On compatible stuctures on Lie algebroids

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On compatible structures on Lie algebroids

- Prehistory of cohomological theory of Lie bialgebras. The big bracket.
- The big bracket for Lie algebroids.
- Compatible structures on Lie algebroids.
- Cmpatibble structures associated with MONGE–AMPÈRE operators.

An even graded Poisson bracket

Bertram Kostant and Shlomo Sternberg,
Symplectic reduction, BRS cohomology, and infinite-dimensional Clifford algebras, *Ann. Physics* 176, 1987, 49–113.

Voir aussi Marc Henneaux, *Physics Rep.* 126, 1985.

Let E be a vector space and ϕ a symmetric bilinear form over E.
Define $\{u, v\}_\phi = \phi(u, v)$, for $u, v \in E$.
Extend $\{\ ,\ \}_\phi$ to $\wedge^\bullet E = \oplus_j \wedge^j E$ as a biderivation.

Then $\{\ ,\ \}_\phi$ is a Poisson bracket of degree -2 on $\wedge^\bullet E$.

In the theory of Lie bialgebras

Pierre Lecomte, Claude Roger
Modules et cohomologies des bigèbres de Lie, *C. R. Acad. Sci. Paris Sér. I Math.* 310, 1990, 405–410.

Let V be a vector space. Let $E = V^* \oplus V$.
Let ϕ be the canonical symmetric bilinear form.
For $x, y \in V$, $\xi, \eta \in V^*$,

$$\boxed{\phi(x,y) = 0, \quad \phi(\xi,\eta) = 0, \quad \phi(\xi,x) =< \xi, x >,}$$

where $<\ ,\ >$ is the duality between V and V^*.

Then $\{\ ,\ \}_\phi = \{\ ,\ \}$ is a Poisson bracket of bidegree $(-1,-1)$ on $\wedge^\bullet(V^* \oplus V) = \oplus_{k\geq 0, \ell\geq 0} \wedge^\ell V^* \otimes \wedge^k V$, called the big bracket.

Supermanifold interpretation

$V^* \oplus V$ is the cotangent bundle T^*V of V.

Consider the supermanifold $\Pi V = V[1]$ and its cotangent bundle $T^*V[1]$. The algebra of functions on $T^*V[1]$ is $\wedge^\bullet(V^* \oplus V)$.

The Poisson bracket $\{\ ,\ \}$ is the Poisson bracket of the canonical symplectic structure of this cotangent bundle.

In Lie algebra theory

In restriction to $\wedge^\bullet V^* \otimes V$, the even Poisson bracket $\{\ ,\ \}$ is the Nijenhuis-Richardson bracket of vector-valued forms on V, and similarly for vector-valued forms on V^* (up to a sign).

Example 1:
A Lie algebra structure on V is an element $\mu \in \wedge^2 V^* \otimes V$ such that

$$\{\mu, \mu\} = 0.$$

Example 2:
A Lie coalgebra structure on V is an element $\gamma \in V^* \otimes \wedge^2 V$ such that

$$\{\gamma, \gamma\} = 0.$$

The cohomological approach to Lie bialgebras

A Lie bialgebra structure on $(V, V^)$ is an element $\mu + \gamma \in (\wedge^2 V^* \otimes V) \oplus (V^* \otimes \wedge^2 V)$ such that $\boxed{\{\mu + \gamma, \mu + \gamma\} = 0.}$

$$\begin{cases} \{\mu, \mu\} = 0 \ , \\ \{\mu, \gamma\} = 0 \ , \\ \{\gamma, \gamma\} = 0 \ . \end{cases}$$

The compatibility conditon,

γ is a cocycle of (V, μ) or μ is a cocycle of (V^*, γ),

is simply expressed as

$$\{\mu, \gamma\} = 0.$$

The Lie algebra cohomology operators of (V, μ) and (V^*, γ) are $d_\mu = \{\mu, .\}$ and $d_\gamma = \{\gamma, .\}$: bicomplex of the Lie bialgebra.

Two ways to generalize

- Algebraic:

Quasi-Lie-bialgebras

See Drinfeld, Quasi-Hopf algebras, *Leningrad Math. J.* 1, 1990.

