De Lecomte–Roger à Monge–Ampère
From Lecomte–Roger to Monge–Ampère

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Algèbres de Lie de dimension infinie
Géométrie et cohomologie
Congrès international en l'honneur de Claude Roger
pour son 60^{e} anniversaire
Lyon, Novembre 2009

Pour Claude Roger :
Bon anniversaire! Happy birthday

Où il sera question d'algèbres de Lie (graduées de dimension infinie) et de l'usage des crochets

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A story of (infinite-dimensional graded) Lie algebras and brackets

LR to MA: the path from the LECOMTE–ROGER cohomological theory of Lie bialgebras to the properties of MONGE–AMPÈRE structures

- Prehistory of cohomological theory of Lie bialgebras (LECOMTE–ROGER). The big bracket.
- The big bracket for Lie algebroids.
- Quick review of compatible structures on Lie algebroids.
- Determine the compatibble structures associated with MONGE–AMPÈRE operators.

An even graded Poisson bracket

Back in 1987...

Bertram Kostant and Shlomo Sternberg,
Symplectic reduction, BRS cohomology, and infinite-dimensional Clifford algebras, *Ann. Physics* 176, 1987, 49–113.

Voir aussi Marc Henneaux, *Physics Rep.* 126, 1985.

Let E be a vector space and ϕ a symmetric bilinear form over E.
Define $\{u, v\}_\phi = \phi(u, v)$, for u, $v \in E$.
Extend $\{\ ,\ \}_\phi$ to $\wedge^\bullet E = \oplus_j \wedge^j E$ as a biderivation.

Then $\{\ ,\ \}_\phi$ is a Poisson bracket of degree -2 on $\wedge^\bullet E$.

Then came Pierre Lecomte and Claude Roger

Application of Kostant–Sternberg's result in

Pierre Lecomte, Claude Roger
Modules et cohomologies des bigèbres de Lie, *C. R. Acad. Sci. Paris Sér. I Math.* 310, 1990, 405–410.

Let V be a vector space. Let $E = V^* \oplus V$.
Let ϕ be the canonical symmetric bilinear form.
For $x, y \in V$, $\xi, \eta \in V^*$,

$$\boxed{\phi(x,y) = 0, \quad \phi(\xi,\eta) = 0, \quad \phi(\xi,x) = <\xi, x>,}$$

where $< \ , \ >$ is the duality between V and V^*.

Then $\{\ ,\ \}_\phi = \{\ ,\ \}$ is a Poisson bracket of bidegree $(-1,-1)$ on $\wedge^\bullet(V^* \oplus V) = \oplus_{k\geq 0, \ell \geq 0} \wedge^\ell V^* \otimes \wedge^k V$.

I called this bracket the big bracket because other brackets are expressible in terms of it.

Super!

$V^* \oplus V$ is the cotangent bundle T^*V of V.

Consider the supermanifold $\Pi V = V[1]$ and its cotangent bundle $T^*V[1]$. The algebra of functions on $T^*V[1]$ is $\wedge^\bullet(V^* \oplus V)$.

The Poisson bracket $\{\ ,\ \}$ is the Poisson bracket of the canonical symplectic structure of this cotangent bundle!

Remark: Towards Lie algebra theory

In restriction to $\wedge^\bullet V^* \otimes V$, the even Poisson bracket $\{\ ,\ \}$ is the Nijenhuis-Richardson bracket of vector-valued forms on V, and similarly for vector-valued forms on V^* (up to a sign).

Example 1:
A Lie algebra structure on V is an element $\mu \in \wedge^2 V^* \otimes V$ such that

$$\{\mu, \mu\} = 0.$$

Example 2:
A Lie coalgebra structure on V is an element $\gamma \in V^* \otimes \wedge^2 V$ such that

$$\{\gamma, \gamma\} = 0.$$

The cohomological approach to Lie bialgebras
LECOMTE and ROGER, 1990

A Lie bialgebra structure on (V, V^*) is an element $\mu + \gamma \in (\wedge^2 V^* \otimes V) \oplus (V^* \otimes \wedge^2 V)$ such that $\boxed{\{\mu+\gamma, \mu+\gamma\} = 0.}$

$$\begin{cases} \{\mu,\mu\} = 0\ , \\ \{\mu,\gamma\} = 0\ , \\ \{\gamma,\gamma\} = 0\ . \end{cases}$$

