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Représentations des algèbres affinisées quantiques : q, t-caractères et produit de fusion

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Résumé

Dans cette thèse nous proposons plusieurs contributions à l'étude des groupes quantiques et de leurs représentations.

Dans le cadre de l'étude des représentations de dimension finie des algèbres affines quantiques, nous proposons une nouvelle construction algébrique générale des q, t-caractères (t-déformations des q-caractères de Frenkel-Reshetikhin), indépendante de la construction géométrique de Nakajima (cette dernière n'est valable que pour le cas ADE). Cela nous permet d'étendre la quantification de l'anneau de Grothendieck et la définition des analogues des polynômes de Kazhdan-Lusztig aux cas non simplement lacés.

Par ailleurs nous établissons une décomposition triangulaire des affinisées quantiques générales (incluant les algèbres affines et toroïdales quantiques) et classifions leurs représentations intégrables de plus haut poids. Nous proposons une nouvelle construction d'un produit de fusion en définissant une déformation du “nouveau coproduit de Drinfel'd”.

Mots clés : Algèbre non-commutative, groupes quantiques, théorie des représentations, algèbres affines quantiques, algèbres toroïdales quantiques, q, t-caractères, polynômes de Kazhdan-Lusztig, produit de fusion.

Abstract

In this thesis we propose several new developments in the study of quantum groups and their representations.

In the framework of the study of finite dimensional representations of quantum affine algebras, we give a new and more general algebraic construction of q, t-characters (t-deformations of Frenkel-Reshetikhin's q-characters), independent from the geometric construction of Nakajima (which holds only in the ADE-case). It allows us to extend the quantization of the Grothendieck ring and the definition of analogs of Kazhdan-Lusztig polynomials to non simply laced cases.

Besides we establish a triangular decomposition of general quantum affinizations (including quantum affine and toroidal algebras) and classify their integrable highest weight representations. We propose a new construction of a fusion product by defining a deformation of the “new Drinfel'd coproduct”.

Key words : Non-commutative algebra, quantum groups, representation theory, quantum affine algebras, quantum toroidal algebras, q, t-characters, Kazhdan-Lusztig polynomials, fusion product.

Introduction

Dans cette thèse nous proposons plusieurs contributions à l'étude des groupes quantiques et de leurs représentations. D'une part dans le cadre de l'étude des représentations de dimension finie des algèbres affines quantiques, nous donnons une nouvelle construction algébrique générale des q, t-caractères (t-déformations des q-caractères de Frenkel et Reshetikhin), indépendante de la construction géométrique de Nakajima (qui n'est valable que pour le cas ADE). D'autre part nous établissons une décomposition triangulaire des affinisées quantiques générales, étudions leurs représentations intégrables et proposons pour ces algèbres une nouvelle construction d'un produit de fusion.

Le texte est constitué de cinq parties :

1) t-analogues des opérateurs d'écrantage associés aux q-caractères (article [He1], publié dans *International Mathematics Research Notices*)

2) Algebraic approach to q, t-characters (article [He2], publié dans *Advances in Mathematics*)

3) The t-analogs of q-characters at roots of unity for quantum affine algebras and beyond (article [He3], publié dans *Journal of Algebra*)

4) Representations of quantum affinizations and fusion product (article [He4], à paraître dans *Transformation Groups*)

5) Monomials of q and q, t-characters for non-simply laced quantum affinizations (article [He5], à paraître dans *Mathematische Zeitschrift*)

L'approche algébrique des q, t-caractères est traitée dans les parties 1, 2, 3 et 5, et l'étude des affinisées quantiques, de leurs représentations et du nouveau produit de fusion est traitée dans les parties 3, 4 et 5.

Les notations différant parfois légèrement d'une partie à l'autre, nous proposons à la fin de la thèse un index des notations. De plus, au début de chaque partie, les résultats nécessaires des parties précédentes sont explicités dans le cadre approprié.

Dans le résumé suivant, après quelques rappels sur les algèbres affines quantiques, nous présentons les résultats de cette thèse plus précisément.

Rappels sur les algèbres affines quantiques et leurs représentations

La notion de quantification prend sa source en physique ; elle correspond au passage continu d'un espace d'observables classique à un espace d'observables quantique, moins commutatif. Le procédé est paramétré par un nombre complexe $q = e^h$ (h "constante" de Planck).

Les algèbres de Kac-Moody (généralisations des algèbres semi-simples complexes) jouent un rôle majeur en mathématiques et en physique (voir [Kac]). Il y a moins de vingt ans, Drinfel'd et Jimbo ont découvert une quantification des algèbres de Kac-Moody en associant à toute algèbre de Kac-Moody symétrisable $\mathfrak{g}$ et tout nombre complexe non nul $q \in \mathbb{C}^*$ une algèbre de Hopf $\mathcal{U}_q(\mathfrak{g})$ (algèbre de Kac-Moody quantique). Ces algèbres, aussi appelées groupes quantiques, sont depuis lors au centre de nombreux développements en mathématiques et en physique (voir par exemple les ouvrages [CP4, ES, Gu, Ja, Jo, Kas, KRT, L]).

Supposons que $q \in \mathbb{C}^*$ n'est pas une racine de l'unité. Pour $\mathcal{U}_q(\mathfrak{g})$ une algèbre quantique de type fini (avec $\mathfrak{g}$ de type fini de rang n ; voir par exemple la définition p 130) la structure de l'anneau de Grothendieck $\mathrm{Rep}(\mathcal{U}_q(\mathfrak{g}))$ des représentations de dimension finie est tout à fait analogue à celle du cas classique $q = 1$. On a en particulier des isomorphismes d'anneaux :

$$\mathrm{Rep}(\mathcal{U}_q(\mathfrak{g})) \simeq \mathrm{Rep}(\mathfrak{g}) \simeq \mathbb{Z}[\Lambda]^W \simeq \mathbb{Z}[T_1, ..., T_n]$$

construits à partir d'un morphisme χ tel que pour une représentation V de sous-espaces de poids V_λ :

$$\chi(V) = \sum_{\lambda \in \Lambda} \dim(V_\lambda)\lambda$$

où Λ désigne l'ensemble des poids de V.

Pour une algèbre affine quantique $\mathcal{U}_q(\hat{\mathfrak{g}})$ (avec $\hat{\mathfrak{g}}$ de type affine de rang n ; voir par exemple la définition p 131) la situation est différente car χ n'est pas injectif (dans ce qui suit les algèbres affines quantiques sont supposées non tordues). Les représentations de dimension finie ont été classifiées par Chari et Pressley en s'appuyant sur une propriété importante de $\mathcal{U}_q(\hat{\mathfrak{g}})$; en effet on en connaît deux réalisations, celle de Drinfel'd-Jimbo et une nouvelle comme affinisée d'une algèbre quantique de type fini (nouvelle réalisation de Drinfel'd [Dr2]). Frenkel et Reshetikhin [FR3] ont introduit un morphisme de q-caractères adapté à ces représentations. Cette application décrit la décomposition en sous-espaces de Jordan (sous espaces de l-poids) pour une certaine famille commutante $(\phi_{i,m}^{\pm})_{m \geq 0, i \in I}$ d'éléments de $\mathcal{U}_q(\hat{\mathfrak{g}})$ correspondant à une sous-algèbre de Cartan : il s'agit d'un morphisme d'anneau χ_q injectif ($I = \{1, ..., n\}$) :

$$\chi_q : \mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}})) \to \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^*} = \mathcal{Y}$$

$$\chi_q(V) = \sum_{\gamma} \dim(V_{(\gamma)}) m_\gamma$$

où $V_{(\gamma)}$ désigne le sous-espace de l-poids γ et $m_\gamma \in \mathcal{Y}$ est un monôme associé à γ. Lorsqu'on regarde la limite classique $q = 1$ on retrouve l'application de caractères usuelle. L'application χ_q permet d'établir $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}})) \simeq \mathbb{Z}[X_{i,a}]_{i \in I, a \in \mathbb{C}^*}$ (où $X_{i,a} \in \mathcal{Y}$ est le q-caractère d'une représentation fondamentale) et de montrer en particulier que $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ est commutatif.

Le morphisme de q-caractères vérifie une propriété de symétrie analogue au cas classique de l'action du groupe de Weyl : $\mathrm{Im}(\chi) = \mathbb{Z}[\Lambda]^W$. En effet Frenkel et Reshetikhin ont défini n opérateurs d'écrantage ($i \in I$) :

$$S_i : \mathcal{Y} \to \bigoplus_{a \in \mathbb{C}^*} \mathcal{Y}.S_{i,a} / \sum_{a \in \mathbb{C}^*} \mathcal{Y}.(S_{i,aq_i^2} - A_{i,aq_i}.S_{i,a})$$

où pour $i \in I$ et $a \in \mathbb{C}^*$, $A_{i,a} \in \mathcal{Y}$ est un certain monôme. S_i est une dérivation vérifiant $S_i(Y_a) = Y_a.S_a$. On a la propriété de symétrie [FR3, FM1] :

$$\mathrm{Im}(\chi_q) = \bigcap_{i \in I} \mathrm{Ker}(S_i)$$

Approche algébrique de la théorie des q,t-caractères

Dans le cas où $\mathfrak{g}$ est de type ADE, Nakajima a raffiné la théorie en introduisant des t-analogues des q-caractères [N2, N3] appelés q,t-caractères. Le morphisme de q,t-caractères est une application $\mathbb{Z}[t^{\pm}]$-linéaire :

$$\chi_{q,t} : \mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}})) \otimes_{\mathbb{Z}} \mathbb{Z}[t^{\pm}] \to \mathcal{Y}_t = \mathbb{Z}[Y_{i,a}^{\pm}, t^{\pm}]_{i \in I, a \in \mathbb{C}^*}$$

qui devient χ_q à $t = 1$. Du point de vue des représentations, elle permet de mieux comprendre la structure de chaque sous espace de l-poids. La définition de $\chi_{q,t}$ est donnée sous forme de propriétés combinatoires que nous appellerons propriétés axiomatiques des q,t-caractères. Elles font intervenir une multiplication t-déformée $*_N$ sur $\mathcal{Y}_t$, c'est à dire telle que pour des monôme $m_1, m_2 \in \mathcal{Y}_t$ on a $m_1 *_N m_2 = t^{2d(m_1,m_2)} m_1 m_2$ avec $d(m_1, m_2) \in \mathbb{Z}$ (voir aussi [VV3]). L'existence de $\chi_{q,t}$ vérifiant les propriétés axiomatiques est non-triviale et est prouvée par Nakajima grâce à la théorie géométrique des variétés carquois qui n'existe que dans le cas ADE.

Nous proposons une nouvelle construction algébrique des q,t-caractères indépendante de l'approche géométrique. L'existence d'une telle construction répond positivement à une conjecture de Nakajima [N3] et est de plus valable pour toutes les algèbres affines quantiques. Elle est également étendue aux cas où le paramètre de quantification q est une racine de l'unité, et pour une grande classe de matrices de Cartan non finies.

Dans la partie 1 ([He1]) nous construisons des t-analogues des opérateurs d'écrantage $S_{i,t}$ tels que $\chi_{q,t}$ a une propriété de symétrie relativement aux $S_{i,t}$ analogue à celle de χ_q. En effet :

Théorème 1. *([He1]) Pour $\mathfrak{g}$ de type ADE, on a :* $\bigcap_{i \in I} Ker(S_{i,t}) = Im(\chi_{q,t})$.

La construction des $S_{i,t}$ est purement algébrique et est aussi valable pour les cas non-simplement lacés.

Ensuite dans la partie 2 ([He2]) la multiplication $*_N$ est déduite de la structure d'une certaine sous-algèbre de Heisenberg $\mathcal{H}$ de $\mathcal{U}_q(\hat{\mathfrak{g}})$: $\mathcal{H}$ est générée par des $y_i[m] \in \mathcal{H}$ ($i \in I, m \in \mathbb{Z}$) et des éléments centraux $c_m \in \mathcal{H}$ ($m \in \mathbb{Z}$) tels que :

$$[y_i[m], y_j[r]] = \delta_{m,-r} \alpha_{i,j}(m) c_m$$

avec $\alpha_{i,j}(m) \in \mathbb{C}$. Considérons alors les séries (ou courants) :

$$\tilde{Y}_{i,a} = \exp(\sum_{m \geq 1} y_i[m] a^m z^m) \exp(\sum_{m \leq 1} y_i[m] a^m z^{-m}) \in \mathcal{H}[[z]]$$

Alors l'étude des relations de commutation entre les $\tilde{Y}_{i,a}$ donne une structure multiplicative $*_t$ sur $\mathcal{Y}_t$: $\mathcal{Y}_t$ est identifiée à un quotient d'une sous-algèbre de $\mathcal{H}[[z]]$, le paramètre t étant analogue à la charge centrale $c \in \mathcal{U}_q(\hat{\mathfrak{g}})$:

$$t = \exp(\sum_{m\geq 1} z^m \frac{c^m - c^{-m}}{m}) \in \mathcal{H}[[z]]$$

On obtient alors :

Théorème 2. *([He2]) Pour $\mathfrak{g}$ de type ADE les multiplications $*_N$ et $*_t$ coïncident.*

Les opérateurs $S_{i,t}$ sont également obtenus grâce à certains courants $\tilde{S}_i \in \mathcal{H}[[h]]$: on a, pour $\lambda \in \mathcal{Y}_t$,

$$S_{i,t}(\lambda) = \frac{1}{1-t^2}[\tilde{S}_i, \lambda]$$

Ainsi on peut étendre la définition des propriétés axiomatiques des q,t-caractères à tous les cas finis grâce à $*_t$ et $S_{i,t}$. Nous montrons l'existence d'une unique application vérifiant ces propriétés dans ce cadre général (ci-dessous, $\mathcal{Y}_t^\infty$ est une complétion de $\mathcal{Y}_t$ où certaines sommes infinies sont autorisées) :

Théorème 3. *([He2]) Pour toute algèbre affine quantique $\mathcal{U}_q(\hat{\mathfrak{g}})$, il existe une unique application $\mathbb{Z}[t^\pm]$-linéaire $\chi_{q,t} : Rep(\mathcal{U}_q(\hat{\mathfrak{g}})) \rightarrow \mathcal{Y}_t^\infty$ vérifiant les propriétés axiomatiques des q,t-caractères. On a de plus $Im(\chi_{q,t}) = \bigcap_{i\in I} Ker(S_{i,t})$.*

Dans le cas ADE, $\chi_{q,t}$ est l'application de Nakajima [N3]. Notre preuve repose sur la construction d'une t-déformation d'un algorithme que Frenkel et Mukhin [FM1] définirent pour construire les q-caractères. Une fois l'algorithme t-déformé explicité, le point technique est de montrer qu'il est bien défini (c'est à dire qu'à chaque étape les résultats sont indépendants des choix) et qu'il produit des q,t-caractères vérifiant les propriétés axiomatiques. Pour ce faire on montre qu'on peut se ramener aux algèbres de rang 2 pour lesquelles on peut faire une vérification "à la main".

Les q,t-caractères permettent dans le cadre général de construire une quantification de l'anneau de Grothendieck ainsi que des analogues des polynômes de Kazhdan-Lusztig, dans le même esprit que Nakajima le fit pour le cas ADE. Nous pouvons en particulier formuler une conjecture relative à la multiplicité des représentations irréductibles dans un produit tensoriel, le cas ADE étant un résultat établi par Nakajima : pour $m = \prod_{i\in I, a\in\mathbb{C}^*} Y_{i,a}^{u_{i,a}}$ mônome dominant (c'est à dire avec les $u_{i,a} \geq 0$) considérons $L(m)$ le $\mathcal{U}_q(\hat{\mathfrak{g}})$-module simple de plus haut l-poids m et $M(m) = \bigotimes_{i\in I, a\in\mathbb{C}^*} L(Y_{i,a})^{u_{i,a}}$ le $\mathcal{U}_q(\hat{\mathfrak{g}})$-module standard de plus haut poids m (le produit tensoriel est commutatif dans l'anneau de Grothendieck). Lorsque $\mathfrak{g}$ est de type ADE, Nakajima [N3] a défini pour $m, m' \in \mathcal{Y}_t$ des monômes dominants des polynômes $P_{m,m'}(t) \in \mathbb{Z}[t]$ (analogues des polynômes de Kazhdan-Lusztig) et pour m dominant des $L_t(m) \in \mathcal{Y}_t$, invariant par $t \mapsto t^{-1}$, tels que :

$$\chi_{q,t}(M(m)) = L_t(m) + \sum_{m'<m} P_{m,m'}(t^{-1}) L_t(m') \tag{1}$$

Nakajima [N3] montre que dans le groupe de Grothendieck :

$$M(m) = L(m) + \sum_{m'<m} P_{m,m'}(1)L(m') \tag{2}$$

Comme nous avons construit le morphisme de q,t-caractères pour toutes les algèbres affines quantiques, on peut définir de même (avec la formule (1)) des analogues des polynômes de Kazhdan-Lusztig $P_{m,m'}(t) \in \mathbb{Z}[t]$ pour les cas non simplement lacés. On conjecture que la formule (2) sur la décomposition des produits tensoriels en modules irréductibles est vraie dans le cas général.

Les ϵ-caractères ont été construits par Frenkel et Mukhin [FM2] pour étudier les représentations de dimension finie de diverses spécialisations de $\mathcal{U}_q(\hat{\mathfrak{g}})$ à $q = \epsilon$ racine de l'unité de rang s. Pour les cas ADE Nakajima [N3] définit les propriétés axiomatiques des ϵ,t-caractères et prouve l'existence du morphisme de ϵ,t-caractères $\hat{\chi}_{\epsilon,t}$. Dans la partie 3 ([He3]) nous construisons $\hat{\chi}_{\epsilon,t}$ en général en développant notre approche algébrique : pour $s > 2$ l'ordre de ϵ (ou $s = 0$), nous munissons $\hat{\mathcal{Y}}_t^s = \mathbb{Z}[W_{i,a}, V_{i,a}]_{1\leq i\leq n, a\in\mathbb{C}^*}$ d'une multiplication t-déformée $*_t^s$ (en utilisant des courants comme pour le cas générique [He2]). Nous définissons les opérateurs d'écrantage t-déformés aux racines de l'unité $\hat{S}_{i,t}^s$ et donnons les propriétés axiomatiques des ϵ,t-caractères pour tous les cas finis. Puis nous construisons une application $\tau_s : \hat{\mathcal{Y}}_t^0 \to \hat{\mathcal{Y}}_t^s$ telle que $\tau_s(\mathrm{Ker}(\hat{S}_{i,t}^0)) \subset \mathrm{Ker}(S_{i,t}^s)$. De plus on peut relever $\chi_{q,t}$ en $\hat{\chi}_{q,t} : \mathbb{Z}[X_{i,a}]_{i\in I, a\in\mathbb{C}^*} \to \hat{\mathcal{Y}}_t^0$ (voir [N3]), et nous montrons :

Théorème 4. *([He3]) Pour toute algèbre affine quantique, il existe une unique application $\mathbb{Z}[t^{\pm}]$-linéaire $\hat{\chi}_{\epsilon,t}$ vérifiant les propriétés axiomatiques des ϵ,t-caractères. De plus $\hat{\chi}_{\epsilon,t} = \tau_s \circ \hat{\chi}_{q,t}$.*

Nous généralisons en particulier la construction des analogues des polynômes de Kazhdan-Lusztig aux racines de l'unités de [N3] à ces situations. De plus nous construisons les q-caractères et q,t-caractères pour une grande classe de matrices de Cartan généralisées (incluant les cas finis et affines exceptés $A_2^{(2)}, A_1^{(1)}$) en étendant l'approche de [He2] :

Théorème 5. *([He3]) Soit C une matrice de Cartan généralisée telle que $i \neq j \Rightarrow C_{i,j}C_{j,i} \leq 3$. Alors il existe une unique application $\mathbb{Z}[t^{\pm}]$-linéaire $\chi_{q,t} : \mathbb{Z}[X_{i,a}, t^{\pm}]_{i\in I, a\in\mathbb{C}^*} \to \mathcal{Y}_t^{\infty}$ vérifiant les propriétés axiomatiques des q,t-caractères.*

Dans la partie 5 ([He5]), une étude combinatoire au cas par cas permet de montrer que les monômes apparaissant dans les q et q,t-caractères des représentations standards (produits tensoriels de représentations fondamentales) sont les mêmes dans les cas non simplement lacés (le cas simplement lacé a été traité dans [N3]) et que les coefficients sont positifs. En particulier ces q,t-caractères peuvent être considérés comme des t-déformations des q-caractères.

Théorème 6. *([He5]) Soit $\mathcal{U}_q(\hat{\mathfrak{g}})$ une algèbre affine quantique non simplement lacée et M un module standard de $\mathcal{U}_q(\hat{\mathfrak{g}})$. Les coefficients de $\chi_{q,t}(M)$ sont dans $\mathbb{N}[t^{\pm}]$ et les monômes de $\chi_{q,t}(M)$ sont les monômes de $\chi_q(M)$.*

Représentations et produit de fusion des algèbres affinisées quantiques

Le procédé d'affinisation quantique (que Drinfel'd [Dr2] décrit pour construire la seconde réalisation d'une algèbre affinisée quantique) peut être appliqué à toute algèbre de Kac-Moody quantique $\mathcal{U}_q(\mathfrak{g})$. On obtient ainsi une nouvelle classe d'algèbres appelées affinisées quantiques : l'affinisée quantique de $\mathcal{U}_q(\mathfrak{g})$ est notée $\mathcal{U}_q(\hat{\mathfrak{g}})$. Les algèbres affines quantiques $\mathcal{U}_q(\hat{\mathfrak{g}})$ (avec $\mathcal{U}_q(\mathfrak{g})$ de type fini) en sont les plus simples exemples et sont très particulières puisque ce sont aussi des algèbres de Kac-Moody quantiques (deuxième réalisation de Drinfel'd) et en particulier des algèbres de Hopf (avec un coproduit permettant de considérer des produits tensoriels de représentations). Si $\mathcal{U}_q(\mathfrak{g})$ est affine, l'affinisée quantique $\mathcal{U}_q(\hat{\mathfrak{g}})$ est appelée une algèbre toroïdale quantique. Ce ne sont pas des algèbres de Kac-Moody quantiques, mais elles sont également très étudiées. Dans la partie 4 ([He4]) nous étudions les affinisées quantiques générales $\mathcal{U}_q(\hat{\mathfrak{g}})$ ($\mathcal{U}_q(\mathfrak{g})$ Kac-Moody quantique générale) et nous développons leur théorie des représentations. Nous établissons d'abord une décomposition triangulaire : pour les sous-algèbres $\mathcal{U}_q(\hat{\mathfrak{g}})^-,\mathcal{U}_q(\hat{\mathfrak{h}}),\mathcal{U}_q(\hat{\mathfrak{g}})^+$ de $\mathcal{U}_q(\hat{\mathfrak{g}})$ (générées par certains éléments de $\mathcal{U}_q(\hat{\mathfrak{g}})$) on a :

Théorème 7. *([He4]) La multiplication $\mathcal{U}_q(\hat{\mathfrak{g}})^- \otimes \mathcal{U}_q(\hat{\mathfrak{h}}) \otimes \mathcal{U}_q(\hat{\mathfrak{g}})^+ \to \mathcal{U}_q(\hat{\mathfrak{g}})$ définit un isomorphisme d'espace vectoriel.*

Dans la preuve la difficulté est de vérifier la compatibilité des relations de Serre quantiques.

Ce résultat permet de considérer des modules de Verma. Nous donnons une classification des représentations intégrables de plus haut poids (de type 1) *à la* Drinfel'd-Chari-Pressley (l'intégrabilité est définie au sens habituel en considérant la décomposition en sous espaces de poids relativement au réseau des poids de $\mathfrak{g}$). Une généralisation du morphisme de q-caractères que Frenkel et Reshetikhin ont défini pour les algèbres affines quantiques se révèle être un outil puissant dans cette étude : on définit $\chi_q : \mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}})) \to \mathcal{Y}$ où $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ est le groupe de Grothendieck des représentations intégrables que nous avons classifiées. Bien qu'à priori on n'ait pas, en général, de structure d'anneau sur $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$, pour une grande classe d'affinisées quantiques (incluant les algèbres affines et toroïdales quantiques) la combinatoire des q-caractères donne :

Théorème 8. *([He4]) L'image de χ_q est $\bigcap_{i\in I} Ker(S_i)$ où pour $i \in I$ l'opérateur d'écrantage S_i est une dérivation. En particulier $Im(\chi_q)$ est un sous-anneau de $\mathcal{Y}$.*

Notons que ce résultat montre que ces q-caractères obtenus par l'étude des représentations coïncident avec les q-caractères obtenus combinatoirement dans le théorème 5 (évaluation à $t = 1$ de $\chi_{q,t}$).

Comme χ_q est injectif on a une structure d'anneau induite $*$ sur $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$. Nous en proposons une interprétation : aucun coproduit n'a été défini en général pour $\mathcal{U}_q(\hat{\mathfrak{g}})$ mais pour les algèbres affines quantiques on a les formules du "nouveau coproduit" Δ de Drinfel'd (qui fait intervenir des sommes infinies). Nous leur donnons un sens en général et proposons une nouvelle construction de produits tensoriels dans des catégories plus grandes $\mathcal{O}^R(\tilde{\mathcal{U}}_q'(\hat{\mathfrak{g}}))$ ($R \geq 0$; $\tilde{\mathcal{U}}_q'(\hat{\mathfrak{g}})$ fait intervenir un paramètre formel u) en définissant une déformation Δ_u de Δ (qui ne peut pas être directement utilisé pour $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ à cause des sommes infinies).

Théorème 9. *([He4]) Pour $R \geq 0$, le coproduit Δ_u définit une application bilinéaire $\otimes_R : \mathcal{O}(\tilde{\mathcal{U}}_q(\hat{\mathfrak{g}})) \times \mathcal{O}^R(\tilde{\mathcal{U}}_q(\hat{\mathfrak{g}})) \to \mathcal{O}^{R+1}(\tilde{\mathcal{U}}_q(\hat{\mathfrak{g}}))$.*

Nous introduisons un procédé de spécialisation permettant de revenir à la catégorie $\mathcal{O}(\mathcal{U}_q(\hat{\mathfrak{g}}))$. Nous démontrons en particulier que $*$ est un produit de fusion, ce qui n'apparaissait pas dans l'étude des q-caractères :

Théorème 10. *([He4]) Le sous-monoïde $Rep^+(\mathcal{U}_q(\hat{\mathfrak{g}})) \subset Rep(\mathcal{U}_q(\hat{\mathfrak{g}}))$ des vraies représentations est stable par $*$.*

Dans la partie 5 ([He5]) nous démontrons des propriétés complémentaires plus précises (ces résultats sont utilisés dans la preuve des théorèmes 5 et 7). D'abord nous démontrons, puis utilisons, une généralisation d'un résultat de [FR3, FM1, N1] : pour une affinisée quantique générale, les l-poids d'un module simple de plus haut l-poids sont inférieurs au plus haut l-poids au sens des monômes ($m' \leq m$ signifie que m' s'écrit comme produit de m par certains monômes $A_{i,a}^{-1}$) :

Théorème 11. *([He5]) Pour m' monôme apparaissant dans le q-caractère d'un module simple de plus haut poids m, on a $m' \leq m$.*

Enfin nous montrons que pour certaines affinisées quantiques (incluant les cas A, B, C) les espaces de l-poids des représentations fondamentales sont de dimension 1 :

Théorème 12. *([He5]) Soit $\mathfrak{g}$ de type A_n ($n \geq 1$), $A_l^{(1)}$ ($l \geq 2$), B_n ($n \geq 2$) ou C_n ($n \geq 2$). Soient $i \in I, a \in \mathbb{C}^*$ et V une représentation simple de plus haut poids l-poids $Y_{i,a}$). Alors les espaces de l-poids de V sont de dimension 1.*

Première partie

t-analogues des opérateurs d'écrantage

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Résumé. Nous proposons des opérateurs d'écrantage adaptés aux q,t-caractères définis par Nakajima [N2, N3] pour les représentations de dimension finie des algèbres affines quantifiées. Ce sont des analogues des opérateurs d'écrantage de Frenkel et Reshetikhin relatifs à la théorie des q-caractères [FR3], avec en particulier les mêmes propriétés de symétrie. Comme la théorie de Nakajima utilise des anneaux non-commutatifs, nous aurons à considérer des bimodules adaptés à ces structures. Notre construction étant purement algébrique et s'appuyant sur la définition combinatoire des q,t-caractères, elle est étendue aux cas non-simplement lacés.

Abstract. We propose screening operators for the theory of q,t-characters of Nakajima [N2, N3] for finite dimensional representations of quantum affine algebras. They are analogs of the screening operators introduced by Frenkel and Reshetikhin for the q-characters [FR3], with the same properties of symmetry. Nakajima introduced non-commutative rings, so we propose bimodules. Since our construction is only algebraic and uses the combinatorial definition of q,t-characters, it can therefore be extended to the non-simply laced cases.

1 Introduction

Dans ce qui suit $q \in \mathbb{C}^*$ est supposé ne pas être une racine de l'unité.

Dans le cas d'une algèbre de Lie semi-simple $\mathfrak{g}$, la structure de l'anneau de Grothendieck $\mathrm{Rep}(\mathcal{U}_q(\mathfrak{g}))$ des représentations de dimension finie de l'algèbre semi-simple quantifiée $\mathcal{U}_q(\mathfrak{g})$ est bien comprise, voir [R]. En fait on a pu montrer qu'elle est tout à fait analogue à celle du cas classique $q = 1$ déjà bien connu. On a en particulier des isomorphismes d'anneaux :

$$\mathrm{Rep}(\mathcal{U}_q(\mathfrak{g})) \simeq \mathrm{Rep}(\mathfrak{g}) \simeq \mathbb{Z}[\Lambda]^W \simeq \mathbb{Z}[T_1, ..., T_n]$$

construits à partir d'un morphisme de caractères χ tel que pour une représentation V de sous-espaces de poids V_λ :

$$\chi(V) = \sum_{\lambda \in \Lambda} \dim(V_\lambda)\lambda$$

où Λ désigne l'ensemble des poids de V.