Proto-Lie-bialgebras

See yks, *Contemp. Math.* 132, 1992.

- Geometric:

Lie bialgebroids

See Mackenzie and Xu, *Duke* 73, 1994: infinitesimal of a Poisson groupoid.

See also yks, 1995.

- Geometric version of quasi-Lie-bialgebras, proto-Lie- bialgebras:

Quasi-Lie-bialgebroids, proto-Lie- bialgebroids

See Dmitry Roytenberg, *LMP* 61, 2002.

See also yks, Prog. Math., 2010... arXiv 0711.2043.

An important formula

(V, μ) is a Lie algebra, μ is the Lie algebra structure. Then $\wedge^\bullet V$ is equipped with the Schouten bracket, $[\ ,\ ]^\mu$.
Let X, $Y \in \wedge^\bullet V$. Then

$$\boxed{[X, Y]^\mu = \{\{X, \mu\}, Y\}.}$$

This result shows that the Schouten bracket is a derived bracket.
(yks, *Ann. Fourier* 46, 1996 and LMP 69, 2004)
(Ted Voronov, *Contemp. Math.* 315, 2002)

Lie algebroids

$A \to M$ vector bundle,
• Lie algebra bracket, $[\ ,\ ]$, on ΓA,
• anchor, $\rho : A \to TM$, vector bundle morphism,
• Leibniz rule, $[X, fY] = f[X, Y] + (\rho(X) \cdot f)Y$,
X, $Y \in \Gamma A$, $f \in C^\infty(M)$.

Then ρ induces a Lie algebra homomorphism from ΓA to $\Gamma(TM)$.

Examples

- Lie algebra,
- TM for M any manifold,
- T^*M when M is a Poisson manifold,
- action Lie algebroids,
- etc.

The big bracket for vector bundles
Roytenberg, 2002

When $V \to M$ is a vector bundle,
let $V[1]$ be the graded manifold obtained from V by assigning

- degree 0 to the coordinates on the base,
- degree 1 to the coordinates on the fibers.

Let $\mathcal{F}$ be the bigraded commutative algebra of smooth functions on $T^*V[1]$.

Remark: When M is a point, V is just a vector space.
Then $\mathcal{F} = \wedge^\bullet(V^* \oplus V)$. See above.

Local coordinates on $T^*V[1]$, and their bidegrees:

$$\begin{array}{cccc} x^i & \xi^a & p_i & \theta_a \\ (0,0) & (0,1) & (1,1) & (1,0) \end{array}$$

The bigraded Poisson algebra $\mathcal{F}$

As the cotangent bundle of a graded manifold, $T^*V[1]$ is canonically equipped with an even Poisson structure.
Denote the even Poisson bracket on $\mathcal{F}$ by $\{\ ,\ \}$.
We call it the big bracket because it generalizes the big bracket on $\wedge^\bullet(V^* \oplus V)$.

- It is of bidegree $(-1, -1)$.
- It is skew-symmetric, $\{u, v\} = -(-1)^{|u|\,|v|}\{v, u\}$, u, $v \in \mathcal{F}$,
- It satisfies the Jacobi identity,

$$\{u, \{v, w\}\} = \{\{u, v\}, w\} + (-1)^{|u|\,|v|}\{v, \{u, w\}\} \ ,$$

u, v, $w \in \mathcal{F}$.
In local coordinates,

$$\boxed{\{x^i, p_j\} = \delta^i_j \quad \text{and} \quad \{\xi^a, \theta_b\} = \delta^a_b\,.}$$

Consequence: $\{f, p_j\} = \frac{\partial f}{\partial x^j}$, for $f \in C^\infty(M)$.

Lie algebroids

A Lie algebroid structure on $A \to M$ is an element μ of $\mathcal{F}$ of bidegree $(1, 2)$ such that

$$\{\mu, \mu\} = 0 \ .$$

Schouten bracket of multivectors (sections of $\wedge^\bullet A$) X and Y:

$$[X, Y]_\mu = \{\{X, \mu\}, Y\}, \ X, Y \in \Gamma \wedge^\bullet A.$$

In particular,

• the Lie bracket of $X, \ Y \in \Gamma A$,

$$[X, Y] = [X, Y]_\mu = \{\{X, \mu\}, Y\} \ .$$

• the anchor of A, $\rho : A \to TM$,

$$\rho(X) f = [X, f]_\mu = \{\{X, \mu\}, f\} \ ,$$

for $X \in \Gamma A$, $f \in C^\infty(M)$.

