Diagonal arithmetics : exercises

1. Examples of vanishing of the Chow group of zero-cycles:

   (a) Let X be a smooth projective retract rational variety over a field k. Show that X is CH_0-trivial. (Hint: one can use a moving lemma here.)

   (b) Let X be a smooth projective rationally connected variety over an algebraically closed field k. Show that $CH_0(X) = 0$.

2. **Proof of the moving lemma for zero-cycles**. Let k be an infinite perfect field and let X be a smooth irreducible quasi-projective variety over k. Let $U \subset X$ be a nonempty Zariski open of X. Let $F = X \setminus U$.

   (a) Show that if $\dim X = 1$, then any zero-cycle in X is rationally equivalent to a zero-cycle with support in U. (One can use that the semi-local ring $\mathcal{O}_{X,F}$ is a principal ideal domain).

   (b) Let $d = \dim X$. Let $x \in X$ be a closed point. Show that there is a function $g \in \mathcal{O}_{X,x}$ such that the condition $g = 0$ defines locally a closed subset containing F.

   (c) Show that one can find a system $f_1, \ldots f_{d-1}$ of regular parameters of $\mathcal{O}_{X,x}$ such that the image of g in $\mathcal{O}_{X,x}/(f_1, \ldots f_{d-1})$ is nonzero.

   (d) Let C be a curve defined as a closure in X of the locus $f_1 = \ldots = f_{d-1} = 0$. Let $\pi : D \to X$ be the normalisation of C. Show that there exists a point $y \in D$ such that $x = \pi_*(y)$.

   (e) use the case of $\dim X = 1$ to show that x is rationally equivalent to a zero-cycle supported on U. Conclude that any zero-cycle on X is rationally equivalent to a zero-cycle supported on U.

3. CH_0**-universal triviality and conditions on fibers**. Let k be a field and let $f : Z \to Y$ be a proper map between algebraic varieties over k. Assume that for any M a (scheme) point of Y, the fiber Z_M is CH_0-universally trivial.

   (a) Show that the push-forward $f_* : CH_0(Z) \to CH_0(Y)$ is surjective.

   (b) Let $z \in ker(f_*)$. Show that there exists some integral curves $C_i \subset Y$ closed in Y such that $f_*(z) = \sum div_{\tilde{C}_i}(g_i)$ for some functions g_i on C_i ($\tilde{C}_i$ is the normalization of C_i.)

   (c) Show that one can find finite surjective maps $f_i^j : D_i^j \to C_i$ from integral curves $D_i^j \subset Z$ such that $\sum_j n_i^j deg(f_i^j) = 1$ for some $n_i^j \in \mathbb{Z}$.

   (d) Consider $z' = z - \sum_i \sum_j n_i^j div_{D_i^j}(g_i)$. Show that $f_*(z') = 0$ as a zero-cycle. Deduce that z' is a sum of finitely many zero-cycles of degree zero included in fibers of f. Conclude that z is rationally equivalent to zero.

4. Stable birational invariance of the unramified cohomology groups:

(a) Let k be a field and let $F \subset \mathbb{A}^1_k$ be a finite subset of an affine line over k. Show that there exists an exact sequence

$$0 \to H^i(k, \mu_n^{\otimes j}) \to H^i(\mathbb{A}^1_k \setminus F, \mu_n^{\otimes j}) \to \oplus_{P \in F} H^{i-1}(k(P), \mu_n^{\otimes (j-1)}) \to 0$$

(hint : one can use purity and Gysin exact sequence)

(b) Deduce that the following sequence is exact :

$$0 \to H^i(k, \mu_n^{\otimes j}) \to H^i(k(t), \mu_n^{\otimes j}) \to \oplus_{P \in {\mathbb{A}^1_k}^{(1)}} H^{i-1}(k(P), \mu_n^{\otimes (j-1)}) \to 0$$