The compatibility conditon,

γ is a cocycle of (V, μ) or μ is a cocycle of (V^*, γ),

is simply expressed as

$$\{\mu,\gamma\} = 0.$$

The Lie algebra cohomology operators of (V, μ) and (V^*, γ) are $d_\mu = \{\mu, .\}$ and $d_\gamma = \{\gamma, .\}$: bicomplex of the Lie bialgebra.

Two ways to generalize

- Algebraic:

Quasi-Lie-bialgebras

See DRINFELD, Quasi-Hopf algebras, *Leningrad Math. J.* 1, 1990.

Proto-Lie-bialgebras

See YKS, *Contemp. Math.* 132, 1992.

- Geometric:

Lie bialgebroids

See MACKENZIE and XU, *Duke* 73, 1994: infinitesimal of a Poisson groupoid.

See YKS, *Acta. Appl. Math.* 41, 1995: differential Gerstenhaber algebra.

- Geometric version of quasi-Lie-bialgebras, proto-Lie- bialgebras:

Quasi-Lie-bialgebroids, proto-Lie- bialgebroids

See DMITRY ROYTENBERG, *LMP* 61, 2002.

See YKS, Prog. Math. in honor of Gerstenhaber and Stasheff, 2010... arXiv 0711.2043.

Claude Roger
Prog. Math. 99 (in honor of Souriau), 1991

[p. 411, middle of the page, I modified the notation]

(V, μ) is a Lie algebra, μ is the Lie algebra structure. Then $\wedge^\bullet V$ is equipped with the Schouten bracket, $[\ ,\ ]^\mu$.
Let X, $Y \in \wedge^\bullet V$. Then

$$\boxed{[X, Y]^\mu = \{\{X, \mu\}, Y\}.}$$

Cf. also YKS, *Contemp. Math.* 132, 1992, formula (2.15), p. 473.

This result shows that the Schouten bracket is a derived bracket.
(YKS, *Ann. Fourier* 46, 1996 and LMP 69, 2004)
(Ted Voronov, *Contemp. Math.* 315, 2002)

Lie algebroids

$A \to M$ vector bundle,
• Lie algebra bracket, $[\ ,\ ]$, on ΓA,
• anchor, $\rho : A \to TM$, vector bundle morphism,
• Leibniz rule, $[X, fY] = f[X, Y] + (\rho(X) \cdot f)Y$,
$X,\ Y \in \Gamma A$, $f \in C^\infty(M)$.

Then ρ induces a Lie algebra homomorphism from ΓA to $\Gamma(TM)$.

Examples

- Lie algebra,
- TM for M any manifold,
- T^*M when M is a Poisson manifold,
- action Lie algebroids,
- etc.

The big bracket for vector bundles
ROYTENBERG, 2002

When $V \to M$ is a vector bundle,
let $V[1]$ be the graded manifold obtained from V by assigning
- degree 0 to the coordinates on the base,
- degree 1 to the coordinates on the fibers.

Let $\mathcal{F}$ be the bigraded commutative algebra of smooth functions on $T^*V[1]$.

Remark: When M is a point, V is just a vector space.
Then $\mathcal{F} = \wedge^\bullet(V^* \oplus V)$. See above!

Local coordinates on $T^*V[1]$, and their bidegrees:

x^i	ξ^a	p_i	θ_a
$(0,0)$	$(0,1)$	$(1,1)$	$(1,0)$

The bigraded Poisson algebra $\mathcal{F}$

As the cotangent bundle of a graded manifold, $T^*V[1]$ is canonically equipped with an even Poisson structure.
Denote the even Poisson bracket on $\mathcal{F}$ by $\{\ ,\ \}$.
We call it the big bracket because it generalizes the big bracket on $\wedge^\bullet(V^* \oplus V)$.