En revanche la quantification modifie la théorie des représentations lorsqu'on s'intéresse au cas général des algèbres de Kac-Moody. Dans le cas affine $\mathcal{U}_q(\hat{\mathfrak{g}})$, Frenkel et Reshetikhin [FR3], motivés par la théorie des W-algèbres déformées, ont récemment introduit un morphisme d'anneaux injectif, dit de q-caractères, à valeurs dans un anneau de polynômes de Laurent :

$$\chi_q : \mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}})) \to \mathbb{Z}[Y_{i,a}^{\pm 1}]_{1\leq i\leq n, a\in\mathbb{C}^*} = \mathcal{Y}$$

La construction de χ_q repose sur l'existence d'une R-matrice universelle. L'application χ_q permet de décrire l'anneau $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}})) \simeq \mathbb{Z}[X_{i,a}]_{i\in I, a\in\mathbb{C}^*}$ et lorsqu'on regarde la limite classique $q = 1$ on retrouve l'application de caractères usuelle. En fait cette application prend en compte la décomposition en sous-espaces de Jordan pour une certaine famille commutante $(\phi_{i,m}^{\pm})_{m\in\mathbb{Z}, i\in I}$ d'éléments de $\mathcal{U}_q(\hat{\mathfrak{g}})$:

$$\chi_q(V) = \sum_{\gamma} \mathrm{dim}(V_{(\gamma)}) \prod_{i\in I} \prod_{r=1\dots k_{\gamma i}} Y_{i,a_{\gamma ir}} \prod_{s=1\dots l_{\gamma i}} Y_{i,b_{\gamma ir}}^{-1}$$

où $V_{(\gamma)}$ désigne le sous-espace de Jordan de V de poids :

$$\sum_{m\geq 0} \gamma_{i,\pm m}^{\pm} u^{\pm m} = \gamma_i^{\pm}(u) = q_i^{k_{\gamma i} - l_{\gamma i}} \frac{Q_i(uq_i^{-1})R_i(uq_i)}{Q_i(uq_i)R_i(uq_i^{-1})}$$

avec $a_{\gamma ir}$ les racines du polynôme Q_i et $b_{\gamma ir}$ les racines du polynôme R_i.

Le morphisme de q-caractères vérifie une propriété de symétrie analogue au cas classique de l'action du groupe de Weyl qui veut $\mathrm{Im}(\chi) = \mathbb{Z}[\Lambda]^W$. En effet Frenkel et Reshetikhin ont défini n opérateurs dits d'écrantage (avec $A_{i,a} \in \mathcal{Y}$ monômes) :

$$S_i : \mathbb{Z}[Y_{i,a}^{\pm 1}]_{a\in\mathbb{C}^*, i\in I} = \mathcal{Y} \to \bigoplus_{a\in\mathbb{C}^*} \mathcal{Y}.S_{i,a} / \sum_{a\in\mathbb{C}^*} \mathcal{Y}.(S_{i,aq_i^2} - A_{i,aq_i}.S_{i,a})$$

qui sont des dérivations ($S_i(UV) = US_i(V) + VS_i(U)$), vérifiant $S_i(Y_a) = Y_a.S_a$, et ont conjecturé :

$$\mathrm{Im}(\chi_q) = \bigcap_{i\in I} \mathrm{Ker}(S_i)$$

Ils l'ont montré dans le cas sl_2 [FR3] puis Frenkel et Mukhin ont obtenu le résultat dans le cas général [FM1].

Ces opérateurs permettent de comprendre la structure combinatoire des q-caractères. Par exemple :

$$\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}})) \simeq \mathrm{Im}(\chi_q) = \mathfrak{K} = \bigcap_{i\in I} (\mathbb{Z}[Y_{j,a}^{\pm}]_{j\neq i, a\in\mathbb{C}^*} \mathbb{Z}[Y_{i,b} + Y_{i,b}A_{i,bq_i}^{-1}]_{b\in\mathbb{C}^*})$$

Dans le cas où $\mathfrak{g}$ est de type ADE, Nakajima a raffiné la théorie en introduisant un t-analogue des q-caractères grâce à un point de vue géométrique lié aux variétés carquois [N2, N3]. Il considère des applications $\chi_{q,t}$ et $\hat{\chi}_{q,t}$ de $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}})) \otimes_{\mathbb{Z}} \mathbb{Z}[t^{\pm}]$ vers les anneaux de polynômes respectivement $\mathcal{Y}_t = \mathbb{Z}[Y_{i,a}^{\pm}, t^{\pm}]_{i\in I, a\in\mathbb{C}^*}$ et $\hat{\mathcal{Y}}_t = \mathbb{Z}[V_{i,a}, W_{i,a}, t^{\pm}]_{i\in I, a\in\mathbb{C}^*}$. D'un point de vue des représentations, elles permettent de mieux comprendre la structure de chaque sous-espace de Jordan. Au passage il introduit une nouvelle multiplication $*$ sur $\hat{\mathcal{Y}}_t$ qui n'est pas commutative.

Nous proposons dans cet article des t-analogues des opérateurs d'écrantage, adaptés aux applications $\chi_{q,t}$ et $\hat{\chi}_{q,t}$. Cet article est organisé de la manière suivante :

Dans la section 2, on rappelle la propriété fondamentale de symétrie des opérateurs d'écrantage de Frenkel et Reshetikhin (Théorème 2.1) ainsi que quelques résultats élémentaires sur l'anneau $\hat{\mathcal{Y}}_t$ de Nakajima. On définit dans la section 3 les opérateurs $\hat{S}_{t,i}^l$ qui peuvent être interprétés comme des dérivations pour la multiplication usuelle et une certaine structure de bimodule. Dans la section 4 on définit les t-analogues des opérateurs d'écrantage $\hat{S}_{t,i}$ qui dans le cas où $\mathfrak{g}$ est de type ADE peuvent être interprétés comme des dérivations en utilisant la loi $*$ de Nakajima. Ces opérateurs vérifient la propriété attendue dans le théorème 4.1 :

$$\bigcap_{i\in I}\mathrm{Ker}(\hat{S}_{t,i}) = \hat{\mathfrak{K}}_t \supseteq \mathrm{Im}(\hat{\chi}_{q,t})$$

On définit dans la partie 5 des opérateurs $S_{i,t}$ pour l'anneau $\mathcal{Y}_t$ rendant le diagramme commutatif :

$$\begin{array}{ccccc} & \hat{\mathcal{Y}}_t & \xrightarrow{\hat{S}_{t,i}} & \hat{\mathcal{Y}}_{t,i} & \\ \hat{\Pi}_t\downarrow & & & \downarrow & \hat{\Pi}_{t,i} \\ & \mathcal{Y}_t & \xrightarrow{S_{t,i}} & \mathcal{Y}_{t,i} & \\ \Pi_t\downarrow & & & \downarrow & \Pi_{t,i} \\ & \mathcal{Y} & \xrightarrow{S} & \mathcal{Y}_1 & \end{array}$$

avec une propriété de symétrie dans le théorème 5.1 :

$$\bigcap_{i\in I}\mathrm{Ker}(S_{t,i}) = \mathfrak{K}_t = \mathrm{Im}(\chi_{q,t})$$

où $\chi_{q,t}$ est égal au $\tilde{\chi}_{q,t}$ de [N2]. Dans la partie 6 on donne la construction d'involutions analogues à celle de Nakajima.

Notons que la construction repose sur l'existence d'une structure de bimodule sur le module libre à gauche $\bigoplus_{a\in\mathbb{C}^*}\hat{\mathcal{Y}}_t S_{i,a}$ telle que le t-analogue de $\sum_{a\in\mathbb{C}^*}\mathcal{Y}.(S_{i,aq_i^2} - A_{i,aq_i}.S_{i,a})$ soit un sous-bimodule. Cette structure est caractérisée par les relations suivantes, où $m\in\hat{\mathcal{Y}}_t$ est un monôme :

$$S_{i,a}m = t^{2u_{i,a}(m)}mS_{i,a}$$

Nous montrerons dans un prochain article[1] comment ces opérateurs permettent d'étendre la théorie des q,t-caractères au cas général.

[1]Voir [He2].

2 Rappels

Soit $\mathfrak{g}$ une algèbre de Lie simple. On note n le rang de $\mathfrak{g}$, $(C_{i,j})_{1\leq i,j\leq n}$ sa matrice de Cartan, $I = \{1, ..., n\}$ et $D = \mathrm{diag}(r_1, ..., r_n)$ telle que $B = DC$ est symétrique (voir [FR3]). On note $q_i = q^{r_i}$. Pour z une indeterminée et $l \in \mathbb{Z}$ on note $[l]_z = \frac{z^l - z^{-l}}{z - z^{-1}} \in \mathbb{Z}[z^{\pm}]$. Pour $l \geq 0$ on note $[l]_z! = [l]_z[l-1]_z...[1]_z$ et pour $0 \leq r \leq m$: $\begin{bmatrix} m \\ r \end{bmatrix}_z = [m]_z!/[r]_z![m-r]_z$.

2.1 Opérateurs d'écrantage [FR3, FM1]

On considère l'anneau :

$$\mathcal{Y} = \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{C}^*}$$

les $\mathcal{Y}$-modules libres $(i \in I)$:

$$\mathcal{Y}_i^l = \bigoplus_{a\in\mathbb{C}^*} \mathcal{Y}.S_{i,a}$$

et les $\mathcal{Y}$-modules $\mathcal{Y}_i$ définis respectivement comme $\mathcal{Y}$-module quotient de $\mathcal{Y}_i^l$:

$$\mathcal{Y}_i = \bigoplus_{a\in\mathbb{C}^*} \mathcal{Y}.S_{i,a} / \sum_{a\in\mathbb{C}^*} \mathcal{Y}.(S_{i,aq_i^2} - A_{i,aq_i}.S_{i,a})$$

par le sous-module $F_i = \sum_{a\in\mathbb{C}^*} \mathcal{Y}.(S_{i,aq_i^2} - A_{i,aq_i}.S_{i,a})$, avec :

$$A_{i,a} = Y_{i,aq_i^{-1}} Y_{i,aq_i} \prod_{j/C_{j,i}=-1} Y_{j,a}^{-1} \prod_{j/C_{j,i}=-2} Y_{j,aq}^{-1} Y_{j,aq^{-1}}^{-1} \prod_{j/C_{j,i}=-3} Y_{j,aq^2}^{-1} Y_{j,a}^{-1} Y_{j,aq^{-2}}^{-1}$$

On a alors les opérateurs d'écrantage $S_i : \mathcal{Y} \to \mathcal{Y}_i$ qui sont des dérivations pour le produit de $\mathcal{Y}$:

$$S_i(U.V) = U.S_i(V) + V.S_i(U)$$

et qui vérifient pour $a \in \mathbb{C}^*$:

$$S_i(Y_{j,a}^{\pm}) = \pm\delta_{i,j} Y_{i,a}^{\pm} S_{i,a}$$

On peut définir de manière analogue $S_i^l : \mathcal{Y} \to \mathcal{Y}_i^l$.

Théorème 2.1. *(***Frenkel-Reshetikhin** *[FR3]) Le noyau de S_i est le sous-anneau de $\mathcal{Y}$:*

$$Ker(S_i) = \mathfrak{K}_i = \mathbb{Z}[Y_{j,a}^{\pm}]_{j\neq i, a\in\mathbb{C}^*}.\mathbb{Z}[Y_{i,b}(1 + A_{i,bq_i}^{-1})]_{b\in\mathbb{C}^*}$$

2.2 L'anneau $\hat{\mathcal{Y}}_t$ [N3]

On considère à présent l'anneau :

$$\hat{\mathcal{Y}}_t = \mathbb{Z}[V_{i,a}, W_{i,a}, t^{\pm}]_{i\in I, a\in\mathbb{C}^*}$$

C'est un $\mathbb{Z}[t^{\pm}]$-module libre de base l'ensemble des $\prod_{i\in I, a\in\mathbb{C}^*} V_{i,a}^{v_{i,a}(m)} W_{i,a}^{w_{i,a}(m)}$ qu'on appelera monômes.
On définit un morphisme d'anneau $\tilde{\Pi}_t : \hat{\mathcal{Y}}_t \to \mathcal{Y}$ par :

$$m = \prod_{i\in I, a\in\mathbb{C}^*} V_{i,a}^{v_{i,a}(m)} W_{i,a}^{w_{i,a}(m)} \mapsto \prod_{i\in I, a\in\mathbb{C}^*} Y_{i,a}^{u_{i,a}(m)} \text{ et } t \mapsto 1$$

avec pour un tel monôme m :

$$u_{i,a}(m) = w_{i,a}(m) - v_{i,aq_i^{-1}}(m) - v_{i,aq_i}(m)$$

$$+ \sum_{j/C_{i,j}=-1} v_{j,a}(m) + \sum_{j/C_{i,j}=-2} (v_{j,aq}(m) + v_{j,aq^{-1}}(m)) + \sum_{j/C_{i,j}=-3} (v_{j,aq^2}(m) + v_{j,a}(m) + v_{j,aq^{-2}}(m))$$

Remarquer que $\tilde{\Pi}_t$ est construit pour que $\tilde{\Pi}_t(W_{i,a}) = Y_{i,a}$ et $\tilde{\Pi}_t(V_{i,a}) = A_{i,a}^{-1}$. On peut définir pour un monôme $m = \prod_{i\in I, a\in\mathbb{C}^*} Y_{i,a}^{u_{i,a}(m)} \in \mathcal{Y}$ les $u_{i,a}(m)$ de manière évidente, et alors ces quantités sont conservées par $\tilde{\Pi}_t$.
Pour $m \in \hat{\mathcal{Y}}$ monôme i-dominant, c'est à dire vérifiant $\forall a \in \mathbb{C}^*, u_{i,a}(m) \geq 0$, on pose :

$$E_i(m) = m \prod_{a\in\mathbb{C}^*} \sum_{r_a=0\ldots u_{i,a}(m)} t^{r_a(u_{i,a}(m)-r_a)} \begin{bmatrix} u_{i,a}(m) \\ r_a \end{bmatrix}_t V_{i,aq_i}^{r_a}$$

et on note $\hat{\mathfrak{K}}_{t,i}$ le sous $\mathbb{Z}[t^{\pm}]$-module de $\hat{\mathcal{Y}}_t$ engendré par ces $E_i(m)$. On pose alors $\hat{\mathfrak{K}}_t = \bigcap_{i\in I} \hat{\mathfrak{K}}_{t,i}$, et on note $\hat{A}$ l'ensemble des monômes de $\hat{\mathcal{Y}}_t$, $\hat{B}_i \subset \hat{A}$ l'ensemble des monômes i-dominants de $\hat{\mathcal{Y}}_t$.

Lemme 2.1. *Pour chaque $i \in I$, on a une décomposition en somme directe de $\mathbb{Z}[t^{\pm}]$-modules :*

$$\hat{\mathcal{Y}}_t = \hat{\mathfrak{K}}_{t,i} \oplus \bigoplus_{m\in\hat{A}-\hat{B}_i} \mathbb{Z}[t^{\pm}]m = (\bigoplus_{m\in\hat{B}_i} \mathbb{Z}[t^{\pm}]E_i(m)) \oplus (\bigoplus_{m\in\hat{A}-\hat{B}_i} \mathbb{Z}[t^{\pm}]m)$$

Démonstration :

Notons d'abord que pour $m \in \hat{B}_i$, on peut écrire $E_i(m) = m + f(m)$ avec

$$f(m) = m((\prod_{a\in\mathbb{C}^*} \sum_{r_a=0\ldots u_{i,a}} t^{r_a(u_{i,a}-r_a)} \begin{bmatrix} u_{i,a} \\ r_a \end{bmatrix}_t V_{i,aq_i}^{r_a}) - 1)$$

qui ne fait intervenir que des monômes de i-poids $wt_i(m') = \sum_{a\in\mathbb{C}^*} u_{i,a}(m') < wt_i(m)$.

Considérons une combinaison linéaire qui s'annule :

$$\sum_{m\in\hat{B}_i} \lambda_m(t) E_i(m) + \sum_{m\in\hat{A}-\hat{B}_i} \mu_m(t) m = 0$$

avec les $\lambda_m(t), \mu_m(t) \in \mathbb{Z}[t^{\pm}]$. Si on suppose qu'un des $\lambda(t) \neq 0$, soit $m_1 \in \hat{B}_i$ un monôme dominant de i-poids maximal parmi ceux qui vérifient $\lambda_m(t) \neq 0$. Alors le monôme m_1 ne peut apparaître que dans $E_i(m_1)$ puisque si il apparaissait dans $E_i(m_2)$, le i-poids de m_2 serait strictement plus grand que le sien. Donc $\lambda_{m_1}(t) = 0$, contradiction. Donc tous les $\lambda_m(t)$ sont nuls, et alors $\sum\limits_{m\in\hat{A}-\hat{B}_i} \mu_m(t)m = 0$ implique la nullité des $\mu_m(t)$.

Il nous reste à montrer que tout $m \in \hat{A}$ est dans $F = (\bigoplus\limits_{m\in\hat{B}_i} \mathbb{Z}[t^{\pm}]E_i(m)) \oplus (\bigoplus\limits_{m\in\hat{A}-\hat{B}_i} \mathbb{Z}[t^{\pm}]m)$. C'est clair si $m \in \hat{A} - \hat{B}_i$. Dans le cas $m \in \hat{B}_i$, montrons le par récurrence sur le i-poids de m. Si m est de i-poids 0, tous les $u_{i,a}(m)$ sont nuls et $m = E_i(m)$. Puis dans le cas général, on a $E_i(m) = m + f(m)$. □

3 Les opérateurs $\hat{S}^l_{t,i}$

3.1 Définition

On considère les $\hat{\mathcal{Y}}_t$-modules libres suivants ($i \in I$) :

$$\hat{\mathcal{Y}}^l_{t,i} = \bigoplus_{a\in\mathbb{C}^*} \hat{\mathcal{Y}}_t.S_{i,a}$$

On a alors une application naturelle déduite de $\tilde{\Pi}_t$:

$$\tilde{\Pi}^l_{t,i} : \hat{\mathcal{Y}}^l_{t,i} \to \mathcal{Y}^l_i \ , \ \tilde{\Pi}_{t,i}(\sum_{a\in\mathbb{C}^*} \lambda_a.S_{i,a}) = \sum_{a\in\mathbb{C}^*} \tilde{\Pi}_t(\lambda_a).S_{i,a}$$

Definition 3.1. *On note $\hat{S}^l_{t,i}$ l'application $\mathbb{Z}[t^{\pm}]$-linéaire $\hat{S}^l_{t,i} : \hat{\mathcal{Y}}_t \to \hat{\mathcal{Y}}^l_{t,i}$ qui prend sur un monôme $m \in \hat{A}$ la valeur :*

$$\hat{S}^l_{t,i}(m) = m(\sum_{a\in\mathbb{C}^*/u_{i,a}(m)\geq 0} (1 + ... + t^{2(u_{i,a}(m)-1)})S_{i,a} - \sum_{a\in\mathbb{C}^*/u_{i,a}(m)<0} (t^{-2} + ... + t^{2u_{i,a}(m)})S_{i,a})$$

Lemme 3.2. *Le diagramme (1) suivant est commutatif :*

$$\begin{array}{ccc} \hat{\mathcal{Y}}_t & \xrightarrow{\hat{S}^l_{t,i}} & \hat{\mathcal{Y}}^l_{t,i} \\ \tilde{\Pi}_t \downarrow & & \downarrow \ \tilde{\Pi}^l_{t,i} \\ \mathcal{Y} & \xrightarrow{S^l_i} & \mathcal{Y}^l_{1,i} \end{array}$$

Démonstration :

Toutes les applications sont $\mathbb{Z}$-linéaires, il suffit donc de regarder un monôme $m \in \hat{A}$ et $\lambda(t) \in \mathbb{Z}[t^{\pm}]$:

$$\begin{aligned} & (\tilde{\Pi}^l_{t,i} \circ \hat{S}^l_{t,i})(\lambda(t)m) \\ = \ & \lambda(1)\tilde{\Pi}^l_{t,i}(m(\sum_{a\in\mathbb{C}^*/u_{i,a}\geq 0} (1 + ... + t^{2(u_{i,a}-1)})S_{i,a} - \sum_{a\in\mathbb{C}^*/u_{i,a}<0} (t^{-2} + ... + t^{2u_{i,a}})S_{i,a}) \\ = \ & \lambda(1)\tilde{\Pi}_t(m)(\sum_{a\in\mathbb{C}^*} u_{i,a}S_{i,a})) = S^l_i(\lambda(1)\tilde{\Pi}_t(m)) = S^l_i(\tilde{\Pi}_t(\lambda(t)m)) \end{aligned}$$

□

3.2 Interprétation de $\hat{S}_{t,i}^l$ en terme de dérivation

3.2.1 Des lois de bimodule sur $\hat{\mathcal{Y}}_{t,i}^l$

Comme $u_{i,a}$ est additive ($u_{i,a}(m_1.m_2) = u_{i,a}(m_1) + u_{i,a}(m_2)$), on a :

Lemme 3.3. *Il existe sur $\hat{\mathcal{Y}}_{t,i}^l$ une unique structure de bimodule pour la multiplication usuelle de $\hat{\mathcal{Y}}_t$ telle que la structure à gauche soit la structure naturelle, et que pour tout $a \in \mathbb{C}^*$ et tout monôme $m \in \hat{A}$:*

$$S_{i,a}.m = t^{2u_{i,a}(m)} m.S_{i,a} \ , \ S_{i,a}.t = t.S_{i,a}$$

On peut généraliser ce qui précède au cas d'une multiplication tordue sur $\hat{\mathcal{Y}}_t$: pour tout bicaractère $d : \hat{A} \times \hat{A} \to \mathbb{Z}$, c'est-à-dire vérifiant :

$$d(m_1.m_2, m_3) = d(m_1, m_3) + d(m_2, m_3) \ , \ d(m_1, m_2.m_3) = d(m_1, m_2) + d(m_1, m_3)$$

on a une loi de composition interne $*_d$ associative et $\mathbb{Z}[t^{\pm}]$-linéaire sur $\hat{\mathcal{Y}}_t$ en posant :

$$m_1 *_d m_2 = t^{2d(m_1,m_2)} m_1.m_2$$

Remarquons que pour obtenir une loi associative, il suffit de demander que d vérifie sur des monômes m_1, m_2, m_3, la propriété de cocycle (vérifiée par exemple par les bicaractères) :

$$-d(m_2, m_3) + d(m_1 m_2, m_3) - d(m_1, m_2 m_3) + d(m_1, m_2) = 0$$

On peut munir $\hat{\mathcal{Y}}_{t,i}^l$ d'une structure de $\hat{\mathcal{Y}}_t$-bimodule pour la multiplication $*_d$ en posant ($U \in \hat{\mathcal{Y}}_t$) :

$$U *_d (\sum_{a\in\mathbb{C}^*} \lambda_a.S_a) = \sum_{a\in\mathbb{C}^*} (U *_d \lambda_a).S_{i,a} \ , \ S_{i,a} *_d m = t^{2u_{i,a}(m)} m *_d S_{i,a} \ , \ S_{i,a} *_d t = t *_d S_{i,a}$$

3.2.2 $\hat{S}_{t,i}^l$ comme dérivation pour $*_d$

Pour d bicaratère :

Proposition 3.4. *L'application $\hat{S}_{t,i}^l$ est une dérivation par rapport à la multiplication $*_d$:*

$$\forall U, V \in \hat{\mathcal{Y}}_t, \hat{S}_t^l(U *_d V) = U *_d \hat{S}_t^l(V) + \hat{S}_t^l(U) *_d V$$

Démonstration :

Pour vérifier la propriété de dérivation, et il suffit de montrer que pour deux monômes $m, m' \in \hat{A}$ on a $\hat{S}^l_{t,i}(m *_d m') = \hat{S}^l_{t,i}(m) *_d m' + m *_d \hat{S}^l_{t,i}(m')$. Calculons en effet :

$$\begin{array}{rl} & \hat{S}^l_{t,i}(m) * m' + m * \hat{S}^l_{t,i}(m') \\ = & m *_d m' \\ & (\sum_{a\in\mathbb{C}^*/u_{i,a}\geq 0}(1+...+t^{2(u_{i,a}-1)})t^{2u_{i,a'}}S_{i,a} - \sum_{a\in\mathbb{C}^*/u_{i,a}<0}(t^{-2}+...+t^{2u_{i,a}})t^{2u_{i,a'}}S_{i,a} \\ & + \sum_{b\in\mathbb{C}^*/u'_{i,b}\geq 0}(1+...+t^{2(u'_{i,b}-1)})S_{i,b} - \sum_{b\in\mathbb{C}^*/u'_{i,b}<0}(t^{-2}+...+t^{2u'_{i,b}})S_{i,b}) \\ = & m *_d m'(\sum_{a\in\mathbb{C}^*/u_{i,a}+u'_{i,a}\geq 0}(1+...+t^{2(u_{i,a}+u'_{i,a}-1)})S_{i,a} \\ & - \sum_{a\in\mathbb{C}^*/u_{i,a}+u'_{i,a}<0}(t^{-2}+...+t^{2(u_{i,a}+u'_{i,a})})S_{i,a}) = \hat{S}^l_{t,i}(m *_d m') \end{array}$$

□

4 Les t-opérateurs d'écrantage $\hat{S}_{t,i}$

4.1 Définition de $\hat{S}_{t,i}$

Definition 4.1. *On considère le $\mathbb{Z}[t^{\pm}]$-sous module de $\hat{\mathcal{Y}}^l_{t,i}$:*

$$\hat{F}_{t,i} = \sum_{a\in\mathbb{C}^*, m\in\hat{A}} \mathbb{Z}[t^{\pm}]m(V_{i,aq_i}t^{2u_{i,aq_i^2}(m)}S_{i,aq_i^2} - t^2S_{i,a})$$

Le module quotient obtenu est noté $\hat{\mathcal{Y}}_{t,i} = \hat{\mathcal{Y}}^l_{t,i}/\hat{F}_{t,i}$, et l'application obtenue à partir de $\hat{S}^l_{t,i}$ par composition avec la projection $\hat{p}_{t,i}$ de $\hat{\mathcal{Y}}^l_{t,i}$ sur $\hat{\mathcal{Y}}_{t,i}$ est notée $\hat{S}_{t,i}$.

On obtient alors la proposition suivante en remarquant :

$$\tilde{\Pi}_{t,i}(\sum_{a\in\mathbb{C}^*, m\in\hat{A}} \lambda_{m,a}m(V_{i,aq_i}t^{2u_{i,aq_i^2}(m)}S_{i,aq_i^2} - t^2S_{i,a}))$$

$$= \sum_{a\in\mathbb{C}^*}\lambda_{m,a}(1)\Pi_t(m)(A^{-1}_{i,aq_i}.S_{i,aq_i^2} - S_{i,a}) \in F_i$$

Proposition 4.2. *L'application $\tilde{\Pi}^l_{t,i}$ donne naturellement une application $\tilde{\Pi}_{t,i}$, et on a les diagrammes commutatifs :*

$$\begin{array}{ccccc} & \hat{\mathcal{Y}}^l_{t,i} & \longrightarrow & \hat{\mathcal{Y}}_{t,i} & \\ \tilde{\Pi}^l_{t,i} & \downarrow & & \downarrow & \tilde{\Pi}_{t,i} \\ & \mathcal{Y}^l_i & \longrightarrow & \mathcal{Y}_i & \end{array}$$

$$\begin{array}{ccccc} & \hat{\mathcal{Y}}_t & \xrightarrow{\hat{S}_{t,i}} & \hat{\mathcal{Y}}_{t,i} & \\ \tilde{\Pi}_t & \downarrow & & \downarrow & \tilde{\Pi}_{t,i} \\ & \mathcal{Y} & \xrightarrow{S_i} & \mathcal{Y}_i & \end{array}$$

Soit $(\hat{\mathcal{Y}}_t)_i = \mathbb{Z}[V_{i,a}, W_{i,a}, t^{\pm}]_{a\in\mathbb{C}^*} \subset \hat{\mathcal{Y}}_t$. On définit alors $\pi_i : \hat{\mathcal{Y}}_t \to (\hat{\mathcal{Y}}_t)_i$ comme l'unique morphisme d'anneaux $\mathbb{Z}[t^{\pm}]$-linéaire tel que :

$$\pi_i(W_{i,a}) = W_{i,a} \ , \ \pi_i(V_{i,a}) = V_{i,a} \ , \ \pi_i(W_{j,a}) = 1 \text{ si } j \neq i$$

$$\pi_i(V_{j,a}) = 1 \text{ si } C_{i,j} = 0 \ , \ \pi_i(V_{j,a}) = W_{i,a} \text{ si } C_{i,j} = -1$$

$$\pi_i(V_{j,a}) = W_{i,aq}W_{i,aq^{-1}} \text{ si } C_{i,j} = -2 \ , \ \pi_i(V_{j,a}) = W_{i,aq^2}W_{i,a}W_{i,aq^{-2}} \text{ si } C_{i,j} = -3$$

Pour un monôme $m \in \hat{A}$, on a alors $u_{i,a}(m) = u_{i,a}(\pi_i(m))$ pour $a \in \mathbb{C}^*$.