The differential of a Lie algebroid

The operator $d_\mu = \{\mu, .\}$ is a differential on $\Gamma(\wedge^\bullet A^*)$ which defines the Lie algebroid cohomology of A.

Lie algebroid cohomology generalizes
• Chevalley–Eilenberg cohomology (when M is a point, A is a Lie algebra), and
• de Rham cohomology (when $A = TM$).

The Lichnerowicz–Poisson cohomology of a Poisson manifold is another example.

Lie bialgebroids

A Lie bialgebroid is defined by μ of bidegree $(1,2)$ and γ of bidegree $(2,1)$ such that $\{\mu+\gamma, \mu+\gamma\} = 0$.

(A, μ, γ) is a Lie bialgebroid iff $(\Gamma(\wedge^\bullet A), [\ ,\ ]_\mu, d_\gamma)$ is a differential Gerstenhaber algebra.

A Lie-quasi bialgebroid is defined by μ of bidegree $(1,2)$, γ of bidegree $(2,1)$, and $\phi \in \Gamma(\wedge^3 A)$ of bidegree $(3,0)$ such that $\{\phi+\mu+\gamma, \phi+\mu+\gamma\} = 0$.

A quasi-Lie bialgebroid is defined by μ of bidegree $(1,2)$, γ of bidegree $(2,1)$, and $\psi \in \Gamma(\wedge^3 A^*)$ of bidegree $(0,3)$ such that $\{\mu+\gamma+\psi, \mu+\gamma\ +\psi\} = 0$.

A proto-Lie bialgebroid is defined by μ of bidegree $(1,2)$, γ of bidegree $(2,1)$, $\phi \in \Gamma(\wedge^3 A)$ of bidegree $(3,0)$, and $\psi \in \Gamma(\wedge^3 A^*)$ of bidegree $(0,3)$ such that $\{\phi+\mu+\gamma+\psi, \phi+\mu+\gamma+\psi\} = 0$.

Examples: twist of a Lie bialgebra, twisted Poisson structures (with a 3-form background).

Cf. yks and Vladimir Rubtsov,
Compatible structures on Lie algebroids and Monge–Ampère operators,
Acta Appl. Math. 2010.

PN, $P\Omega$, ΩN, Hitchin pairs, complementary 2-forms

Magri–Morosi 1984, yks–Magri 1990, Vaisman 1996, Crainic 2004.

Nijenhuis structures

Let (A, μ) be a Lie algebroid (μ of bidegree $(1, 2)$ denotes the Lie algebroid structure of the vector bundle A).
Let $N \in \Gamma(A^* \otimes A)$ be a $(1, 1)$-tensor on A, an element of bidegree $(1, 1)$. Then the deformed structure,

$$\mu_N = \{N, \mu\} \ ,$$

defines an anchor, $\rho \circ N$, and a skew-symmetric bracket on A.
Denote it by $[\ ,\ ]_N^\mu$, Explicitly,

$$[X, Y]_N^\mu = \{\{X, \{N, \mu\}\}, Y\} \ ,$$

for X and $Y \in \Gamma A$.
Proposition. For X, $Y \in \Gamma A$,

$$[X, Y]_N^\mu = [NX, Y]_\mu + [X, NY]_\mu - N[X, Y]_\mu \ .$$

Torsion

The Nijenhuis torsion of N is defined by

$$(\mathcal{T}_\mu N)(X, Y) = [NX, NY]_\mu - N([NX, Y]_\mu + [X, NY]_\mu) + N^2[X, Y]_\mu ,$$

for all X and $Y \in \Gamma A$.

It is clear that $(\mathcal{T}_\mu N)(X, Y) = [NX, NY]_\mu - N([X, Y]^\mu_N)$.