- It is of bidegree $(-1, -1)$.
- It is skew-symmetric, $\{u, v\} = -(-1)^{|u|\,|v|}\{v, u\}$, u, $v \in \mathcal{F}$,
- It satisfies the Jacobi identity,

$$\{u, \{v, w\}\} = \{\{u, v\}, w\} + (-1)^{|u|\,|v|}\{v, \{u, w\}\}\ ,$$

u, v, $w \in \mathcal{F}$.
In local coordinates,

$$\boxed{\{x^i, p_j\} = \delta^i_j \quad \text{and} \quad \{\xi^a, \theta_b\} = \delta^a_b\,.}$$

Consequence: $\{f, p_j\} = \frac{\partial f}{\partial x^j}$, for $f \in C^\infty(M)$.

A new look at Lie algebroids

A Lie algebroid structure on $A \to M$ is an element μ of $\mathcal{F}$ of bidegree $(1, 2)$ such that

$$\{\mu, \mu\} = 0 \ .$$

Schouten bracket of multivectors (sections of $\wedge^\bullet A$) X and Y:

$$[X, Y]_\mu = \{\{X, \mu\}, Y\}, \ X, Y \in \Gamma \wedge^\bullet A.$$

In particular,
• the Lie bracket of $X, \ Y \in \Gamma A$,

$$[X, Y] = [X, Y]_\mu = \{\{X, \mu\}, Y\} \ .$$

• the anchor of A, $\rho : A \to TM$,

$$\rho(X)f = [X, f]_\mu = \{\{X, \mu\}, f\} \ ,$$

for $X \in \Gamma A$, $f \in C^\infty(M)$.

The differential of a Lie algebroid

The operator $d_\mu = \{\mu, .\}$ is a differential on $\Gamma(\wedge^\bullet A^*)$ which defines the Lie algebroid cohomology of A.

Lie algebroid cohomology generalizes
• Chevalley–Eilenberg cohomology (when M is a point, A is a Lie algebra), and
• de Rham cohomology (when $A = TM$).

The Lichnerowicz–Poisson cohomology of a Poisson manifold is another example.

Lie bialgebroids

A Lie bialgebroid is defined by μ of bidegree $(1,2)$ and γ of bidegree $(2,1)$ such that $\{\mu+\gamma,\mu+\gamma\}=0$.

(A,μ,γ) is a Lie bialgebroid iff $(\Gamma(\wedge^\bullet A),[\ ,\ ]_\mu,d_\gamma)$ is a differential Gerstenhaber algebra.

A Lie-quasi bialgebroid is defined by μ of bidegree $(1,2)$, γ of bidegree $(2,1)$, and $\phi\in\Gamma(\wedge^3 A)$ of bidegree $(3,0)$ such that $\{\phi+\mu+\gamma,\phi+\mu+\gamma\}=0$.

A quasi-Lie bialgebroid is defined by μ of bidegree $(1,2)$, γ of bidegree $(2,1)$, and $\psi\in\Gamma(\wedge^3 A^*)$ of bidegree $(0,3)$ such that $\{\mu+\gamma+\psi,\mu+\gamma\ +\psi\}=0$.

A proto-Lie bialgebroid is defined by μ of bidegree $(1,2)$, γ of bidegree $(2,1)$, $\phi\in\Gamma(\wedge^3 A)$ of bidegree $(3,0)$, and $\psi\in\Gamma(\wedge^3 A^*)$ of bidegree $(0,3)$ such that $\{\phi+\mu+\gamma+\psi,\phi+\mu+\gamma+\psi\}=0$.

Examples: twist of a Lie bialgebra, twisted Poisson structures (with a 3-form background).

Compatible structures on Lie algebroids

Cf. YKS and VLADIMIR RUBTSOV,
Compatible structures on Lie algebroids and Monge–Ampère operators,
Acta Appl. Math. (to appear), arXiv:0812.4838

PN, $P\Omega$, ΩN, Hitchin pairs, complementary 2-forms

MAGRI–MOROSI 1984, YKS–MAGRI 1990, VAISMAN 1996, CRAINIC 2004.

