Moduli spaces of stable curves and R-equivalence

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In these notes we present some known results on R-equivalence on rationally connected varieties defined over function fields in one variable over $\mathbb{C}$ and, more generally, over C_1 fields or over fields of cohomological dimension at most one. We will mostly focus on rationally simply connected varieties and explain in detail how to show that R-equivalence on such varieties is trivial, as soon as we are over a function field in one variable over $\mathbb{C}$ (cf.[Pi]).

The notion of rationally simply connected varieties has been introduced by de Jong and Starr in [dJS]. They proved that a smooth complete intersection of r hypersurfaces in $\mathbb{P}^n_k$ of respective degrees $d_1, \ldots, d_r$ and of dimension at least 3 is rationally simply connected if $\sum_{i=1}^{r} d_i^2 \leq n+1$. We sketch some of their arguments. Then we discuss how, using their ideas, one can get the triviality of R-equivalence over a function field in one variable over $\mathbb{C}$.

The study of R-equivalence on rationally simply connected varieties requires some understanding of rational points on moduli space of stable curves and we discuss it in section 2. In section 3 we proceed to the study of R-equivalence on RSC varieties. To finish we sketch some results on R-equivalence on other rationally connected varieties, over fields as above (cf.[CTSa], [CTSk], [Ma]).

Contents

1 Introduction

Let X be a (separably) rationally connected variety, defined over a field k. Recall that two rational points x_1, x_2 of X are called *directly R-equivalent* if there is a morphism $f : \mathbb{P}^1_k \to X$ such that x_1 and x_2 belong to the image of $\mathbb{P}^1_k(k)$. This generates an equivalence relation called *R-equivalence*.

Assuming one of the following hypotheses :

(i) $k = \mathbb{C}(C)$ a function field of a complex curve;

(ii) $k = \mathbb{C}((t))$ a formal power series field;

(iii) k is a C_1 field;

(iv) $cd\, k \leq 1$

one wonders ([CT], 10.11) if the set $X(k)/R$ is trivial. In general, the answer is not expected to be positive, as pointed out in [CT], see also the remark below.

Nevertheless, it turns out that this is the case in all results we know :

1. X is a smooth compactification of a linear algebraic group and $cd\, k \leq 1$ ([CTSa]);
2. X is a surface fibered in conics of degree 4 over the projective line and $cd\, k \leq 1$ ([CTSk]);
3. X is a smooth intersection of two quadrics in $\mathbb{P}^n_k$ with $n \geq 5$ and $cd\, k \leq 1$([CTSaSD]);
4. X is a smooth cubic hypersurface in $\mathbb{P}^n_k$ with $n \geq 5$ and k is C_1 ([Ma]);
5. $k = \mathbb{C}(C)$ or $k = \mathbb{C}((t))$ and X is a smooth complete intersection of r hypersurfaces in $\mathbb{P}^n_k$ of dergrees $d_1, \dots d_r$ satisfying $\sum d_i^2 \leq n+1$. More generally, the same holds for a k-rationally simply connected variety ([Pi]).

Remark 1.1. The triviality of R-equivalence for smooth projective geometrically rational surfaces over $\mathbb{C}(t)$ would imply the unirationality of (smooth projective) varieties of dimension 3, fibered in conics over $\mathbb{P}^2_{\mathbb{C}}$, which is an open question.

In fact, let $p : X \to \mathbb{P}^2_{\mathbb{C}}$ be a morphism from a smooth projective variety X of dimension 3, such that the general fiber of p is a conic. As $\mathbb{P}^2_{\mathbb{C}}$ is birational to $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, we can replace X by $p' : X' \to \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ with the same assumption : the general fiber of p' is a conic. Let $\eta : \operatorname{Spec} \mathbb{C}(t) \to \mathbb{P}^1_{\mathbb{C}}$ be the generic point. Base change by η on the second factor gives a morphism $p'_\eta : X'_\eta \to \mathbb{P}^1_{\mathbb{C}(t)}$. The variety X'_η is a (geometrically) rational surface over $k = \mathbb{C}(t)$. If the R-equivalence on $X'_\eta(k)$ is trivial, one can find a rational curve $f : C = \mathbb{P}^1_{\mathbb{C}(t)} \to X'_\eta$ joining two rational points in different fibers of p'_η. This means that the induced map $p'_\eta \circ f : C \to \mathbb{P}^1_{\mathbb{C}(t)}$ is surjective. Base change by $p'_\eta \circ f$ gives the following diagram :

$$\begin{array}{ccc} Y = X'_\eta \times_{\mathbb{P}^1_{\mathbb{C}(t)}} C & \xrightarrow{g} & C \simeq \mathbb{P}^1_{\mathbb{C}(t)} \\ \downarrow & & \\ X'_\eta & & \end{array}$$

The general fiber of g is a conic, having a rational point as the map g has a section by our construction. This means that Y is rational over $\mathbb{C}(t)$. The projection $Y \to X'_\eta$ now shows that X' is unirational.

In what follows we explain the proof of 5 and related questions. We will also give a sketch of ideas for other results. First, we need to establish some facts on moduli spaces of stable curves.

2 Rational points on moduli space of curves

In this section we work with the moduli space $\bar{M}_{0,2}(X,d)$ of stable curves of genus zero. We analyse what one can say about an object representing a rational point of this space. In particular, we are interested in applications to R-equivalence.

2.1 Basic facts on $\bar{M}_{0,n}(X,d)$

Let us first make precise some facts about the moduli space $\bar{M}_{0,n}(X,d)$. Let X be a projective variety over a field k. Let H be an ample divisor on X. Let $\bar{k}$ be an algebraic closure of k. The space of rational curves of fixed degree[1] on X is not compact in general. One way to compactify it, due to Kontsevich, is to use stable curves.

Definition 2.1. A *stable* curve over X of degree d with n marked points is a datum $(C, p_1, \dots, p_n, f)$ of

(i) a proper geometrically connected and geometrically reduced k-curve C with only nodal singularities,

(ii) an ordered collection $p_1, \dots, p_n$ of distinct smooth k-rational points of C,

(iii) a k-morphism $f : C \to X$ with $\deg_C f^*H = d$,

such that the stability condition is satisfied :

(iv) C has only finitely many $\bar{k}$-automorphisms fixing the points $p_1, \dots, p_n$ and commuting with f.

We say that two stable curves $(C, p_1, \dots, p_n, f)$ and $(C', p'_1, \dots, p'_n, f')$ are *isomorphic* if there exists an isomorphism $\phi : C \to C'$ such that $\phi(p_i) = p'_i$, $i = 1, \dots, n$ and $f' \circ \phi = f$.