Proposition. In terms of the big bracket,

$$\boxed{\mathcal{T}_\mu N = \frac{1}{2}\left(\{N, \{N, \mu\}\} - \{N^2, \mu\}\right) \ .}$$

(yks 1996, Grabowski 2006, Antunes 2008)

Nijenhuis structures, complex structures

Corollary 1.

• A necessary and sufficient condition for the deformed structure $\mu_N = \{N, \mu\}$ to be a Lie algebroid structure on A is $\{\mu, \mathcal{T}_\mu N\} = 0$.

• A sufficient condition for the deformed structure $\mu_N = \{N, \mu\}$ to be a Lie algebroid structure on A is $\boxed{\mathcal{T}_\mu N = 0.}$

Proof. In fact, $\frac{1}{2}\{\{N, \mu\}, \{N, \mu\}\} = \{\mu, \mathcal{T}_\mu N\}$.

Corollary 2. An almost complex structure N on A is a complex structure if and only if

$$\{\{N, \mu\}, N\} = \mu \ .$$

Complementary 2-forms for Poisson structures

Let (A, μ) be a Lie algebroid. If $\pi \in \Gamma(\wedge^2 A)$, then

$$\gamma_\pi = \{\pi, \mu\}$$

is of bidegree $(2, 1)$.
γ_π is a Lie algebroid structure on A^* if and only if

$$\boxed{\{\mu, [\pi, \pi]_\mu\} = 0.}$$

We now dualize this construction.

Let (A^*, γ) be a Lie algebroid. If $\omega \in \Gamma(\wedge^2 A^*)$, then

$$\widetilde{\mu} = \{\gamma, \omega\}$$

is of bidegree $(1, 2)$.
$\widetilde{\mu}$ is a Lie algebroid structure on A if and only if

$$\boxed{\{[\omega, \omega]_\gamma, \gamma\} = 0.}$$

Dualization and composition

Combine the two preceding constructions:

$$\boxed{(A,\mu) \overset{(\pi)}{\rightsquigarrow} (A^*,\gamma_\pi) \overset{(\omega)}{\rightsquigarrow} (A,\widetilde{\mu})\ ,}$$

with $\gamma_\pi = \{\pi,\mu\}$, and $\widetilde{\mu} = \{\gamma_\pi,\omega\} = \{\{\pi,\mu\},\omega\}$.

Definition (Vaisman). A 2-form satisfying $[\omega,\omega]_\gamma = 0$ where $\gamma = \gamma_\pi = \{\pi,\mu\}$ is called a complementary 2-form for π.

Proposition. A sufficient condition for $\widetilde{\mu} = \{\gamma_\pi,\omega\} = \{\{\pi,\mu\},\omega\}$ to be a Lie algebroid structure on A is

$$\begin{cases} [\pi,\pi]_\mu = 0 & (\pi \text{ is Poisson})\ , \\ [\omega,\omega]_\pi = 0 & (\omega \text{ is a complementary 2-form for } \pi)\ . \end{cases}$$

More on complementary 2-forms

Let π be a bivector, and ω a 2-form on (A, μ). Let

$$\boxed{N : TM \stackrel{\omega^\flat}{\to} T^*M \stackrel{\pi^\sharp}{\to} TM.}$$

Proposition.

• If π is a Poisson bivector and ω is a complementary 2-form for π, then $[X, Y]_{\widetilde{\mu}} = [X, Y]^{\mu}_{N} - \pi^\sharp(i_{X \wedge Y} d_\mu \omega)$, for all $X,\ Y \in \Gamma A$, defines a Lie bracket on the space of sections of A.

• If, in addition, ω is closed, the bracket $[\ ,\ ]^{\mu}_{N}$ is a Lie bracket.