Nijenhuis structures

Let (A, μ) be a Lie algebroid (μ of bidegree $(1, 2)$ denotes the Lie algebroid structure of the vector bundle A).
Let $N \in \Gamma(A^* \otimes A)$ be a $(1, 1)$-tensor on A, an element of bidegree $(1, 1)$. Then the deformed structure,

$$\mu_N = \{N, \mu\} \ ,$$

defines an anchor, $\rho \circ N$, and a skew-symmetric bracket on A.
Denote it by $[\ ,\ ]^\mu_N$, Explicitly,

$$[X, Y]^\mu_N = \{\{X, \{N, \mu\}\}, Y\} \ ,$$

for X and $Y \in \Gamma A$.
Proposition. For X, $Y \in \Gamma A$,

$$[X, Y]^\mu_N = [NX, Y]_\mu + [X, NY]_\mu - N[X, Y]_\mu \ .$$

Grabowski's formula

The Nijenhuis torsion of N is defined by

$$(\mathcal{T}_\mu N)(X,Y) = [NX,NY]_\mu - N([NX,Y]_\mu + [X,NY]_\mu) + N^2[X,Y]_\mu \ ,$$

for all X and $Y \in \Gamma A$.

It is clear that $(\mathcal{T}_\mu N)(X,Y) = [NX,NY]_\mu - N([X,Y]^\mu_N)$.

Proposition. In terms of the big bracket,

$$\boxed{\mathcal{T}_\mu N = \frac{1}{2}\left(\{N,\{N,\mu\}\} - \{N^2,\mu\}\right) \ .}$$

Grabowski 2006, Antunes 2008, yks–Rubtsov and also yks 1996.

Nijenhuis structures, complex structures

Corollary 1.
• A necessary and sufficient condition for the deformed structure $\mu_N = \{N, \mu\}$ to be a Lie algebroid structure on A is $\{\mu, \mathcal{T}_\mu N\} = 0$.
• A sufficient condition for the deformed structure $\mu_N = \{N, \mu\}$ to be a Lie algebroid structure on A is $\boxed{\mathcal{T}_\mu N = 0.}$

Proof. In fact, $\frac{1}{2}\{\{N, \mu\}, \{N, \mu\}\} = \{\mu, \mathcal{T}_\mu N\}$.

Corollary 2. An almost complex structure N on A is a complex structure if and only if

$$\{\{N, \mu\}, N\} = \mu \ .$$

Complementary 2-forms for Poisson structures

Let (A, μ) be a Lie algebroid. If $\pi \in \Gamma(\wedge^2 A)$, then

$$\gamma_\pi = \{\pi, \mu\}$$

is of bidegree $(2, 1)$.
γ_π is a Lie algebroid structure on A^* if and only if

$$\boxed{\{\mu, [\pi, \pi]_\mu\} = 0.}$$

We now dualize this construction.
Let (A^*, γ) be a Lie algebroid. If $\omega \in \Gamma(\wedge^2 A^*)$, then

$$\widetilde{\mu} = \{\gamma, \omega\}$$

is of bidegree $(1, 2)$.
$\widetilde{\mu}$ is a Lie algebroid structure on A if and only if

$$\boxed{\{[\omega, \omega]_\gamma, \gamma\} = 0.}$$

Dualization and composition

Combine the two preceding constructions:

$$\boxed{(A,\mu) \overset{(\pi)}{\leadsto} (A^*,\gamma_\pi) \overset{(\omega)}{\leadsto} (A,\widetilde{\mu})\ ,}$$

with $\gamma_\pi = \{\pi,\mu\}$, and $\widetilde{\mu} = \{\gamma_\pi,\omega\} = \{\{\pi,\mu\},\omega\}$.

Definition (Vaisman). A 2-form satisfying $[\omega,\omega]_\gamma = 0$ where $\gamma = \gamma_\pi = \{\pi,\mu\}$ is called a complementary 2-form for π.

