Algebraic cycles on varieties over finite fields

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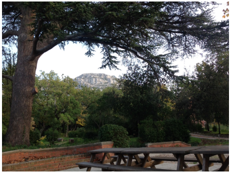

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Question: what objects one can associate to X?

Subvarieties of smaller dimension

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Look at all $Y \subset X$ irreducibles of dimension $d - i$.
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The *group of cycles* of codimension i is

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In general difficult to determine!

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1. $H^i_{\acute{e}t}(X, \mu_n^{\otimes j})$ are finite, $H^i_{\acute{e}t}(X, \mathbb{Z}_\ell(j))$ are $\mathbb{Z}_\ell$-modules of finite type (resp. with $\bar{X}$); $H^i_{\acute{e}t}(\bar{X}, \mathbb{Z}_\ell)$ have no torsion for almost all ℓ (Gabber, difficult);

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2. Hochschild-Serre spectral sequence relates X and $\bar{X}$:
   $0 \to H^1(G, H^{i-1}_{\acute{e}t}(\bar{X}, \mathbb{Z}_\ell(j)) \to H^i_{\acute{e}t}(X, \mathbb{Z}_\ell(j)) \to H^i_{\acute{e}t}(\bar{X}, \mathbb{Z}_\ell(j))^G \to 0$

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3. there is a *cycle class map* $CH^i(X) \otimes \mathbb{Z}_\ell \to H^{2i}_{ét}(X, \mathbb{Z}_\ell(i))$.

Computing cohomology

- $H^{2d}_{ét}(\bar{X}, \mu_n^{\otimes d}) \overset{\simeq}{\to} \mathbb{Z}/n$; $H^i_{ét}(\bar{X}, \mu_n^{\otimes j}) = 0, i > 2n$; $H^i_{ét}(\bar{X}, \mu_n^{\otimes j})$ and $H^{2d-i}_{ét}(\bar{X}, \mu_n^{\otimes (d-j)})$ are dual (resp. with $\mathbb{Q}_\ell$-coefficients).

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- $H^{i}_{ét}(\mathbb{P}^n_{\bar{\mathbb{F}}}, \mu_r^{\otimes j}) = \begin{cases} \mu_r^{\otimes j - \frac{i}{2}} & i \text{ even}, i \leq 2n \\ 0 & i \text{ otherwise.} \end{cases}$

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- $X \subset \mathbb{P}^n$ is a hypersurface. Same formulas as above for $\bar{X}$, but for $i = d$:

$$H^d_{ét}(\bar{X}, \mu_r^{\otimes j}) = H^d_{ét}(\mathbb{P}^n_{\bar{\mathbb{F}}}, \mu_r^{\otimes j}) \oplus H^d_{ét}(\bar{X}, \mu_r^{\otimes j})',$$

$H^d_{ét}(\bar{X}, \mu_r^{\otimes j})'$ is of HUGE rank $\frac{(deg\, X-1)^{d+2}+(-1)^d(deg\, X-1)}{deg\, X}$.

Computing cohomology

In general :

Theorem (D.Madore and F. Orgogozo) *There exists an algorithm which allows to compute the groups $H^i_{ét}(\bar{X}, \mathbb{Z}/\ell)$ (so that the étale cohomology groups are computable in the sense of Church-Turing.)*

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- another geometric version :

$$cl^i_{\mathbb{Q}_\ell} : CH^i(\bar{X}) \otimes \mathbb{Q}_\ell \to \bigcup H^{2i}_{ét}(\bar{X}, \mathbb{Q}_\ell(i))^H$$

where the union is over all open subgroups $H \subset G$.

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Integral versions : understand if we have the surjectivity with $\mathbb{Z}_\ell$-coefficients (counterexamples exist).

Remark: using Weil conjectures, one can show that the map $H^{2i}_{ét}(X,\mathbb{Q}_\ell(i)) \to H^{2i}_{ét}(\bar{X},\mathbb{Q}_\ell(i))^G$ is an isomorphism (in fact the kernel $H^1(G, H^{2i-1}_{ét}(\bar{X},\mathbb{Z}_\ell(i))$ of the map with $\mathbb{Z}_\ell$-coefficients is finite). So that we can identify $c^i_{\mathbb{Q}_\ell}$ and $\bar{c}^i_{\mathbb{Q}_\ell}$.

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- the kernel of $Z^i(X) \to H^{2i}_{ét}(X, \mathbb{Z}_\ell(i))$ consists of classes *numerically* equivalent to zero, i.e. having zero intersection with any cycle of complimentary dimension (Tate); with $\mathbb{Q}_\ell$-coefficients rational and numerical equivalence coincide (Beilinson conjecture), so that $c^i_{\mathbb{Q}_\ell}$ is also injective (conjecturally).