Definitions

PN $(N \circ \pi^\sharp = \pi^\sharp \circ N^*)$

$$\boxed{\{\{\pi,\mu\},\pi\}=0, \{\{\pi,\mu\},N\}+\{\{N,\mu\},\pi\}=0, \{N,\{N,\mu\}\}-\{N^2,\mu\}=0}$$

$P\Omega$

$$\boxed{\{\{\pi,\mu\},\pi\} = 0, \quad \{\mu,\omega\} = 0, \quad \{\{\{\pi,\omega\},\mu\},\omega\} = 0}$$

ΩN $(\omega^\flat \circ N = N^* \circ \omega^\flat)$

$$\boxed{\{\mu,\omega\} = 0, \quad \{N,\{N,\mu\}\} - \{N^2,\mu\} = 0, \quad \{\mu,\{N,\omega\}\} = 0}$$

Hitchin pair $(\omega^\flat \circ N = N^* \circ \omega^\flat)$

$$\boxed{\{\mu,\omega\} = 0, \quad \{\mu,\{N,\omega\}\} = 0}$$

Complementary 2-form

$$\boxed{\{\{\pi,\mu\},\pi\} = 0, \quad \{\{\omega,\{\pi,\mu\}\},\omega\} = 0}$$

Relationships

$$\boxed{P\Omega \Longrightarrow PN \quad (N = \pi \circ \omega)}$$

$$\boxed{PN \quad and \quad \pi \quad non\!-\!degenerate \Longrightarrow P\Omega \quad (\omega = \pi^{-1} \circ N)}$$

$$\boxed{\Omega N \quad and \quad \omega \quad non\!-\!degenerate \Longrightarrow PN \quad (\pi = N \circ \omega^{-1})}$$

$$\boxed{PN \quad and \quad \pi \quad non\!-\!degenerate \Longrightarrow \Omega N \quad (\omega = \pi^{-1} \circ N)}$$

$$\boxed{P\Omega \Longrightarrow \Omega N \quad (N = \pi \circ \omega)}$$

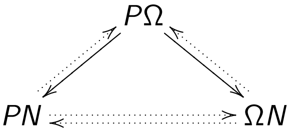
$$\boxed{\Omega N \quad and \quad \omega \quad non\!-\!degenerate \Longrightarrow P\Omega \quad (\pi = N \circ \omega^{-1})}$$

$$\boxed{Hitchin\ pair\ and\ N\ Nijenhuis \Longleftrightarrow \Omega N\ and\ \omega\ non\!-\!degenerate}$$

$$\boxed{\omega \quad closed \quad complementary \quad 2\!-\!form \quad for \quad \pi \Longleftrightarrow P\Omega}$$

Diagram of relationships

The implications can be summarized in a diagram.



The dotted arrows represent implications under a non-degeneracy assumption.

Poisson quasi-Nijenhuis structures

Poisson quasi-Nijenhuis structures in the sense of Mathieu Stiénon and Ping Xu (Comm. Math. Phys. 270 (2007), no. 3, 709–725) can be treated by the same method. More generally, Poisson quasi-Nijenhuis structures with background in the sense of Paulo Antunes (Lett. Math. Phys. 86 (2008), no. 1, 33–45) are defined by a quadruple (π, N, ψ, H), $\psi, H \in \Gamma(\wedge^3 A^*)$, satisfying

$\{\{\pi, \mu\}, \pi\} = 0,$

$\{\{\pi, \mu\}, N\} + \{\{N, \mu\}, \pi\} = \{\{H, \pi\}, \pi\},$

$\{N, \{N, \mu\}\} - \{N^2, \mu\} = -2\{\pi, \psi\} + \{\{\pi, H\}, N\} + \{\{N, H\}, \pi\},$

$\{\{N, \{N, H\}\} - \{N^2, H\}, \mu\} = 2\{\{N, \mu\}, \psi\}.$

PqN and Lie quasi-bialgebroids

PN implies (A^*_π, A_N) is a Lie bialgebroid.

PqN implies (A^*_π, A_N, ψ) is a Lie quasi-bialgebroid.

PqN with background implies $(A^*_\pi, A^H_N, \psi + i_N H)$ is a Lie quasi-bialgebroid, where $d^H_N(\alpha) = d_N\alpha - i_{\pi^\sharp\alpha}H$, for $\alpha \in \Gamma(A^*)$.

Let $S = \{\pi + N, \mu + H\} + \psi$. If (π, N, ψ, H) is a PqN structure with background on A, then $\{S, S\} = 0$, i.e., S is a Courant algebroid structure on $A + A^*$.