Proposition 4.3. *Le noyau de l'application $\hat{S}_{t,i}$ contient $\hat{\mathfrak{K}}_{t,i}$.*

Démonstration :

Soit $m \in \hat{A}$ un monôme i-dominant. Dans $S^l_{t,i}(E_i(m))$, on peut factoriser tous les termes par m, et notons $\frac{\hat{S}^l_{t,i}(E_i(m))}{m} \in \hat{\mathcal{Y}}^l_{t,i}$ la quantité obtenue. Elle ne dépend que des $u_{i,a}(m)$ $(a \in \mathbb{C}^*)$, et donc :

$$\frac{\hat{S}^l_{t,i}(E_i(m))}{m} = \frac{\hat{S}^l_{t,i}(E_i(\pi_i(m)))}{\pi_i(m)}$$

Remarquons de plus que les $u_{i,a}$ étant conservés, on a :

$$\hat{S}^l_{t,i}(E_i(m)) \in \hat{F}_{t,i} \Leftrightarrow \hat{S}^l_{t,i}(E_i(\pi_i(m))) \in \hat{F}_{t,i}$$

En conséquence il nous suffit de montrer $\hat{S}_{t,i}(E_i(m)) = 0$ pour $m \in \hat{B}_i \cap (\hat{\mathcal{Y}}_t)_i$. Mais alors tout se passe comme si on travaillait avec $\mathfrak{g} = \mathcal{U}_{q_i}(sl_2)$. On est ainsi ramené au cas ADE qui sera établie plus bas, indépendamment de ce qui précède, dans la proposition 4.8. $\square$

4.2 Interprétation de $\hat{S}_{t,i}$ comme dérivation dans le cas ADE

Dans cette sous-partie on se restreint au cas où $\mathfrak{g}$ est de type ADE. On a alors tous les $q_i = q$ et la matrice de Cartan est symétrique.

4.2.1 Rappels [N3] et compléments sur la loi $*$ de Nakajima

On pose pour deux monômes $m_1, m_2 \in \hat{A}$:

$$\begin{aligned} d_N(m_1,m_2) &= \sum_{i\in I, a\in\mathbb{C}^*} (v_{i,aq}(m_1)u_{i,a}(m_2) + w_{i,aq}(m_1)v_{i,a}(m_2)) \\ &= \sum_{i\in I, a\in\mathbb{C}^*} (u_{i,a}(m_1)v_{i,aq^{-1}}(m_2) + v_{i,a}(m_1)w_{i,aq^{-1}}(m_2)) \end{aligned}$$

Ce bicaractère, introduit par Nakajima dans [N2, N3], permet comme précédemment de définir une nouvelle multiplication sur $\hat{\mathcal{Y}}_t$ en posant pour m_1, m_2 deux monômes :

$$m_1 *_{d_N} m_2 = t^{2d_N(m_1,m_2)} m_1.m_2$$

avec . la multiplication usuelle. On notera dans la suite simplement d et $*$. Cette nouvelle multiplication $*$ n'est pas commutative.

Notons que $\hat{\mathfrak{K}}_t$ est une partie de $\hat{\mathcal{Y}}_t$ stable pour la multiplication $*$ ([N3]).

Lemme 4.4. *Soit $m \in \hat{A}$ un monôme, $i \in I$ et $a \in \mathbb{C}^*$. On a alors :*

$$V_{i,aq} * m = t^{2(u_{i,a}(m)-u_{i,aq^2}(m))} m * V_{i,aq} = t^{2u_{i,a}(m)} V_{i,aq}.m$$

C'est une conséquence immédiate de $d(V_{i,aq}, m) = u_{i,a}(m)$ et $d(m, V_{i,aq}) = u_{i,aq^2}(m)$.

Lemme 4.5. *Soit $(m_1, ..., m_p)$ des monômes tels qu'il existe un $a \in \mathbb{C}^*$ vérifiant pour tout r, $u_{i,a}(m_r) = 1$ et $u_{i,b}(m_r) = 0$ pour $b \neq a$. Alors il existe $\alpha \in \mathbb{Z}$ tel que :*

$$(m_1 * (1 + V_{i,aq})) * (m_2 * (1 + V_{i,aq})) * ... * (m_p * (1 + V_{i,aq})) = t^\alpha m_1 ... m_p \sum_{r=0...p} t^{r(p-r)} \begin{bmatrix} p \\ r \end{bmatrix}_t V_{i,aq}^r$$

Démonstration :

On procède par récurrence sur p en s'appuyant sur le lemme 4.4. Pour $p = 1$, on a $m_1 * V_{i,aq} = m_1 V_{i,aq}$ et on a le résultat avec $\alpha = 1$. Ensuite dans le cas général :

$$\begin{aligned} &(m_1 * (1 + V_{i,aq})) * (m_2 * (1 + V_{i,aq})) * ... * (m_{p+1} * (1 + V_{i,aq})) \\ &= t^\alpha (m_1 + m_1 V_{i,aq}) * (m_2 ... m_{p+1} \sum_{r=0...p} t^{r(p-r)} \begin{bmatrix} p \\ r \end{bmatrix}_t V_{i,aq}^r) \\ &= t^{\alpha + 2d(m_1, m_2 ... m_{p+1})} m_1 m_2 ... m_{p+1} \sum_{r=0...p} t^{r(p-r)} \begin{bmatrix} p \\ r \end{bmatrix}_t V_{i,aq}^r \\ &+ t^{\alpha + 2d(m_1, m_2 ... m_{p+1})} m_1 m_2 ... m_{p+1} \sum_{r=0...p} t^{r(p-r)} \begin{bmatrix} p \\ r \end{bmatrix}_t t^{2p-2r} V_{i,aq}^{r+1} \\ &= t^{\alpha + 2d(m_1, m_2 ... m_{p+1})} m_1 m_2 ... m_{p+1} \sum_{r=0...p+1} (t^{r(p-r)} \begin{bmatrix} p \\ r \end{bmatrix}_t \\ &+ t^{(r-1)(p-r+1)} \begin{bmatrix} p \\ r-1 \end{bmatrix}_t t^{2p-2r+2}) V_{i,aq}^r \end{aligned}$$

Et on conclut en remarquant :

$$t^{r(p-r)} \begin{bmatrix} p \\ r \end{bmatrix}_t + t^{(r-1)(p-r+1)} \begin{bmatrix} p \\ r-1 \end{bmatrix}_t t^{2p-2r+2} = t^{r(p+1-r)} \begin{bmatrix} p+1 \\ r \end{bmatrix}_t$$

□

On peut alors exprimer les $E_i(m)$ en utilisant la loi $*$:

Proposition 4.6. *On fixe un $i \in I$. Soit $m \in \hat{B}_i$ un monôme i-dominant. Pour $a \in \mathbb{C}^*$, on considère la suite $(Z_{i,a}) = (Z_{i,a,l})_{1 \leq l \leq z_{i,a}}$ formée de*

$$z_{i,a} = w_{i,a} + \sum_{j / C_{j,i} = -1} v_{j,a} = u_{i,a} + v_{i,aq} + v_{i,aq^{-1}}$$

termes où $W_{i,a}$ apparaît $w_{i,a}$ fois et pour j tel que $C_{j,i} = -1$, $V_{j,a}$ apparaît $v_{j,a}$ fois :

$$\{W_{i,a}, ..., W_{i,a}, V_{j_1,a}, ..., V_{j_1,a}, V_{j_2,a}, ..., V_{j_m,a}\} = \{Z_{i,a,1}, ..., Z_{i,a,z_{i,a}}\}$$

Alors il existe un unique $\beta \in \mathbb{Z}$ tel que :

$$t^\beta E_i(m) = (\prod^*_{j\neq i, a\in\mathbb{C}^*} W_{j,a}) * (\prod^{\overrightarrow{*}}_{j/C_{j,i}=0, a\in(\mathbb{C}^*/q^{2\mathbb{Z}}), r\in\mathbb{Z}} V_{j,aq^{2r}}) * (\prod^*_{a\in(\mathbb{C}^*/q^{2\mathbb{Z}})} \prod^{\overrightarrow{*}}_{r\in\mathbb{Z}} m_{i,a,r})$$

avec :

$$m_{i,a,r} = (Z_{i,a,1} * (1+V_{i,aq})) * ... * (Z_{i,a,u_{i,a}} * (1+V_{i,aq}))$$
$$*(Z_{i,a,u_{i,a}+1} * V_{i,aq} * Z_{i,aq^2,u_{i,aq^2}+v_{i,aq^3}+1}) * ... * (Z_{i,a,u_{i,a}+v_{i,aq}} * V_{i,aq} * Z_{i,aq^2,u_{i,aq^2}+v_{i,aq^3}+v_{i,aq}})$$

Démonstration :

On commence par expliciter $E_i(m)$:

$$E_i(m) = m\prod_{a\in\mathbb{C}^*}(\sum_{r_a=0...u_{i,a}(m)} t^{r_a(u_{i,a}(m)-r_a)} \begin{bmatrix} u_{i,a}(m) \\ r_a \end{bmatrix}_t V_{i,aq}^{r_a})$$

Si on ne tient pas compte des t, l'expression annoncée est correcte puisqu'on a le bon nombre $v_{i,aq}$ de $V_{i,aq}$ et tous les $Z_{i,a,l}$ pour $l = 1...u_{i,a} + v_{i,aq} + v_{i,aq^{-1}}$. Le seul problème est l'inhomogènéïté de $E_i(m)$ du fait des puissances de $V_{i,aq}$.

Les seuls facteurs de m qui contribuent aux $u_{i,b}(m)$ sont les $W_{i,a}, V_{i,a}$ et les $V_{j,a}$ avec $C_{j,i} = -1$. Mais ce sont exactement les facteurs qui posent problème avec $V_{i,aq}$ d'après le lemme 4.4. On en déduit une première expression :

$$t^\alpha E_i(m) = (\prod^*_{j\neq i, a\in\mathbb{C}^*} W_{j,a}) * (\prod^{\overrightarrow{*}}_{j/C_{j,i}=0, a\in(\mathbb{C}^*/q^{2\mathbb{Z}}), r\in\mathbb{Z}} V_{j,aq^{2r}})$$
$$*(m' \sum_{r_a=0...u_{i,a}(m)} t^{r_a(u_{i,a}(m)-r_a)} \begin{bmatrix} u_{i,a}(m) \\ r_a \end{bmatrix}_t V_{i,aq}^{r_a})$$

avec

$$\begin{aligned} m' &= (\prod_{a\in\mathbb{C}^*} W_{i,a}^{w_{i,a}} V_{i,a}^{v_{i,a}})(\prod_{a\in\mathbb{C}^*, j/C_{i,j}=-1} V_{j,a}^{v_{j,a}}) \\ &= \prod_{a\in\mathbb{C}^*}(\prod_{l=1...u_{i,a}} Z_{i,a,l})(\prod_{r=1...v_{i,aq}} Z_{i,a,u_{i,a}+r} V_{i,aq} Z_{i,aq^2,u_{i,aq^2}+v_{i,aq^3}+r}) \end{aligned}$$

Il nous suffit donc de montrer qu'il existe $\gamma \in \mathbb{Z}$ tel que :

$$E_i(m') = m' \sum_{r_a=0...u_{i,a}(m)} t^{r_a(u_{i,a}(m)-r_a)} \begin{bmatrix} u_{i,a}(m) \\ r_a \end{bmatrix}_t V_{i,aq}^{r_a} = t^{-\gamma}(\prod^*_{a\in\mathbb{C}^*/q^{2\mathbb{Z}}} \prod^{\overrightarrow{*}}_{r\in\mathbb{Z}} m_{i,a,r})$$

Or d'après le lemme 4.5, le facteur $Z_{i,a,1}...Z_{i,a,u_{i,a}(m)} \sum_{r_a=0...u_{i,a}(m)} t^{r_a(u_{i,a}(m)-r_a)} \begin{bmatrix} u_{i,a}(m) \\ r_a \end{bmatrix}_t V_{i,aq}^{r_a}$ est égal à une puissance de t près à $(Z_{i,a,1} * (1+V_{i,aq})) * ... * (Z_{i,a,u_{i,a}} * (1+V_{i,aq}))$. Il ne reste plus qu'à vérifier que les facteurs restant $\prod_{a\in\mathbb{C}^*}(\prod_{r=1...v_{i,aq}} Z_{i,a,u_{i,a}+r} V_{i,aq} Z_{i,aq^2,u_{i,aq^2}+v_{i,aq^3}+r})$ ne posent pas de problème vis à vis de l'inhomogènéïté en puissances de $V_{i,aq}$, mais c'est le cas car pour tout $r \in \mathbb{Z}$:

$$u_{i,aq^{2r}}(Z_{i,a,u_{i,a}+l} * V_{i,aq} * Z_{i,aq^2,u_{i,aq^2}+v_{i,aq^3}+l}) = 0$$

$\square$

4.2.2 Une structure de bimodule sur $\hat{\mathcal{Y}}_{t,i}$ pour la loi $*$

Lemme 4.7. *Le sous $\mathbb{Z}[t^{\pm}]$-module $\hat{F}_{t,i}$ de $\hat{\mathcal{Y}}^l_{t,i}$ est en fait un sous-module à gauche pour la loi $*$:*

$$\hat{F}_{t,i} = \sum_{a\in\mathbb{C}^*} \hat{\mathcal{Y}}_t * (V_{i,aq}.S_{i,aq^2} - t^2 S_{i,a})$$

C'est aussi un sous-bimodule $\hat{\mathcal{Y}}_t * \hat{F}_{t,i} = \hat{F}_{t,i} * \hat{\mathcal{Y}} = \hat{F}_{t,i}$ et $\hat{\mathcal{Y}}_{t,i}$ hérite d'une structure de bimodule.

Démonstration :

La première propriété découle directement du lemme 4.4 qui donne pour $m \in \hat{A}$:

$$m * (V_{i,aq}.S_{i,aq^2} - t^2 S_{i,a}) = t^{2u_{i,aq_i^2}(m)} m V_{i,aq_i} S_{i,aq_i^2} - t^2 m S_{i,a}$$

Pour la propriété de sous-bimodule, soit $m \in \hat{A}$ un monôme. En utilisant le lemme 4.4, on a pour $\lambda_a \in \hat{\mathcal{Y}}_t$:

$$\begin{aligned} \lambda_a * (V_{i,aq}.S_{i,aq^2} - t^2 S_{i,a}) * m &= \lambda_a * (t^{2u_{i,aq^2}(m)} V_{i,aq} * m.S_{i,aq^2} - t^{2+2u_{i,a}(m)} m.S_{i,a}) \\ &= t^{2u_{i,a}(m)} \lambda_a * m * (V_{i,aq}.S_{i,aq^2} - t^2 S_{i,a}) \in \hat{F}_{t,i} \end{aligned}$$

□

4.2.3 $\hat{S}_{t,i}$ est une dérivation

Le résultat suivant permet d'obtenir la proposition 4.3 :

Proposition 4.8. *L'application $\hat{S}_{t,i}$ est une dérivation pour le produit $*$ et son noyau contient $\hat{\mathfrak{K}}_{t,i}$.*

Démonstration :

La propriété de dérivation est conservée, en effet pour $U, V \in \hat{\mathcal{Y}}_t$ on a :

$$\hat{S}_{t,i}(U * V) = \hat{p}_{t,i}(U * \hat{S}^l_{t,i}(V)) + \hat{p}_{t,i}(\hat{S}_{t,i}(U) * V)$$

$$= U * \hat{p}_{t,i}(\hat{S}^l_{t,i}(V)) + \hat{p}_{t,i}(\hat{S}_{t,i}(U)) * V = U * \hat{S}_{t,i}(V) + \hat{S}_{t,i}(U) * V$$

Pour montrer que $\hat{\mathfrak{K}}_{t,i} \subset \mathrm{Ker}(\hat{S}_{t,i})$, considérons un monôme i-dominant m, et décomposons $E_i(m)$ en utilisant la proposition 4.6 sous la forme d'un produit pour $*$. En utilisant la propriété de dérivation de $\hat{S}_{t,i}$, il nous suffit d'obtenir que chacun des termes est annulé. Or pour $a \in \mathbb{C}^*$:

$$\hat{S}_{t,i}(Z_{i,a,k} * (1 + V_{i,aq})) = Z_{i,a,k}.S_{i,a} - t^{-2} Z_{i,a,k} * V_{i,aq}.S_{i,aq^2} = Z_{i,a,k} * (S_{i,a} - t^{-2} V_{i,aq} S_{i,aq^2}) = 0$$

$$\hat{S}^l_{t,i}(Z_{i,a,k} * V_{aq} * Z_{i,aq^2,k'}) = 0$$

car pour tout $b \in \mathbb{C}^*$, $u_{i,b}(Z_{i,a,k} * V_{aq} * Z_{i,aq^2,k'}) = 0$ □

4.3 Interprétation de $\hat{S}_{t,i}$ dans le cas général

Dans le cas général, on ne dispose pas de bicaratère vérifiant les deux relations fondamentales du cas ADE pour tout $i \in I$

$$d(V_{i,aq_i}, m) = u_{i,a}(m) \text{ et } d(m, V_{i,aq_i}) = u_{i,aq_i^2}(m)$$

Par exemple, pour $\mathfrak{g}$ de type B_2, on a $C = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$ (la matrice de Cartan dans [FR3] est la transposée de celle de [Bo]), $q_1 = q^2$, $q_2 = q$ et :

$$0 = u_{1,aq^{-2}}(V_{2,a}) \neq u_{2,aq}(V_{1,a}) = 1$$

On peut seulement définir pour chaque $i \in I$ un bicaractère d_i qui vérifie une propriété partielle. Nous étudierons une multiplication $*$ pour le cas général dans un prochain article[2].

4.4 Démonstration du théorème 4.1

On retourne au cas général $\mathfrak{g}$ simple quelconque.

Théorème 4.1. *On a* $\hat{\mathfrak{K}}_{t,i} = Ker(\hat{S}_{t,i})$.

Démonstration :

La première inclusion $\hat{\mathfrak{K}}_{t,i} \subset \mathrm{Ker}(\hat{S}_{t,i})$ est déjà connue dans la proposition 4.3.

Supposons par l'absurde qu'on n'ait pas égalité. Alors on considère un $x \in \mathrm{Ker}(\hat{S}_{t,i}) - \hat{\mathfrak{K}}_{t,i}$ qu'on décompose en utilisant le lemme 2.1 sur $\hat{\mathcal{Y}}_t = \hat{\mathfrak{K}}_{t,i} \oplus \bigoplus_{m \in \hat{A} - \hat{B}_i} \mathbb{Z}[t^{\pm}]m$ sous la forme $x = v + u$ avec $u \neq 0$. On note $u = \sum_{m \in M} \lambda_m m$ avec $M \subset \hat{A} - \hat{B}_i$, $\lambda_m \in \mathbb{Z}[t^{\pm}]$ et $\lambda_m \neq 0$ pour $m \in M$. Alors $x, v \in \mathrm{Ker}(\hat{S}_{t,i})$, donc u est un élément non nul de $\bigoplus_{m \in \hat{A} - \hat{B}_i} \mathbb{Z}[t^{\pm}]m \cap \mathrm{Ker}(\hat{S}_{t,i})$.

Pour $m \in \hat{A} - \hat{B}_i$, notons N_m le nombre de classe $R \in \mathbb{C}^*/q^{2\mathbb{Z}}$ tel qu'il existe $a \in R$ vérifiant $u_{i,a}(m) < 0$. Tous les monômes m de $\hat{A} - \hat{B}_i$ vérifient $N_m \geq 1$. Soit $m_0 \in M$ avec N_{m_0} minimal parmi les N_m pour $m \in M$. Soit alors $a \in \mathbb{C}^*$ tel que $u_{i,a}(m_0) < 0$ et pour $r < 0$, $u_{i,aq^{2r}}(m_0) \geq 0$. Lorsqu'on calcule

$$\hat{S}_{t,i}(u) = 0 = \sum_{m \in M} \lambda_m m \Big(\sum_{b \in \mathbb{C}^* / u_{i,b}(m) \geq 0} (1 + t^2 + ... + t^{2(u_{i,b}(m)-1)}) S_{i,b}$$
$$- \sum_{b \in \mathbb{C}^* / u_{i,b}(m) < 0} (t^{-2} + ... + t^{2u_{i,b}(m)}) S_{i,b} \Big)$$

on voit appaître le terme $-\lambda_{m_0} m_0 (t^{-2} + ... + t^{2u_{i,a}(m)}) S_{i,a}$. Ce terme doit être annulé par projection sur $\hat{\mathcal{Y}}_{t,i}$. Les termes qui vont l'annuler peuvent provenir soit d'un $S_{i,aq_i^{2r}}$

[2] Voir [He2, He3].

avec $r<0$, soit d'un $S_{i,aq_i^{2r}}$ avec $r>0$. Dans le premier cas on a un monôme $m_1\in M$ tel que $m_1 V_{i,aq_i^{-1}}V_{i,aq_i^{-3}}...V_{i,aq_i^{2r+1}}=m_0$, dans le deuxième on a un momôme $m_1=m_0V_{i,aq_i}...V_{i,aq_i^{2r-1}}\in M$. On peut ainsi définir une suite de monômes m_p tant que $u_{i,a}(m_p)<0$. Les termes de la suite sont distincts deux à deux, car à chaque opération soit on ajoute des $V_{i,aq_i^{2r+1}}$ avec $r\geq 0$, soit on enlève des $V_{i,aq_i^{2r+1}}$ avec $r<0$. Notons aussi qu'à chaque opération on ne diminue pas les $u_{i,aq_i^{2r}}$ avec $r<0$, et on n'augmente pas N. Comme M est fini, la suite se termine sur un $m_P\in M$ qui vérifie $u_{i,aq_i^{2r}}(m_P)\geq 0$ pour $r\leq 0$, et $N_{m_P}=N_{m_0}$. En notant $m^0=m_0$ et $m^1=m_P$, ce nouveau procédé donne une suite m^j telle que $\min\{r/u_{i,aq^{2r}}<0\}$ est strictement croissante. Par finitude de M, la suite se termine sur un $m^{P'}\in M$ tel que $u_{i,aq^{2r}}\geq 0$ pour tout $r\in\mathbb{Z}$ et les autres classes de $\mathbb{C}^*/q_i^{2\mathbb{Z}}$ n'ont pas été modifiées. Donc $N_{m_{P'}}<N_{m_0}$, contradiction. □

5 Opérateurs d'écrantage pour l'anneau $\mathcal{Y}_t$

5.1 Rappels et compléments

5.1.1 L'anneau $\mathcal{Y}_t$

En suivant Nakajima [N2, N3] on considère l'anneau "intermédiaire" entre $\hat{\mathcal{Y}}_t$ et $\mathcal{Y}$:

$$\mathcal{Y}_t=\mathbb{Z}[t^{\pm},Y_{i,a},Y_{i,a}^{-1}]_{i\in I,a\in\mathbb{C}^*}$$

On a un morphisme d'anneaux canonique $\Pi_t:\mathcal{Y}_t\rightarrow\mathcal{Y}$ et que $Y_{i,a}^{\pm}\mapsto Y_{i,a}^{\pm}$ et $t\mapsto 1$. On peut considérer pour tout bicaractère d l'application $\mathbb{Z}[t^{\pm}]$-linéaire $\hat{\Pi}_d:\hat{\mathcal{Y}}_t\rightarrow\mathcal{Y}_t$:

$$m=\prod_{i\in I,a\in\mathbb{C}^*}V_{i,a}^{v_{i,a}(m)}W_{i,a}^{w_{i,a}(m)}\mapsto t^{-d(m,m)}\prod_{i\in I,a\in\mathbb{C}^*}Y_{i,a}^{u_{i,a}(m)}\text{ et }t\mapsto 1$$

On a toujours $\tilde{\Pi}_t=\Pi_t\circ\hat{\Pi}_d$.

Dans le cas du bicaractère trivial $d=0$, on note $\hat{\Pi}_0=\hat{\Pi}_t$ et c'est alors un morphisme d'anneaux. Dans le cas ADE, on peut prendre d_N et on retrouve l'application $\hat{\Pi}=\hat{\Pi}_{d_N}$ de [N3].

Lemme 5.1. *Un produit $p=\prod_{i\in I,a\in\mathbb{C}^*}A_{i,a}^{v_{i,a}}\in\mathcal{Y}$ vaut 1 si et seulement si tous les $v_{i,a}=0$.*

En conséquence on définit une relation d'ordre partiel sur l'ensemble A des monômes de $\mathcal{Y}$ en posant :

$$m\leq m'\Leftrightarrow m'/m\text{ est un monôme en }A_{i,a}^{-1}$$

Démonstration :

Supposons par l'absurde qu'un tel produit p peut être égal à 1 avec des $v_{i,a}\neq 0$. Considérons alors un a tel qu'il existe un $i\in I$ avec $v_{i,a}\neq 0$ mais pour $m\in\mathbb{Z}$ stritement positif, pour $j\in I$, $v_{j,aq^m}=0$. Parmi ces i, on en choisit un tel que la longueur de la racine associée soit maximale. Dans p, le facteur $Y_{i,aq_i}^{-v_{i,a}}$ doit se simplifier avec un autre facteur.

Cependant par définition de a il ne peut pas venir de $A_{i,aq_i^2}^{v_{i,aq_i^2}}$. Il reste donc les possibilités suivantes :

il provient d'un $A_{j,aq_i}^{v_{j,aq_i}}$ avec $C_{i,j}=-1$, $j\neq i$. Alors $v_{j,aq_i}\neq 0$, contradiction.

il provient d'un $A_{j,aq_iq^{-1}}^{v_{j,aq_iq^{-1}}}$ avec $C_{i,j}=-2$, $j\neq i$, ce qui impose $v_{j,aq_iq^{-1}}\neq 0$. Comme $C_{i,j}=-2$, les racines associées à i et j ne sont pas de même longueur, et donc en utilisant l'hypothèse sur i, on a $r_i>r_j\geq 1$. Alors $q_iq^{-1}=q$ ou q^2, donc $v_{j,aq}\neq 0$ ou $v_{j,aq^2}\neq 0$, ce qui n'est pas possible d'après le choix de i.

il provient d'un $A_{j,aq_iq^{-2}}^{v_{j,aq_iq^{-2}}}$ avec $C_{i,j}=-3$, $j\neq i$, ce qui impose $v_{j,aq_iq^{-2}}\neq 0$. On est dans le cas où $\mathfrak{g}$ est de type G_2. Les racines associées à i et j ne sont pas de même longueur, et $r_i/r_j=3$ ou $\frac{1}{3}$. Si $r_i=3$, on a $v_{j,aq}\neq 0$ ce qui est contraire au choix de i. Si $r_i=1$, on a $v_{j,aq^{-1}}\neq 0$ et $v_{j,aq^m}=0$ pour $m\geq 0$. Mais alors on ne peut pas annuler $Y_{j,aq^{-1}q_j}^{v_{j,aq^{-1}}}=Y_{j,aq^2}^{v_{j,aq^{-1}}}$.

Pour que $\leq$ soit bien une relation d'ordre, la propriété la moins évidente est l'antisymétrie qui est assurée par ce qui précède. □

5.1.2 Quelques notations

On note A l'ensemble des monômes de $\mathcal{Y}$, $B_i\subset A$ l'ensemble des monômes i-dominants de $\mathcal{Y}$. Pour $\prod_{a\in\mathbb{C}^*,j} Y_{j,a}^{u_{j,a}}=m\in B_i$, on pose :

$$E_{0,i}(m)=m\prod_{a\in\mathbb{C}^*}\sum_{r_a=0\dots u_{i,a}} t^{r_a(u_{i,a}-r_a)}\begin{bmatrix}u_{i,a}\\ r_a\end{bmatrix}_t A_{i,aq_i}^{-r_a}$$

Notons que si $m\in B_i\cap\hat{\Pi}_t(\hat{B}_i)=B_i'$, on a $E_{0,i}(m)=\hat{\Pi}_t(E_i(m'))$ avec $m=\hat{\Pi}_t(m')$.

On note $\mathfrak{K}_{t,i}$ le $\mathbb{Z}[t^\pm]$-module engendré par les $E_{0,i}(m)$ avec $m\in B_i$. On a $\hat{\Pi}_t(\hat{\mathfrak{K}}_i)\subset\mathfrak{K}_{t,i}$ mais on n'a pas égalité dans le cas général.

Pour $i\in I$, on a comme dans le lemme 2.1 une somme directe de $\mathbb{Z}[t^\pm]$-modules :

$$\mathcal{Y}_t=\mathfrak{K}_{t,i}\oplus\bigoplus_{m\in A-B_i}\mathbb{Z}[t^\pm]m=(\bigoplus_{m\in B_i}\mathbb{Z}[t^\pm]E_{i,0}(m))\oplus(\bigoplus_{m\in A-B_i}\mathbb{Z}[t^\pm]m)$$

Lemme 5.2. *On a l'égalité :*

$$\hat{\Pi}_t(\hat{\mathfrak{K}}_t)=\bigcap_{i\in I}\mathfrak{K}_{t,i}$$

et on notera $\mathfrak{K}_t$ cette sous-partie de $\mathcal{Y}_t$.