Zeta functions

If $\mathbb{F} = F_q$ is a finite field with q elements, define

$$Z(X, T) = exp(\sum_{n \geq 1} |X(F_{q^n})| \frac{T^n}{n})$$

$$\zeta(X, s) = Z(X, q^{-s}),$$

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Tate conjecture, the strong form
$ord_{s=i}\zeta(X,s) = -dim(Z^i(X)/\sim_{num}) \otimes \mathbb{Q}.$

The case of divisors

- One has an exact sequence

$$0 \to \mathrm{Pic}\, X \otimes \mathbb{Z}_\ell \to H^2_{\acute{e}t}(X, \mathbb{Z}_\ell(1)) \to Hom(\mathbb{Q}_\ell/\mathbb{Z}_\ell, BrX) \to 0$$

  where the last group has NO torsion : it follows that $\mathrm{Pic}\, X \otimes \mathbb{Z}_\ell \to H^2_{\acute{e}t}(X, \mathbb{Z}_\ell(1))$ is surjective $\Leftrightarrow$ $\mathrm{Pic}\, X \otimes \mathbb{Q}_\ell \to H^2_{\acute{e}t}(X, \mathbb{Q}_\ell(1))$ is surjective $\Leftrightarrow$ BrX is finite.

Zero-cycles

Theorem

(J.-L. Colliot-Thélène, J.-J. Sansuc, C.Soulé)
The cycle class induces an isomorphism

$$CH^d(X) \otimes \mathbb{Z}_\ell \xrightarrow{\sim} H^{2d}(X, \mathbb{Z}_\ell(d)).$$

Torsion

- (J.-L. Colliot-Thélène, J.-J. Sansuc, C.Soulé) the torsion subgroup $CH^2(X)_{tors}$ is finite and the map $CH^2(X)_{tors} \to H^4(X, \mathbb{Z}_\ell(2))$ is injective.
- could one have that the kernel of the map $CH^i(X)\{\ell\} \to H^{2i}(X, \mathbb{Z}_\ell(i))$ is nonzero?

Known cases of Tate's conjecture

- Divisors ($i = 1$) on abelian varieties, precise version:

$$Hom(A, B) \otimes \mathbb{Z}_\ell \to Hom_{\mathbb{Q}_\ell}(T_\ell(A), T_\ell(B))$$

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- K3 surfaces in caracteristic different from 2 (F. Charles, D. Maulik, K. Madapusi Pera), examples : $X \subset \mathbb{P}^3$ a quartic; X a double cover $w^2 = f_6(x, y, z)$ with f_6 of degree 6.
- some other specific varieties.

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- Example : $a_0x_0^n + a_1x_1^n + a_2x_2^n + a_3x_3^n = 0$ is dominated by the product of $a_0x_0^n + a_1x_1^n = y^n$ and $a_2x_2^n + a_3x_3^n = z^n$.

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- *Remark 2*, reductions : if E, E' are two elliptic curves over a number field k, then there are infinitely many places where the reductions of E and E' are geometrically isogeneous (F. Charles). In particular, for a given elliptic curve E over k either E is supersingular at infinitely many places, or has complex multiplication at inifinitely many places.

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None of these maps need be surjective!

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- With more work one can produce a projective variety (by some hyperplane sections).
- for non-torsion classes: take exceptional G (such as G_2, F_4, E_8) containing $(\mathbb{Z}/\ell)^3$.

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- Define the unramified elements in these cohomology groups : $\xi \in H^i(\mathbb{F}(X), \mathbb{Z}/\ell)$ having no residus (there are formulas to compute) for all valuations on $\mathbb{F}(X)$ (discrete rank one) :

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  (or with $\mu_\ell^{\otimes j}$; by llimit, with $\mathbb{Q}_\ell/\mathbb{Z}_\ell(j)$ coefficients).
- Then $Coker(c^2)_{tors} = H^3_{nr}(\mathbb{F}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ if this last group is finite.

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- We do not know what happens in dimension 4.

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- Equation of the generic fiber (a quadric with coefficients in $\mathbb{F}(x,y) = \mathbb{F}(\mathbb{P}^2)$):

$$x_0^2 - ax_1^2 - fx_2^2 + afx_3^2 + g_1 g_2 x_4^2 = 0,$$

  with $a \in \mathbb{F}$ non-square, $f = x/y$ and g_i are fractions of products of 8 linear forms (configuration is specific to get residues we want!)

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- From the discussion on divisors it follows easily that $\mathrm{Pic}\, X \otimes \mathbb{Z}_\ell \xrightarrow{\sim} H^2_{\acute{e}t}(X, \mathbb{Z}_\ell(1)) \subset \mathbb{Z}_\ell^7$. (but for different cubics surfaces one can get different submodules of $\mathbb{Z}_\ell^7$).

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  - but we still do not know if $CH^2(X) \otimes \mathbb{Z}_\ell \to H^4_{ét}(X, \mathbb{Z}_\ell(2))$ is surjective...

The End