Example of PqN with background (Antunes).
Let π be a Poisson bivector, and ω a 2-form. Set $N = \pi^\sharp \circ \omega^\flat$ and $\omega_N = \frac{1}{2}\{N, \omega\}$. Then (π, N, ψ, H) with $\psi = d\omega_N$ and $H = -d\omega$ is a PqN structure with background.

Monge–Ampère structures on manifolds

A. Kushner, V. Lychagin, V. Rubtsov
Contact Geometry and Non-Linear Differential Equations,
CUP, 2007.

Let M be a smooth manifold of dimension n and let T^*M be its cotangent bundle.

Denote the canonical symplectic 2-form on T^*M by Ω.

A Monge–Ampère structure on M is defined by an n-form ω on T^*M such that $\boxed{\omega \wedge \Omega = 0.}$

Effective forms

Denote the canonical bivector on T^*M (the inverse of the canonical symplectic 2-form Ω) by π_Ω.
An effective form on T^*M of degree k, $2 \leq k \leq n$, is a k-form ω such that $i_{\pi_\Omega}\omega = 0$.
An n-form ω defines a Monge–Ampère structure on M iff it is effective.

Monge–Ampère equations

Define the Monge–Ampère equation associated with ω by

$$\Delta_\omega(f) = 0 \ , \qquad (1)$$

where

$$\Delta_\omega(f) = (df)^*(\omega) \ , \qquad (2)$$

for $f \in C^\infty(M)$.

Classical Monge–Ampère equations

Question: What is the relation between classical Monge–Ampère equation $rt - s^2 = F$ and Monge–Ampère structures on manifolds?

$$f, F : \mathbb{R}^2 \to \mathbb{R} \text{ and } r = \frac{\partial^2 f}{\partial x^2},\ t = \frac{\partial^2 f}{\partial y^2},\ s = \frac{\partial^2 f}{\partial x \partial y},$$

$$rt - s^2 = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2.$$

Answer: Let $M = \mathbb{R}^2$ with coordinates (x, y).
Let T^*M have coordinates (x, y, p, q).
Then $df : (x, y) \mapsto (x, y, p, q)$, where $p = \frac{\partial f}{\partial x}$, $q = \frac{\partial f}{\partial y}$.
The 2-form $\omega = dp \wedge dq$ is effective. In fact
$\Omega = dx \wedge dp + dy \wedge dq$, and therefore $\omega \wedge \Omega = 0$.
Compute $(df)^*\omega = (\frac{\partial^2 f}{\partial x^2} dx + \frac{\partial}{\partial y}\frac{\partial f}{\partial x} dy) \wedge (\frac{\partial}{\partial x}\frac{\partial f}{\partial y} dx + \frac{\partial^2 f}{\partial y^2} dy)$
$= (\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - (\frac{\partial^2 f}{\partial x \partial y})^2) dx \wedge dy$.

Conclusion: the Monge–Ampère equations on manifolds are a vast class of generalizations of the classical Monge–Ampère equation $rt - s^2 = 0$.

Monge–Ampère structures in dimension 2

- Define the Pfaffian $Pf(\omega)$ of the 2-form ω by

$$Pf(\omega)\Omega \wedge \Omega = \omega \wedge \omega \ ,$$

- Define the $(1,1)$-tensor N_ω on T^*M by

$$\omega(X,Y) = \Omega(N_\omega X, Y) \ ,$$

$X, Y \in \mathcal{X}(T^*M)$.

- Then

$$N_\omega^2 + Pf(\omega)\mathrm{Id} = 0.$$

A Monge–Ampère structure $T^*(M)$ defined by a 2-form ω is called non-degenerate if $Pf(\omega)$ is nowhere-vanishing.

Non-degenerate Monge–Ampère structures in dimension 2

If the Monge–Ampère structure defined by ω on $T^*(M)$ is non-degenerate, consider

- the normalized 2-form $\widetilde{\omega}$ defined by

$$\widetilde{\omega} = \frac{\omega}{\sqrt{|Pf(\omega)|}},$$

- the normalized $(1,1)$-tensor $\widetilde{N}_\omega$ defined by

$$\widetilde{N}_\omega = \frac{N_\omega}{\sqrt{|Pf(\omega)|}} \ .$$

Then $\widetilde{N}_\omega$ has square $-\mathrm{Id}$ (elliptic case) or Id (hyperbolic case), an almost complex or an almost product structure on T^*M.