Proposition. A sufficient condition for $\widetilde{\mu} = \{\gamma_\pi,\omega\} = \{\{\pi,\mu\},\omega\}$ to be a Lie algebroid structure on A is

$$\begin{cases} [\pi,\pi]_\mu = 0 & (\pi \text{ is Poisson})\ , \\ [\omega,\omega]_\pi = 0 & (\omega \text{ is a complementary 2-form for } \pi)\ . \end{cases}$$

More on complementary 2-forms

Let π be a bivector, and ω a 2-form on (A, μ). Let

$$\boxed{N : TM \stackrel{\omega^\flat}{\to} T^*M \stackrel{\pi^\sharp}{\to} TM.}$$

Proposition.

• If π is a Poisson bivector and ω is a complementary 2-form for π, then $[X, Y]_{\widetilde{\mu}} = [X, Y]_N^\mu - \pi^\sharp(i_{X\wedge Y} d_\mu \omega)$, for all $X,\ Y \in \Gamma A$, defines a Lie bracket on the space of sections of A.

• If, in addition, ω is closed, the bracket $[\ ,\ ]_N^\mu$ is a Lie bracket.

Definitions

PN $(N \circ \pi^\sharp = \pi^\sharp \circ N^*)$

$$\boxed{\{\{\pi,\mu\},\pi\}=0, \{\{\pi,\mu\},N\}+\{\{N,\mu\},\pi\}=0, \{N,\{N,\mu\}\}-\{N^2,\mu\}=0}$$

$P\Omega$

$$\boxed{\{\{\pi,\mu\},\pi\} = 0, \quad \{\mu,\omega\} = 0, \quad \{\{\{\pi,\omega\},\mu\},\omega\} = 0}$$

ΩN $(\omega^\flat \circ N = N^* \circ \omega^\flat)$

$$\boxed{\{\mu,\omega\} = 0, \quad \{N,\{N,\mu\}\} - \{N^2,\mu\} = 0, \quad \{\mu,\{N,\omega\}\} = 0}$$

Hitchin pair $(\omega^\flat \circ N = N^* \circ \omega^\flat)$

$$\boxed{\{\mu,\omega\} = 0, \quad \{\mu,\{N,\omega\}\} = 0}$$

Complementary 2-form

$$\boxed{\{\{\pi,\mu\},\pi\} = 0, \quad \{\{\omega,\{\pi,\mu\}\},\omega\} = 0}$$

Relationships

$$\boxed{P\Omega \Longrightarrow PN \quad (N = \pi \circ \omega)}$$

$$\boxed{PN \quad and \quad \pi \quad non\!-\!degenerate \Longrightarrow P\Omega \quad (\omega = \pi^{-1} \circ N)}$$

$$\boxed{\Omega N \quad and \quad \omega \quad non\!-\!degenerate \Longrightarrow PN \quad (\pi = N \circ \omega^{-1})}$$

$$\boxed{PN \quad and \quad \pi \quad non\!-\!degenerate \Longrightarrow \Omega N \quad (\omega = \pi^{-1} \circ N)}$$

$$\boxed{P\Omega \Longrightarrow \Omega N \quad (N = \pi \circ \omega)}$$

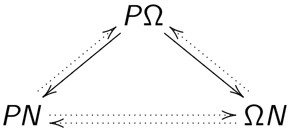
$$\boxed{\Omega N \quad and \quad \omega \quad non\!-\!degenerate \Longrightarrow P\Omega \quad (\pi = N \circ \omega^{-1})}$$

$$\boxed{Hitchin\ pair\ and\ N\ Nijenhuis \Longleftrightarrow \Omega N\ and\ \omega\ non\!-\!degenerate}$$

$$\boxed{\omega \quad closed \quad complementary \quad 2\!-\!form \quad for \quad \pi \Longleftrightarrow P\Omega}$$

Diagram of relationships

The implications can be summarized in a diagram.



The dotted arrows represent implications under a non-degeneracy assumption.

Monge–Ampère structures on manifolds

A. KUSHNER, V. LYCHAGIN, V. RUBTSOV
Contact Geometry and Non-Linear Differential Equations,
CUP, 2007.

Let M be a smooth manifold of dimension n and let T^*M be its cotangent bundle.

Denote the canonical symplectic 2-form on T^*M by Ω.