The precise construction of the moduli space of stable curves in this sense can be found in the article of Araujo and Kollár [AK]. Here are some important points from [AK] which we will use in what follows :

1. There exists a *coarse moduli space* $\bar{M}_{g,n}(X,d)$ for all stable curves over X of arithmetic genus g of degree d with n marked points, which is a *projective k-scheme* ([AK], Thm. 50).[2]
2. Saying that $\bar{M}_{g,n}(X,d)$ is a *coarse* moduli space means the following :

   (i) there is a bijection of sets :

$$\Phi : \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{genus } g \text{ stable curves over } \bar{k} \\ f : C \to X_{\bar{k}} \text{ with } n \text{ marked points,} \\ \deg_C f^*H = d \end{array} \right\} \xrightarrow{\sim} \bar{M}_{g,n}(X,d)(\bar{k});$$

   (ii) if $\mathcal{C} \to S$ is a family of genus g stable curves of degree d with n marked points, parametrized by a k-scheme S, then there exists a unique morphism $M_S : S \to \bar{M}_{g,n}(X,d)$ such that for every $s \in S(\bar{k})$ we have

$$M_S(s) = \Phi(\mathcal{C}_s).$$

[1] We fix the degree of the curves we consider in order to have a space of finite type.

[2] Over $\mathbb{C}$ the construction was first given in [FP]. In this paper, the authors fix a *curve class* $\beta \in H^{2\dim X - 2}(X, \mathbb{Z})$ rather than the degree. They consider stable curves $f : C \to X$ such that the class of $f_*[C]$ is β. They prove that there exists a coarse moduli space $\bar{M}_{g,n}(X, \beta)$ parametrizing all genus g stable curves of class β with n marked points. The result in [AK] holds over arbitrary, not necessarily algebraically closed field and, more generally, over a noetherian base.

3. Note that over nonclosed fields we do not have a bijection between isomorphism classes of stable curves and rational points of the corresponding moduli space, see [AK] p.31. In particular, a k-point of $\bar{M}_{g,n}(X,d)$ does not in general correspond to a stable curve defined over k.

We denote by $M_{g,n}(X,d)$ the open locus corresponding to irreducible curves and by

$$ev_n : \bar{M}_{g,n}(X,d) \to \underbrace{X \times \ldots \times X}_{n}$$

the evaluation morphism which sends a stable curve to the image of its marked points. In what follows we will focus on stable curves of genus zero.

Let P and Q be two k-points of X. Suppose there exists a stable curve of genus zero $f : C \to X_{\bar{k}}$ over $\bar{k}$ with two marked points mapping to P and Q, such that the corresponding point $\Phi(f)$ is a k-point of $\bar{M}_{0,2}(X,d)$. Our goal is to deduce that P and Q are R-equivalent over k. We will explain two ways how to proceed. The first one, quite elementary, makes use of the combinatorics particular to stable curves of genus *zero*. The second way requires more sophisticated tools and applies to fields of cohomological dimension at most one. Let us state the main result of this section.

Proposition 2.2. *Let X be a projective variety over a field k of characteristic zero. Let P and Q be k-points of X. Let $f : C \to X_{\bar{k}}$ be a stable curve over $\bar{k}$ of genus zero with two marked points mapping to P and Q. Let H be a fixed ample divisor on X and let $d = \deg_C f^*H$. If the corresponding point $\Phi(f) \in \bar{M}_{0,2}(X,d)$ is a k-point of $\bar{M}_{0,2}(X,d)$, then the points P and Q are R-equivalent over k.*

2.2 Combinatorial arguments

2.2.1 Notations

Let k be a field of characteristic zero. Let us fix an algebraic closure $\bar{k}$ of k. Let $L \overset{i}{\hookrightarrow} \bar{k}$ be a finite Galois extension of k, and let $G = \mathrm{Aut}_k(L)$. For any $\sigma \in G$ we denote $\sigma^* : \mathrm{Spec}\, L \to \mathrm{Spec}\, L$ the induced morphism. If Y is an L-variety, denote ${}^{\sigma}Y$ the base change of Y by σ^* and ${}^{\sigma}Y_{\bar{k}}$ the base change by $(i \circ \sigma)^*$. We denote the projection ${}^{\sigma}Y \to Y$ by σ^* too. If $f : Z \to Y$ is an L-morphism of L-varieties, then we denote ${}^{\sigma}f : {}^{\sigma}Z \to {}^{\sigma}Y$ and ${}^{\sigma}f_{\bar{k}} : {}^{\sigma}Z_{\bar{k}} \to {}^{\sigma}Y_{\bar{k}}$ the induced morphisms.

Note that if $Y \subset \mathbb{P}^n_L$ is a projective variety, then ${}^{\sigma}Y$ can be obtained by applying σ to each coefficient in the equations defining Y. Thus, if Y is defined over k, then the subvarieties Y, ${}^{\sigma}Y$ of $\mathbb{P}^n_L$ are given by the same embedding for all $\sigma \in G$. In this case the collection of morphisms $\{\sigma^* : Y \to Y\}_{\sigma \in G}$ defines a right action of G on Y. By Galois descent ([BLR], 6.2), if a subvariety $Z \subset Y$ is stable under this action of G, then Z also is defined over k.

2.2.2 Some lemmas on graphs

Let us first give a proof of the following well-known lemma :

Lemma 2.3. *Let C be a projective geometrically connected and geometrically reduced curve of arithmetic genus $p_a(C) = h^1(C, O_C) = 0$ over a perfect field k. Assume C has only nodal singularities. Then any two smooth k-points a, b of C are R-equivalent.*

Proof. For any field extension F of k let us call an *F-path joining a and b* a closed F-subcurve $C' \subset C_F$ such that

(i) $C' = C'_1 \cup C'_2 \cup \ldots \cup C'_r$ where C'_i, $i = 1, \ldots r$ are smooth F-rational curves;

(ii) $a \in C'_1$, $b \in C'_r$;

(iii) if $1 \leq i \leq r-1$, the intersection $C'_i \cap C'_{i+1}$ is an F-point and the curves C'_i and C'_{i+j} do not intersect for $j > 1$.

Since the arithmetic genus of C is zero, its geometric irreducible components are smooth rational curves over $\bar{k}$ intersecting transversally. As C is geometrically connected, there exists a $\bar{k}$-path C' joining a and b. We may assume that C' is an L-path for some finite Galois extension L of k. Moreover, such a path is unique : if there were two different paths we would have a cycle formed by components of C_L, which is impossible as $p_a(C) = 0$. We would like to find a k-path, thus we will achieve the proof.

Let us write $C' = C'_1 \cup C'_2 \cup \ldots \cup C'_r$, $a \in C'_1$, $b \in C'_r$. We will show that C' comes from a k-path by base extension. Let us take $\sigma \in \mathrm{Aut}_k(L)$. Then ${}^{\sigma}C'_1, \ldots, {}^{\sigma}C'_r$ is an L-path joining a and b. Since such a path is unique, for every $i = 1, \ldots r$ the components C'_i and ${}^{\sigma}C'_i$ of C_L are equal and $C'_i \cap C'_{i+1} = {}^{\sigma}C'_i \cap {}^{\sigma}C'_{i+1} = \sigma(C'_i \cap C'_{i+1})$, $i = 1, \ldots r$. This means that every component of the path $C'_i \subset C_L$ is stable over the action of $\mathrm{Aut}_k(L)$ on C_L, hence it is defined over k, that is $C'_i = D_i \times_k L$, for some k-curve $D_i \subset C$. By the same argument, the intersection points of D_i and D_{i+1}, $i = 1, \ldots r-1$ are k-points. We deduce that $\{D_1, \ldots D_r\}$ is a k-path joining a and b, so the points a and b are R-equivalent over k. □

The next lemma will be used in the proof of Proposition 2.2. The necessity of all the hypotheses, which amounts to saying that we have a rational point on the moduli space, will be clear from the context.