Démonstration :

On sait déjà :

$$\hat{\Pi}_t(\hat{\mathfrak{K}}_t)=\hat{\Pi}_t(\bigcap_{i\in I}\hat{\mathfrak{K}}_i)\subset\bigcap_{i\in I}\hat{\Pi}_t(\hat{\mathfrak{K}}_{t,i})\subset\bigcap_{i\in I}\mathfrak{K}_{t,i}$$

Considérons à présent $x\in\bigcap_{i\in I}\mathfrak{K}_i$. Soit $m\in A$ un monôme maximal parmi ceux qui interviennent dans x pour la relation d'ordre $\leq$ du lemme 5.1. Pour chaque $i\in I$, m provient d'un certain $E_{0,i}(m')$ avec m' i-dominant, ce qui impose $m \leq m'$. On a donc $m = m'$ et m est i-dominant pour tout $i \in I$. Il est donc de la forme :

$$m = \prod_{i\in I, a\in\mathbb{C}^*} Y_{i,a}^{u_{i,a}(m)} = \hat{\Pi}_t(\prod_{i\in I, a\in\mathbb{C}^*} W_{i,a}^{u_{i,a}(m)}) \in \hat{\Pi}_t(\hat{\mathcal{Y}}_t)$$

car les $u_{i,a}(m) \geq 0$. Si on suppose $m \neq 1$ (soit $x \notin \mathbb{Z}[t^{\pm}]$) et on considère $i_0 \in I$ tel que $wt_{i_0}(m) \neq 0$, on a dans l'écriture de x dans $\mathfrak{K}_{i_0}$ le monôme m qui ne peut provenir que de

$$E_{0,i_0}(m) = \hat{\Pi}_t(E_{i_0}(\prod_{i\in I, a\in\mathbb{C}^*} W_{i,a}^{u_{i,a}(m)})) \in \hat{\Pi}_t(\hat{\mathfrak{K}}_t)$$

On peut alors enlever de x le terme $E_{0,i_0}(m)$ avec son coefficient de $\mathbb{Z}[t^{\pm}]$, et on se ramène à un élément de $\bigcap_{i\in I}\mathfrak{K}_{t,i}$ faisant intervenir strictement moins de monôme, ce qui permet de conclure par récurrence. □

5.2 Les opérateurs $S^l_{t,i}$

5.2.1 Définition

On considère les $\mathcal{Y}_t$-modules libres :

$$\mathcal{Y}^l_{t,i} = \bigoplus_{a\in\mathbb{C}^*} \mathcal{Y}_t.S_{i,a}$$

On déduit respectivement de $\hat{\Pi}_t$, $\hat{\Pi}$ (dans le cas ADE), Π_t des applications $\hat{\Pi}^l_{t,i}$, $\hat{\Pi}^l_i$, $\Pi^l_{t,i}$. On note $S^l_{t,i}$ l'application $\mathbb{Z}[t^{\pm}]$-linéaire $S^l_{t,i} : \mathcal{Y}_t \to \mathcal{Y}^l_{t,i}$ qui prend sur un monôme $m \in \mathcal{Y}_t$ la valeur :

$$S^l_{t,i}(m) = m(\sum_{a\in\mathbb{C}^*/u_{i,a}(m)\geq 0} (1 + ... + t^{2(u_{i,a}(m)-1)})S_{i,a} - \sum_{a\in\mathbb{C}^*/u_{i,a}(m)<0} (t^{-2} + ... + t^{2u_{i,a}(m)})S_{i,a})$$

On voit immédiatement $S^l_{t,i}(Y_{j,a}) = \delta_{j,i}Y_{i,a}S_{i,a}$ et $S^l_{t,i}(Y^{-1}_{j,a}) = -\delta_{j,i}t^{-2}Y^{-1}_{i,a}S_{i,a}$, et :

Lemme 5.3. *Le diagramme suivant est commutatif :*

$$\begin{array}{ccc} \hat{\mathcal{Y}}_t & \xrightarrow{\hat{S}^l_{t,i}} & \hat{\mathcal{Y}}^l_{t,i} \\ \hat{\Pi}_t \downarrow & & \downarrow \quad \hat{\Pi}^l_{t,i} \\ \mathcal{Y}_t & \xrightarrow{S^l_{t,i}} & \mathcal{Y}^l_{t,i} \\ \Pi_t \downarrow & & \downarrow \quad \Pi^l_{t,i} \\ \mathcal{Y} & \xrightarrow{S^l_i} & \mathcal{Y}^l_{1,i} \end{array}$$

Dans le cas ADE, le diagramme (1)' obtenu en utilisant respectivement $\hat{\Pi}, \hat{\Pi}^l_i$ à la place de $\hat{\Pi}_t, \hat{\Pi}^l_{t,i}$ est commutatif également.

5.2.2 Interprétation des S_i^l en terme de dérivation

On munit comme précédemment $\mathcal{Y}_{t,i}^l$ d'une unique structure de $\mathcal{Y}$-bimodule telle que $S_{i,a}.m = t^{2u_{i,a}(m)}m.S_{i,a}$, et :

Proposition 5.4. *L'application $S_{t,i}^l$ a alors une propriété de dérivation :*

$$\forall U, V \in \hat{\mathcal{Y}}_t, S_{t,i}^l(U.V) = U.S_{t,i}^l(V) + S_{t,i}^l(U).V$$

5.3 t-analogues des opérateurs d'écrantage pour $\mathcal{Y}_t$

5.3.1 Définition des opérateurs $S_{t,i}$

On considère le sous $\mathbb{Z}[t^\pm]$-module $\hat{\Pi}_{t,i}^l(\hat{F}_{t,i})$ de $\mathcal{Y}_{t,i}^l$. Il découle du fait que $\hat{\Pi}_t$ est un morphisme d'anneaux qui conserve les quantités $u_{i,a}$:

Lemme 5.5. *Les éléments de $\hat{\Pi}_{t,i}^l(\hat{F}_{t,i})$ sont les éléments de $\mathcal{Y}_{t,i}^l$ de la forme ($\lambda_{m,a} \in \mathbb{Z}[t^\pm]$ presque tous nuls) :*

$$\sum_{a\in\mathbb{C}^*, m\in\hat{\Pi}_t(\hat{A})} \lambda_{m,a} m(A_{i,aq_i}^{-1} t^{2u_{i,aq_i^2}(m)} S_{i,aq_i^2} - t^2 S_{i,a})$$

Soit à présent :

$$F_{t,i} = \sum_{a\in\mathbb{C}^*, m\in A} \mathbb{Z}[t^\pm].m(A_{i,aq_i}^{-1} t^{2u_{i,aq_i^2}(m)} S_{i,aq_i^2} - t^2 S_{i,a})$$

C'est un sous $\mathbb{Z}[t^\pm]$-module de $\mathcal{Y}_{t,i}^l$ qui contient $\hat{\Pi}_{t,i}^l(\hat{F}_{t,i})$,dont les éléments de $F_{t,i}$ sont caractérisés par l'écriture :

$$\sum_{a\in\mathbb{C}^*} (A_{i,aq_i}^{-1} S_{i,aq_i^2}.U_a - t^2 U_a.S_{i,a}) \text{ avec } U_a \in \mathcal{Y}_t$$

Definition 5.6. *On note $\mathcal{Y}_{t,i}$ le $\mathbb{Z}[t^\pm]$-module quotient de $\mathcal{Y}_{t,i}^l$ par $F_{t,i}$, et $S_{t,i}$ l'application obtenue à partir de $S_{t,i}^l$ par projection sur $\mathcal{Y}_{t,i}$.*

Notons que $F_{t,i}$ n'est pas un $\mathcal{Y}_t$-sous module de $\mathcal{Y}_{t,i}^l$, mais c'est une partie de $\mathcal{Y}_{t,i}^l$ stable par multiplication par des éléments de $\mathbb{Z}[Y_{j,a}^\pm]_{j\neq i}$. En particulier on peut définir une multiplication à gauche sur $\mathcal{Y}_{t,i}$ par les élements de $\mathbb{Z}[Y_{j,a}^\pm]_{j\neq i}$. Elle commute alors avec la projection p_i de $\mathcal{Y}_{t,i}^l$ sur $\mathcal{Y}_{t,i}$.

Proposition 5.7. *Les applications $\Pi_{t,i}^l, \hat{\Pi}_{t,i}^l$ donnent naturellement des applications $\Pi_{t,i}, \hat{\Pi}_{t,i}$ rendant le diagramme suivant commutatif (les flêches sans nom représentent les projections canoniques) :*

$$\begin{array}{ccc} \hat{\mathcal{Y}}_{t,i}^l & \longrightarrow & \hat{\mathcal{Y}}_{1,i,t} \\ \hat{\Pi}_{t,i}^l \downarrow & & \downarrow \quad \hat{\Pi}_{t,i} \\ \mathcal{Y}_{t,i}^l & \longrightarrow & \mathcal{Y}_{t,i} \\ \Pi_{t,i}^l \downarrow & & \downarrow \quad \Pi_{t,i} \\ \mathcal{Y}_i^l & \longrightarrow & \mathcal{Y}_i \end{array}$$

On a alors le diagramme commutatif suivant :

$$\begin{array}{ccccc} & \hat{\mathcal{Y}}_t & \xrightarrow{\hat{S}_{t,i}} & \hat{\mathcal{Y}}_{t,i} & \\ \hat{\Pi}_t \downarrow & & & \downarrow & \hat{\Pi}_{t,i} \\ & \mathcal{Y}_t & \xrightarrow{S_{t,i}} & \mathcal{Y}_{t,i} & \\ \Pi_t \downarrow & & & \downarrow & \Pi_{t,i} \\ & \mathcal{Y} & \xrightarrow{S_i} & \mathcal{Y}_i & \end{array}$$

et :

$$\mathfrak{K}_{t,i} \subset Ker(S_{t,i})$$

Démonstration :

Montrons le dernier point : $\hat{\Pi}_{t,i} \circ \hat{S}_{t,i} = S_{t,i} \circ \hat{\Pi}_t$ et $\hat{\mathfrak{K}}_{t,i} \subset \mathrm{Ker}(\hat{S}_{t,i})$ implique $\hat{\Pi}_t(\hat{\mathfrak{K}}_{t,i}) \subset \mathrm{Ker}(S_{t,i})$. Soit alors $m \in B_i$ qu'on décompose $m = m_i \prod_{j \neq i} m_j$ avec les $m_j \in \mathbb{Z}[Y_{j,a}^{\pm}]_{a \in \mathbb{C}^*}$. On a :

$$E_{0,i}(m) = E_{0,i}(m_i) \prod_{j \neq i} m_j$$

et comme pour tout $a \in \mathbb{C}^*$, $u_{i,a}(\prod_{j \neq i} m_j) = 0$, on a :

$$S_{t,i}^l(E_{0,i}(m)) = (\prod_{j \neq i} m_j) S_{t,i}^l(E_{0,i}(m_i))$$

Alors pour la multiplication à gauche sur $\mathcal{Y}_{t,i}$ par des éléments de $\mathbb{Z}[Y_{j,a}^{\pm}]_{j \neq i}$, on a :

$$S_{t,i}(E_{0,i}(m)) = (\prod_{j \neq i} m_j) S_{t,i}(E_{0,i}(m_i))$$

Mais comme m est i-dominant, on a $m_i = \prod_{a \in \mathbb{C}^*} Y_{i,a}^{u_{i,a}} = \hat{\Pi}_t(\prod_{a \in \mathbb{C}^*} W_{i,a}^{u_{i,a}})$ avec les $u_{i,a} = u_{i,a}(m_i) \geq 0$. En conséquence :

$$E_{0,i}(m_i) = \hat{\Pi}_t(E_i(\prod_{a \in \mathbb{C}^*} W_{i,a}^{u_{i,a}})) \in \hat{\Pi}_t(\hat{\mathfrak{K}}_{t,i}) \subset \mathrm{Ker}(S_{t,i})$$

□

5.4 Remarques sur les opérateurs $S'_{t,i}$

Pour définir des $E'_{0,i}(m)$ analogues des $E_{0,i}(m)$ relatifs à $\hat{\Pi}$, on considère pour $i \in I$ en suivant [N2] $\phi_i : \mathfrak{K}_{t,i} \to \mathcal{Y}_t$ telle que $(m \in B_i)$:

$$E_{0,i}(m) = \sum \lambda_M(t) M \mapsto \sum \lambda_M(t) t^{-\alpha(m,M)} M$$

avec pour $M = m \prod_{a\in\mathbb{C}^*} A_{i,a}^{-r_a}$ ($r_a \geq 0$) qui intervient effectivement dans $E_{0,i}(m)$:

$$\alpha(m,M) = \sum_{a\in\mathbb{C}^*} r_a(u_{i,aq_i^{-1}}(m) + u_{i,aq_i}(m) - r_a - r_{aq_i^{-2}})$$

On pose alors pour $m \in B_i$, $E'_{0,i}(m) = \phi_i(E_{0,i}(m))$, et :

Lemme 5.8. *Dans le cas ADE, soit $m \in B'_i$. On a, si $m = \hat{\Pi}_t(m')$, l'égalité :*

$$E'_{0,i}(m) = t^{-d(m',m')}\hat{\Pi}(E_i(m'))$$

On note alors $\mathfrak{K}'_{t,i}$ le $\mathbb{Z}[t^\pm]$-module engendré par les $E_{0,i}(m)'$ avec $m \in B_i$.
Pour $i \in I$, on obtient de la même manière que dans le lemme 2.1 une décomposition :

$$\mathcal{Y}_t = \mathfrak{K}'_{t,i} \oplus \bigoplus_{m\in A-B_i} \mathbb{Z}[t^\pm]m = (\bigoplus_{m\in B_i} \mathbb{Z}[t^\pm]E'_{i,0}(m)) \oplus (\bigoplus_{m\in A-B_i} \mathbb{Z}[t^\pm]m)$$

Dans le cas ADE, on a $\hat{\Pi}(\hat{\mathfrak{K}}_i) \subset \mathfrak{K}'_{t,i}$, on n'a pas égalité dans le cas général, mais :

$$\hat{\Pi}(\hat{\mathfrak{K}}_t) = \bigcap_{i\in I} \mathfrak{K}'_{t,i}$$

comme dans le lemme 5.2 et on notera $\mathfrak{K}'_t$ cette sous-partie de $\mathcal{Y}_t$. Alors on pose :

$$F'_{t,i} = \sum_{a\in\mathbb{C}^*, m\in A} \mathbb{Z}[t^\pm].m(A_{i,aq_i}^{-1} t^{u_{i,aq_i^2}(m)-u_{i,a}(m)} S_{i,aq_i^2} - tS_{i,a})$$

puis $\mathcal{Y}'_{t,i} = \mathcal{Y}^l_{t,i}/F'_{t,i}$, et $S'_{t,i}$ la composée de $S^l_{t,i}$ avec la projection de $\mathcal{Y}^l_{t,i}$ sur $\mathcal{Y}'_{t,i}$.
Dans le cas ADE, on a $\hat{\Pi}^l_{t,i}(\hat{F}_{t,i}) \subset F'_{t,i}$, le diagramme commutatif :

$$\begin{array}{ccc} \hat{\mathcal{Y}}_t & \xrightarrow{\hat{S}_{t,i}} & \hat{\mathcal{Y}}_{t,i} \\ \hat{\Pi}\downarrow & & \downarrow \quad \hat{\Pi}_i \\ \mathcal{Y}_t & \xrightarrow{S'_{t,i}} & \mathcal{Y}'_{t,i} \\ \Pi_t \downarrow & & \downarrow \quad \Pi'_{t,i} \\ \mathcal{Y} & \xrightarrow{S_i} & \mathcal{Y}_i \end{array}$$

et $\mathfrak{K}'_{t,i} \subset \mathrm{Ker}(S'_{t,i})$.

5.5 Noyau des t-opérateurs d'écrantage $S_{t,i}$

Théorème 5.1. *On a $\mathfrak{K}_{t,i} = Ker(S_{t,i})$.*

On pourrait montrer ce résultat de la même manière que $\hat{\mathfrak{K}}_{t,i} = \mathrm{Ker}(\hat{S}_{t,i})$ en utilisant la décomposition de $\mathcal{Y}_t$ du lemme 2.1. Cette méthode permet aussi retrouver le résultat du théorème 2.1 en utilisant la décomposition de $\mathcal{Y}$:

$$\mathcal{Y} = \mathfrak{K}_i \oplus \bigoplus_{m\in A-B_i} \mathbb{Z}m$$

On obtient également dans le cas ADE $\mathfrak{K}'_{t,i} = \mathrm{Ker}(S'_{t,i})$.

On peut également déduire le résultat du théorème 5.1 de celui du théorème 2.1 en notant :

Lemme 5.9. *Le noyau de Π_t est $Ker(\Pi_t) = (t-1)\mathcal{Y}_t$. Si $\alpha \in \mathcal{Y}^l_{t,i}$ vérifie $(t-1)\alpha \in F_{t,i}$ alors $\alpha \in F_{t,i}$.*

6 Compléments relatifs aux involutions

On définit ([N3]) sur $\mathcal{Y}_t$ une involution par $\overline{t} = t^{-1}, \overline{Y^{\pm}_{i,a}} = Y^{\pm}_{i,a}$, et sur $\hat{\mathcal{Y}}_t$ par :

$$\overline{t} = t^{-1} \ , \overline{m} = t^{2d(m,m)}m \ (d \text{ bicaratère})$$

Dans le cas ADE avec d_N, elle est anti multiplicative relativement à $*$ et commute avec $\hat{\Pi}$.

On étend ces involutions à $\mathcal{Y}^l_{t,i}$ (respectivement à $\hat{\mathcal{Y}}^l_{t,i}$) en posant $\overline{S_{i,a}} = t^{-2}S_{i,a}$, soit ($U_a \in \mathcal{Y}_t$ resp $\in \hat{\mathcal{Y}}_t$) :

$$\overline{\sum_{a\in\mathbb{C}^*} U_a S_{i,a}} = \sum_{a\in\mathbb{C}^*} t^{-2}S_{i,a}\overline{U_a}$$

Lemme 6.1. *On a pour $x \in \hat{\mathcal{Y}}_t$ ($y \in \mathcal{Y}_t$), $\overline{S^l_{t,i}(x)} = S^l_{t,i}(\overline{x})$ ($\overline{\hat{S}^l_{t,i}(y)} = \hat{S}^l_{t,i}(\overline{y})$). De plus $\hat{F}_{t,i}, F'_{t,i}$ sont stables par les involutions correpondantes.*

Démonstration :

Les deux résultats s'obtiennent de manière analogue, en considérant par exemple un monôme $a \in \mathcal{Y}_t$:

$\overline{S^l_{t,i}(\lambda(t)m)}$
$= \lambda(t^{-1})(\sum_{a\in\mathbb{C}^* u_{i,a}(m)\geq 0} (1 + t^{-2} + ... + t^{-2(u_{i,a}(m)-1)})t^{-2}S_{i,a}m$
$+ \sum_{a\in\mathbb{C}^* u_{i,a}(m)<0} (t^2 + ... + t^{-2u_{i,a}(m)})t^{-2}S_{i,a}m)$
$= \lambda(t^{-1})m(\sum_{a\in\mathbb{C}^* u_{i,a}(m)\geq 0} (t^{2(u_{i,a}(m)-1)} + ... + t^2 + 1)S_{i,a}$
$+ \sum_{a\in\mathbb{C}^* u_{i,a}(m)<0} (t^{2u_{i,a}(m)} + ... + t^{-2})S_{i,a})$
$= \lambda(t^{-1})S^l_{t,i}(m) = S^l_{t,i}(\overline{\lambda(t)m})$ Considérons ensuite, par exemple dans le cas ADE :

$$\overline{\lambda(t)m(V_{i,aq_i}t^{2u_{i,aq_i^2}(m)}S_{i,aq_i^2} - t^2S_{i,a})}$$

$$= \lambda(t^{-1})t^{-4+2u_{i,a}(m)}m(A_{i,aq_i}^{-1}t^{u_{i,aq_i^2}(m)-u_{i,a}(m)}S_{i,aq_i^2} - tS_{i,a}) \in F'_{t,i}$$

□

Ainsi $\hat{\mathcal{Y}}_{t,i}, \mathcal{Y}'_{t,i}$ héritent d'involutions qui commutent avec les t-opérateurs d'écrantage.

Deuxième partie

Algebraic approach to q, t-characters

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Résumé. Frenkel et Reshetikhin [FR3] ont introduit les q-caractères pour étudier les représentations de dimension finie de l'algèbre affine quantique $\mathcal{U}_q(\hat{\mathfrak{g}})$. Dans le cas simplement lacé, Nakajima [N2, N3] a défini des déformations des q-caracères appelées q, t-caractères. La définition est combinatoire mais la preuve de l'existence utilise la théorie des variétés carquois qui n'existe que dans le cas simplement lacé. Dans cet article nous proposons une nouvelle approche algébrique générale (non nécessairement simplement lacée) pour les q, t-caractères, motivée par les opérateurs d'écrantage déformés [He1]. Les t-déformations sont déduites de la structure de $\mathcal{U}_q(\hat{\mathfrak{g}})$: le paramètre t est analogue à la charge centrale $c \in \mathcal{U}_q(\hat{\mathfrak{g}})$. Ainsi les q, t-caractères permettent de construire une quantification de l'anneau de Grothendieck et des analogues généraux des polynômes de Kazhdan-Lusztig dans le même esprit que Nakajima le fit pour le cas simplement lacé.

Abstract. Frenkel and Reshetikhin [FR3] introduced q-characters to study finite dimensional representations of the quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$. In the simply laced case Nakajima [N2, N3] defined deformations of q-characters called q, t-characters. The definition is combinatorial but the proof of the existence uses the geometric theory of quiver varieties which holds only in the simply laced case. In this article we propose an algebraic general (non necessarily simply laced) new approach to q, t-characters motivated by the deformed screening operators [He1]. The t-deformations are naturally deduced from the structure of $\mathcal{U}_q(\hat{\mathfrak{g}})$: the parameter t is analog to the central charge $c \in \mathcal{U}_q(\hat{\mathfrak{g}})$. The q, t-characters lead to the construction of a quantization of the Grothendieck ring and to general analogues of Kazhdan-Lusztig polynomials in the same spirit as Nakajima did for the simply laced case.

1 Introduction

We suppose $q \in \mathbb{C}^*$ is not a root of unity. In the case of a semi-simple Lie algebra $\mathfrak{g}$, the structure of the Grothendieck ring $\mathrm{Rep}(\mathcal{U}_q(\mathfrak{g}))$ of finite dimensional representations of the quantum algebra $\mathcal{U}_q(\mathfrak{g})$ is well understood. It is analogous to the classical case $q = 1$. In particular we have ring isomorphisms :

$$\mathrm{Rep}(\mathcal{U}_q(\mathfrak{g})) \simeq \mathrm{Rep}(\mathfrak{g}) \simeq \mathbb{Z}[\Lambda]^W \simeq \mathbb{Z}[T_1, ..., T_n]$$

deduced from the injective homomorphism of characters χ :

$$\chi(V) = \sum_{\lambda \in \Lambda} \dim(V_\lambda)\lambda$$

where V_λ are weight spaces of a representation V and Λ is the weight lattice.

For the general case of Kac-Moody algebras the picture is less clear. In the affine case $\mathcal{U}_q(\hat{\mathfrak{g}})$, Frenkel and Reshetikhin [FR3] introduced an injective ring homomorphism of q-characters :

$$\chi_q : \mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}})) \to \mathbb{Z}[Y_{i,a}^{\pm}]_{1 \leq i \leq n, a \in \mathbb{C}^*} = \mathcal{Y}$$

The homomorphism χ_q allows to describe the ring

$$\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}})) \simeq \mathbb{Z}[X_{i,a}]_{i \in I, a \in \mathbb{C}^*}$$

where the $X_{i,a}$ are fundamental representations. It particular $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ is commutative.

The morphism of q-characters has a symmetry property analogous to the classical action of the Weyl group $\mathrm{Im}(\chi) = \mathbb{Z}[\Lambda]^W$: Frenkel and Reshetikhin defined n screening operators S_i such that $\mathrm{Im}(\chi_q) = \bigcap_{i \in I} \mathrm{Ker}(S_i)$ (the result was proved by Frenkel and Mukhin for the general case in [FM1]).

In the simply laced case Nakajima introduced t-analogues of q-characters ([N2], [N3]) : it is a $\mathbb{Z}[t^{\pm}]$-linear map

$$\chi_{q,t} : \mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}})) \otimes_{\mathbb{Z}} \mathbb{Z}[t^{\pm}] \to \mathcal{Y}_t = \mathbb{Z}[Y_{i,a}^{\pm}, t^{\pm}]_{i \in I, a \in \mathbb{C}^*}$$

which is a deformation of χ_q and multiplicative in a certain sense. A combinatorial axiomatic definition of q, t-characters is given. But the existence is non-trivial and is proved with the geometric theory of quiver varieties which holds only in the simply laced case.

In [He1] we introduced t-analogues of screening operators $S_{i,t}$ such that in the simply laced case :

$$\bigcap_{i \in I} \mathrm{Ker}(S_{i,t}) = \mathrm{Im}(\chi_{q,t})$$

It is a first step in the algebraic approach to q, t-characters proposed in this article : we define and construct q, t-characters in the general (non necessarily simply laced) case. The motivation of the construction appears in the non-commutative structure of the Cartan subalgebra $\mathcal{U}_q(\hat{\mathfrak{h}}) \subset \mathcal{U}_q(\hat{\mathfrak{g}})$, the study of screening currents and of deformed screening operators. In particular this article proves a conjecture that Nakajima made for the simply laced case (remark 3.10 in [N3]) : there exists a purely combinatorial proof of the existence of q, t-characters.

As an application we construct a deformed algebra structure, an involution of the Grothendieck ring, and analogues of Kazhdan-Lusztig polynomials in the general case in the same spirit as Nakajima did for the simply laced case.

This article is organized as follows : after some backgrounds in section 2, we define a deformed non-commutative algebra structure on $\mathcal{Y}_t = \mathbb{Z}[Y_{i,a}^{\pm}, t^{\pm}]_{i \in I, a \in \mathbb{C}^*}$ (section 3) : it is naturally deduced from the relations of $\mathcal{U}_q(\hat{\mathfrak{h}}) \subset \mathcal{U}_q(\hat{\mathfrak{g}})$ (theorem 3.11) by using the quantization in the direction of the central element c. In particular in the simply laced case it can be used to construct the deformed multiplication of Nakajima [N3] (proposition 3.16) and of Varagnolo-Vasserot [VV3] (section 3.5.3).

This picture allows us to introduce the deformed screening operators of [He1] as commutators of Frenkel-Reshetikhin's screening currents of [FR2] (section 4). In [He1] we gave explicitly the kernel of each deformed screening operator (theorem 4.10).

In analogy to the classical case where $\mathrm{Im}(\chi_q) = \bigcap_{i \in I} \mathrm{Ker}(S_i)$, we have to describe the intersection of the kernels of deformed screening operators. We introduce a completion of this intersection (section 5.2) and give its structure in proposition 5.17. It is easy to see that it is not too big (lemma 5.7) ; but the point is to prove that it contains enough elements : it is the main result of our construction in theorem 5.11 which is crucial for us. It is proved by induction on the rank n of $\mathfrak{g}$.

We define a t-deformed algorithm (section 5.7.1) analog to the Frenkel -Mukhin's algorithm [FM1] to construct q, t-characters in the completion of $\mathcal{Y}_t$. An algorithm was also used by Nakajima in the simply laced case in order to compute the q, t-characters for some examples ([N2]) assuming they exist (which was geometrically proved). Our aim is different : we do not know *a priori* the existence in the general case. That is why we have to show that the algorithm is well defined, never fails (lemma 5.22) and gives a convenient element (lemma 5.23).

This construction gives q, t-characters for fundamental representations ; we deduce from them the injective morphism of q, t-characters $\chi_{q,t}$ (definition 6.1). We study the properties of $\chi_{q,t}$ (theorem 6.2). Some of them are generalization of the axioms that Nakajima defined in the simply laced case ([N3]) ; in particular we have constructed the morphism of [N3].

We have some applications : the morphism gives a deformation of the Grothendieck ring because the image of $\chi_{q,t}$ is a subalgebra for the deformed multiplication (section 6.2). Moreover we define an antimultiplicative involution of the deformed Grothendieck ring (section 6.3) ; the construction of this involution is motivated by the new point view adopted in this paper : it is just replacing c by $-c$ in $\mathcal{U}_q(\hat{\mathfrak{g}})$. In particular we define constructively analogues of Kazhdan-Lusztig polynomials and a canonical basis (theorem 6.9) motivated by the introduction of [N3]. We compute explicitly the polynomials for some examples.

In section 7 we raise some questions : we conjecture[3] that the coefficients of q, t-characters of fundamental representations are in $\mathbb{N}[t^{\pm}] \subset \mathbb{Z}[t^{\pm}]$. In the ADE-case it a result of Nakajima ; we give an alternative elementary proof for the A-cases in section 7.1. The cases G_2, B_2, C_2 are also checked in the appendix. The cases F_4, B_n, C_n $(n \leq 10)$ have been checked on a computer.

We also conjecture that the generalized analogues to Kazhdan-Lusztig polynomials give at $t = 1$ the multiplicity of simple modules in standard modules. We propose some generalizations and further applications which will be studied elsewhere.

[3]This conjecture is proved later in [He5].

In the appendix we give explicit computations of q,t-characters for Lie algebras of rank 2. They are used in the proof of theorem 5.11.

2 Background

2.1 Cartan matrix

A generalized Cartan matrix of rank n is a matrix $C = (C_{i,j})_{1\leq i,j\leq n}$ such that $C_{i,j} \in \mathbb{Z}$ and $(i \neq j)$:

$$C_{i,i} = 2 \ , \ C_{i,j} \leq 0 \ , \ C_{i,j} = 0 \Leftrightarrow C_{j,i} = 0$$

Let $I = \{1, ..., n\}$. We say that C is symmetrizable if there is a matrix $D = \mathrm{diag}(r_1, ..., r_n)$ $(r_i \in \mathbb{N}^*)$ such that $B = DC$ is symmetric.

Let $q \in \mathbb{C}^*$ be the parameter of quantization. In the following we suppose it is not a root of unity. z is an indeterminate. If C is symmetrizable, let $q_i = q^{r_i}$, $z_i = z^{r_i}$ and $C(z) = (C(z)_{i,j})_{1\leq i,j\leq n}$ the matrix with coefficients in $\mathbb{Z}[z^{\pm}]$ such that $(i \neq j)$:

$$C(z)_{i,j} = [C_{i,j}]_z \ , \ C(z)_{i,i} = [C_{i,i}]_{z_i} = z_i + z_i^{-1}$$

where for $l \in \mathbb{Z}$ we use the notation $[l]_z = \frac{z^l - z^{-l}}{z - z^{-1}}$. In particular, the coefficients of $C(z)$ are symmetric Laurent polynomials (invariant under $z \mapsto z^{-1}$). We define the diagonal matrix $D_{i,j}(z) = \delta_{i,j}[r_i]_z$ and the matrix $B(z) = D(z)C(z)$.