A Monge–Ampère structure on M is defined by an n-form ω on T^*M such that $\boxed{\omega \wedge \Omega = 0.}$

Effective forms

Denote the canonical bivector on T^*M (the inverse of the canonical symplectic 2-form Ω) by π_Ω.
An effective form on T^*M of degree k, $2 \leq k \leq n$, is a k-form ω such that $i_{\pi_\Omega}\omega = 0$.
An n-form ω defines a Monge–Ampère structure on M iff it is effective.

Monge–Ampère equations

Define the Monge–Ampère equation associated with ω by

$$\Delta_\omega(f) = 0 \ ,$$

where

$$\Delta_\omega(f) = (df)^*(\omega) \ ,$$

for $f \in C^\infty(M)$.

Classical Monge–Ampère equations

Question: What is the relation between classical Monge–Ampère equation $rt - s^2 = F$ and Monge–Ampère structures on manifolds?

$$f, F : \mathbb{R}^2 \to \mathbb{R} \text{ and } r = \frac{\partial^2 f}{\partial x^2},\ t = \frac{\partial^2 f}{\partial y^2},\ s = \frac{\partial^2 f}{\partial x \partial y},$$

$$rt - s^2 = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2.$$

Answer: Let $M = \mathbb{R}^2$ with coordinates (x, y).
Let T^*M have coordinates (x, y, p, q).
Then $df : (x, y) \mapsto (x, y, p, q)$, where $p = \frac{\partial f}{\partial x}$, $q = \frac{\partial f}{\partial y}$.
The 2-form $\omega = dp \wedge dq$ is effective. In fact
$\Omega = dx \wedge dp + dy \wedge dq$, and therefore $\omega \wedge \Omega = 0$.
Compute $(df)^*\omega = (\frac{\partial^2 f}{\partial x^2} dx + \frac{\partial}{\partial y}\frac{\partial f}{\partial x} dy) \wedge (\frac{\partial}{\partial x}\frac{\partial f}{\partial y} dx + \frac{\partial^2 f}{\partial y^2} dy)$
$= (\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - (\frac{\partial^2 f}{\partial x \partial y})^2) dx \wedge dy.$

Conclusion: the Monge–Ampère equations on manifolds are a vast class of generalizations of the classical Monge–Ampère equation $rt - s^2 = 0$.

Monge–Ampère structures in dimension 2

- Define the Pfaffian $Pf(\omega)$ of the 2-form ω by

$$Pf(\omega)\Omega \wedge \Omega = \omega \wedge \omega \ ,$$

- Define the $(1,1)$-tensor N_ω on T^*M by

$$\omega(X,Y) = \Omega(N_\omega X, Y) \ ,$$

$X, Y \in \mathcal{X}(T^*M)$.

- Then

$$N_\omega^2 + Pf(\omega)\mathrm{Id} = 0.$$

A Monge–Ampère structure $T^*(M)$ defined by a 2-form ω is called non-degenerate if $Pf(\omega)$ is nowhere-vanishing.

Non-degenerate Monge–Ampère structures in dimension 2

If the Monge–Ampère structure defined by ω on $T^*(M)$ is non-degenerate, consider

- the normalized 2-form $\widetilde{\omega}$ defined by

$$\widetilde{\omega} = \frac{\omega}{\sqrt{|Pf(\omega)|}},$$

- the normalized $(1,1)$-tensor $\widetilde{N}_\omega$ defined by

$$\widetilde{N}_\omega = \frac{N_\omega}{\sqrt{|Pf(\omega)|}} .$$

Then $\widetilde{N}_\omega$ has square $-\mathrm{Id}$ (elliptic case) or Id (hyperbolic case), an almost complex or an almost product structure on T^*M.

When is the normalized $(1,1)$-tensor integrable?

Proposition. The following properties are equivalent:

• $\widetilde{N}_\omega$ is integrable (the Nijenhuis torsion of $\widetilde{N}_\omega$ vanishes),

• $\widetilde{\omega}$ is closed,

• the differential operator Δ_ω is equivalent to an operator with constant coefficients.

(See Kushner–Lychagin–Rubtsov.)

Compatible structures associated to Monge–Ampère structures in dimension 2

If α is a non-degenerate 2-form, let π_α be its inverse bivector.

