero. Let X be a smooth complete intersection of r hypersurfaces in $\mathbb{P}^n_k$ of respective degrees $d_1, \ldots, d_r$ with $\sum_{i=1}^{r} d_i^2 \leq n+1$. Suppose that $\dim X \geq 3$. Then for every $e \geq 2$ there exists a geometrically irreducible k-component $M_{e,2} \subset \overline{M}_{0,2}(X, e)$ such that the restriction of the evaluation morphism*

$$ev_2 : M_{e,2} \to X \times X$$

is dominant with rationally connected generic fibre.

Proof. Let us first recall the construction of [dJS] in the case $k = \mathbb{C}$. Note that, as $\dim X \geq 3$, we know that $H_2(X, \mathbb{Z}) = \mathbb{Z}\alpha$ where the degree of α equals to 1 ([V], 13.25).

In [dJS], de Jong and Starr prove that for every integer $e \geq 2$ there exists an irreducible component $M_{e,2} \subset \overline{M}_{0,2}(X, e)$ such that the restriction of the evaluation morphism $ev_2 : M_{e,2} \to X \times X$ is dominant with rationally connected generic fibre. In order to convince ourselves that $M_{e,2}$ is in fact the unique component satisfying the above property, we will specify the construction of $M_{e,2}$ more precisely:

1. One first shows that there exists a *unique* irreducible component $M_{1,1} \subset \overline{M}_{0,1}(X, 1)$ such that the restriction of the evaluation $ev_1|_{M_{1,1}} : M_{1,1} \to X$ is dominant ([dJS], 1.7).

2. The component $M_{1,0} \subset \overline{M}_{0,0}(X, 1)$ is constructed as the image of $M_{1,1}$ under the morphism $\overline{M}_{0,1}(X, 1) \to \overline{M}_{0,0}(X, 1)$ forgetting the marked point. Then one constructs the component of higher degree $M_{e,0}$ as the *unique* component of $\overline{M}_{0,0}(X, e)$ which intersects the subvariety of $\overline{M}_{0,0}(X, e)$ parametrizing a degree e cover of the smooth, free curve parametrized by $M_{1,0}$ ([dJS], 3.3).

3. The component $M_{e,2} \subset \overline{M}_{0,2}(X, e)$ is the *unique* component such that its image under the morphism $\overline{M}_{0,2}(X, e) \to \overline{M}_{0,0}(X, e)$, which forgets about the marked points, is $M_{e,0}$.

Let us now consider the general case. Let $\bar{k}$ be an algebraic closure of k. As k is of finite type over $\mathbb{Q}$, we may assume that $\bar{k} \subset \mathbb{C}$. Since the decomposition into geometrically irreducible components does not depend on which algebraically closed field we choose, by the first step above there exists a unique irreducible component $M_{1,1} \subset \overline{M}_{0,1}(X_{\bar{k}}, 1)$ such that the restriction of the evaluation $ev_1|_{M_{1,1}}$ is dominant. As this component is unique, it is defined over k. Hence, from the construction above, the component $M_{e,2}$ is also defined over k, which completes the proof. □

Let us now prove the corollary. Let X be a smooth complete intersection of r hypersurfaces in $\mathbb{P}^n_k$ of respective degrees $d_1, \ldots, d_r$ with $\sum_{i=1}^{r} d_i^2 \leq n + 1$. Thus, if $\dim X = 1$ then X is a line and the corollary is obvious. If $\dim X = 2$ then X is a quadric surface in $\mathbb{P}^3_k$. We have that X is birational to $\mathbb{P}^2_k$ as it has a k-point and the corollary follows. If $\dim X \geq 3$, we have that X is k-rationally simply connected by Proposition 4.1. If k is a function field in one variable over $\mathbb{C}$ or the field $\mathbb{C}((t))$ we have $X(k)/R = 1$ and $A_0(X) = 0$ by Theorem 1.5. This completes the proof of the corollary.

Remark 4.2. The next argument, due to Jason Starr, gives a simpler way to prove the corollary in the case $\sum d_i^2 \leq n$. More precisely, we have :

Proposition 4.3. *Let k be a C_1 field. Let $X \overset{i}{\hookrightarrow} \mathbb{P}^n_k$ be the vanishing set of r polynomials $f_1, \dots f_r$ of respective degrees $d_1, \dots d_r$. If $\sum d_i^2 \leq n$ then any two points $x_1, x_2 \in X(k)$ can be joined by two lines defined over k: there is a point $x \in X(k)$ such that $l(x, x_i) \subset X$, $i = 1, 2$, where $l(x, x_i)$ denote the line through x and x_i.*

Proof. We may assume that $x_1 = (1 : 0 : \dots : 0)$ and $x_2 = (0 : 1 : 0 : \dots : 0)$ via the embedding i. The question is thus to find a point $x = (x_0 : \dots : x_n)$ with coordinates in k such that

$$\begin{cases} f_i(tx_0 + s, tx_1, \dots tx_n) = 0 \\ f_i(tx_0, tx_1 + s, \dots tx_n) = 0, \end{cases} \quad i = 1, \dots r.$$

As x_1, x_2 are in $X(k)$ these conditions are satisfied for $t = 0$. Thus we may assume $t = 1$. Writing $f_i(x_0 + s, x_1, \dots x_n) = \sum_{j=0}^{d_i} P^i_j(x_0, \dots x_n) s^j$ with $\deg P^i_j = d_i - j$ we see that each equation $f_i(x_0 + s, x_1, \dots x_n) = 0$ gives us d_i conditions on $x_0, \dots x_n$ of degrees $1, \dots d_i$. By the same argument, each equation $f_i(x_0, x_1 + s, \dots x_n) = 0$ gives $d_i - 1$ conditions of degrees $1, \dots d_i - 1$ as we know from the previous equation that we have no term of degree zero. The sum of the degrees of all these conditions on $x_0, \dots x_n$ is $\sum_{i=0}^{r} d_i^2$. As $\sum_{i=0}^{r} d_i^2 \leq n$ by Tsen-Lang theorem we can find a solution over k, which completes the proof. □

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