In the following we suppose that C is of finite type, in particular $\det(C) \neq 0$. In this case C is symmetrizable; if C is indecomposable there is a unique choice of $r_i \in \mathbb{N}^*$ such that $r_1 \wedge ... \wedge r_n = 1$. We have $B_{i,j}(z) = [B_{i,j}]_z$ and $B(z)$ is symmetric. See [Bo] or [Kac] for a classification of those finite Cartan matrices.

C is said to be simply-laced if $r_1 = ... = r_n = 1$. In this case C and $C(z) = B(z)$ are symmetric. In the classification those matrices are of type ADE.

Denote by $\mathfrak{U} \subset \mathbb{Q}(z)$ the subgroup $\mathbb{Z}$-linearly spanned by the $\frac{P(z)}{Q(z^{-1})}$ such that $P(z) \in \mathbb{Z}[z^{\pm}]$, $Q(z) \in \mathbb{Z}[z]$, the zeros of $Q(z)$ are roots of unity and $Q(0) = 1$. It is a subring of $\mathbb{Q}(z)$, and for $R(z) \in \mathfrak{U}, m \in \mathbb{Z}$ we have $R(z^m) \in \mathfrak{U}$ and $R(q^m) \in \mathbb{C}$ makes sense. It follows from lemma 1.1 of [FM1] that $C(z)$ has inverse $\tilde{C}(z)$ with coefficients of the form $R(z) \in \mathfrak{U}$.

2.2 Finite quantum algebras

We refer to [R] for the definition of the finite quantum algebra $\mathcal{U}_q(\mathfrak{g})$ associated to a finite Cartan matrix, the definition and properties of the type 1-representations of $\mathcal{U}_q(\mathfrak{g})$, the Grothendieck ring $\mathrm{Rep}(\mathcal{U}_q(\mathfrak{g}))$ and the injective ring morphism of characters $\chi : \mathrm{Rep}(\mathcal{U}_q(\mathfrak{g})) \to \mathbb{Z}[y_i^{\pm}]$.

2.3 Quantum affine algebras

The quantum affine algebra associated to a finite Cartan matrix C is a $\mathbb{C}$-algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$ with generators $x_{i,m}^{\pm}$ $(i \in I, m \in \mathbb{Z})$, $k_i^{\pm}$ $(i \in I)$, $h_{i,m}$ $(i \in I, m \in \mathbb{Z}^*)$, central elements $c^{\pm\frac{1}{2}}$. We refer to [FR3] for the relations and the definition of the $\phi_{i,m}^{\pm} \in \mathcal{U}_q(\hat{\mathfrak{g}})$ $(i \in I, m \in \mathbb{Z})$. In particular we will use the relations :

$$[h_{i,m}, h_{j,m'}] = \delta_{m,-m'}\frac{1}{m}[mB_{ij}]_q\frac{c^m - c^{-m}}{q - q^{-1}}$$

One has an embedding $\mathcal{U}_q(\mathfrak{g}) \subset \mathcal{U}_q(\hat{\mathfrak{g}})$ and a Hopf algebra structure on $\mathcal{U}_q(\hat{\mathfrak{g}})$ (see [FR3] for example). The Cartan subalgebra $\mathcal{U}_q(\hat{\mathfrak{h}}) \subset \mathcal{U}_q(\hat{\mathfrak{g}})$ is the $\mathbb{C}$-subalgebra of $\mathcal{U}_q(\hat{\mathfrak{g}})$ generated by the $h_{i,m}, c^{\pm}$ $(i \in I, m \in \mathbb{Z} - \{0\})$.

2.4 Finite dimensional representations of $\mathcal{U}_q(\hat{\mathfrak{g}})$

A finite dimensional representation V of $\mathcal{U}_q(\hat{\mathfrak{g}})$ is called of type 1 if c acts as Id and V is of type 1 as a representation of $\mathcal{U}_q(\mathfrak{g})$. Denote by $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ the Grothendieck ring of finite dimensional representations of type 1. The operators $\{\phi_{i,\pm m}^{\pm}, i \in I, m \in \mathbb{Z}\}$ commute on a such V. So we have a pseudo-weight space decomposition :

$$V = \bigoplus_{\gamma \in \mathbb{C}^{I\times\mathbb{Z}} \times \mathbb{C}^{I\times\mathbb{Z}}} V_\gamma$$

where for $\gamma = (\gamma^+, \gamma^-)$, V_γ is a simultaneous generalized eigenspace :

$$V_\gamma = \{x \in V / \exists p \in \mathbb{N}, \forall i \in \{1, ..., n\}, \forall m \in \mathbb{Z}, (\phi_{i,m}^{\pm} - \gamma_{i,m}^{\pm})^p.x = 0\}$$

The $\gamma_{i,m}^{\pm}$ are called pseudo-eigen values of V.

Theorem 2.1. *(***Chari, Pressley** *[CP3, CP4]) Every simple representation V in Rep($\mathcal{U}_q(\hat{\mathfrak{g}})$) is a highest weight representation, that is to say there is $v_0 \in V$ (highest weight vector) $\gamma_{i,m}^{\pm} \in \mathbb{C}$ (highest weight) such that :*

$$V = \mathcal{U}_q(\hat{\mathfrak{g}}).v_0 \ , \ c^{\frac{1}{2}}.v_0 = v_0$$

$$\forall i \in I, m \in \mathbb{Z}, x_{i,m}^{+}.v_0 = 0 \ , \ \ , \ \phi_{i,m}^{\pm}.v_0 = \gamma_{i,m}^{\pm} v_0$$

Moreover we have an I-uplet $(P_i(u))_{i\in I}$ of (Drinfel'd-)polynomials such that $P_i(0) = 1$ and :

$$\gamma_i^{\pm}(u) = \sum_{m\in\mathbb{N}} \gamma_{i,\pm m}^{\pm} u^{\pm} = q_i^{\deg(P_i)}\frac{P_i(uq_i^{-1})}{P_i(uq_i)} \in \mathbb{C}[[u^{\pm}]]$$

and $(P_i)_{i\in I}$ parameterizes simple modules in Rep($\mathcal{U}_q(\hat{\mathfrak{g}})$).

Theorem 2.2. *(***Frenkel, Reshetikhin** *[FR3]) The eigenvalues $\gamma_i(u)^{\pm} \in \mathbb{C}[[u^{\pm}]]$ of a representation $V \in$ Rep($\mathcal{U}_q(\hat{\mathfrak{g}})$) have the form :*

$$\gamma_i^{\pm}(u) = q_i^{deg(Q_i)-deg(R_i)}\frac{Q_i(uq_i^{-1})R_i(uq_i)}{Q_i(uq_i)R_i(uq_i^{-1})}$$

where $Q_i(u), R_i(u) \in \mathbb{C}[u]$ and $Q_i(0) = R_i(0) = 1$.

Note that the polynomials Q_i, R_i are uniquely defined by γ. Denote by $Q_{\gamma,i}$, $R_{\gamma,i}$ the polynomials associated to γ.

2.5 q-characters

Let $\mathcal{Y}$ be the commutative ring $\mathcal{Y} = \mathbb{Z}[Y_{i,a}^{\pm}]_{i\in I, a\in\mathbb{C}^*}$.

Definition 2.3. *For $V \in Rep(\mathcal{U}_q(\hat{\mathfrak{g}}))$ a representation, the q-character $\chi_q(V)$ of V is :*

$$\chi_q(V) = \sum_{\gamma} dim(V_\gamma) \prod_{i\in I, a\in\mathbb{C}^*} Y_{i,a}^{\lambda_{\gamma,i,a}-\mu_{\gamma,i,a}} \in \mathcal{Y}$$

where for $\gamma \in \mathbb{C}^{I\times\mathbb{Z}} \times \mathbb{C}^{I\times\mathbb{Z}}$, $i \in I$, $a \in \mathbb{C}^*$ the $\lambda_{\gamma,i,a}, \mu_{\gamma,i,a} \in \mathbb{Z}$ are defined by :

$$Q_{\gamma,i}(z) = \prod_{a\in\mathbb{C}^*} (1 - za)^{\lambda_{\gamma,i,a}} \ , \ R_{\gamma,i}(z) = \prod_{a\in\mathbb{C}^*} (1 - za)^{\mu_{\gamma,i,a}}$$

Theorem 2.4. *(**Frenkel, Reshetikhin** [FR3]) The map $\chi_q : Rep(\mathcal{U}_q(\hat{\mathfrak{g}})) \to \mathcal{Y}$ is an injective ring homomorphism and the following diagram is commutative :*

$$\begin{array}{ccc} Rep(\mathcal{U}_q(\hat{\mathfrak{g}})) & \xrightarrow{\chi_q} & \mathbb{Z}[Y_{i,a}^{\pm}]_{i\in I,a\in\mathbb{C}^*} \\ \downarrow res & & \downarrow \beta \\ Rep(\mathcal{U}_q(\mathfrak{g})) & \xrightarrow{\chi} & \mathbb{Z}[y_i^{\pm}]_{i\in I} \end{array}$$

where β is the ring homomorphism such that $\beta(Y_{i,a}) = y_i$ ($i \in I, a \in \mathbb{C}^*$).

For $m \in \mathcal{Y}$ of the form $m = \prod_{i\in I,a\in\mathbb{C}^*} Y_{i,a}^{u_{i,a}(m)}$ ($u_{i,a}(m) \geq 0$), denote by $V_m \in \text{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ the simple module with Drinfel'd polynomials $P_i(u) = \prod_{a\in\mathbb{C}^*} (1 - ua)^{u_{i,a}(m)}$. In particular for $i \in I, a \in \mathbb{C}^*$ let $V_{i,a} = V_{Y_{i,a}}$ and $X_{i,a} = \chi_q(V_{i,a})$. The simple modules $V_{i,a}$ are called fundamental representations.

Let $M_m = \bigotimes_{i\in I,a\in\mathbb{C}^*} V_{i,a}^{\otimes u_{i,a}(m)}$. It is called a standard module and his q-character is $\prod_{i\in I,a\in\mathbb{C}^*} X_{i,a}^{u_{i,a}(m)}$.

Corollary 2.5. *(**Frenkel, Reshetikhin** [FR3]) The ring $Rep(\mathcal{U}_q(\hat{\mathfrak{g}}))$ is commutative and isomorphic to $\mathbb{Z}[X_{i,a}]_{i\in I,a\in\mathbb{C}^*}$.*

Proposition 2.6. *(**Frenkel, Mukhin** [FM1]) For $i \in I, a \in \mathbb{C}^*$, we have $X_{i,a} \in \mathbb{Z}[Y_{j,aq^l}^{\pm}]_{j\in I,l\geq 0}$.*

In particular for $a \in \mathbb{C}^*$ we have an injective ring homomorphism :

$$\chi_q^a : \text{Rep}_a = \mathbb{Z}[X_{i,aq^l}]_{i\in I,l\in\mathbb{Z}} \to \mathcal{Y}_a = \mathbb{Z}[Y_{i,aq^l}^{\pm}]_{i\in I,l\in\mathbb{Z}}$$

For $a, b \in \mathbb{C}^*$ let $\alpha_{b,a} : \text{Rep}_a \to \text{Rep}_b$ and $\beta_{b,a} : \mathcal{Y}_a \to \mathcal{Y}_b$ be the canonical ring homomorphism. As a consequence of theorem 4.2 (or see [FR3, FM1]), we have $\beta_{b,a} \circ \chi_q^a = \chi_q^b \circ \alpha_{b,a}$. In particular it suffices to study χ_q^1. In the following denote $\text{Rep} = \text{Rep}_1$, $X_{i,l} = X_{i,q^l}$, $\mathcal{Y} = \mathcal{Y}_1$ and $\chi_q = \chi_q^1 : \text{Rep} \to \mathcal{Y}$.

3 Twisted polynomial algebras related to quantum affine algebras

The aim of this section is to define the t-deformed algebra $\mathcal{Y}_t$ and to describe its structure (theorem 3.11). We define the Heisenberg algebra $\mathcal{H}$, the subalgebra $\mathcal{Y}_u \subset \mathcal{H}[[h]]$ and eventually $\mathcal{Y}_t$ as a quotient of $\mathcal{Y}_u$.

3.1 Heisenberg algebras related to quantum affine algebras

Definition 3.1. *$\mathcal{H}$ is the $\mathbb{C}$-algebra defined by generators $a_i[m]$ ($i \in I, m \in \mathbb{Z} - \{0\}$), central elements c_r ($r > 0$) and relations ($i,j \in I, m,r \in \mathbb{Z} - \{0\}$) :*

$$[a_i[m], a_j[r]] = \delta_{m,-r}(q^m - q^{-m})B_{i,j}(q^m)c_{|m|}$$

This definition is motivated by the structure of $\mathcal{U}_q(\hat{\mathfrak{g}})$: in $\mathcal{H}$ the c_r are algebraically independent, but we have a surjective homomorphism from $\mathcal{H}$ to $\mathcal{U}_q(\hat{\mathfrak{h}})$ such that $a_i[m] \mapsto (q-q^{-1})h_{i,m}$ and $c_r \mapsto \frac{c^r - c^{-r}}{r}$.
For $j \in I, m \in \mathbb{Z}$ we set $y_j[m] = \sum\limits_{i \in I} \tilde{C}_{i,j}(q^m)a_i[m] \in \mathcal{H}$.

Lemma 3.2. *We have the Lie brackets in $\mathcal{H}$ ($i,j \in I, m,r \in \mathbb{Z}$) :*

$$[a_i[m], y_j[r]] = (q^{mr_i} - q^{-r_i m})\delta_{m,-r}\delta_{i,j}c_{|m|}$$

$$[y_i[m], y_j[r]] = \delta_{m,-r}\tilde{C}_{j,i}(q^m)(q^{mr_j} - q^{-mr_j})c_{|m|}$$

Let π_+ and π_- be the $\mathbb{C}$-algebra endomorphisms of $\mathcal{H}$ such that ($i \in I$, $m > 0$, $r < 0$) :

$$\pi_+(a_i[m]) = a_i[m] \ , \ \pi_+(a_i[r]) = 0 \ , \ \pi_+(c_m) = 0$$

$$\pi_-(a_i[m]) = 0 \ , \ \pi_-(a_i[r]) = a_i[r] \ , \ \pi_-(c_m) = 0$$

They are well-defined because the relations are preserved. We set $\mathcal{H}^+ = \mathrm{Im}(\pi_+) \subset \mathcal{H}$ and $\mathcal{H}^- = \mathrm{Im}(\pi_-) \subset \mathcal{H}$. Note that $\mathcal{H}^+$ (resp. $\mathcal{H}^-$) is the subalgebra of $\mathcal{H}$ generated by the $a_i[m]$, $i \in I, m > 0$ (resp. $m < 0$). So $\mathcal{H}^+$ and $\mathcal{H}^-$ are commutative algebras, and :

$$\mathcal{H}^+ \simeq \mathcal{H}^- \simeq \mathbb{C}[a_i[m]]_{i \in I, m > 0}$$

A product of the $a_i[m], c_r$ is called a $\mathcal{H}$-monomial.

Lemma 3.3. *There is a unique $\mathbb{C}$-linear endomorphism $::$ of $\mathcal{H}$ such that for all m $\mathcal{H}$-monomials : $m := \pi_+(m)\pi_-(m)$.*

In particular there is a triangular decomposition $\mathcal{H} \simeq \mathcal{H}^+ \otimes \mathbb{C}[c_r]_{r>0} \otimes \mathcal{H}^-$.

Proof : The $\mathcal{H}$-monomials span the $\mathbb{C}$-vector space $\mathcal{H}$, so the map is unique. But there are non trivial linear combinations between them because of the relations of $\mathcal{H}$: it suffices to show that for m_1, m_2 $\mathcal{H}$-monomials the definition of :: is compatible with the relations $(i, j \in I,\, l, k \in \mathbb{Z} - \{0\})$:

$$m_1 a_i[k]a_j[l]m_2 - m_1 a_j[l]a_i[k]m_2 = \delta_{k,-l}(q^k - q^{-k})B_{i,j}(q^k)m_1 c_{|k|}m_2$$

We can conclude because $\mathcal{H}^+$ and $\mathcal{H}^-$ are commutative and :

$$\pi_+(m_1 c_{|k|}m_2) = \pi_-(m_1 c_{|k|}m_2) = 0$$

□

3.2 The deformed algebra $\mathcal{Y}_u$

3.2.1 Construction of $\mathcal{Y}_u$

Consider the $\mathbb{C}$-algebra $\mathcal{H}_h = \mathcal{H}[[h]]$ and the application $\exp : h\mathcal{H}_h \to \mathcal{H}_h$. For $l \in \mathbb{Z}$, $i \in I$, introduce $\tilde{A}_{i,l}, \tilde{Y}_{i,l} \in \mathcal{H}_h$ such that :

$$\tilde{A}_{i,l} = \exp(\sum_{m>0} h^m a_i[m]q^{lm})\exp(\sum_{m>0} h^m a_i[-m]q^{-lm})$$

$$\tilde{Y}_{i,l} = \exp(\sum_{m>0} h^m y_i[m]q^{lm})\exp(\sum_{m>0} h^m y_i[-m]q^{-lm})$$

Note that $\tilde{A}_{i,l}$ and $\tilde{Y}_{i,l}$ are invertible in $\mathcal{H}_h$. Recall the definition $\mathfrak{U} \subset \mathbb{Q}(z)$ of section 2.1. For $R \in \mathfrak{U}$, introduce $t_R \in \mathcal{H}_h$:

$$t_R = \exp(\sum_{m>0} h^{2m} R(q^m)c_m)$$

Definition 3.4. *$\mathcal{Y}_u$ is the $\mathbb{Z}$-subalgebra of $\mathcal{H}_h$ generated by the $\tilde{Y}_{i,l}^{\pm}, \tilde{A}_{i,l}^{\pm}, t_R$ ($i \in I, l \in \mathbb{Z}, R \in \mathfrak{U}$).*

In this section we give properties of $\mathcal{Y}_u$ and subalgebras of $\mathcal{Y}_u$ which will be useful in section 3.3.

3.2.2 Relations in $\mathcal{Y}_u$

Lemma 3.5. *We have the following relations in $\mathcal{Y}_u$ ($i, j \in I$ $l, k \in \mathbb{Z}$) :*

$$\tilde{A}_{i,l}\tilde{Y}_{j,k}\tilde{A}_{i,l}^{-1}\tilde{Y}_{j,k}^{-1} = t_{\delta_{i,j}(z^{-r_i} - z^{r_i})(-z^{(l-k)} + z^{(k-l)})} \tag{3}$$

$$\tilde{Y}_{i,l}\tilde{Y}_{j,k}\tilde{Y}_{i,l}^{-1}\tilde{Y}_{j,k}^{-1} = t_{\tilde{C}_{j,i}(z)(z^{r_j} - z^{-r_j})(-z^{(l-k)} + z^{(k-l)})} \tag{4}$$

$$\tilde{A}_{i,l}\tilde{A}_{j,k}\tilde{A}_{i,l}^{-1}\tilde{A}_{j,k}^{-1} = t_{B_{i,j}(z)(z^{-1} - z)(-z^{(l-k)} + z^{(k-l)})} \tag{5}$$

Proof : For $A, B \in h\mathcal{H}_h$ such that $[A, B] \in h\mathbb{C}[c_r]_{r>0}$, we have :

$$\exp(A)\exp(B) = \exp(B)\exp(A)\exp([A, B])$$

So a straightforward computation and lemma 3.2 give the result. □

3.2.3 Commutative subalgebras of $\mathcal{H}_h$

The $\mathbb{C}$-algebra endomorphisms π_+, π_- of $\mathcal{H}$ are naturally extended to $\mathbb{C}$-algebra endomorphisms of $\mathcal{H}_h$. As $\mathcal{Y}_u \subset \mathcal{H}_h$, we have by restriction the $\mathbb{Z}$-algebra morphisms $\pi_\pm : \mathcal{Y}_u \to \mathcal{H}_h$. Introduce $\mathcal{Y} = \pi_+(\mathcal{Y}_u) \subset \mathcal{H}^+[[h]]$. In this section 3.2.3 we study $\mathcal{Y}$. In particular we will see in proposition 3.8 that the notation $\mathcal{Y}$ is consistent with section 2.5. For $i \in I, l \in \mathbb{Z}$, denote :

$$Y_{i,l}^{\pm} = \pi_+(\tilde{Y}_{i,l}^{\pm}) = \exp(\pm\sum_{m>0} h^m y_i[m] q^{lm})$$

$$A_{i,l}^{\pm} = \pi_+(\tilde{A}_{i,l}^{\pm}) = \exp(\pm\sum_{m>0} h^m a_i[m] q^{lm})$$

Lemma 3.6. *For $i \in I, l \in \mathbb{Z}$, we have :*

$$A_{i,l} = Y_{i,l-r_i} Y_{i,l+r_i} (\prod_{j/C_{j,i}=-1} Y_{j,l}^{-1})(\prod_{j/C_{j,i}=-2} Y_{j,l+1}^{-1} Y_{j,l-1}^{-1})(\prod_{j/C_{j,i}=-3} Y_{j,l+2}^{-1} Y_{j,l}^{-1} Y_{j,l-2}^{-1})$$

In particular $\mathcal{Y}$ is generated by the $Y_{i,l}^{\pm}$ ($i \in I, l \in \mathbb{Z}$).

The formula already appeared in [FR3]. We have the last point because $\mathcal{Y}_u$ is generated by the $\tilde{Y}_{i,l}^{\pm}, \tilde{A}_{i,l}^{\pm}, t_R$. We need a technical lemma to describe $\mathcal{Y}$:

Lemma 3.7. *Let $J = \{1, ..., r\}$ and let Λ be the polynomial commutative algebra $\Lambda = \mathbb{C}[\lambda_{j,m}]_{j \in J, m \geq 0}$. For $R = (R_1, ..., R_r) \in \mathfrak{U}^r$, consider :*

$$\Lambda_R = exp(\sum_{j \in J, m>0} h^m R_j(q^m) \lambda_{j,m}) \in \Lambda[[h]]$$

Then the $(\Lambda_R)_{R \in \mathfrak{U}^r}$ are $\mathbb{C}$-linearly independent. In particular the $\Lambda_{j,l} = \Lambda_{(0,...,0,z^l,0,...,0)}$ ($j \in J$, $l \in \mathbb{Z}$) are $\mathbb{C}$-algebraically independent.

Proof : Suppose we have a linear combination ($\mu_R \in \mathbb{C}$, only a finite number of $\mu_R \neq 0$) :

$$\sum_{R \in \mathfrak{U}^r} \mu_R \Lambda_R = 0$$

The coefficients of h^L in Λ_R are of the form

$$R_{j_1}(q^{l_1})^{L_1} R_{j_2}(q^{l_2})^{L_2} ... R_{j_N}(q^{l_N})^{L_N} \lambda_{j_1,l_1}^{L_1} \lambda_{j_2,l_2}^{L_2} ... \lambda_{j_N,l_N}^{L_N}$$

where $l_1 L_1 + ... + l_N L_N = L$. So for $N \geq 0$, $j_1, ..., j_N \in J$, $l_1, ..., l_N > 0$, $L_1, ..., L_N \geq 0$ we have :

$$\sum_{R \in \mathfrak{U}^r} \mu_R R_{j_1}(q^{l_1})^{L_1} R_{j_2}(q^{l_2})^{L_2} ... R_{j_N}(q^{l_N})^{L_N} = 0$$

If we fix $L_2, ..., L_N$, we have for all $L_1 = l \geq 0$:

$$\sum_{\alpha_1 \in \mathbb{C}} \alpha_1^l \sum_{R \in \mathfrak{U}^r / R_{j_1}(q^{l_1}) = \alpha_1} \mu_R R_{j_2}(q^{l_2})^{L_2} ... R_{j_N}(q^{l_N})^{L_N} = 0$$

We get a Van der Monde system which is invertible, so for all $\alpha_1 \in \mathbb{C}$:

$$\sum_{R\in\mathfrak{U}^r/R_{j_1}(q^{l_1})=\alpha_1} \mu_R R_{j_2}(q^{l_2})^{L_2}...R_{j_N}(q^{l_N})^{l_N} = 0$$

By induction we get for $r' \leq N$ and all $\alpha_1, ..., \alpha_{r'} \in \mathbb{C}$:

$$\sum_{R\in\mathfrak{U}^r/R_{j_1}(q^{l_1})=\alpha_1,...,R_{j_{r'}}(q^{l_{r'}})=\alpha_{r'}} \mu_R R_{j_{r'+1}}(q^{l_{r'+1}})^{L_{r'+1}}...R_{j_N}(q^{l_N})^{L_N} = 0$$

And so for $r' = N$:

$$\sum_{R\in\mathfrak{U}^r/R_{j_1}(q^{l_1})=\alpha_1,...,R_{j_N}(q^{l_N})=\alpha_N} \mu_R = 0$$

Let be $S \geq 0$ such that for all $\mu_R, \mu_{R'} \neq 0$, $j \in J$ we have $R_j - R'_j = 0$ or $R_j - R'_j$ has at most $S-1$ roots. We set $N = Sr$ and :

$$((j_1, l_1), ..., (j_S, l_S)) = ((1,1), (1,2), ..., (1,S), (2,1), ..., (2,S), (3,1), ..., (r,S))$$

We get for all $\alpha_{j,l} \in \mathbb{C}$ $(j \in J, 1 \leq l \leq S)$:

$$\sum_{R\in\mathfrak{U}^r/\forall j\in J, 1\leq l\leq S, R_j(q^l)=\alpha_{j,l}} \mu_R = 0$$

It suffices to show that there is at most one term is this sum. But consider $P, Q \in \mathfrak{U}$ such that for all $1 \leq l \leq S$, $P(q^l) = P'(q^l)$. As q is not a root of unity the q^l are different and $P - P'$ has S roots, so is 0.

For the last assertion, we can write a monomial $\prod\limits_{j\in J, l\in\mathbb{Z}} \Lambda_{j,l}^{u_{j,l}} = \Lambda_{\sum\limits_{l\in\mathbb{Z}} u_{1,l}z^l, ..., \sum\limits_{l\in\mathbb{Z}} u_{r,l}z^l}$. In particular there is no trivial linear combination between those monomials. $\square$

It follows from lemma 3.6 and lemma 3.7 :

Proposition 3.8. *The $Y_{i,l} \in \mathcal{Y}$ are $\mathbb{Z}$-algebraically independent and generate the $\mathbb{Z}$-algebra $\mathcal{Y}$. In particular, $\mathcal{Y}$ is the commutative polynomial algebra $\mathbb{Z}[Y_{i,l}^{\pm}]_{i\in I, l\in\mathbb{Z}}$.*

The $A_{i,l}^{-1} \in \mathcal{Y}$ are $\mathbb{Z}$-algebraically independent. In particular the subalgebra of $\mathcal{Y}$ generated by the $A_{i,l}^{-1}$ is the commutative polynomial algebra $\mathbb{Z}[A_{i,l}^{-1}]_{i\in I, l\in\mathbb{Z}}$.

3.2.4 Generators of $\mathcal{Y}_u$

The $\mathbb{C}$-linear endomorphism :: of $\mathcal{H}$ is naturally extended to a $\mathbb{C}$-linear endomorphism of $\mathcal{H}_h$. As $\mathcal{Y}_u \subset \mathcal{H}_h$, we have by restriction a $\mathbb{Z}$-linear morphism :: from $\mathcal{Y}_u$ to $\mathcal{H}_h$. We say that $m \in \mathcal{Y}_u$ is a $\mathcal{Y}_u$-monomial if it is a product of generators $\tilde{A}_{i,l}^{\pm}, \tilde{Y}_{i,l}^{\pm}, t_R$. In the following, for a product of non commuting terms, denote $\overrightarrow{\prod\limits_{s=1..S}} U_s = U_1 U_2 ... U_S$.

Lemma 3.9. *The algebra $\mathcal{Y}_u$ is generated by the $\tilde{Y}_{i,l}^{\pm}, t_R$ $(i \in I, l \in \mathbb{Z}, R \in \mathfrak{U})$.*

Proof : Let be $i \in I$, $l \in \mathbb{Z}$. It follows from proposition 3.8 that $\pi_+(\tilde{A}_{i,l})$ is of the form $\pi_+(\tilde{A}_{i,l}) = \prod_{i\in I, l\in\mathbb{Z}} Y_{i,l}^{u_{i,l}}$ and that $: m :=: \overrightarrow{\prod_{l\in\mathbb{Z}}}\prod_{i\in I}\tilde{Y}_{i,l}^{u_{i,l}} :$. So it suffices to show that for m a $\mathcal{Y}_u$-monomial, there is a unique $R_m \in \mathfrak{U}$ such that $m = t_{R_m} : m :$. Let us write $m = t_R \overrightarrow{\prod_{s=1..S}} U_s$ where $U_s \in \{\tilde{A}_{i,l}^{\pm}, \tilde{Y}_{i,l}^{\pm}\}_{i\in I, l\in\mathbb{Z}}$ are generators. Then :

$$: m := (\prod_{s=1..S} \pi_+(U_s))(\prod_{s=1..S} \pi_-(U_s))$$

And we can conclude because it follows from the proof of lemma 3.5 that for $1 \leq s, s' \leq S$, there is $R_{s,s'} \in \mathfrak{U}$ such that

$$\pi_+(U_s)\pi_-(U_{s'}) = t_{R_{s,s'}}\pi_-(U_{s'})\pi_+(U_s)$$

□

In particular it follows from this proof that $: \mathcal{Y}_u :\subset \mathcal{Y}_u$.