When is the normalized $(1,1)$-tensor integrable?

Proposition. The following properties are equivalent:

- $\widetilde{N}_\omega$ is integrable (the Nijenhuis torsion of $\widetilde{N}_\omega$ vanishes),
- $\widetilde{\omega}$ is closed,
- the differential operator Δ_ω is equivalent to an operator with constant coefficients.

(See Kushner–Lychagin–Rubtsov.)

Compatible structures associated to Monge–Ampère structures in dimension 2

If α is a non-degenerate 2-form, let π_α be its inverse bivector.

Assume ω defines a non-degenerate Monge–Ampère structure on M such that the 2-form $\widetilde{\omega}$ is closed. Then

- The pair $(\pi_\Omega, \widetilde{N}_\omega)$ is a PN-structure on T^*M.
- The pair $(\pi_{\widetilde{\omega}}, \widetilde{N}_\omega)$ is a PN-structure on T^*M.
- The pair $(\pi_\Omega, \widetilde{\omega})$ is a $P\Omega$-structure on T^*M.
- The pair $(\widetilde{\omega}, \widetilde{N}_\omega)$ is an ΩN-structure on T^*M.

Unimodularity

The modular class of a Lie algebroid is a class in the degree 1 Lie algebroid cohomology that generalizes the modular class of a Poisson manifold (the class of the divergence of the Poisson bivector).
(See Evens, Lu, Weinstein, *Quart. J. Math* 50, 1999.)

The modular class of a Lie algebroid structure obtained by deformation by a Nijenhuis tensor N is the class of the 1-form $d_\mu(TrN)$.

Proposition. Assume ω defines a non-degenerate Monge–Ampère structure on M such that the 2-form $\widetilde{\omega}$ is closed. Then, the Lie algebroid structure of $T(T^*M)$ obtained by deformation by $\widetilde{N}_\omega$ is unimodular.

Proof. Since ω is effective, $0 = i_{\pi_\Omega}\omega = 4\,TrN_\omega$.

More compatible structures

Assume ω defines a non-degenerate Monge–Ampère structure on M such that $d\omega = 0$. Then

- the pair (Ω, N_ω) is a Hitchin pair on T^*M,
- if, in addition, ω is equivalent to a 2-form with constant coefficients, the pair (Ω, N_ω) is an ΩN-structure on T^*M.

Let (A, μ, γ) be a Lie bialgebroid. Consider the Dorfman bracket on $A \oplus A^*$ defined by

$$[u, v]_D = \{\{u, \mu + \gamma\}, v\} \ ,$$

for u and $v \in \Gamma(A \oplus A^*)$.
The skew-symmetrized Dorfman bracket is called the Courant bracket.
In particular, if A is a tangent bundle, $A = TQ$, and if $\gamma = 0$, the Dorfman bracket on $TQ \oplus T^*Q$ is explicitly,

$$\boxed{[X + \alpha, Y + \beta]_D = [X, Y] + \mathcal{L}_X \beta - i_Y(d\alpha),}$$

for all vector fields, X and Y, and all 1-forms, α and β, on Q.
The original Courant bracket (1990) is recovered as the skew-symmetrized Dorfman bracket.

Generalized geometry

A generalized almost complex structure on Q is a vector bundle endomorphism J of $TQ \oplus T^*Q$ of square $-\mathrm{Id}$.

A generalized complex structure on Q is a generalized almost complex structure with vanishing Nijenhuis torsion (defined in terms of the Dorfman bracket).

Replace $-\mathrm{Id}$ by Id to define generalized almost product structures and generalized product structures.

Generalized complex structures and Monge–Ampère structures

Generalized complex structures and generalized product structures on T^*M appear in the case of Monge–Ampère structures "of divergence type" (such that $\omega + \varphi\Omega$ is closed for a funcion φ) on 2-dimensional manifolds, and in the case of Monge–Ampère structures on 3-dimensional manifolds.