Lemma 2.4. *Let X be a projective variety over a perfect field k. Let L be a finite Galois extension of k. Denote $G = \mathrm{Aut}_k(L)$. Let P and Q be k-points of X. Suppose we can find an L-stable curve of genus zero $f : C \to X_L$ with two marked points $a, b \in C(L)$, satisfying the following conditions :*

(i) $f(a) = P$, $f(b) = Q$;

(ii) for every $\sigma \in G$ there exists a $\bar{k}$-morphism $\phi_\sigma : C_{\bar{k}} \to {}^{\sigma}C_{\bar{k}}$ such that

$$\phi_\sigma(a) = \sigma(a),\ \phi_\sigma(b) = \sigma(b) \text{ and } {}^{\sigma}f_{\bar{k}} \circ \phi_\sigma = f_{\bar{k}}.$$

Then the points P and Q are R-equivalent over k.

Proof. By Lemma 2.3, we have a unique L-path $\{C_1, \ldots, C_m\}$ joining $a \in C_1(L)$ and $b \in C_m(L)$, where the C_i are irreducible components of C. We will use the curves $f(C_1), \ldots f(C_m)$ to show that P and Q are R-equivalent over k. Let us first show that these curves are defined over k and not only over L.

For every $\sigma \in G$ we have an L-path $\{{}^{\sigma}C_1, \ldots, {}^{\sigma}C_m\}$ joining $\sigma(a) \in {}^{\sigma}C_1(L)$ and $\sigma(b) \in {}^{\sigma}C_m(L)$. On the other hand, $\{\phi_\sigma(C_{1,\bar{k}}), \ldots, \phi_\sigma(C_{m,\bar{k}})\}$ is a $\bar{k}$-path joining $\sigma(a)$ and $\sigma(b)$. Since the arithmetic genus of ${}^{\sigma}C_{\bar{k}}$ is zero, such a path is unique. That is, it coincides with the path $\{{}^{\sigma}C_{1,\bar{k}}, \ldots, {}^{\sigma}C_{m,\bar{k}}\}$. So we have

$$\phi_\sigma(C_{i,\bar{k}}) = {}^{\sigma}C_{i,\bar{k}},\ i = 1, \ldots, m.$$

Let us fix $1 \leq i \leq m$. Denote the image $f(C_i)$ of C_i in X_L by Z_i. As ${}^{\sigma}f$ is a base change by σ^*, we have the following commutative diagram :

$$\begin{array}{ccc} {}^{\sigma}C_i & \xrightarrow{{}^{\sigma}f} & {}^{\sigma}X_L \\ \downarrow & & \downarrow \\ C_i & \xrightarrow{f} & X_L \end{array}$$

We thus see that ${}^{\sigma}f({}^{\sigma}C_i) = {}^{\sigma}Z_i$. Using base change by $i : L \to \bar{k}$ in the first line of the diagram above we obtain that ${}^{\sigma}f_{\bar{k}}({}^{\sigma}C_{i,\bar{k}}) = {}^{\sigma}Z_{i,\bar{k}}$. On the other hand, since $\phi_\sigma(C_{i,\bar{k}}) = {}^{\sigma}C_{i,\bar{k}}$ and ${}^{\sigma}f_{\bar{k}} \circ \phi_\sigma = f_{\bar{k}}$, we have ${}^{\sigma}Z_{i,\bar{k}} = {}^{\sigma}f_{\bar{k}}({}^{\sigma}C_{i,\bar{k}}) = {}^{\sigma}f_{\bar{k}}(\phi_\sigma(C_{i,\bar{k}})) = f_{\bar{k}}(C_{i,\bar{k}}) = Z_{i,\bar{k}}$. Since ${}^{\sigma}Z_i$ and Z_i are L-subvarieties of X_L, we deduce that ${}^{\sigma}Z_i = Z_i$ for all $\sigma \in G$. By Galois descent, this means that the curve Z_i is defined over k, that is, there exists a k-curve $D_i \subset X$ such that $Z_i = D_i \times_k L$.

To conclude, we will show that the curve D_i is a k-rational curve on X, that is, it is the image of some morphism from $\mathbb{P}^1_k$ to X, and that the point $f(C_i \cap C_{i+1})$ is the image of a k-point. Note that it may be not so obvious if $f(C_i \cap C_{i+1})$ is a singular point of D_i.

Let $\tilde{D}_i \to D_i$ be the normalisation morphism. It induces an isomorphism over the smooth locus D_i^{sm}. Since C_i is smooth, the morphism $f|_{C_i} : C_i \to D_i \times_k L$ extends to a morphism $f_i : C_i \to \tilde{D}_i \times_k L$:

$$\begin{tikzcd} & \tilde{D}_i \times_k L \arrow[d] \\ C_i \arrow[ur, "f_i"] \arrow[r, "f|_{C_i}"] & D_i \times_k L. \end{tikzcd}$$

This implies that $\tilde{D}_i$ is an L-rational curve. We have ${}^{\sigma}f_{i,\bar{k}} \circ \phi_\sigma = f_{i,\bar{k}}$, as this is true over a Zariski open subset D_i^{sm}. Moreover, for every $1 \leq i \leq m-1$ we have $\phi_\sigma(C_i \cap C_{i+1}) = {}^{\sigma}C_i \cap {}^{\sigma}C_{i+1}$. Using the same argument as above, we deduce that the point $f_i(C_i \cap C_{i+1})$ is a k-point of $\tilde{D}_i$. This implies that $\tilde{D}_i$ is a k-rational curve as it is L-rational and has a k-point. Moreover, the point $f(C_i \cap C_{i+1})$ is a k-point of X as the image of $f_i(C_i \cap C_{i+1})$. Hence P is R-equivalent to $f(C_1 \cap C_2)$ as there is a rational curve $\tilde{D}_1 \to X$ connecting them. By the same argument, $f(C_{i-1} \cap C_i)$ is R-equivalent to $f(C_i \cap C_{i+1})$ for all $1 < i < m-1$ and $f(C_{m-1} \cap C_m)$ is R-equivalent to Q. Therefore P and Q are R-equivalent. $\square$

2.2.3 Proof of Proposition 2.2.

We will show that the hypotheses of Lemma 2.4 are satisfied. We call a and b the marked points of C. We may assume that C, f, a and b are defined over a finite Galois extension $L \overset{i}{\hookrightarrow} \bar{k}$ of k. That is, we may assume that C is an L-curve, $a, b \in C(L)$ and that we have an L-morphism $f : C \to X_L$. Let us denote $T = \operatorname{Spec} L$. We view L as a k-scheme and $f : C \to X \times_k T$ as a family of stable curves parametrized by T. Thus we have a moduli map $M_T : T = \operatorname{Spec} L \to \bar{M}_{0,2}(X, d)$ defined over k and such that for every $t \in T(\bar{k})$ we have

$$M_T(t) = \Phi(C_t)$$

where $f_t : C_t \to X_{\bar{k}}$ is the fibre of $f : C \to X \times_k T$ over t.