3.3 The deformed algebra $\mathcal{Y}_t$

Denote by $\mathbb{Z}((z^{-1}))$ the ring of series of the form $P = \sum_{r\leq R_P} P_r z^r$ where $R_P \in \mathbb{Z}$ and the coefficients $P_r \in \mathbb{Z}$. Recall the definition $\mathfrak{U}$ of section 2.1. We have an embedding $\mathfrak{U} \subset \mathbb{Z}((z^{-1}))$ by expanding $\frac{1}{Q(z^{-1})}$ in $\mathbb{Z}[[z^{-1}]]$ for $Q(z) \in \mathbb{Z}[z]$ such that $Q(0) = 1$. So we can introduce maps :

$$\pi_r : \mathfrak{U} \to \mathbb{Z} \text{ , } P = \sum_{k\leq R_P} P_k z^k \mapsto P_r$$

Definition 3.10. *We define $\mathcal{Y}_t$ (resp. $\mathcal{H}_t$) as the algebra quotient of $\mathcal{Y}_u$ (resp. $\mathcal{H}_h$) by relations :*

$$t_R = t_{R'} \textit{ if } \pi_0(R) = \pi_0(R')$$

We keep the notations $\tilde{Y}_{i,l}^{\pm}, \tilde{A}_{i,l}^{\pm}$ for their image in $\mathcal{Y}_t$. Denote by t the image of $t_1 = \exp(\sum_{m>0} h^{2m} c_m)$ in $\mathcal{Y}_t$. As π_0 is additive, the image of t_R in $\mathcal{Y}_t$ is $t^{\pi_o(R)}$. In particular $\mathcal{Y}_t$ is generated by the $\tilde{Y}_{i,l}^{\pm}, \tilde{A}_{i,l}^{\pm}, t^{\pm}$.

As the defining relations of $\mathcal{H}_t$ involve only the c_l and $\pi_+(c_l) = \pi_-(c_l) = 0$, the algebra endomorphisms π_+, π_- of $\mathcal{H}_t$ are well-defined. So we can define $\mathcal{H}_t^+, \mathcal{H}_t^-, \mathcal{Y}_t^+, \mathcal{Y}_t^-$ in the same way as in section 3.2.3 and :: a $\mathbb{C}$-linear endomorphism of $\mathcal{H}_t$ as in section 3.2.4. The $\mathbb{Z}[t^{\pm}]$-subalgebra $\mathcal{Y}_t \subset \mathcal{H}_t$ verifies $: \mathcal{Y}_t :\subset \mathcal{Y}_t$ (proof of lemma 3.9). We have $\mathcal{Y}_t^+ \simeq \mathcal{Y}$. We say that $m \in \mathcal{Y}_t$ (resp. $m \in \mathcal{Y}$) is a $\mathcal{Y}_t$-monomial (resp. a $\mathcal{Y}$-monomial) if it is a product of the generators $\tilde{Y}_{i,m}^{\pm}, t^{\pm}$ (resp. $Y_{i,m}^{\pm}$).

Theorem 3.11. *The algebra $\mathcal{Y}_t$ is defined by generators $\tilde{Y}_{i,l}^{\pm}$ ($i \in I, l \in \mathbb{Z}$), central elements $t^{\pm}$ and relations ($i, j \in I, k, l \in \mathbb{Z}$) :*

$$\tilde{Y}_{i,l}\tilde{Y}_{j,k} = t^{\gamma(i,l,j,k)}\tilde{Y}_{j,k}\tilde{Y}_{i,l}$$

where $\gamma : (I \times \mathbb{Z})^2 \to \mathbb{Z}$ *is given by :*

$$\gamma(i,l,j,k) = \sum_{r\in\mathbb{Z}} \pi_r(\tilde{C}_{j,i}(z))(-\delta_{l-k,-r_j-r} - \delta_{l-k,r-r_j} + \delta_{l-k,r_j-r} + \delta_{l-k,r_j+r})$$

Proof : As the image of t_R in $\mathcal{Y}_t$ is $t^{\pi_o(R)}$, we can deduce the relations from lemma 3.5. For example we use formula 8 (p. 92) :

$$\pi_0(\tilde{C}_{j,i}(z)(z^{r_j} - z^{-r_j})(-z^{(l-k)} + z^{(k-l)})) = \gamma(i,l,j,k)$$

It follows from lemma 3.6 that $\mathcal{Y}_t$ is generated by the $\tilde{Y}_{i,l}^{\pm}, t^{\pm}$.

It follows from lemma 3.7 that the $t_R \in \mathcal{Y}_u$ ($R \in \mathfrak{U}$) are $\mathbb{Z}$-linearly independent. So the $\mathbb{Z}$-algebra $\mathbb{Z}[t_R]_{R\in\mathfrak{U}}$ is defined by generators $(t_R)_{R\in\mathfrak{U}}$ and relations $t_{R+R'} = t_R t_{R'}$ for $R, R' \in \mathfrak{U}$. In particular the image of $\mathbb{Z}[t_R]_{R\in\mathfrak{U}}$ in $\mathcal{Y}_t$ is $\mathbb{Z}[t^{\pm}]$.

Let A be the classes of $\mathcal{Y}_t$-monomials modulo $t^{\mathbb{Z}}$. So we have :

$$\sum_{m\in A} \mathbb{Z}[t^{\pm}].m = \mathcal{Y}_t$$

We prove the sum is direct : suppose we have a linear combination $\sum_{m\in A} \lambda_m(t)m = 0$ where $\lambda_m(t) \in \mathbb{Z}[t^{\pm}]$. We saw in proposition 3.8 that $\mathcal{Y} \simeq \mathbb{Z}[Y_{i,l}^{\pm}]_{i\in I, l\in\mathbb{Z}}$. So $\lambda_m(1) = 0$ and $\lambda_m(t) = (t-1)\lambda_m^{(1)}(t)$ where $\lambda_m^{(1)}(t) \in \mathbb{Z}[t^{\pm}]$. In particular $\sum_{m\in A} \lambda_m(t)^{(1)}(t)m = 0$ and we get by induction $\lambda_m(t) \in (t-1)^r\mathbb{Z}[t^{\pm}]$ for all $r \geq 0$. This is possible if and only if all $\lambda_m(t) = 0$. □

In the same way using the last assertion of proposition 3.8, we have :

Proposition 3.12. *The sub* $\mathbb{Z}[t^{\pm}]$*-algebra of* $\mathcal{Y}_t$ *generated by the* $\tilde{A}_{i,l}^{-1}$ *is defined by generators* $\tilde{A}_{i,l}^{-1}, t^{\pm}$ $(i \in I, l \in \mathbb{Z})$ *and relations :*

$$\tilde{A}_{i,l}^{-1}\tilde{A}_{j,k}^{-1} = t^{\alpha(i,l,j,k)}\tilde{A}_{j,k}^{-1}\tilde{A}_{i,l}^{-1}$$

where $\alpha : (I \times \mathbb{Z})^2 \to \mathbb{Z}$ *is given by :*

$$\alpha(i,l,i,k) = 2(-\delta_{l-k,2r_i} + \delta_{l-k,-2r_i})$$

$$\alpha(i,l,j,k) = 2\sum_{r=C_{i,j}+1, C_{i,j}+3, \ldots, -C_{i,j}-1} (-\delta_{l-k,-r_i+r} + \delta_{l-k,r_i+r}) \text{ (if } i \neq j)$$

Moreover we have the following relations in $\mathcal{Y}_t$:

$$\tilde{A}_{i,l}\tilde{Y}_{j,k} = t^{\beta(i,l,j,k)}\tilde{Y}_{j,k}\tilde{A}_{i,l}$$

where $\beta : (I \times \mathbb{Z})^2 \to \mathbb{Z}$ is given by :

$$\beta(i,l,j,k) = 2\delta_{i,j}(-\delta_{l-k,r_i} + \delta_{l-k,-r_i})$$

3.4 Notations and technical complements

In this section we study some technical properties of the $\mathcal{Y}$-monomials and the $\mathcal{Y}_t$-monomials which will be used in the following. Denote by A the set of $\mathcal{Y}$-monomials. It is a $\mathbb{Z}$-basis of $\mathcal{Y}$ (proposition 3.8). Let us define an analog $\mathbb{Z}[t^{\pm}]$-basis of $\mathcal{Y}_t$: denote A' the set of $\mathcal{Y}_t$-monomials of the form $m =: m :$. It follows from theorem 3.11 that :

$$\mathcal{Y}_t = \bigoplus_{m\in A'} \mathbb{Z}[t^{\pm}]m$$

The map $\pi : A' \to A$ defined by $\pi(m) = \pi_+(m)$ is a bijection. In the following we identify A and A'. In particular we have an embedding $\mathcal{Y} \subset \mathcal{Y}_t$ and an isomorphism of $\mathbb{Z}[t^{\pm}]$-modules $\mathcal{Y} \otimes_{\mathbb{Z}} \mathbb{Z}[t^{\pm}] \simeq \mathcal{Y}_t$. We say that $\chi_1 \in \mathcal{Y}_t$ has the same monomials as $\chi_2 \in \mathcal{Y}$ if in the decompositions $\chi_1 = \sum\limits_{m\in A} \lambda_m(t)m$, $\chi_2 = \sum\limits_{m\in A} \mu_m m$ we have $\lambda_m(t) = 0 \Leftrightarrow \mu_m = 0$.

For m a $\mathcal{Y}$-monomial we set $u_{i,l}(m) \in \mathbb{Z}$ such that $m = \prod\limits_{i\in I, l\in\mathbb{Z}} Y_{i,l}^{u_{i,l}(m)}$ and $u_i(m) = \sum\limits_{l\in\mathbb{Z}} u_{i,l}(m)$. For m a $\mathcal{Y}_t$-monomial, we set $u_{i,l}(m) = u_{i,l}(\pi_+(m))$ and $u_i(m) = u_i(\pi_+(m))$. Note that $u_{i,l}$ is invariant by multiplication by t and compatible with the identification of A and A'. Section 3.3 implies that for $i \in I, l \in \mathbb{Z}$ and m a $\mathcal{Y}_t$-monomial we have :

$$\tilde{A}_{i,l} m = t^{-2u_{i,l-r_i}(m)+2u_{i,l+r_i}(m)} m \tilde{A}_{i,l}$$

Denote by $B_i \subset A$ the set of i-dominant $\mathcal{Y}$-monomials, that is to say $m \in B_i$ if $\forall l \in \mathbb{Z}$, $u_{i,l}(m) \geq 0$. For $J \subset I$ denote by $B_J = \bigcap\limits_{i\in J} B_i$ the set of J-dominant $\mathcal{Y}$-monomials. In particular, $B = B_I$ is the set of dominant $\mathcal{Y}$-monomials.

We recall that we can define a partial ordering on A by putting $m \leq m'$ if there is a $\mathcal{Y}$-monomial M which is a product of $A_{i,l}^{\pm}$ ($i \in I, l \in \mathbb{Z}$) such that $m = Mm'$ (see for example [He1]). A maximal (resp. lowest, higher...) weight $\mathcal{Y}$-monomial is a maximal (resp. minimal, higher...) element of A for this ordering. We deduce from π_+ a partial ordering on the $\mathcal{Y}_t$-monomials.

Following [FM1], a $\mathcal{Y}$-monomial m is said to be right negative if the factors $Y_{j,l}$ appearing in m, for which l is maximal, have negative powers. A product of right negative $\mathcal{Y}$-monomials is right negative. It follows from lemma 3.6 that the $A_{i,l}^{-1}$ are right negative. A $\mathcal{Y}_t$-monomial is said to be right negative if $\pi_+(m)$ is right negative.

Lemma 3.13. *Let $(i_1, l_1), ..., (i_K, l_K)$ be in $(I \times \mathbb{Z})^K$. For $U \geq 0$, the set of the $m = \prod\limits_{k=1...K} A_{i_k,l_k}^{-v_{i_k,l_k}(m)}$ ($v_{i_k,l_k}(m) \geq 0$) such that $\underset{i\in I,k\in\mathbb{Z}}{min} u_{i,k}(m) \geq -U$ is finite.*

Proof : Suppose it is not the case : let be $(m_p)_{p\geq 0}$ such that $\min\limits_{i\in I,k\in\mathbb{Z}} u_{i,k}(m_p) \geq -U$ but $\sum\limits_{k=1...K} v_{i_k,l_k}(m_p) \underset{p\to\infty}{\to} +\infty$. So there is at least one k such that $v_{i_k,l_k}(m_p) \underset{p\to\infty}{\to} +\infty$. Denote by $\mathfrak{R}$ the set of such k. Among those $k \in \mathfrak{R}$, such that l_k is maximal suppose that r_{i_k} is maximal (recall the definition of r_i in section 2.1). In particular, we have $u_{i_k,l_k+r_{i_k}}(m_p) = -v_{i_k,l_k}(m_p) + f(p)$ where $f(p)$ depends only of the $v_{i_{k'},l_{k'}}(m_p)$, $k' \notin \mathfrak{R}$. In particular, $f(p)$ is bounded and $u_{i_k,l_k+r_{i_k}}(m_p) \underset{p\rightarrow\infty}{\rightarrow} -\infty$. □

Lemma 3.14. *For $M \in B$, $K \geq 0$ the following set is finite :*

$$\{MA_{i_1,l_1}^{-1}...A_{i_R,l_R}^{-1}/R \geq 0, l_1, ..., l_R \geq K\} \cap B$$

Proof : Let us write $M = Y_{i_1,l_1}...Y_{i_R,l_R}$ such that $l_1 = \min_{r=1...R} l_r$, $l_R = \max_{r=1...R} l_r$ and consider m in the set. It is of the form $m = MM'$ where $M' = \prod_{i\in I, l\geq K} A_{i,l}^{-v_{i,l}}$ $(v_{i,l} \geq 0)$. Let $L = \max\{l \in \mathbb{Z}/\exists i \in I, u_{i,l}(M') < 0\}$. M' is right negative so for all $i \in I$, $l > L \Rightarrow v_{i,l} = 0$. But m is dominant, so $L \leq l_R$. In particular $M' = \prod_{i\in I, K\leq l\leq l_R} A_{i,l}^{-v_{i,l}}$. It suffices to prove that the $v_{i,l}(m_r)$ are bounded under the condition m dominant. This follows from lemma 3.13. □

3.5 Presentations of deformed algebras

Our construction of $\mathcal{Y}_t$ using $\mathcal{H}_h$ (section 3.3) is a "concrete" presentation of the deformed structure. Let us look at another approach : in this section we define two bicharacters $\mathcal{N}, \mathcal{N}_t$ related to basis of $\mathcal{Y}_t$. All the information of the multiplication of $\mathcal{Y}_t$ is contained is those bicharacters because we can construct a deformed $*$ multiplication on the "abstract" $\mathbb{Z}[t^{\pm}]$-module $\mathcal{Y} \otimes_{\mathbb{Z}} \mathbb{Z}[t^{\pm}]$ by putting for $m_1, m_2 \in A$ $\mathcal{Y}$-monomials (for example with $\mathcal{N}$) :

$$m_1 * m_2 = t^{\mathcal{N}(m_1,m_2)-\mathcal{N}(m_2,m_1)} m_2 * m_1$$

Those presentations appeared earlier in other papers [N3, VV3] for the simply laced case. In particular this section identifies our approach with those articles and gives an algebraic motivation of the deformed structures of [N3, VV3] related to the structure of $\mathcal{U}_q(\hat{\mathfrak{g}})$.

3.5.1 The bicharacters $\mathcal{N}$ and $\mathcal{N}_t$

It follows from the proof of lemma 3.9 that for m a $\mathcal{Y}_t$-monomial, there is $N(m) \in \mathbb{Z}$ such that $m = t^{N(m)} : m :$. For m_1, m_2 $\mathcal{Y}_t$-monomials we define

$$\mathcal{N}(m_1, m_2) = N(m_1m_2) - N(m_1) - N(m_2)$$

We have $N(Y_{i,l}) = N(A_{i,l}) = 0$. Note that for $\alpha, \beta \in \mathbb{Z}$ we have $N(t^\alpha m) = \alpha + N(m)$ and $\mathcal{N}(t^\alpha m_1, t^\beta m_2) = \mathcal{N}(m_1, m_2)$. In particular the map $\mathcal{N} : A \times A \rightarrow \mathbb{Z}$ is well-defined and independent of the choice of a representant in $\pi_+^{-1}(A)$.

Lemma 3.15. *The map $\mathcal{N} : A\times A \rightarrow \mathbb{Z}$ is a bicharacter, that is to say for $m_1, m_2, m_3 \in A$, we have :*

$$\mathcal{N}(m_1m_2, m_3) = \mathcal{N}(m_1, m_3) + \mathcal{N}(m_2, m_3)$$

$$\mathcal{N}(m_1, m_2m_3) = \mathcal{N}(m_1, m_2) + \mathcal{N}(m_1, m_3)$$

Moreover for $m_1,...,m_k$ $\mathcal{Y}_t$-monomials, we have :

$$N(m_1m_2...m_k)=N(m_1)+N(m_2)+...+N(m_k)+\sum_{1\leq i<j\leq k}\mathcal{N}(m_i,m_j)$$

Proof : It is a consequence of the relations (m_1,m_2 $\mathcal{Y}_t$-monomials) :

$$\pi_-(m_1)\pi_+(m_2)=t^{\mathcal{N}(m_1,m_2)}\pi_+(m_2)\pi_-(m_1)$$

□

For m a $\mathcal{Y}_t$-monomial and $l\in\mathbb{Z}$, denote $\pi_l(m)=\prod_{j\in I}\tilde{Y}_{j,l}^{u_{j,l}(m)}$. It is well defined because for $i,j\in I$ and $l\in\mathbb{Z}$ we have $\tilde{Y}_{i,l}\tilde{Y}_{j,l}=\tilde{Y}_{j,l}\tilde{Y}_{i,l}$ (theorem 3.11). For m a $\mathcal{Y}_t$-monomial denote $\tilde{m}=\overrightarrow{\prod_{l\in\mathbb{Z}}}\pi_l(m)$, and A_t the set of $\mathcal{Y}_t$-monomials of the form $\tilde{m}$. From theorem 3.11 there is a unique $N_t(m)\in\mathbb{Z}$ such that $m=t^{N_t(m)}\tilde{m}$, and :

$$\mathcal{Y}_t=\bigoplus_{m\in A_t}\mathbb{Z}[t^{\pm}]m$$

We define $\mathcal{N}_t:A\times A\rightarrow\mathbb{Z}$ as $\mathcal{N}$ and we have analogues result of lemma 3.15 for N_t because (m_1,m_2 $\mathcal{Y}_t$-monomials) :

$$\mathcal{N}_t(m_1,m_2)=\sum_{l>l'}(\mathcal{N}(\pi_l(m_1),\pi_{l'}(m_2))-\mathcal{N}(\pi_{l'}(m_2),\pi_l(m_1)))$$

3.5.2 Presentation related to the basis A_t and identification with [N3]

We suppose we are in the ADE-case.

Let $m_1=:\prod_{i\in I,l\in\mathbb{Z}}\tilde{Y}_{i,l}^{y_{i,l}}\tilde{A}_{i,l}^{-v_{i,l}}:$, $m_2=:\prod_{i\in I,l\in\mathbb{Z}}\tilde{Y}_{i,l}^{y'_{i,l}}\tilde{A}_{i,l}^{-v'_{i,l}}:\in\mathcal{Y}_t$. We set $m_1^y=:\prod_{i\in I,l\in\mathbb{Z}}\tilde{Y}_{i,l}^{y_{i,l}}:$ and $m_2^y=:\prod_{i\in I,l\in\mathbb{Z}}\tilde{Y}_{i,l}^{y'_{i,l}}:$.

Proposition 3.16. *We have $\mathcal{N}_t(m_1,m_2)=\mathcal{N}_t(m_1^y,m_2^y)+2d(m_1,m_2)$, where :*

$$d(m_1,m_2)=\sum_{i\in I,l\in\mathbb{Z}}v_{i,l+1}u'_{i,l}+y_{i,l+1}v'_{i,l}=\sum_{i\in I,l\in\mathbb{Z}}u_{i,l+1}v'_{i,l}+v_{i,l+1}y'_{i,l}$$

where $u_{i,l}=y_{i,l}-v_{i,l-1}-v_{i,l+1}+\sum_{j/C_{i,j}=-1}v_{j,l}$ and $u'_{i,l}=y'_{i,l}-v'_{i,l-1}-v'_{i,l+1}+\sum_{j/C_{i,j}=-1}v'_{j,l}$.

Proof :

First note that we have ($i\in I,l\in\mathbb{Z}$) :

$$\mathcal{N}_t(Y_{i,l},A_{i,l-1}^{-1})=\mathcal{N}_t(A_{i,l+1}^{-1},Y_{i,l})=\mathcal{N}_t(A_{i,l+1}^{-1},Y_{i,l})=2$$

$$\mathcal{N}_t(Y_{i,l+1}^{-1},Y_{i,l-1}^{-1})=\mathcal{N}_t(A_{i,l+1}^{-1},A_{i,l-1}^{-1})=-2$$

We have $\mathcal{N}_t(m_1, m_2) = A + B + C + D$ where :

$A = \mathcal{N}_t(m_1^y, m_2^y)$

$B = \sum_{i,j\in I, l,k\in\mathbb{Z}} y_{i,l}v'_{j,k}\mathcal{N}_t(Y_{i,l}, A_{j,k}^{-1}) = 2\sum_{i\in I,l\in\mathbb{Z}} y_{i,l}v'_{i,l-1}$

$C = \sum_{i,j\in I, l,k\in\mathbb{Z}} v_{i,l}y'_{j,k}\mathcal{N}_t(A_{i,l}^{-1}, Y_{j,k}) = 2\sum_{i\in I,l\in\mathbb{Z}} v_{i,l+1}y'_{i,l}$

$D = \sum_{i,j\in I, l,k\in\mathbb{Z}} v_{i,l}v'_{j,k}\mathcal{N}_t(A_{i,l}^{-1}, A_{j,k}^{-1})$

$= -2\sum_{i\in I,l\in\mathbb{Z}} (v_{i,l+1}v'_{i,l-1} + v_{i,l}v_{i,l'}) + 2\sum_{C_{j,i}=-1,l\in\mathbb{Z}} v_{i,l+1}v'_{j,l}$

And we have the annonced formula for $B + C + D$. □

The bicharacter d was introduced for the ADE-case by Nakajima in [N3] motivated by geometry. It particular this proposition 3.16 gives a new motivation for this deformed structure.

3.5.3 Presentation related to the basis A and identification with [VV3]

Lemma 3.17. *For $m_1, m_2 \in A$, we have :*

$$\mathcal{N}(m_1, m_2) = \sum_{i,j\in I,l,k\in\mathbb{Z}} u_{i,l}(m_1)u_{j,k}(m_2)((\tilde{C}_{j,i}(z))_{r_j+l-k} - (\tilde{C}_{j,i}(z))_{-r_j+l-k})$$

Proof : Indeed we have $\tilde{Y}_{i,l}\tilde{Y}_{j,k} = t_{\tilde{C}_{j,i}(z)z^{k-l}(z^{-r_j}-z^{r_j})} : \tilde{Y}_{i,l}\tilde{Y}_{j,k} :$. □

In sl_2-case we have $C(z) = z + z^{-1}$ and $\tilde{C}(z) = \frac{1}{z+z^{-1}} = \sum_{r\geq 0}(-1)^r z^{-2r-1}$. So $\tilde{Y}_l\tilde{Y}_k = t^s : \tilde{Y}_l\tilde{Y}_k :$ where :

$s = 0$ if $l - k = 1 + 2r$, $r \in \mathbb{Z}$

$s = 0$ if $l - k = 2r$, $r > 0$

$s = 2(-1)^{r+1}$ if $l - k = 2r$, $r < 0$

$s = -1$ if $l = k$

It is analogous to the multiplication introduced for the ADE-case by Varagnolo-Vasserot (see the notations in [VV3]) :

$$Y_{i,l}Y_{j,m} = t^{2\epsilon_{z^l\omega_i,z^m\omega_j} - 2\epsilon_{z^m\omega_j,z^l\omega_i}} Y_{j,m}Y_{i,l} \ , \ \epsilon_{\lambda,\mu} = \pi_0((z^{-1}\Omega^{-1}(\bar{\lambda})|\mu))$$

But we have $\epsilon_{z^l\omega_i,z^m\omega_j} = (\tilde{C}_{i,j}(z))_{l+1-m}$ and if we set $\epsilon'_{\lambda,\mu} = \pi_0((z\Omega^{-1}(\bar{\lambda})|\mu))$ then we have $\epsilon'_{z^l\omega_i,z^m\omega_j} = (\tilde{C}_{i,j}(z))_{l-1-m}$ and

$$\epsilon_{z^l\omega_i,z^m\omega_j} - \epsilon'_{z^l\omega_i,z^m\omega_j} = \mathcal{N}(Y_{i,l}, Y_{j,m})$$

4 Deformed screening operators

Motivated by the screening currents of [FR2] we give in this section a "concrete" approach to deformations of screening operators. In particular the t-analogues of screening operators defined in [He1] will appear as commutators in $\mathcal{H}_h$. Let us begin with some background about classical screening operators.

4.1 Reminder : classical screening operators ([FR3, FM1])

4.1.1 Classical screening operators and symmetry property of q-characters

Definition 4.1. *The i^{th}-screening operator is the $\mathbb{Z}$-linear map defined by :*

$$S_i : \mathcal{Y} \to \mathcal{Y}_i = \frac{\bigoplus_{l\in\mathbb{Z}} \mathcal{Y}.S_{i,l}}{\sum_{l\in\mathbb{Z}} \mathcal{Y}.(S_{i,l+2r_i} - A_{i,l+r_i}.S_{i,l})}$$

$$\forall m \in A, S_i(m) = \sum_{l\in\mathbb{Z}} u_{i,l}(m) S_{i,l}$$

Note that the i^{th}-screening operator can also be defined as the derivation such that $S_i(1) = 0$ and $S_i(Y_{j,l}) = \delta_{i,j} Y_{i,l}.S_{i,l}$ $(j \in I, l \in \mathbb{Z})$.

Theorem 4.2. *(***Frenkel, Reshetikhin, Mukhin** *[FR3, FM1]) The image of $\chi_q : \mathbb{Z}[X_{i,l}]_{i\in I, l\in\mathbb{Z}} \to \mathcal{Y}$ is :*

$$Im(\chi_q) = \bigcap_{i\in I} Ker(S_i)$$

It is analogous to the classical symmetry property of χ : $\mathrm{Im}(\chi) = \mathbb{Z}[y_i^{\pm}]_{i\in I}^W$.

4.1.2 Structure of the kernel of S_i

Let $\mathfrak{K}_i = \mathrm{Ker}(S_i)$. It is a $\mathbb{Z}$-subalgebra of $\mathcal{Y}$.

Theorem 4.3. *(***Frenkel, Reshetikhin, Mukhin** *[FR3],[FM1]) The $\mathbb{Z}$-subalgebra $\mathfrak{K}_i$ of $\mathcal{Y}$ is generated by the $Y_{i,l}(1 + A_{i,l+r_i}^{-1}), Y_{j,l}^{\pm}$ $(j \neq i, l \in \mathbb{Z})$.*

For $m \in B_i$, let $E_i(m) = m \prod_{l\in\mathbb{Z}} (1 + A_{i,l+r_i}{}^{-1})^{u_{i,l}(m)} \in \mathfrak{K}_i$.

Corollary 4.4. *The $\mathbb{Z}$-module $\mathfrak{K}_i$ is freely generated by the $E_i(m)$ $(m \in B_i)$.*

4.1.3 Examples in the sl_2-case

We suppose in this section that $\mathfrak{g} = sl_2$. For $m \in B$, let $L(m) = \chi_q(V_m)$ be the q-character of the $\mathcal{U}_q(\hat{sl_2})$-irreducible representation of highest weight m. In particular $L(m) \in \mathfrak{K}$ and $\mathfrak{K} = \bigoplus_{m \in B} \mathbb{Z}L(m)$.

In [FR3] an explicit formula for $L(m)$ is given : a $\sigma \subset \mathbb{Z}$ is called a 2-segment if σ is of the form $\sigma = \{l, l+2, ..., l+2k\}$. Two 2-segment are said to be in special position if their union is a 2-segment that properly contains each of them. All finite subset of $\mathbb{Z}$ with multiplicity $(l, u_l)_{l \in \mathbb{Z}}$ ($u_l \geq 0$) can be broken in a unique way into a union of 2-segments which are not in pairwise special position.

For $m \in B$ we decompose $m = \prod_j \prod_{l \in \sigma_j} Y_l \in B$ where the $(\sigma_j)_j$ is the decomposition of the $(l, u_l(m))_{l \in \mathbb{Z}}$. We have $L(m) = \prod_j L(\prod_{l \in \sigma_j} Y_l)$. So it suffices to give the formula for a 2-segment :

$$L(Y_l Y_{l+2} Y_{l+4} ... Y_{l+2k}) = Y_l Y_{l+2} Y_{l+4} ... Y_{l+2k} + Y_l Y_{l+2} ... Y_{l+2(k-1)} Y_{l+2(k+1)}^{-1}$$
$$+ Y_l Y_{l+2} ... Y_{l+2(k-2)} Y_{l+2k}^{-1} Y_{l+2(k+1)}^{-1} + ... + Y_{l+2}^{-1} Y_{l+2}^{-1} ... Y_{l+2(k+1)}^{-1}$$

We say that m is irregular if there are $j_1 \neq j_2$ such that :

$$\sigma_{j_1} \subset \sigma_{j_2} \text{ and } \sigma_{j_1} + 2 \subset \sigma_{j_2}$$

Lemma 4.5. (**Frenkel, Reshetikhin** *[FR3]*) *There is a dominant $\mathcal{Y}$-monomial other than m in $L(m)$ if and only if m is irregular.*

4.1.4 Complements : another basis of $\mathfrak{K}_i$

Let us go back to the general case. Let $\mathcal{Y}_{sl_2} = \mathbb{Z}[Y_l^{\pm}]_{l \in \mathbb{Z}}$ the ring $\mathcal{Y}$ for the sl_2-case. Let i be in I and for $0 \leq k \leq r_i - 1$, let $\omega_k : A \to \mathcal{Y}_{sl_2}$ be the map defined by $\omega_k(m) = \prod_{l \in \mathbb{Z}} Y_l^{u_{i,k+lr_i}(m)}$ and $\nu_k : \mathbb{Z}[(Y_{l-1}Y_{l+1})^{-1}]_{l \in \mathbb{Z}} \to \mathcal{Y}$ be the ring homomorphism such that $\nu_k((Y_{l-1}Y_{l+1})^{-1}) = A_{i,k+lr_i}^{-1}$. For $m \in B_i$, $\omega_k(m)$ is dominant in $\mathcal{Y}_{sl_2}$ and so we can define $L(\omega_k(m))$ (see section 4.1.3). We have $L(\omega_k(m))\omega_k(m)^{-1} \in \mathbb{Z}[(Y_{l-1}Y_{l+1})^{-1}]_{l \in \mathbb{Z}}$. We introduce :

$$L_i(m) = m \prod_{0 \leq k \leq r_i - 1} \nu_k(L(\omega_k(m))\omega_k(m)^{-1}) \in \mathfrak{K}_i$$

In analogy with the corollary 4.4 we have $\mathfrak{K}_i = \bigoplus_{m \in B_i} \mathbb{Z}L_i(m) \simeq \mathbb{Z}^{(B_i)}$.

