Diagonal Arithmetics.
An introduction : Chow groups.

Alena Pirutka

CNRS, École Polytechnique

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- (**push-forward**) for $f : X \to Y$ proper define $f_* : Z_i(X) \to Z_i(Y)$ by $f_*([V]) = 0$ if $dim\, f(V) < i$ and $f_*([V]) = a \cdot f(V)$ if $dim\, f(V) = i$, where a is the degree of the field extension $k(V)/k(f(V))$.

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- (**intersections**) $V \subset X$ and $W \subset X$ intersect *properly* if all irreducible components of $V \times_X W$ have codimension $codim_X V + codim_X W$. One then defines $V \cdot W$ as the sum of these components (with some multiplicities!).

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- For $X/\mathbb{C}$ smooth projective one has a cycle class map $c^i : Z^i(X) \to H^{2i}(X, \mathbb{Z})$, giving $Z^i(X) \otimes \mathbb{Q} \to Hdg^i(X)$ where $Hdg^i(X) = H^{2i}(X, \mathbb{Q}) \cap H^{i,i}(X)$ (the Hodge classes). The Hodge conjecture predicts that this last map should be surjective.

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$\sim$ an equivalence relation on algebraic cycles is *adequate* if

- $\sim$ is compatible with addition of cycles;
- for any X/k smooth projective and $\alpha, \beta \in Z^*(X)$ one can find $\alpha' \sim \alpha$ and $\beta' \sim \beta$ such that α' and β' intersect properly (i.e. all components have *right* codimension)
- if $X, Y/k$ are smooth projective, pr_X (resp. pr_Y) $X \times Y \to X$ (resp. Y) is the first (resp. second) projection and $\alpha \in Z^*(X)$, $\beta = pr_X^{-1}(\alpha)$ and $\gamma \in Z^*(X \times Y)$ intersecting β properly, then $\alpha \sim 0 \Rightarrow pr_{Y*}(\beta \cdot \gamma) \sim 0$.

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- if K/k is a finite field extension of degree m, $\pi : X_K \to X$, then the composition $\pi_* \circ \pi^* : CH_i(X) \to CH_i(X_K) \to CH_i(X)$ is the multiplication by m.
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- (**cycle class**) for X smooth projective, we have $CH^i(X) \to Hdg^i(X)$ $(k = \mathbb{C})$.
- (**localisation sequence**) $\tau : Z \subset X$ closed, $j : U \subset X$ the complement. Then we have an exact sequence

$$CH_i(Z) \xrightarrow{\tau_*} CH_i(X) \xrightarrow{j^*} CH_i(U) \to 0.$$

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- More : any $\alpha \in CH_*(X \times Y)$ gives $\alpha_* : CH_*(X) \to CH_*(Y)$: if $\gamma \in CH_*(X)$, then $\alpha_*(\gamma) = pr_{Y*}(\alpha \cdot pr_X^*(\gamma))$, i.e. α_* is the composition

$$CH_*(X) \to CH_*(X \times Y) \stackrel{\cdot\alpha}{\to} CH_*(X \times Y) \to CH_*(Y).$$

- On cohomology: any $\alpha \in CH^i(X \times Y)$ gives $\alpha_* : H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q}) : \alpha_*(\gamma) = pr_{Y*}(c^i(\alpha) \cup pr_X^*(\gamma))$, $\gamma \in H^*(X, \mathbb{Q})$.
- An important example : consider $\Delta_X \subset X \times X$ the diagonal. Then $[\Delta_X]_*$ is the identity map.

Other classical equivalence relations

Let $\alpha \in Z^i(X)$

- **(algebraic)** $\alpha \sim_{alg} 0$ if there exists a smooth projective curve C and two points $c_1, c_2 \in C(k)$ and $\beta \in Z^i(X \times C)$ such that $\alpha = \tau_{c_1}^* \beta - \tau_{c_2}^* \beta$, where τ_{c_i} is the inclusion of c_i in C.
- **(homological)** $\alpha \sim_{hom} 0$ if $c^i(\alpha) = 0$ (over $\mathbb{C}$, over k take another (Weil) cohomology)
- **(numerical)** $\alpha \sim_{num} 0$ if for any $\beta \in Z^{d-i}(X)$ one has $\alpha \cdot \beta$ (is well-defined!) is zero.

one has $\{\alpha \sim_{rat} 0\} \subset \{\alpha \sim_{alg} 0\} \subset \{\alpha \sim_{hom} 0\} \subset \{\alpha \sim_{num} 0\}$.

Plan of the lectures

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- (**Bloch-Srinivas**) triviality of CH_0 and equivalence relations;
- (**Voisin, Colliot-Thélène – Pirutka, Beauville, Totaro, Hassett-Kresch-Tschinkel**) universal triviality of CH_0 and stable rationality.