Note that $T \times_k \bar{k} = \prod_G \operatorname{Spec} \bar{k}$ where the product is indexed by $G = \operatorname{Aut}_k(L)$ and the morphism $\prod_G \bar{k} \to T = \operatorname{Spec} L$ is given by $(i \circ \sigma)^*$ on the corresponding component. This implies that a $\bar{k}$-point $t \in T(\bar{k})$ corresponds to some $\sigma \in G$ and the morphism f_t is the base change by $(i \circ \sigma)^*$. Hence the morphism f_t is the morphism ${}^{\sigma}f_{\bar{k}} : {}^{\sigma}C_{\bar{k}} \to X_{\bar{k}}$ and the marked points of ${}^{\sigma}C_{\bar{k}}$ are $\sigma(a)$ and $\sigma(b)$.

Since the curve $f_{\bar{k}} : C_{\bar{k}} \to X_{\bar{k}}$ corresponds to a k-point of $\bar{M}_{0,2}(X, d)$, we can factor M_T as

$$T = \operatorname{Spec} L \to \operatorname{Spec} k \overset{\Phi(f_{\bar{k}})}{\to} \bar{M}_{0,2}(X, d).$$

We thus see that for every $t \in T(\bar{k})$ the point $M_T(t)$ is the same point $\Phi(f_{\bar{k}})$ of $\bar{M}_{0,2}(X, d)$. Hence for every $\sigma \in G$ the curves ${}^{\sigma}C_{\bar{k}}$ and $C_{\bar{k}}$ are isomorphic as stable curves. This means that there exists a $\bar{k}$-morphism $\phi_\sigma : C_{\bar{k}} \to {}^\sigma C_{\bar{k}}$, such that

$$\phi_\sigma(a) = \sigma(a),\ \phi_\sigma(b) = \sigma(b) \text{ and } {}^\sigma f_{\bar{k}} \circ \phi_\sigma = f_{\bar{k}}.$$

Now the proposition follows from lemma 2.4.

2.3 Stack-theoretical arguments

We will give a second proof of Proposition 2.2 in the case when the base field k is of cohomological dimension at most 1. In fact, using this assumption and the fact that the automorphism group a stable curve is *finite*, we will show that any rational point of a moduli space corresponds to an object defined over the base field. We will present a point of view of [DDE], all the arguments can be found there.

Definition 2.5. Let k be a perfect field, $\bar{k}$ an algebraic closure and $G = Gal(\bar{k}/k)$. We say that k is of *cohomological dimension at most* 1 if for any (continuous) finite G-module M and any integer $i \geq 2$ we have $H^i(G, M) = 0$.

Example 2.6. Any C_1 field is of cohomological dimension at most 1. Note that the converse is not true ([A]). Thus finite fields, function fields in one variable over an algebraically closed field and formal series fields in one variable over an algebraically closed field give examples of a field of $cd \leq 1$.

Let us give a short sketch of the argument. It is a general fact that, given a point x on a moduli space, corresponding to an object over $\bar{k}$ with the automorphisms group G, the obstruction to lift x to an object over k lives in a certain (non-abelian, as G is not necessarily abelian) 2-cohomology set (and not a group in general), in the sense of [Gi]. Now, the fact that G is finite, allows us to reduce to the abelian case and to deduce that H^2 vanishes under the hypothesis $cd\,k \leq 1$.

We first give some notions from non-abelian cohomology.

2.3.1 Gerbes and non-abelian cohomology

Let S be an étale site.

Definition 2.7. An *S-gerbe* is a stack[3] satisfying the following conditions :

(i) any two sections over an open set U are locally isomorphic, i.e. there exists an open subset $V \subset U$ such that the restrictions to V of the two given section to $\mathcal{G}(V)$ are isomorphic;

(ii) locally each fiber is nonempty : each open set U admits an open subset V such that the fiber above V is nonempty.

Example 2.8. Let us consider the following stack $\mathcal{G}_f$ over the étale site $\mathrm{Spec}\, k_{\acute{e}t}$:

(i) the objects of $\mathcal{G}_f$ over an extension L of k are stable curves $D \to X_L$ over L which become isomorphic over $\bar{k}$ to the $\bar{k}$-curve $C \to X_{\bar{k}}$ corresponding to the point $\Phi(f)$ as in Proposition 2.2.

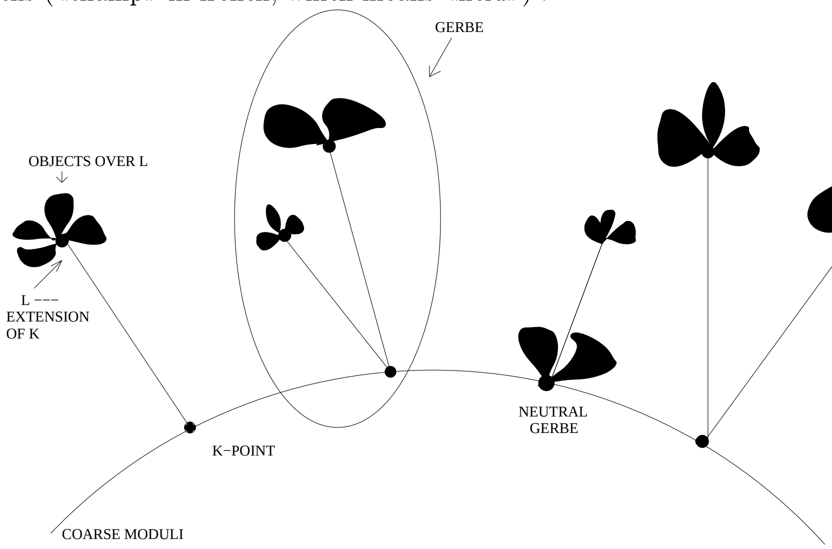
(ii) the morphisms between two stable curves over L are L-isomorphisms.

[3]Let us briefly recall the definition of a stack. It is a category X fibered on groupoids over S (i.e. for each open $U \subset S$ the fiber $X(U)$ is a groupoid), such that for each open $U \subset S$ and $x, y \in X(U)$, $Hom(x, y)$ is a sheaf and the following glueing condition is satisfied : given an open $U \subset S$, an open covering $(U_i)_{i \in I}$ of U and elements $x_i \in X(U_i), i \in I$, if for all i, j there exists an isomorphism ϕ_{ij} between the restrictions of x_i and x_j to $U_i \cap U_j$ such that $\phi_{ij} = \phi_{ik}\phi_{kj}$ over $U_i \cap U_j \cap U_k$, then there exists $x \in X(U)$ restricting to x_i over U_i.