4.2 Screening currents

Following [FR2], for $i \in I, l \in \mathbb{Z}$, introduce $\tilde{S}_{i,l} \in \mathcal{H}_h$:

$$\tilde{S}_{i,l} = \exp(\sum_{m>0} h^m \frac{a_i[m]}{q_i^m - q_i^{-m}} q^{lm}) \exp(\sum_{m>0} h^m \frac{a_i[-m]}{q_i^{-m} - q_i^m} q^{-lm})$$

Lemma 4.6. *We have the following relations in $\mathcal{H}_h$:*

$$\tilde{A}_{i,l}\tilde{S}_{i,l-r_i} = t_{-z^{-2r_i}-1}\tilde{S}_{i,l+r_i}$$

$$\tilde{S}_{i,l}\tilde{A}_{j,k} = t_{C_{i,j}(z)(z^{(k-l)}+z^{(l-k)})}\tilde{A}_{j,k}\tilde{S}_{i,l}$$

$$\tilde{S}_{i,l}\tilde{Y}_{j,k} = t_{\delta_{i,j}(z^{(k-l)}+z^{(l-k)})}\tilde{Y}_{j,k}\tilde{S}_{i,l}$$

The proof is a straightforward computation as for lemma 3.5.

4.3 Deformed bimodules

In this section we define and study a t-analogue $\mathcal{Y}_{i,t}$ of the module $\mathcal{Y}_i$.

For $i \in I$, let $\mathcal{Y}_{i,u}$ be the sub $\mathcal{Y}_u$-left-module of $\mathcal{H}_h$ generated by the $\tilde{S}_{i,l}$ ($l \in \mathbb{Z}$). It follows from lemma 4.6 that $(\tilde{S}_{i,l})_{-r_i \leq l < r_i}$ generate $\mathcal{Y}_{i,u}$ and that it is also a subbimodule of $\mathcal{H}_h$. Denote by $\tilde{S}_{i,l} \in \mathcal{H}_t$ the image of $\tilde{S}_{i,l} \in \mathcal{H}_h$ in $\mathcal{H}_t$.

Definition 4.7. *$\mathcal{Y}_{i,t}$ is the sub left-module of $\mathcal{H}_t$ generated by the $\tilde{S}_{i,l}$ ($l \in \mathbb{Z}$).*

In particular it is the image of $\mathcal{Y}_{i,u}$ in $\mathcal{H}_t$. It follows from lemma 4.6 that for $l \in \mathbb{Z}$, we have in $\mathcal{Y}_{i,t}$:

$$\tilde{A}_{i,l}\tilde{S}_{i,l-r_i} = t^{-1}\tilde{S}_{i,l+r_i}$$

So $\mathcal{Y}_{i,t}$ is generated by the $(\tilde{S}_{i,l})_{-r_i \leq l < r_i}$. It follows from lemma 4.6 that for m a $\mathcal{Y}_t$-monomial

$$\tilde{S}_{i,l}.m = t^{2u_{i,l}(m)}m.\tilde{S}_{i,l}$$

and so $\mathcal{Y}_{i,t}$ a sub $\mathcal{Y}_t$-bimodule of $\mathcal{H}_t$.

Proposition 4.8. *The $\mathcal{Y}_t$-left-module $\mathcal{Y}_{i,t}$ is freely generated by $(\tilde{S}_{i,l})_{-r_i \leq l < r_i}$.*

Proof : We saw that $(\tilde{S}_{i,l})_{-r_i \leq l < r_i}$ generate $\mathcal{Y}_{i,t}$. We prove they are $\mathcal{Y}_t$-linearly independent : for $(R_1, ..., R_n) \in \mathfrak{U}^n$, introduce :

$$Y_{R_1,...,R_n} = \exp(\sum_{m>0, j \in I} h^m y_j[m] R_j(q^m)) \in \mathcal{H}_t^+$$

It follows from lemma 3.7 that the $(Y_R)_{R \in \mathfrak{U}^n}$ are $\mathbb{Z}$-linearly independent. Note that we have $\pi_+(\mathcal{Y}_{i,t}) \subset \bigoplus_{R \in \mathfrak{U}^n} \mathbb{Z}Y_R$ and that $\mathcal{Y} = \bigoplus_{R \in \mathbb{Z}[z^\pm]^n} \mathbb{Z}Y_R$. Suppose we have a linear combination $(\lambda_r \in \mathcal{Y}_t)$ $\lambda_{-r_i}\tilde{S}_{i,-r_i} + ... + \lambda_{r_i-1}\tilde{S}_{i,r_i-1} = 0$. Introduce $\mu_{k,R} \in \mathbb{Z}$ such that $\pi_+(\lambda_k) = \sum_{R \in \mathbb{Z}[z^\pm]^n} \mu_{k,R} Y_R$ and

$R_{i,k} = (R_{i,k}^1(z), ..., R_{i,k}^n(z)) \in \mathfrak{U}^n$ such that $\pi_+(\tilde{S}_{i,k}) = Y_{R_{i,k}}$. If we apply π_+ to the linear combination, we get :

$$\sum_{R \in \mathbb{Z}[z^\pm]^n, -r_i \leq k \leq r_i - 1} \mu_{k,R} Y_R Y_{R_{i,k}} = 0$$

and we have for all $R' \in \mathfrak{U}$:

$$\sum_{-r_i \leq k \leq r_i - 1 / R' - R_{i,k} \in \mathbb{Z}[z^{\pm}]^n} \mu_{k, R' - R_{i,k}} = 0$$

Suppose we have $-r_i \leq k_1 \neq k_2 \leq r_i - 1$ such that $R' - R_{i,k_1}, R' - R_{i,k_2} \in \mathbb{Z}[z^{\pm}]^n$. So $R_{i,k_1} - R_{i,k_2} \in \mathbb{Z}[z^{\pm}]^n$. But $a_i[m] = \sum_{j \in I} C_{j,i}(q^m) y_j[m]$, so for $j \in I$:

$$C_{j,i}(z) \frac{z^{k_1} - z^{k_2}}{z^{r_i} - z^{-r_i}} = (R^j_{i,k_1}(z) - R^j_{i,k_2}(z)) \in \mathbb{Z}[z^{\pm}]$$

In particular for $j = i$ we have $C_{i,i}(z) \frac{z^{k_1} - z^{k_2}}{z^{r_i} - z^{-r_i}} = \frac{(z^{r_i} + z^{-r_i})(z^{k_1} - z^{k_2})}{z^{r_i} - z^{-r_i}} \in \mathbb{Z}[z^{\pm}]$. This is impossible because $|k_1 - k_2| < 2r_i$. So we have only one term in the sum and all $\mu_{k,R} = 0$. So $\pi_+(\lambda_k) = 0$, and $\lambda_k \in (t-1)\mathcal{Y}_t$. We have by induction for all $m > 0$, $\lambda_k \in (t-1)^m \mathcal{Y}_t$. It is possible if and only if $\lambda_k = 0$. □

Denote by $\mathcal{Y}_i$ the $\mathcal{Y}$-bimodule $\pi_+(\mathcal{Y}_{i,t})$. It is consistent with section 4.1.

4.4 t-analogues of screening operators

We introduced t-analogues of screening operators in [He1]. The picture of the last section enables us to define them from a new point of view. For m a $\mathcal{Y}_t$-monomial, we have :

$$[\tilde{S}_{i,l}, m] = \tilde{S}_{i,l} m - m \tilde{S}_{i,l} = (t^{2u_{i,l}(m)} - 1) m \tilde{S}_{i,l} = t^{u_{i,l}(m)}(t - t^{-1})[u_{i,l}(m)]_t m \tilde{S}_{i,l}$$

So for $\lambda \in \mathcal{Y}_t$ we have $[\tilde{S}_{i,l}, \lambda] \in (t^2 - 1)\mathcal{Y}_{i,t}$, and $[\tilde{S}_{i,l}, \lambda] \neq 0$ only for a finite number of $l \in \mathbb{Z}$. So we can define :

Definition 4.9. *The i^{th} t-screening operator is the map $S_{i,t} : \mathcal{Y}_t \to \mathcal{Y}_{i,t}$ such that ($\lambda \in \mathcal{Y}_t$) :*

$$S_{i,t}(\lambda) = \frac{1}{t^2 - 1} \sum_{l \in \mathbb{Z}} [\tilde{S}_{i,l}, \lambda] \in \mathcal{Y}_{i,t}$$

In particular, $S_{i,t}$ is $\mathbb{Z}[t^{\pm}]$-linear and a derivation. It is the map of [He1]. For m a $\mathcal{Y}_t$-monomial, we have :

$$\pi_+(S_{i,t}(m)) = \pi_+(t^{u_{i,l}(m)-1}[u_{i,l}(m)]_t)\pi_+(m\tilde{S}_{i,l}) = u_{i,l}(m)\pi_+(m\tilde{S}_{i,l})$$

and so $\pi_+ \circ S_{i,t} = S_i \circ \pi_+$.

4.5 Kernel of deformed screening operators

4.5.1 Structure of the kernel

We proved in [He1] a t-analogue of theorem 4.3 :

Theorem 4.10. *([He1]) The kernel of the i^{th} t-screening operator $S_{i,t}$ is the $\mathbb{Z}[t^\pm]$-subalgebra of $\mathcal{Y}_t$ generated by the $\tilde{Y}_{i,l}(1+t\tilde{A}^{-1}_{i,l+r_i}), \tilde{Y}^{\pm}_{j,l}$ $(j\neq i, l\in\mathbb{Z})$.*

Proof : For the first inclusion we compute :

$$S_{i,t}(\tilde{Y}_{i,l}(1+t\tilde{A}^{-1}_{i,l+r_i})) = \tilde{Y}_{i,l}\tilde{S}_{i,l} + t\tilde{Y}_{i,l}\tilde{A}^{-1}_{i,l+r_i}(-t^{-2})\tilde{S}_{i,l+2r_i} = 0$$

For the other inclusion we refer to [He1]. $\square$

Let $\mathfrak{K}_{i,t} = \mathrm{Ker}(S_{i,t})$. It is a $\mathbb{Z}[t^\pm]$-subalgebra of $\mathcal{Y}_t$. In particular we have $\pi_+(\mathfrak{K}_{i,t}) = \mathfrak{K}_i$ (consequence of theorem 4.3 and 4.10).

For $m\in B_i$ introduce : (recall that $\overrightarrow{\prod_{l\in\mathbb{Z}}} U_l$ means $...U_{-1}U_0U_1U_2...$) :

$$E_{i,t}(m) = \overrightarrow{\prod_{l\in\mathbb{Z}}}((\tilde{Y}_{i,l}(1+t\tilde{A}^{-1}_{i,l+r_i}))^{u_{i,l}(m)}\prod_{j\neq i}\tilde{Y}^{u_{j,l}(m)}_{j,l})$$

It is well defined because it follows from theorem 3.11 that for $j\neq i, l\in\mathbb{Z}$, $(\tilde{Y}_{i,l}(1+t\tilde{A}^{-1}_{i,l+r_i}))$ and $\tilde{Y}_{j,l}$ commute. For $m\in B_i$, the formula shows that the $\mathcal{Y}_t$-monomials of $E_{i,t}(m)$ are the $\mathcal{Y}$-monomials of $E_i(m)$ (with identification by π_+). Such elements were used in [N3] for the ADE case. The theorem 4.10 allows us to describe $\mathfrak{K}_{i,t}$:

Corollary 4.11. *For all $m\in B_i$, we have $E_{i,t}(m)\in\mathfrak{K}_{i,t}$. Moreover :*

$$\mathfrak{K}_{i,t} = \bigoplus_{m\in B_i}\mathbb{Z}[t^\pm]E_{i,t}(m) \simeq \mathbb{Z}[t^\pm]^{(B_i)}$$

Proof : First $E_{i,t}(m)\in\mathfrak{K}_{i,t}$ as product of elements of $\mathfrak{K}_{i,t}$. We show easily that the $E_{i,t}(m)$ are $\mathbb{Z}[t^\pm]$-linearly independent by looking at a maximal $\mathcal{Y}_t$-monomial in a linear combination.

Let us prove that the $E_{i,t}(m)$ are $\mathbb{Z}[t^\pm]$-generators of $\mathfrak{K}_{i,t}$: for a product χ of the algebra-generators of theorem 4.10, let us look at the highest weight $\mathcal{Y}_t$-monomial m. Then $E_{i,t}(m)$ is this product up to the order in the multiplication. But for $p=1$ or $p\geq 3$, $Y_{i,l}Y_{i,l+pr_i}$ is the unique dominant $\mathcal{Y}$-monomial of $E_i(Y_{i,l})E_i(Y_{i,l+pr_i})$, so :

$$\tilde{Y}_{i,l}(1+t\tilde{A}^{-1}_{i,l+r_i})\tilde{Y}_{i,l+pr_i}(1+t\tilde{A}^{-1}_{i,l+pr_i+r_i}) \in t^{\mathbb{Z}}\tilde{Y}_{i,l+pr_i}(1+t\tilde{A}^{-1}_{i,l+pr_i+r_i})\tilde{Y}_{i,l}(1+t\tilde{A}^{-1}_{i,l+r_i})$$

And for $p=2$:

$$\tilde{Y}_{i,l}(1+t\tilde{A}^{-1}_{i,l+r_i})\tilde{Y}_{i,l+2r_i}(1+t\tilde{A}^{-1}_{i,l+3r_i}) - \tilde{Y}_{i,l+2r_i}(1+t\tilde{A}^{-1}_{i,l+3r_i})\tilde{Y}_{i,l}(1+t\tilde{A}^{-1}_{i,l+r_i})$$

$$\in \mathbb{Z}[t^\pm] + t^{\mathbb{Z}}\tilde{Y}_{i,l}(1+t\tilde{A}^{-1}_{i,l+r_i})\tilde{Y}_{i,l+2r_i}(1+t\tilde{A}^{-1}_{i,l+3r_i})$$

$\square$

4.5.2 Elements of $\mathfrak{K}_{i,t}$ with a unique i-dominant $\mathcal{Y}_t$-monomial

Proposition 4.12. *For $m \in B_i$, there is a unique $F_{i,t}(m) \in \mathfrak{K}_{i,t}$ such that m is the unique i-dominant $\mathcal{Y}_t$-monomial of $F_{i,t}(m)$. Moreover :*

$$\mathfrak{K}_{i,t} = \bigoplus_{m \in B_i} F_{i,t}(m)$$

Proof : It follows from corollary 4.11 that an element of $\mathfrak{K}_{i,t}$ has at least one i-dominant $\mathcal{Y}_t$-monomial. In particular we have the uniqueness of $F_{i,t}(m)$.

For the existence, let us look at the sl_2-case. Let m be in B. It follows from the lemma 3.14 that $\{MA_{i_1,l_1}^{-1}...A_{i_R,l_R}^{-1}/R \geq 0, l_1, ..., l_R \geq l(M)\} \cap B$ is finite (where $l(M) = \min\{l \in \mathbb{Z}/\exists i \in I, u_{i,l}(M) \neq 0\}$). We define on this set a total ordering compatible with the partial ordering : $m_L = m > m_{L-1} > ... > m_1$. Let us prove by induction on l the existence of $F_t(m_l)$. The unique dominant $\mathcal{Y}_t$-monomial of $E_t(m_1)$ is m_1 so $F_t(m_1) = E_t(m_1)$. In general let $\lambda_1(t), ..., \lambda_{l-1}(t) \in \mathbb{Z}[t^{\pm}]$ be the coefficient of the dominant $\mathcal{Y}_t$-monomials $m_1, ..., m_{l-1}$ in $E_t(m_l)$. We put :

$$F_t(m_l) = E_t(m_l) - \sum_{r=1...l-1} \lambda_r(t) F_t(m_r)$$

Notice that this construction gives $F_t(m) \in m\mathbb{Z}[\tilde{A}_l^{-1}, t^{\pm}]_{l \in \mathbb{Z}}$.

For the general case, let i be in I and m be in B_i. Consider $\omega_k(m)$ as in section 4.1.4. The study of the sl_2-case allows us to set $\chi_k = \omega_k(m)^{-1} F_t(\omega_k(m)) \in \mathbb{Z}[\tilde{A}_l^{-1}, t^{\pm}]_l$. And using the $\mathbb{Z}[t^{\pm}]$-algebra homomorphism $\nu_{k,t} : \mathbb{Z}[\tilde{A}_l^{-1}, t^{\pm}]_{l \in \mathbb{Z}} \to \mathbb{Z}[\tilde{A}_{i,l}^{-1}, t^{\pm}]_{i \in I, l \in \mathbb{Z}}$ defined by $\nu_{k,t}(\tilde{A}_l^{-1}) = \tilde{A}_{i,k+lr_i}^{-1}$, we set (the terms of the product commute) :

$$F_{i,t}(m) = m \prod_{0 \leq k \leq r_i - 1} \nu_{k,t}(\chi_k) \in \mathfrak{K}_{i,t}$$

For the last assertion, we have $E_{i,t}(m) = \sum_{l=1...L} \lambda_l(t) F_{i,t}(m_l)$ where $m_1, ..., m_L$ are the i-dominant $\mathcal{Y}_t$-monomials of $E_{i,t}(m)$ with coefficients $\lambda_1(t), ..., \lambda_L(t) \in \mathbb{Z}[t^{\pm}]$. $\square$

In the same way there is a unique $F_i(m) \in \mathfrak{K}_i$ such that m is the unique i-dominant $\mathcal{Y}$-monomial of $F_i(m)$. Moreover $F_i(m) = \pi_+(F_{i,t}(m))$.

4.5.3 Examples in the sl_2-case

In this section we suppose that $\mathfrak{g} = sl_2$ and we compute $F_t(m) = F_{1,t}(m)$ in some examples with the help of section 4.1.3.

Lemma 4.13. *Let $\sigma = \{l, l+2, ..., l+2k\}$ be a 2-segment and $m_\sigma = \tilde{Y}_l \tilde{Y}_{l+2} ... \tilde{Y}_{l+2k} \in B$. Then $F_t(m_\sigma)$ is equal to :*

$$m_\sigma(1 + t\tilde{A}_{l+2k+1}^{-1} + t^2 \tilde{A}_{l+(2k+1)}^{-1} \tilde{A}_{l+(2k-1)}^{-1} + ... + t^k \tilde{A}_{l+(2k+1)}^{-1} \tilde{A}_{l+(2k-1)}^{-1} ... \tilde{A}_{l+1}^{-1})$$

If σ_1, σ_2 are 2-segments not in special position, we have :

$$F_t(m_{\sigma_1}) F_t(m_{\sigma_2}) = t^{\mathcal{N}(m_{\sigma_1}, m_{\sigma_2}) - \mathcal{N}(m_{\sigma_2}, m_{\sigma_1})} F_t(m_{\sigma_2}) F_t(m_{\sigma_1})$$

If $\sigma_1, ..., \sigma_R$ are 2-segments such that $m_{\sigma_1}...m_{\sigma_r}$ is regular, we have :

$$F_t(m_{\sigma_1}...m_{\sigma_R}) = F_t(m_{\sigma_1})...F_t(m_{\sigma_R})$$

In particular if $m \in B$ verifies $\forall l \in \mathbb{Z}, u_l(m) \leq 1$ then it is of the form $m = m_{\sigma_1}...m_{\sigma_R}$ where the σ_r are 2-segments such that $\max(\sigma_r) + 2 < \min(\sigma_{r+1})$. So the lemma 4.13 gives an explicit formula $F_t(m) = F_t(m_{\sigma_1})...F_t(m_{\sigma_R})$.

Proof : First we need some relations in $\mathcal{Y}_{1,t}$: we know that for $l \in \mathbb{Z}$ we have $t\tilde{S}_{l-1} = \tilde{A}_l^{-1}\tilde{S}_{l+1} = t^2\tilde{S}_{l+1}\tilde{A}_l^{-1}$, so $t^{-1}\tilde{S}_{l-1} = \tilde{S}_{l+1}\tilde{A}_l^{-1}$. So we get by induction that for $r \geq 0$ $t^{-r}\tilde{S}_{l+1-2r} = \tilde{S}_{l+1}\tilde{A}_l^{-1}\tilde{A}_{l-2}^{-1}...\tilde{A}_{l-2(r-1)}^{-1}$. As
$u_{i,l+1}(\tilde{A}_l^{-1}\tilde{A}_{l-2}^{-1}...\tilde{A}_{l-2(r-1)}^{-1}) = u_{i,l+1}(\tilde{A}_l^{-1}) = -1$, we get :

$$t^{-r}\tilde{S}_{l+1-2r} = t^{-2}\tilde{A}_l^{-1}\tilde{A}_{l-2}^{-1}...\tilde{A}_{l-2(r-1)}^{-1}\tilde{S}_{l+1}$$

For $r' \geq 0$, by multiplying on the left by $\tilde{A}_{l+2r'}^{-1}\tilde{A}_{l+2(r'-1)}^{-1}...\tilde{A}_{l+2}^{-1}$ and puting $r' = 1+R', r = R-R', l = L-1-2R'$, we get for $0 \leq R' \leq R$:

$$t^{R'}\tilde{A}_{L+1}^{-1}\tilde{A}_{L-1}^{-1}...\tilde{A}_{L+1-2R'}^{-1}\tilde{S}_{L-2R} = t^{R-2}\tilde{A}_{L+1}^{-1}\tilde{A}_{L-1}^{-1}...\tilde{A}_{L+1-2R}^{-1}\tilde{S}_{L-2R'}$$

Now let be $m = \tilde{Y}_0\tilde{Y}_2...\tilde{Y}_l$ and $\chi \in \mathcal{Y}_t$ given by the formula in the lemma. The last remark enables us to compute $\tilde{S}_t(\chi) = 0$. So $\chi \in \mathfrak{K}_t$. But we see on the formula that m is the unique dominant monomial of χ. So $\chi = F_t(m)$.

For the second point, we have two cases :

if $m_{\sigma_1}m_{\sigma_2}$ is regular, it follows from lemma 4.5 that $L(m_{\sigma_1})L(m_{\sigma_2})$ has no dominant monomial other than $m_{\sigma_1}m_{\sigma_2}$. But our formula shows that $F_t(m_{\sigma_1})$ (resp. $F_t(m_{\sigma_2})$) has the same monomials than $L(m_{\sigma_1})$ (resp. $L(m_{\sigma_2})$). So

$$F_t(m_{\sigma_1})F_t(m_{\sigma_2}) - t^{\mathcal{N}(m_{\sigma_1},m_{\sigma_2})-\mathcal{N}(m_{\sigma_2},m_{\sigma_1})}F_t(m_{\sigma_2})F_t(m_{\sigma_1})$$

has no dominant $\mathcal{Y}_t$-monomial.

if $m_{\sigma_1}m_{\sigma_2}$ is irregular, we have for example $\sigma_{j_1} \subset \sigma_{j_2}$ and $\sigma_{j_1} + 2 \subset \sigma_{j_2}$. Let us write $\sigma_{j_1} = \{l_1, l_1+2, ..., , p_1\}$ and $\sigma_{j_2} = \{l_2, l_2+2, ..., , p_2\}$. So we have $l_2 \leq l_1$ and $p_1 \leq p_2-2$. Let $m = m_1m_2$ be a dominant $\mathcal{Y}$-monomial of $L(m_{\sigma_1}m_{\sigma_2}) = L(m_{\sigma_1})L(m_{\sigma_2})$ where m_1 (resp. m_2) is a $\mathcal{Y}$-monomial of $L(m_{\sigma_1})$ (resp. $L(m_{\sigma_2})$). If m_2 is not m_{σ_2}, we have $Y_{p_2}^{-1}$ in m_2 which can not be canceled by m_1. So $m = m_1m_{\sigma_2}$. Let us write $m_1 = m_{\sigma_1}A_{p_1+1}^{-1}...A_{p_1+1-2r}^{-1}$. So we just have to prove $[\tilde{A}_{p_1+1}^{-1}...\tilde{A}_{p_1+1-2r}^{-1}, m_{\sigma_2}] = 0$. This follows from $(l \in \mathbb{Z})$: $\tilde{A}_l^{-1}\tilde{Y}_{l-1}\tilde{Y}_{l+1} = \tilde{Y}_{l-1}\tilde{Y}_{l+1}\tilde{A}_l^{-1}$. For the last assertion it suffices to show that $F_t(m_{\sigma_1})...F_t(m_{\sigma_R})$ has a unique dominant $\mathcal{Y}_t$-monomial than $m_{\sigma_1}...m_{\sigma_R}$. But $F_t(m_{\sigma_1})...F_t(m_{\sigma_R})$ has the same monomials than $L(m_{\sigma_1})...L(m_{\sigma_R}) = L(m_{\sigma_1}...m_{\sigma_R})$ and $m_{\sigma_1}...m_{\sigma_R}$ is regular. □

4.5.4 Technical complements

Let us go back to the general case. We give some technical results which will be used in the following to compute $F_{i,t}(m)$ in some cases (see proposition 5.15 and the appendix).

Lemma 4.14. *Let i be in I, $l \in \mathbb{Z}$, $M \in A$ such that $u_{i,l}(M) = 1$ and $u_{i,l+2r_i} = 0$. Then we have $\mathcal{N}(M, \tilde{A}^{-1}_{i,l+r_i}) = -1$. In particular $\pi^{-1}(MA^{-1}_{i,l+r_i}) = tM\tilde{A}^{-1}_{i,l+r_i}$.*

Proof : We can suppose $M =: M :$ and we compute in $\mathcal{Y}_u$:

$$M\tilde{A}^{-1}_{i,l+r_i} = t_R : M\tilde{A}^{-1}_{i,l+r_i} :$$

where $R(z) = -(z^{2r_i} - z^{-2r_i}) + \sum_{r\in\mathbb{Z}} u_{i,r}(M) z^{(l+r_i-r)}(z^{r_i} - z^{-r_i})$. So :

$$\mathcal{N}(M, \tilde{A}^{-1}_{i,l+r_i}) = \sum_{r\in\mathbb{Z}} u_{i,r}(M)(z^{2r_i+l-r} - z^{l-r})_0 = -u_{i,l}(M) + u_{i,l+2r_i}(M) = -1$$

$\square$

Lemma 4.15. *Let m be in B_i such that $\forall l, r \in \mathbb{Z}, u_{i,l}(m) \leq 1$ and the set $\{k \in \mathbb{Z}/u_{i,r+2kr_i}(m) = 1\}$ is a 1-segment. Then we have $F_{i,t}(m) = \pi^{-1}(F_i(m))$.*

Proof : Let us look at the sl_2-case : $m = m_1 m_2 = m_{\sigma_1} m_{\sigma_2}$ where σ_1, σ_2 are 2-segment. So the lemma 4.13 gives an explicit formula for $F_t(m)$ and it follows from lemma 4.14 that $F_t(m) = \pi^{-1}(F(m))$.

We go back to the general case : let us write $m = m' m_1 ... m_{2r_i}$ where $m' = \prod_{j\neq i, l\in\mathbb{Z}} Y_{j,l}^{u_{j,l}(m)}$ and $m_r = \prod_{l\in\mathbb{Z}} Y_{i,r+2lr_i}^{u_{i,r+2lr_i}(m)}$. We have m_r of the form
$m_r = Y_{i,l_r} Y_{i,l_r+2r_i} ... Y_{i,l_r+2n_i r_i}$ and :

$$F_{i,t}(m) = t^{-N(m'm_1...m_r)} m' F_{i,t}(m_1)...F_{i,t}(m_{2r_i})$$

The study of the sl_2-case gives $F_{i,t}(m_r) = \pi^{-1}(F_i(m_r))$. It follows from lemma 4.14 that :

$$t^{-N(m'm_1...m_r)} m' \pi^{-1}(F_i(m_1))...\pi^{-1}(F_i(m_r))$$

$$= \pi^{-1}(m' F_i(m_1)...F_i(m_r)) = \pi^{-1}(F_i(m))$$

$\square$

5 Intersection of kernels of deformed screening operators

Motivated by theorem 4.2 we study the structure of a completion of $\mathfrak{K}_t = \bigcap_{i\in I} \mathrm{Ker}(S_{i,t})$ in order to construct $\chi_{q,t}$ in section 6. Note that in the sl_2-case we have $\mathfrak{K}_t = \mathrm{Ker}(S_{1,t})$ that was studied in section 4.

5.1 Reminder : classical case ([FR3, FM1])

For $J \subset I$, denote the $\mathbb{Z}$-subalgebra $\mathfrak{K}_J = \bigcap_{i\in J} \mathfrak{K}_i \subset \mathcal{Y}$ and $\mathfrak{K} = \mathfrak{K}_I$.