Assume ω defines a non-degenerate Monge–Ampère structure on M such that the 2-form $\widetilde{\omega}$ is closed. Then

- The pair $(\pi_\Omega, \widetilde{N}_\omega)$ is a PN-structure on T^*M.
- The pair $(\pi_{\widetilde{\omega}}, \widetilde{N}_\omega)$ is a PN-structure on T^*M.
- The pair $(\pi_\Omega, \widetilde{\omega})$ is a $P\Omega$-structure on T^*M.
- The pair $(\widetilde{\omega}, \widetilde{N}_\omega)$ is an ΩN-structure on T^*M.

Unimodularity

The modular class of a Lie algebroid is a class in the degree 1 Lie algebroid cohomology that generalizes the modular class of a Poisson manifold (the class of the divergence of the Poisson bivector).
(See Evens, Lu, Weinstein, *Quart. J. Math* 50, 1999.)

The modular class of a Lie algebroid structure obtained by deformation by a Nijenhuis tensor N is the class of the 1-form $d_\mu(TrN)$.

Proposition. Assume ω defines a non-degenerate Monge–Ampère structure on M such that the 2-form $\widetilde{\omega}$ is closed. Then, the Lie algebroid structure of $T(T^*M)$ obtained by deformation by $\widetilde{N}_\omega$ is unimodular.

Proof. Since ω is effective, $0 = i_{\pi_\Omega}\omega = 4\,TrN_\omega$.

More compatible structures

Assume ω defines a non-degenerate Monge–Ampère structure on M such that $d\omega = 0$. Then

- the pair (Ω, N_ω) is a Hitchin pair on T^*M,
- if, in addition, ω is equivalent to a 2-form with constant coefficients, the pair (Ω, N_ω) is an ΩN-structure on T^*M.

Another story: generalized almost complex structures

Let (A, μ, γ) be a Lie bialgebroid. Consider the Dorfman bracket on $A \oplus A^*$ defined by

$$[u, v]_D = \{\{u, \mu + \gamma\}, v\} ,$$

for u and $v \in \Gamma(A \oplus A^*)$.
The skew-symmetrized Dorfman bracket is called the Courant bracket.
In particular, if A is a tangent bundle, $A = TQ$, and if $\gamma = 0$, the Dorfman bracket on $TQ \oplus T^*Q$ is explicitly,

$$\boxed{[X + \alpha, Y + \beta]_D = [X, Y] + \mathcal{L}_X \beta - i_Y(d\alpha),}$$

for all vector fields, X and Y, and all 1-forms, α and β, on Q.
The original Courant bracket (1990) is recovered as the skew-symmetrized Dorfman bracket.

“Generalized geometry”

A generalized almost complex structure on Q is a vector bundle endomorphism J of $TQ \oplus T^*Q$ of square $-\mathrm{Id}$.

A generalized complex structure on Q is a generalized almost complex structure with vanishing Nijenhuis torsion (defined in terms of the Dorfman bracket).

Replace $-\mathrm{Id}$ by Id to define generalized almost product structures and generalized product structures.

Generalized complex structures and Monge–Ampère structures

Generalized complex structures and generalized product structures on T^*M appear in the case of Monge–Ampère structures "of divergence type" (such that $\omega + \varphi\Omega$ is closed for a function φ) on 2-dimensional manifolds, and in the case of Monge–Ampère structures on 3-dimensional manifolds.
(See YKS and RUBTSOV, op.cit.)

What else?

Other applications of the big bracket include the study of various generalized compatible pairs.
See Paulo Antunes
Poisson quasi-Nijenhuis structures with background,
LMP 86, 2008 (also see his forthcoming thesis).

See Ugo Bruzzo and Volodya Rubtsov
On the compatiblity of Lie algebroid structures, in progress
(another double complex).

To conclude

It all started in 1990 with the cohomological interpretation of Lie bialgebras. Then came Lie bialgebroids. Now the fashion is Courant algebroids and generalized geometry...

TO BE CONTINUED – À SUIVRE

(pour l'anniversaire suivant)