One can also view $\mathcal{G}_f$ as the fibre of the morphism $\overline{\mathcal{M}}_{0,2}(X,d) \to \bar{M}_{0,2}(X,d)$ over the point $\Phi(f)$, where $\overline{\mathcal{M}}_{0,2}(X,d)$ is the stack of all genus zero stable curves over X of degree d with two marked points. By this description, the objects of $\mathcal{G}_f$ exist locally (that is, over some finite extension of k). Moreover, any two such objects are locally isomorphic (that is, after taking some finite extension of k). What we want to prove is the existence of objects over k, that is, that the curve $C \to X_{\bar{k}}$ is defined over k.

Definition 2.9. A gerbe which has objects over S is called *neutral.*

Remark 2.10. The following picture may illustrate the terminology of gerbes and stacks («champ» in french, which means «field») :



By considering the automorphism groups of objects of $\mathcal{G}$ one can associate a *band* $\mathcal{L} = \mathcal{L}(\mathcal{G})$ to the gerbe $\mathcal{G}$, see [Gi] Ch.IV for details. For example, if $S = \operatorname{Spec} k_{\text{ét}}$, then an S-band corresponds to a group A endowed with a homomorphism $Gal(\bar{k}/k) \to \operatorname{Aut}(A)/\operatorname{Inn}(A)$. Next, one can define a cohomological set $H^2(S,\mathcal{L})$ parametrizing all classes of gerbes whose associated band is $\mathcal{L}$. Note that in the case of $\mathcal{G}_f$ the automorphism group of objects is locally (that is, starting from some extension of k) a finite constant group consisting of the automorphisms of $C \to X_{\bar{k}}$ over $\bar{k}$.

A morphism $u : \mathcal{L} \to \mathcal{M}$ of bands induces a relation

$$H^2(S,\mathcal{L}) \multimap H^2(S,\mathcal{M})$$

where $p \multimap q$ means that there are gerbes P and Q of classes p and q respectively and an u-morphism $P \to Q$. If p is neutral and if $p \multimap q$ than q is neutral.

2.3.2 Reduction to the abelian case

Let us give a sketch of the proof of the following result of Dèbes, Douai and Emsalem ([DDE], cor. 1.3) :

Theorem 2.11. *If the cohomological dimension of k is at most one, then any gerbe $\mathcal{G}$ over the étale site $S = \operatorname{Spec} k_{ét}$ whose associate band $\mathcal{L}$ is locally a constant finite group C is neutral.*

Proof. As the authors point out, their idea goes back to the work of Springer [Sp]. Let us first suppose that C is not nilpotent. Then one can find a prime p and a p-Sylow subgroup H of C which is not normal. One verifies that the following data defines a gerbe $\mathcal{G}'$ over S :

(i) the objects of $\mathcal{G}'$ over a separable extension L of k are the couples (x, T), $x \in \mathcal{G}(L)$ and T is a subsheaf in p-Sylow of $\mathrm{Aut}_L(x)$;

(ii) the L-morphisms $(x, T) \to (x', T')$ are the L-morphisms $a : x \to x'$ such that the induced morphism $a_* : \mathrm{Aut}_L(x) \to \mathrm{Aut}_L(x')$ maps T to T'.

The band associated to the gerbe G' is locally the normalizer $B = N_C(H)$ and by contruction B is a proper subgroup of C. The morphism of bands $\mathcal{L}_B \to \mathcal{L}_C$ induced by the inclusion gives a relation $H^2(S, \mathcal{L}_B) \multimap H^2(S, \mathcal{L}_C)$. Thus if $\mathcal{G}'$ is neural, then we have the same for $\mathcal{G}$.

Thus we may reduce to the case C is nilpotent. Let

$$0 \to C' \to C \to C'' \to 0$$

be an exact sequence with C'' abelian. The fact that $\pi : C \to C''$ is surjective allows us to define a band $\pi_*\mathcal{L}$ and the fact that C'' is abelian implies that we have a well defined *morphism* $\pi_* : H^2(S, \mathcal{L}) \to H^2(S, \pi_*\mathcal{L})$. As $cd\, k \leq 1$ and C'' is abelian, the image $\pi_*([\mathcal{G}])$ is zero. As one can check, this implies that $[\mathcal{G}]$ comes from an element of $H^2(S, \mathcal{L}_{C'})$. Thus we conclude the proof by an induction argument. □

Now we can apply the previous theorem to $\mathcal{G}_f$. We conclude that $\mathcal{G}_f$ is neutral and thus there is a genus zero k-stable curve $f' : C' \to X$ with two marked points $a, b \in C'(k)$ such that the images of a and b are respectively the points P and Q. By lemma 2.3 we deduce that a and b are R-equivalent in C', thus their images P and Q in X are also R-equivalent.

3 R-equivalence over function fields in one variable

3.1 Case of RSC varieties

One can view rationally connected varieties as an analogue of path connected spaces in topology. From this point of view, de Jong and Starr introduce the notion of rationally simply connected varieties as an algebro-geometric analogue of that of simply connected spaces. We use here the following definition[4] :

Definition 3.1. Let k be a field of characteristic zero. A projective geometrically integral variety X over k is called *k-rationally simply connected* if for any sufficiently large integer e there exists a geometrically irreducible component $M_{e,2} \subset \bar{M}_{0,2}(X, e)$ intersecting the open locus of irreducible curves $M_{0,2}(X, e)$, such that the restriction of the evaluation morphism

$$ev_2 : M_{e,2} \to X \times X$$

is dominant with rationally connected general fiber.

Note that a k-rationally simply connected variety X over a field k is rationally connected as $X \times X$ is dominated by $M_{e,2} \cap M_{0,2}(X, e)$ from the definition above. This implies that two general points of X over any algebraically closed field $\Omega \supset k$ can be connected by a rational curve.

The following result is essentially contained in [dJS] and gives an example of RSC varieties in the sense above (essentially the only one we know). We explain some points

[4] The definition in [dJS] is given over $\mathbb{C}$. Here we emphasize that the distinguished component $M_{e,2}$ should be defined over k. Thus we use the notion of *k-rational* simple connectedness.

of the proof, having in mind that we are interested in applications to not algebraically closed fields.

Proposition 3.2. *Let k be a field of characteristic zero. Let X be a smooth complete intersection of r hypersurfaces in $\mathbb{P}^n_k$ of respective degrees $d_1, \dots, d_r$ with $\sum_{i=1}^{r} d_i^2 \leq n+1$. Suppose that $\dim X \geq 3$. Then for every $e \geq 2$ there exists a geometrically irreducible k-component $M_{e,2} \subset \bar{M}_{0,2}(X, e)$ such that the evaluation morphism*

$$ev_2 : M_{e,2} \to X \times X$$

is dominant with rationally connected generic fibre.

Proof. Let us first recall the construction of [dJS] in the case $k = \mathbb{C}$. In that paper, the authors work with the space $\bar{M}_{0,2}(X, \beta)$ of [FP] which parametrizes stable curves of genus zero over X of class $\beta \in H^{2\dim X - 2}(X, \mathbb{Z})$ with two marked points. Hovever, as $\dim X \geq 3$, we know that $H^{2\dim X-2}(X, \mathbb{Z}) = \mathbb{Z}\alpha$ where the degree of α equals to 1 ([V], 13.25). Thus we can replace β by its degree e and work with the space $\bar{M}_{0,2}(X, e)$ as in [AK].

In [dJS], de Jong and Starr prove that for every integer $e \geq 2$ there exists an irreducible component $M_{e,2} \subset \bar{M}_{0,2}(X, e)$ such that the restriction of the evaluation morphism $ev_2 : M_{e,2} \to X \times X$ is dominant with rationally connected generic fibre. We will specify more precisely how they get the component $M_{e,2}$. It will follow from their construction that $M_{e,2}$ is in fact the unique component satisfying the above property. The construction of $M_{e,2}$ is as follows :

1. One first shows that there exists a *unique* irreducible component $M_{1,1} \subset \bar{M}_{0,1}(X, 1)$ such that the restriction of the evaluation $ev_1|_{M_{1,1}} : M_{1,1} \to X$ is dominant ([dJS], 1.7).

2. The component $M_{1,0} \subset \bar{M}_{0,0}(X, 1)$ is constructed as the image of $M_{1,1}$ under the morphism $\bar{M}_{0,1}(X, 1) \to \bar{M}_{0,0}(X, 1)$ forgetting the marked point. Then one constructs the component of higher degree $M_{e,0}$ as the *unique* component of $\bar{M}_{0,0}(X, e)$ which intersects the subvariety of $\bar{M}_{0,0}(X, e)$ parametrizing degree e covers of smooth, free curves parametrized by $M_{1,0}$ ([dJS], 3.3).

3. The component $M_{e,2} \subset \bar{M}_{0,2}(X, e)$ is the *unique* component such that its image under the morphism $\bar{M}_{0,2}(X, e) \to \bar{M}_{0,0}(X, e)$, which forgets about the marked points, is $M_{e,0}$.

The proof of the fact that the general fiber of the evaluation morphism $ev_2 : M_{e,2} \to X \times X$ is rationally connected uses elaborated arguments, in particular, it uses the techniques of twisting surfaces. We will not discuss it here, see [dJS] for details.

Let us now consider the general case. Let $\bar{k}$ be an algebraic closure of k. As k is of finite type over $\mathbb{Q}$, we may assume that $\bar{k} \subset \mathbb{C}$. Since the decomposition into geometrically irreducible components does not depend on which algebraically closed field we choose, by the first step above there exists a unique irreducible component $M_{1,1} \subset \bar{M}_{0,1}(X_{\bar{k}}, 1)$ such that the restriction of the evaluation $ev_1|_{M_{1,1}}$ is dominant. As this component is unique, it is defined over k. Hence, from the construction above, the component $M_{e,2}$ is also defined over k, which completes the proof. □

Next, we will give a proof of the following theorem :

Theorem 3.3. *Let k be either a function field in one variable over $\mathbb{C}$ or the field $\mathbb{C}((t))$. Let X be a k-rationally simply connected variety over k. Then $X(k)/R = 1$.*

Combined with the theorem of de Jong and Starr, this gives :

Corollary 3.4. *Let k be either a function field in one variable over $\mathbb{C}$ or the field $\mathbb{C}((t))$. Let X be a smooth complete intersection of r hypersurfaces in $\mathbb{P}^n_k$ of respective degrees $d_1, \ldots, d_r$. Assume that $\sum_{i=1}^{r} d_i^2 \leq n+1$. Then $X(k)/R = 1$.*

3.1.1 Sketch of the proof

The hypothesis that X is rationally simply connected implies that there exists a (sufficiently large) integer e and a geometrically irreducible component $M_{e,2} \subset \bar{M}_{0,2}(X, e)$ such that the evaluation morphism $ev_2 : M_{e,2} \to X \times X$ is dominant with rationally connected general fibre.

Let P and Q be two k-points of X. Let us suppose that k is a function field in one variable over $\mathbb{C}$. A general strategy is the following. We would like to apply the theorem of Graber, Harris and Starr [GHS] to deduce that there is a rational point in the fibre over (P, Q). If it is so, we can use Proposition 2.2 to deduce that P and Q are R-equivalent. But we only know that a general fibre of ev_2 is rationally connected. We will use two methods to solve this problem. We will also show how to apply the same argument to the case $k = \mathbb{C}((t))$.

3.1.2 Specialisation arguments

First method. The first possibility is to use the following theorem of Hogadi and Xu [HX] :

Theorem 3.5. *Let k be a field of characteristic zero. Let $h : Y \to Z$ be a dominant proper morphism of k-varieties such that Z is smooth and the generic fibre of h is rationally connected. Then for every point $z \in Z$ there exists a subvariety of the fibre Y_z, defined over $k(z)$, which is geometrically irreducible and rationally connected.* □

Let X be as in Theorem 3.3. Let us, as before, take two k-points P and Q of X. By the theorem above, we can find a rationally connected k-subvariety V in the fibre $ev_2^{-1}(P, Q)$. If $k = \mathbb{C}(C)$, there is a rational point in V by [GHS], hence in $ev_2^{-1}(P, Q)$. By Proposition 2.2, the points P and Q are R-equivalent. So we obtain $X(k)/R = 1$.

Second method. The next argument shows that every fiber in a family with rationally connected general fiber has a rational point, as soon as we are over the function field of a complex curve. See also [Sta] p.25.

Lemma 3.6. *Let $k = \mathbb{C}(C)$ be the function field of a (smooth) complex curve C. Let Z and T be projective k-varieties, with T smooth. Let $f : Z \to T$ be a morphism with rationally connected general fibre. Then for every $t \in T(k)$ there exists a rational point in the fibre Z_t.*

Proof. One can choose proper models $\mathcal{T} \to C$ and $F : \mathcal{Z} \to \mathcal{T}$ of T and Z respectively with $\mathcal{T}$ smooth. We know that any fibre of F over some open set $U \subset \mathcal{T}$ is rationally connected.

The point $t \in T(k)$ corresponds to a section $s : C \to \mathcal{T}$. What we want is to find a section $C \to \mathcal{Z} \times_{\mathcal{T}} C$. One can view the image $s(C)$ in $\mathcal{T}$ as a component of a complete intersection C' of hyperplane sections of $\mathcal{T}$ for some projective embedding. In fact, it is sufficient to take $\dim \mathcal{T} - 1$ functions in the ideal of $s(C)$ in $\mathcal{T}$ generating this ideal over some open subset of $s(C)$. Moreover, one may assume that C' is a special fibre of a family $\mathcal{C}$ of hyperplan sections with general fibre a smooth curve intersecting U. After localization, we may also assume that $\mathcal{C}$ is parametrized by $\mathbb{C}[[t]]$. Let A be any affine open subset in $\mathcal{C}$ containing the generic point ξ of $s(C)$. We have the following diagram:

$$\begin{array}{ccccc} & & \mathcal{Z}\times_{\mathcal{T}}\operatorname{Spec}A & \longrightarrow & \mathcal{Z} \\ & & \downarrow{\scriptstyle F_A} & & \downarrow{\scriptstyle F} \\ \xi \longrightarrow \operatorname{Spec}A\otimes_{\mathbb{C}[[t]]}\mathbb{C} & \longrightarrow & \operatorname{Spec}A & \longrightarrow & \mathcal{T} \\ \downarrow & & \downarrow & & \\ Spec\,\mathbb{C} & \longrightarrow & \operatorname{Spec}\mathbb{C}[[t]]. & & \end{array}$$

Let $K = \mathbb{C}((t))$ and let $\bar{K}$ be an algebraic closure of K. By construction, the generic fibre of $F_{\bar{K}} : \mathcal{Z} \times_{\mathcal{T}} \bar{K} \to \operatorname{Spec} A \otimes_{\mathbb{C}[[t]]} \bar{K}$ is rationally connected. By [GHS] we obtain a rational section of $F_{\bar{K}}$. As $\bar{K}$ is the union of the extensions $\mathbb{C}((t^{1/N}))$ for $N \in \mathbb{N}$, we have a rational section for the morphism $\mathcal{Z} \times_{\mathcal{T}} \mathbb{C}[[t^{1/N}]] \to \operatorname{Spec} A \otimes_{\mathbb{C}[[t]]} \mathbb{C}[[t^{1/N}]]$ for some N. By properness, this section extends to all codimension 1 points of $\operatorname{Spec} A \otimes_{\mathbb{C}[[t]]} \mathbb{C}[[t^{1/N}]]$, in particular, to the point ξ on the special fiber. This extends again to give a section $C \to \mathcal{Z} \times_{\mathcal{T}} C$ as desired. □

3.1.3 Case $k = \mathbb{C}((t))$

The theorem of Graber, Harris and Starr admits the following corollary over the power series fields :

Theorem 3.7. *Let X be a projective rationally connected variety over $\mathbb{C}((t))$. Then X has a rational point.*

Proof. We present here a proof from [CT], 7.5. We will use the following two facts :

Fact 1. (Theorem of Greenberg, [Gr]) Let Z be a variety over a henselian discrete valuation ring $\mathcal{O}$. Let t be a generator of the maximal ideal. Suppose that the residue field of $\mathcal{O}$ is of characteristic zero. Then $Z(\mathcal{O}) \neq \emptyset \Leftrightarrow Z(\mathcal{O}/t^n) \neq \emptyset$ for all $n > 0$.

Fact 2. (cf.[BLR] p.82) Let K be the field of fractions of a henselian discrete valuation ring. Let $\hat{K}$ be a completion of K. Let Z be a smooth K-variety. Then $Z(K)$ is dense in $Z(\hat{K})$.

Let us fix some notations.

1. We denote R the henselisation of $\mathbb{C}[t]$ in $t = 0$ and K the field of fractions of R. Note that $\hat{K}$ can be identified to $\mathbb{C}((t))$. We also have $R/t^n \simeq \mathbb{C}[t]/t^n$.

2. Let $\mathcal{X}$ be the closure of $X \subset \mathbb{P}^n_{\mathbb{C}((t))}$ in $\mathbb{P}^n_{\mathbb{C}[[t]]}$. There exists a $\mathbb{C}$-algebra $i : A \hookrightarrow \mathbb{C}[[t]]$ of finite type and a projective and flat A-scheme $\mathcal{X}'$ such that $\mathcal{X}'_{\mathbb{C}[[t]]} \simeq \mathcal{X}$ (take A to be generated by the coefficients of equations defining $\mathcal{X}$ in $\mathbb{P}^n_{\mathbb{C}[[t]]}$.)

3. Let us denote $S = \operatorname{Spec} A$ and let $\xi \in S(\mathbb{C}[[t]])$ be the point corresponding to i.

4. Let $U \subset S$ be an open set such that $\mathcal{X}'_u$ is rationally connected for all $u \in U$. One may assume that U is smooth.

Now we proceed to the proof of the theorem. By fact 2, $U(K) \subset U(\mathbb{C}((t)))$ is dense. On the other hand, the map $S(\mathbb{C}[[t]]) \to S(\mathbb{C}[t]/t^n)$ has open fibres. Thus there exists a point $\xi_n \in U(K) \cap S(\mathbb{C}[[t]])$ having the same image in $S(\mathbb{C}[t]/t^n) = S(R/t^n)$ as ξ. By considering the valuation at t we observe that ξ_n is in fact an R-point.

Let us denote $\mathcal{X}_n$ the base change of $\mathcal{X}'$ by ξ_n. We view $\mathcal{X}_n$ as an R-scheme. Note that the generic fibre X_n/K of $\mathcal{X}_n$ is rationally connected by the choice of U and that $\mathcal{X}_n \times_R R/t^n \simeq \mathcal{X}' \times \mathbb{C}[t]/t^n$ by the choice of ξ_n. As the fraction field K of $R = \mathbb{C}[t]^h_{(t)}$

is a union of function fields of curves, $X_n(K)$ is not empty by [GHS]. By properness, $\mathcal{X}_n(R)$ is not empty. Thus $\mathcal{X}' \times \mathbb{C}[t]/t^n \simeq \mathcal{X}_n(R/t^n)$ is not empty. By fact 1, we deduce that $X(\mathbb{C}((t))) \neq \emptyset$.

□

3.1.4 Proof of 3.3

The theorem now follows by combining the previous results. Let, as before, $M_{e,2} \subset \bar{M}_{0,2}(X,e)$ be a geometrically irreducible component, such that the restriction of the evaluation morphism $ev_2 : M_{e,2} \to X \times X$ is dominant with rationally connected general fibre.

Let P and Q be two k-points of X. We know that a general fibre of ev_2 is rationally connected. If $k = \mathbb{C}((t))$, one concludes using 3.7 and the fact that R-equivalence classes are Zariski dense in this case by [Ko99]. If k is a function field in one variable over $\mathbb{C}$, $ev_2^{-1}(P,Q)(k)$ is not empty by 3.6. By Proposition 2.2, the points P and Q are R-equivalent. So we obtain $X(k)/R = 1$ in both cases. □

Note that the methods used in the proof of the theorem apply more generally over a field k of characteristic zero such that any rationally connected variety over k has a rational point.

As for the corollary, there is in fact a much simpler proof for any C_1 field in the case $\sum d_i^2 \leq n$. The argument is due to Jason Starr.

Proposition 3.8. *Let k be a C_1 field. Let $X \overset{i}{\hookrightarrow} \mathbb{P}^n_k$ be the vanishing set of r polynomials $f_1, \dots f_r$ of respective degrees $d_1, \dots d_r$. If $\sum d_i^2 \leq n$ then any two points $x_1, x_2 \in X(k)$ can be joined by two lines defined over k : there is a point $x \in X(k)$ such that $l(x, x_i) \subset X$, $i = 1, 2$, where $l(x, x_i)$ denote the line through x and x_i.*

Proof. We may assume that $x_1 = (1 : 0 : \dots : 0)$ and $x_2 = (0 : 1 : 0 : \dots : 0)$ via the embedding i. The question is thus to find a point $x = (x_0 : \dots : x_n)$ with coordinates in k such that

$$\begin{cases} f_i(tx_0 + s, tx_1, \dots tx_n) = 0 \\ f_i(tx_0, tx_1 + s, \dots tx_n) = 0, \end{cases} \quad i = 1, \dots r.$$

As x_1, x_2 are in $X(k)$ these conditions are satisfied for $t = 0$. Thus we may assume $t = 1$. Writing $f_i(x_0 + s, x_1, \dots x_n) = \sum_{j=0}^{d_i} P^i_j(x_0, \dots x_n) s^j$ with P^i_j homogeneous of degree $d_i - j$ we see that each equation $f_i(x_0 + s, x_1, \dots x_n) = 0$ gives us d_i conditions on $x_0, \dots x_n$ of degrees $1, \dots d_i$. By the same argument, each equation $f_i(x_0, x_1 + s, \dots x_n) = 0$ gives $d_i - 1$ conditions of degrees $1, \dots d_i - 1$ as we know from the previous equation that we have no term of degree zero. The sum of the degrees of all these conditions on $x_0, \dots x_n$ is $\sum_{i=0}^{r} d_i^2$. As $\sum_{i=0}^{r} d_i^2 \leq n$ by Tsen-Lang theorem we can find a solution over k, which finishes the proof. □

3.2 Case of cubic hypersurfaces

The case of R-equivalence on cubic hypersurfaces was studied by Madore. Let us sketch the proof of the folowing result ([Ma]) :

Theorem 3.9. *Let k be a C_1 field. Let $X \subset \mathbb{P}^n_k$ be a cubic hypersurface. If $n \geq 5$, then $X(k)/R = 1$.*

Proof. Let P, Q be two rational points of X. Let us first assume that X is a singular at at least one of these points. We may thus assume that $P = (1 : 0 : \ldots : 0)$ is a singular point of X. Thus X is defined by an equation $x_0 q(x_1, \ldots x_n) + c(x_1, \ldots x_n)$ with q and c respectively a quadratic and a cubic forms. Let $Q = (y_0 : \ldots : y_n) \in X(k)$. We will show that P is R-equivalent to Q.

If $P = Q$ or if the line PQ is contained in X, it is clear. Otherwise $q(y_1, \ldots y_n) \neq 0$. Let $S = (z_1 : \ldots : z_n)$ be a zero of q, which exists in k by hypothesis that k is C_1.

Consider the rational map

$$\phi : \mathbb{P}_k^{n-1} \dashrightarrow X, \ (x_1 : \ldots : x_n) \mapsto (-c(x_1, \ldots x_n) : x_1 q(x_1, \ldots x_n) : \ldots : x_n q(x_1, \ldots x_n)).$$

Let h be the restriction of ϕ to any rational curve $C \subset \mathbb{P}_k^{n-1}$ joining $Q' = (y_1 : \ldots : y_n)$ to S. We have $h(Q') = Q$. Thus ϕ is defined at Q', so it is defined on an open subset of C. This implies that h is a well-defined morphism. If $c(S) \neq 0$ then h is defined at S and $h(S) = P$. Thus the points P and Q are R-equivalent.

Otherwise, the line $l = (u : vz_1 : \ldots : vz_n)$ joining P to $(0 : z_1 \ldots : z_n)$ is contained in X. Let us show that $h(S)$ is on this line. This will imply that P is R-equivalent to Q by a chain $Ph(S)Q$. In fact, consider the composite map

$$\mathbb{P}_k^{n-1} \overset{\phi}{\dashrightarrow} X \overset{p}{\dashrightarrow} \mathbb{P}_k^{n-1}$$

where p is given by $(x_0 : x_1 : \ldots : x_n) \mapsto (x_1 : \ldots : x_n)$. Note that the map $p \circ \phi$ is identity on its domain of definition.

The map p is defined at Q and $p(Q) = (y_1 : \ldots : y_n)$. The map ϕ is defined at Q' and $\phi(Q') = Q$. Thus the composite $p \circ \phi$ is defined at Q'. This means that $p \circ \phi$ induces the identity map on C. Thus the image of $h(S) = (z_1 : \ldots : z_n)$ by ϕ is S, which means that $h(S) = (u : z_1 : \ldots : z_n)$ is on the line l, as desired.

Let us now suppose that X is smooth at P and at Q. We want to prove that any two K-points P and Q of X are R-equivalent. Let $T(P)$ and $T(Q)$ by the tangent hyperplans to X at P and at Q respectively. The cubic hypersurface $C(P) = X \cap T(P)$ in $T(P) \simeq \mathbb{P}_k^{n-1}$ has a singular point P. Let us define $C(Q)$ similarly. Then either $Q \in T(P)$ and the result follows from the singular case or $T(P)$ is distinct from $T(Q)$. In the latter case, $X \cap T(P) \cap T(Q)$ is defined by a cubic form in $n-1$ variables in the projective space $T(P) \cap T(Q)$ of dimension $n-2$, thus it has a non trivial zero M. Thus P (resp. Q) is R-equivalent to M, again by the singular case. This finishes the proof. $\square$

Remark 3.10. Note that in the theorem above one can replace the hypothesis that k is C_1 by that any quadratic and any cubic form over k in at least $n-1$ variables has a zero.

3.3 Some other cases

The triviality of R-equivalence in cases $1-3$ p. 2, follows from the explicit description of the set of R-equivalence classes as some cohomology group $H^1(Gal(\bar{k}/k), M)$: one uses the fact that $cd\, k \leq 1$ to establish that this group vanishes. Let us consider, as an example, the case X is a smooth compactification of an algebraic torus T. We know from [CTSa] Th.2 and Prop.13, that $X(k)/R \xrightarrow{\sim} H^1(G, \hat{S})$ where $\hat{S}$ is the character group of some particular torus, coming from so-called flasque resolution of T.

Let $G = Gal(\bar{k}/k)$. Note that $\hat{S}$ is not a finite G-module, so we can not simply use the definition of a field of cohomological dimension at most 1. Consider a finite Galois extension K/k trivialising $\hat{S}$. Let H be an invariant subgroup of G acting trivially on $\hat{S}$ and let L/k be the corresponding extension. The restriction-inflation sequence gives an isomorphism $H^1(G/H, S(L)) \xrightarrow{\sim} H^1(G, S(K))$. As $cd\, k \leq 1$, the first group is zero, so is the second.

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