Lemma 5.1. *([FR3, FM1]) A non zero element of $\mathfrak{K}_J$ has at least one J-dominant $\mathcal{Y}$-monomial.*

Proof : It suffices to look at a maximal weight $\mathcal{Y}$-monomial m of $\chi \in \mathfrak{K}_J$: for $i \in J$ we have $m \in B_i$ because $\chi \in \mathfrak{K}_i$. □

Theorem 5.2. *([FR3, FM1]) For $i \in I$ there is a unique $E(Y_{i,0}) \in \mathfrak{K}$ such that $Y_{i,0}$ is the unique dominant $\mathcal{Y}$-monomial in $E(Y_{i,0})$.*

The uniqueness follows from lemma 5.1. For the existence we have $E(Y_{i,0}) = \chi_q(V_{\omega_i}(1))$ (theorem 4.2).

Note that the existence of $E(Y_{i,0}) \in \mathfrak{K}$ suffices to characterize $\chi_q : \mathrm{Rep} \to \mathfrak{K}$. It is the ring homomorphism such that $\chi_q(X_{i,l}) = s_l(E(Y_{i,0}))$ where $s_l : \mathcal{Y} \to \mathcal{Y}$ is given by $s_l(Y_{j,k}) = Y_{j,k+l}$.

For $m \in B$, we defined M_m and V_m in section 2. We set :

$$E(m) = \prod_{m\in B} s_l(E(Y_{i,0}))^{u_{i,l}(m)} = \chi_q(M_m) \in \mathfrak{K} \ , \ L(m) = \chi_q(V_m) \in \mathfrak{K}$$

We have :

$$\mathfrak{K} = \bigoplus_{m\in B} \mathbb{Z}E(m) = \bigoplus_{m\in B} \mathbb{Z}L(m) \simeq \mathbb{Z}^{(B)}$$

For $m \in B$, we can also define a unique $F(m) \in \mathfrak{K}$ such that m is the unique dominant $\mathcal{Y}$-monomial which appears in $F(m)$ (see for example the proof of proposition 4.12).

For $J \subset I$, let $\mathfrak{g}_J$ be the semi-simple Lie algebra of Cartan Matrix $(C_{i,j})_{i,j\in J}$ and $\mathcal{U}_q(\hat{\mathfrak{g}})_J$ the associated quantum affine algebra with coefficient $(r_i)_{i\in J}$. In analogy with the definition of $E_i(m), L_i(m)$ using the sl_2-case (section 4.1.4), we define for $m \in B_J$: $E_J(m)$, $L_J(m)$, $F_J(m) \in \mathfrak{K}_J$ using $\mathcal{U}_q(\hat{\mathfrak{g}})_J$. We have :

$$\mathfrak{K}_J = \bigoplus_{m\in B_J} \mathbb{Z}E_J(m) = \bigoplus_{m\in B_J} \mathbb{Z}L_J(m) = \bigoplus_{m\in B_J} \mathbb{Z}F_J(m) \simeq \mathbb{Z}^{(B_J)}$$

As a direct consequence of proposition 2.6 we have :

Lemma 5.3. *For $m \in B$, we have $E(m) \in \mathbb{Z}[Y_{i,l}]_{i\in I, l\geq l(m)}$ where $l(m) = min\{l \in \mathbb{Z} / \exists i \in I, u_{i,l}(m) \neq 0\}$.*

5.2 Completion of the deformed algebras

In this section we introduce completions of $\mathcal{Y}_t$ and of $\mathfrak{K}_{J,t} = \bigcap_{i\in J} \mathfrak{K}_{i,t} \subset \mathcal{Y}_t$ $(J \subset I)$. We have the following motivation : we have seen $\pi_+(\mathfrak{K}_{J,t}) \subset \mathfrak{K}_J$ (section 4). In order to prove an analogue of the other inclusion (theorem 5.11) we have to introduce completions where infinite sums are allowed.

5.2.1 The completion $\mathcal{Y}_t^\infty$ of $\mathcal{Y}_t$

Let $\overset{\infty}{A}_t = \prod_{m\in A} \mathbb{Z}[t^\pm].m \simeq \mathbb{Z}[t^\pm]^A$. We have $\bigoplus_{m\in A} \mathbb{Z}[t^\pm].m = \mathcal{Y}_t \subset \overset{\infty}{A}_t$. The algebra structure of $\mathcal{Y}_t$ gives a structure of $\mathcal{Y}_t$-bimodule on $\overset{\infty}{A}_t$ but can not be naturally extended to $\overset{\infty}{A}_t$. We define a $\mathbb{Z}[t^\pm]$-submodule $\mathcal{Y}_t^\infty$ with $\mathcal{Y}_t \subset \mathcal{Y}_t^\infty \subset \overset{\infty}{A}_t$, for which it is the case :

Let $\mathcal{Y}_t^A$ be the $\mathbb{Z}[t^\pm]$-subalgebra of $\mathcal{Y}_t$ generated by the $(\tilde{A}_{i,l}^{-1})_{i\in I, l\in\mathbb{Z}}$. It follows from proposition 3.12 that $\mathcal{Y}_t^A = \bigoplus_{K\geq 0} \mathcal{Y}_t^{A,K}$ where for $K\geq 0$:

$$\mathcal{Y}_t^{A,K} = \bigoplus_{m=:\tilde{A}_{i_1,l_1}^{-1}...\tilde{A}_{i_K,l_K}^{-1}:} \mathbb{Z}[t^\pm].m \subset \mathcal{Y}_t^A$$

Note that for $K_1, K_2 \geq 0$, $\mathcal{Y}_t^{A,K_1}\mathcal{Y}_t^{A,K_2} \subset \mathcal{Y}_t^{A,K_1+K_2}$ for the multiplication of $\mathcal{Y}_t$. So $\mathcal{Y}_t^A$ is a graded algebra if we set $\deg(x) = K$ for $x \in \mathcal{Y}_t^{A,K}$. Denote by $\mathcal{Y}_t^{A,\infty}$ the completion of $\mathcal{Y}_t^A$ for this gradation. It is a sub-$\mathbb{Z}[t^\pm]$-module of $\overset{\infty}{A}_t$.

Definition 5.4. *$\mathcal{Y}_t^\infty$ is the sub $\mathcal{Y}_t$-leftmodule of $\overset{\infty}{A}_t$ generated by $\mathcal{Y}_t^{A,\infty}$.*

In particular, we have : $\mathcal{Y}_t^\infty = \sum_{M\in A} M.\mathcal{Y}_t^{A,\infty} \subset \overset{\infty}{A}_t$.

Lemma 5.5. *There is a unique algebra structure on $\mathcal{Y}_t^\infty$ compatible with the structure of $\mathcal{Y}_t \subset \mathcal{Y}_t^\infty$.*

Proof : The structure is unique because the elements of $\mathcal{Y}_t^\infty$ are infinite sums of elements of $\mathcal{Y}_t$. For $M\in A$, we have $\mathcal{Y}_t^{A,\infty}.M \subset M.\mathcal{Y}_t^{A,\infty}$, so $\mathcal{Y}_t^\infty$ is a sub $\mathcal{Y}_t$-bimodule of $\overset{\infty}{A}_t$. For $M\in A$ and $\lambda \in \mathcal{Y}_t^{A,\infty}$ denote $\lambda^M \in \mathcal{Y}_t^{A,\infty}$ such that $\lambda.M = M.\lambda^M$. We define the $\mathbb{Z}[t^\pm]$-algebra structure on $\mathcal{Y}_t^\infty$ by $(M, M' \in A, \lambda, \lambda' \in \mathcal{Y}_t^{A,\infty})$:

$$(M.\lambda)(M'.\lambda') = MM'.(\lambda^{M'}\lambda')$$

It is well defined because for $M_1, M_2, M \in A, \lambda, \lambda_2 \in \mathcal{Y}_t^A$ we have $M_1\lambda_1 = M_2\lambda_2 \Rightarrow M_1M\lambda_1^M = M_2M\lambda_2^M$. □

5.2.2 The completion $\mathfrak{K}_{i,t}^\infty$ of $\mathfrak{K}_{i,t}$

We define a completion of $\mathfrak{K}_{i,t}$ analog to the completed algebra $\mathcal{Y}_t^\infty$.

For $M\in A$, we define a $\mathbb{Z}[t^\pm]$-linear endomorphism $E_{i,t}^M : M\mathcal{Y}_t^{A,\infty} \rightarrow M\mathcal{Y}_t^{A,\infty}$ such that (m $\mathcal{Y}_t^A$-monomial) :

$$E_{i,t}^M(Mm) = 0 \text{ if : } Mm :\notin B_i$$

$$E_{i,t}^M(Mm) = E_{i,t}(Mm) \text{ if : } Mm :\in B_i$$

It is well-defined because if $m\in \mathcal{Y}_t^{A,K}$ and $: Mm :\in B_i$ we have $E_{i,t}(Mm) \in M \bigoplus_{K'\geq K} \mathcal{Y}^{A,K'}$.

Definition 5.6. *We define* $\mathfrak{K}_{i,t}^\infty = \sum_{M\in A} Im(E_{i,t}^M) \subset \mathcal{Y}_t^\infty$.

For $J\subset I$, we set $\mathfrak{K}_{J,t}^\infty = \bigcap_{i\in J}\mathfrak{K}_{i,t}^\infty$ and $\mathfrak{K}_t^\infty = \mathfrak{K}_{I,t}^\infty$.

Lemma 5.7. *A non zero element of $\mathfrak{K}_{J,t}^\infty$ has at least one J-dominant $\mathcal{Y}_t$-monomial. For $J\subset I$, we have $\mathfrak{K}_{J,t}^\infty \cap \mathcal{Y}_t = \mathfrak{K}_{J,t}$. Moreover $\mathfrak{K}_{J,t}^\infty$ is a $\mathbb{Z}[t^\pm]$-subalgebra of $\mathcal{Y}_t^\infty$.*

Proof : For the first point the proof is analog to the proof of lemma 5.1. Next it suffices to prove the results for $J=\{i\}$. First for $m\in B_i$ we have $E_{i,t}(m) = E_{i,t}^m(m) \in \mathfrak{K}_{i,t}^\infty$ and so $\mathfrak{K}_{i,t} = \bigoplus_{m\in B_i}\mathbb{Z}[t^\pm]E_{i,t}(m) \subset \mathfrak{K}_{i,t}^\infty\cap\mathcal{Y}_t$. Now let χ be in $\mathfrak{K}_{i,t}^\infty$ such that χ has only a finite number of $\mathcal{Y}_t$-monomials. In particular it has only a finite number of i-dominant $\mathcal{Y}_t$-monomials $m_1,...,m_r$ with coefficients $\lambda_1(t),...,\lambda_r(t)$. In particular it follows from the first point that $\chi = \lambda_1(t)F_{i,t}(m_1)+...+\lambda_r(t)F_{i,t}(m_r)\in\mathfrak{K}_{i,t}$ (see proposition 4.12 for the definition of $F_{i,t}(m)$).

For the last assertion, consider M_1, $M_2\in A$ and m_1, m_2 $\mathcal{Y}_t^A$-monomials such that : M_1m_1 : ,: M_2m_2 :$\in B_i$. Then $E_{i,t}(M_1m_1)E_{i,t}(M_2m_2)$ is in the the sub-algebra $\mathfrak{K}_{i,t}\subset\mathcal{Y}_t$ and in $\mathrm{Im}(E_{i,t}^{M_1M_2})$. □

In the same way for $t=1$ we define the $\mathbb{Z}$-algebras $\mathcal{Y}^\infty$ and $\mathfrak{K}_J^\infty\subset\mathcal{Y}^\infty$. The surjective map $\pi_+:\mathcal{Y}_t\to\mathcal{Y}$ is naturally extended to a surjective map $\pi_+:\mathcal{Y}_t^\infty\to\mathcal{Y}^\infty$. For $i\in I$, we have $\pi_+(\mathfrak{K}_{i,t}^\infty)=\mathfrak{K}_i^\infty$ and for $J\subset I$, $\pi_+(\mathfrak{K}_{J,t}^\infty)\subset\mathfrak{K}_J^\infty$. The other inclusion is equivalent to theorem 5.11.

5.2.3 Special submodules of $\mathcal{Y}_t^\infty$

For $m\in A$, $K\geq 0$ we construct a subset $D_{m,K}\subset m\{\tilde{A}_{i_1,l_1}^{-1}...\tilde{A}_{i_K,l_K}^{-1}\}$ stable by the maps $E_{i,t}^m$ such that $\bigcup_{K\geq 0}D_{m,K}$ is countable : we say that $m'\in D_{m,K}$ if and only if there is a finite sequence $(m_0=m,m_1,...,m_R=m')$ of length $R\leq K$, such that for all $1\leq r\leq R$, there is $r'<r$, $J\subset I$ such that $m_{r'}\in B_J$ and for $r'<r''\leq r$, $m_{r''}$ is a $\mathcal{Y}$-monomial of $E_J(m_{r'})$ and $m_{r''}m_{r''-1}^{-1}\in\{A_{j,l}^{-1}/l\in\mathbb{Z},j\in J\}$. The definition means that "there is chain of monomials of some $E_J(m'')$ from m to m''".

Lemma 5.8. *The set $D_{m,K}$ is finite. In particular, the set D_m is countable. Moreover For $m,m'\in A$ such that $m'\in D_m$ we have $D_{m'}\subseteq D_m$. For $M\in A$, the set $B\cap D_M$ is finite.*

Proof : Let us prove by induction on $K\geq 0$ that $D_{m,K}$ is finite : we have $D_{m,0}=\{m\}$ and :

$$D_{m,K+1}\subset \bigcup_{J\subset I, m'\in D_{m,K}\cap B_J}\{\mathcal{Y}\text{-monomials of } E_J(m')\}$$

Consider $(m_0=m,m_1,...,m_R=m')$ a sequence adapted to the definition of D_m. Let m'' be in $D_{m'}$ and $(m_R=m',m_{R+1},...,m_{R'}=m'')$ a sequence adapted to the definition of $D_{m'}$. So $(m_0,m_1,...,m_{R'})$ is adapted to the definition of D_m, and $m''\in D_m$.

Let us look at $m \in B \bigcap D_M$: we can see by induction on the length of a sequence $(m_0 = M, m_1, ..., m_R = m)$ adapted to the definition of D_M that m is of the form $m = MM'$ where $M' = \prod_{i\in I, l\geq l_1} A_{i,l}^{-v_{i,l}}$ $(v_{i,l} \geq 0)$. So the last assertion follows from lemma 3.14. $\square$

Definition 5.9. *$\tilde{D}_m$ is the $\mathbb{Z}[t^\pm]$-submodule of $\mathcal{Y}_t^\infty$ whose elements are of the form $(\lambda_{m'}(t)m)_{m'\in D_m}$.*

For $m \in A$ introduce $m_0 = m > m_1 > m_2 > ...$ the countable set D_m with a total ordering compatible with the partial ordering. For $k \geq 0$ consider an element $F_k \in \tilde{D}_{m_k}$. Note that some infinite sums make sense in $\tilde{D}_m$: for $k \geq 0$, we have $D_{m_k} \subset \{m_k, m_{k+1}, ...\}$. So m_k appears only in the $F_{k'}$ with $k' \leq k$ and the infinite sum $\sum_{k\geq 0} F_k$ makes sense in $\tilde{D}_m$.

5.3 Crucial result for our construction

Definition 5.10. *For $n \geq 1$ denote $P(n)$ the property "for all semi-simple Lie-algebras $\mathfrak{g}$ of rank rk$(\mathfrak{g}) = n$, for all $m \in B$ there is a unique $F_t(m) \in \mathfrak{K}_t^\infty \cap \tilde{D}_m$ such that m is the unique dominant $\mathcal{Y}_t$-monomial of $F_t(m)$.".*

Our construction of q, t-characters is based on theorem 5.11.

Theorem 5.11. *For all $n \geq 1$, the property $P(n)$ is true.*

Note that for $n = 1$, that is to say $\mathfrak{g} = sl_2$, the result follows from section 4. In general the uniqueness follows from lemma 5.7 : if $\chi_1, \chi_2 \in \mathfrak{K}_t^\infty$ are solutions, then $\chi_1 - \chi_2$ has no dominant $\mathcal{Y}_t$-monomial, so $\chi_1 = \chi_2$.

Remark : in the simply-laced case the existence is a consequence of the geometric theory of quivers [N2], [N3], and in A_n, D_n-cases of algebraic explicit constructions [N4]. In the rest of this section 5 we give an algebraic proof of this theorem in the general case.

Let us outline the proof : first we give some preliminary technical results (section 5.4) in which we construct t-analogues of the $E(m)$. Next we prove $P(n)$ by induction on n. Our proof has 3 steps :

Step 1 (section 5.5) : we prove $P(1)$ and $P(2)$ using a more precise property $Q(n)$ such that $Q(n) \Rightarrow P(n)$. The property $Q(n)$ has the following advantage : it can be verified by computation in elementary cases $n = 1, 2$.

Step 2 (section 5.6) : we give some consequences of $P(n)$ which will be used in the proof of $P(r)$ $(r > n)$: we give the structure of $\mathfrak{K}_t^\infty$ (proposition 5.17) for rk$(\mathfrak{g}) = n$ and the structure of $\mathfrak{K}_{J,t}^\infty$ where $J \subset I$, $|J| = n$ and $|I| > n$ (corollary 5.18).

Step 3 (section 5.7) : we prove $P(n)$ $(n \geq 3)$ assuming $P(r)$, $r \leq n$ are true. We give an algorithm (section 5.7.1) to construct explicitly $F_t(m)$. It is called t-algorithm and is a t-analogue of Frenkel-Mukhin algorithm [FM1] (a deformed algorithm was also used by Nakajima in the ADE-case [N2]). As we do not know *a priori* that the algorithm is well defined the general case, we have to show that it never fails (lemma 5.22) and gives a convenient element (lemma 5.23).

5.4 Preliminary : Construction of the $E_t(m)$

Lemma 5.12. *We suppose that for $i\in I$, there is $F_t(\tilde{Y}_{i,0})\in\mathfrak{K}_t^\infty\cap\tilde{D}_{\tilde{Y}_{i,0}}$ such that $\tilde{Y}_{i,0}$ is the unique dominant $\mathcal{Y}_t$-monomial of $F_t(\tilde{Y}_{i,0})$. Then :*

i) All $\mathcal{Y}_t$-monomials of $F_t(\tilde{Y}_{i,0})$, except the highest weight $\mathcal{Y}_t$-monomial, are right negative.

ii) All $\mathcal{Y}_t$-monomials of $F_t(\tilde{Y}_{i,0})$ are products of $\tilde{Y}_{j,l}^{\pm}$ with $l\geq 0$.

iii) The only $\mathcal{Y}_t$-monomial of $F_t(\tilde{Y}_{i,0})$ which contains a $\tilde{Y}_{j,0}^{\pm}$ ($j\in I$) is the highest weight monomial $\tilde{Y}_{i,0}$.

iv) The $F_t(\tilde{Y}_{i,0})$ ($i\in I$) commute.

Note that (i),(ii) and (iii) appeared in [FM1] for $t=1$.

Proof :

i) It suffices to prove that all $\mathcal{Y}_t$-monomials $m_0=\tilde{Y}_{i,0},m_1,...$ of $D_{\tilde{Y}_{i,0}}$ except $\tilde{Y}_{i,0}$ are right negative. But m_1 is the monomial $\tilde{Y}_{i,0}\tilde{A}_{i,1}^{-1}$ of $E_i(\tilde{Y}_{i,0})$ and it is right negative. We can now prove the statement by induction : suppose that m_r is a monomial of $E_J(m_{r'})$, where $m_{r'}$ is right negative. So m_r is a product of $m_{r'}$ by some $\tilde{A}_{j,l}^{-1}$ ($l\in\mathbb{Z}$).Those monomials are right negative because a product of right negative monomial is right negative.

ii) Suppose that $m\in A$ is product of $\tilde{Y}_{k,l}^{\pm}$ with $l\geq 0$. It follows from lemma 5.3 that all monomials of D_m are product of $\tilde{Y}_{k,l}^{\pm}$ with $l\geq 0$.

iii) All $\mathcal{Y}$-monomials of $D_{\tilde{Y}_{i,0}}$ except $\tilde{Y}_{i,0}$ are in $D_{\tilde{Y}_{i,0}\tilde{A}_{i,r_i}^{-1}}$. But $l(\tilde{Y}_{i,0}\tilde{A}_{i,r_i}^{-1})\geq 1$ and we can conclude with the help of lemma 5.3.

iv) Let $i\neq j$ be in I and look at $F_t(\tilde{Y}_{i,0})F_t(\tilde{Y}_{j,0})$. Suppose we have a dominant $\mathcal{Y}_t$-monomial $m_0=m_1m_2$ in $F_t(\tilde{Y}_{i,0})F_t(\tilde{Y}_{j,0})$ different from the highest weight $\mathcal{Y}_t$-monomial $\tilde{Y}_{i,0}\tilde{Y}_{j,0}$. We have for example $m_1\neq\tilde{Y}_{i,0}$, so m_1 is right negative. Let l_1 be the maximal l such that a $\tilde{Y}_{k,l}$ appears in m_1. We have $u_{k,l}(m_1)<0$ and $l>0$. As $u_{k,l}(m_0)\geq 0$ we have $u_{k,l}(m_2)>0$ and $m_2\neq Y_{j,0}$. So m_2 is right negative and there is $k'\in I$ and $l'>l$ such that $u_{k',l'}(m_2)<0$. So $u_{k',l'}(m_1)>0$, contradiction. So the highest weight $\mathcal{Y}_t$-monomial of $F_t(\tilde{Y}_{i,0})F_t(\tilde{Y}_{j,0})$ is the unique dominant $\mathcal{Y}_t$-monomial. In the same way the highest weight $\mathcal{Y}_t$-monomial of $F_t(\tilde{Y}_{j,0})F_t(\tilde{Y}_{i,0})$ is the unique dominant $\mathcal{Y}_t$-monomial. But we have $\tilde{Y}_{i,0}\tilde{Y}_{j,0}=\tilde{Y}_{j,0}\tilde{Y}_{i,0}$, so $F_t(\tilde{Y}_{i,0})F_t(\tilde{Y}_{j,0})-F_t(\tilde{Y}_{j,0})F_t(\tilde{Y}_{i,0})\in\mathfrak{K}_t^\infty$ has no dominant $\mathcal{Y}_t$-monomial, so is equal to 0. □

Denote, for $l\in\mathbb{Z}$, by $s_l:\mathcal{Y}_t^\infty\rightarrow\mathcal{Y}_t^\infty$ the endomorphism of $\mathbb{Z}[t^{\pm}]$-algebra such that $s_l(\tilde{Y}_{j,k})=\tilde{Y}_{j,k+l}$ (it is well-defined because the defining relations of $\mathcal{Y}_t$ are invariant for $k\mapsto k+l$). If the hypothesis of the lemma 5.12 are verified, we can define for $m\in t^{\mathbb{Z}}B$:

$$E_t(m)=m(\overrightarrow{\prod_{l\in\mathbb{Z}}}\prod_{i\in I}\tilde{Y}_{i,l}^{u_{i,l}(m)})^{-1}\overrightarrow{\prod_{l\in\mathbb{Z}}}\prod_{i\in I}s_l(F_t(\tilde{Y}_{i,0}))^{u_{i,l}(m)}\in\mathfrak{K}_t^\infty$$

because for $i\neq j$ $[s_l(F_t(\tilde{Y}_{i,0})),s_l(F_t(\tilde{Y}_{j,0}))]=0$ (lemma 5.12).

5.5 Step 1 : Proof of $P(1)$ and $P(2)$

The aim of this section is to prove $P(1)$ and $P(2)$. First we define a more precise property $Q(n)$ such that $Q(n) \Rightarrow P(n)$.

Definition 5.13. *For $n \geq 1$ denote $Q(n)$ the property "for all semi-simple Lie-algebras $\mathfrak{g}$ of rank $rk(\mathfrak{g}) = n$, for all $i \in I$ there is a unique $F_t(\tilde{Y}_{i,0}) \in \mathfrak{K}_t \cap \tilde{D}_{\tilde{Y}_{i,0}}$ such that $\tilde{Y}_{i,0}$ is the unique dominant $\mathcal{Y}_t$-monomial of $F_t(\tilde{Y}_{i,0})$. Moreover $F_t(\tilde{Y}_{i,0})$ has the same monomials as $E(Y_{i,0})$".*

The property $Q(n)$ is more precise than $P(n)$ because it implies that $F_t(\tilde{Y}_{i,0})$ has only a finite number of monomials.

Lemma 5.14. *For $n \geq 1$, the property $Q(n)$ implies the property $P(n)$.*

Proof : We suppose $Q(n)$ is true. In particular the section 5.4 enables us to construct $E_t(m) \in \mathfrak{K}_t^\infty$ for $m \in B$. The defining formula of $E_t(m)$ shows that it has the same monomials as $E(m)$. So $E_t(m) \in \tilde{D}_m$ and $E_t(m) \in \mathfrak{K}_t$.

Let us prove $P(n)$: let m be in B. The uniqueness of $F_t(m)$ follows from lemma 5.7. Let $m_L = m > m_{L-1} > ... > m_1$ be the dominant monomials of D_m with a total ordering compatible with the partial ordering (it follows from lemma 3.14 that $D_m \cap B$ is finite). Let us prove by induction on l the existence of $F_t(m_l)$. The unique dominant of D_{m_1} is m_1 so $F_t(m_1) = E_t(m_1) \in \tilde{D}_{m_1}$. In general let $\lambda_1(t), ..., \lambda_{l-1}(t) \in \mathbb{Z}[t^\pm]$ be the coefficient of the dominant $\mathcal{Y}_t$-monomials $m_1, ..., m_{l-1}$ in $E_t(m_l)$. We put :

$$F_t(m_l) = E_t(m_l) - \sum_{r=1...l-1} \lambda_r(t) F_t(m_r)$$

We see in the construction that $F_t(m) \in \tilde{D}_m$ because for $m' \in D_m$ we have $E_t(m') \in \tilde{D}_{m'} \subseteq \tilde{D}_m$ (lemma 5.8). □

We need the following general technical result :

Proposition 5.15. *Let m be in B such that all monomial m' of $F(m)$ verifies : $\forall i \in I, m' \in B_i$ implies $\forall l \in \mathbb{Z}, u_{i,l}(m') \leq 1$ and for $1 \leq r \leq 2r_i$ the set $\{l \in \mathbb{Z}/u_{i,r+2lr_i}(m') = 1\}$ is a 1-segment. Then $\pi^{-1}(F(m)) \in \mathcal{Y}_t$ is in $\mathfrak{K}_t$ and has a unique dominant monomial m.*

Proof : Let us write $F(m) = \sum_{m' \in A} \mu(m')m'$ $(\mu(m') \in \mathbb{Z})$. Let i be in I and consider the decomposition of $F(m)$ in $\mathfrak{K}_i$ $F(m) = \sum_{m' \in B_i} \mu(m') F_i(m')$. But $\mu(m') \neq 0$ implies the hypothesis of lemma 4.15 is verified for $m' \in B_i$. So $\pi^{-1}(F_i(m')) = F_{i,t}(m')$. And $\pi^{-1}(F(m)) = \sum_{m' \in B_i} \mu(m') F_{i,t}(m') \in \mathfrak{K}_{i,t}$. □

For $n = 1$ (section 4.5), $n = 2$ (appendix), we can give explicit formula for the $E(Y_{i,0}) = F(Y_{i,0})$. In particular we see that the hypothesis of proposition 5.15 are verified, so :

Corollary 5.16. *The properties $Q(1)$, $Q(2)$ and so $P(1)$, $P(2)$ are true.*

This allow us to start our induction in the proof of theorem 5.11. In section 7.1 we will see other applications of proposition 5.15.

Note that the hypothesis of proposition 5.15 are not verified for fundamental monomials $m = Y_{i,0}$ in general : for example for the D_5-case we have in $F(Y_{2,0})$ the monomial $Y_{3,3}^2 Y_{5,4}^{-1} Y_{2,4}^{-1} Y_{4,4}^{-1}$.

5.6 Step 2 : consequences of the property $P(n)$

Let $n \geq 1$. We suppose in this section that $P(n)$ is proved. We give some consequences of $P(n)$ which will be used in the proof of $P(r)$ $(r > n)$.

Let $\mathfrak{K}_t^{\infty,f}$ be the $\mathbb{Z}[t^\pm]$-submodule of $\mathfrak{K}_t^\infty$ generated by elements with a finite number of dominant $\mathcal{Y}_t$-monomials.

Proposition 5.17. *We suppose $rk(\mathfrak{g}) = n$. We have :*

$$\mathfrak{K}_t^{\infty,f} = \bigoplus_{m \in B} \mathbb{Z}[t^\pm] F_t(m) \simeq \mathbb{Z}[t^\pm]^{(B)}$$

Moreover for $M \in A$, we have :

$$\mathfrak{K}_t^\infty \cap \tilde{D}_M = \bigoplus_{m \in B \cap D_M} \mathbb{Z}[t^\pm] F_t(m) \simeq \mathbb{Z}[t^\pm]^{B \cap D_M}$$

Proof : Let χ be in $\mathfrak{K}_t^{\infty,f}$ and $m_1, ..., m_L \in B$ the dominant $\mathcal{Y}_t$-monomials of χ and $\lambda_1(t), ..., \lambda_L(t) \in \mathbb{Z}[t^\pm]$ their coefficients. It follows from lemma 5.7 that $\chi = \sum_{l=1...L} \lambda_l(t) F_t(m_l)$.

Let us look at the second point : lemma 5.8 shows that $m \in B \cap D_M \Rightarrow F_t(m) \in \tilde{D}_M$. In particular the inclusion $\supseteq$ is clear. For the other inclusion we prove as in the first point that $\mathfrak{K}_t^\infty \cap \tilde{D}_M = \sum_{m \in B \cap D_M} \mathbb{Z}[t^\pm] F_t(m)$. We can conclude because it follows from lemma 3.14 that $D_M \cap B$ is finite. □

We recall that some infinite sum make sense in $\tilde{D}_M$ (section 5.2.3).

Corollary 5.18. *We suppose $rk(\mathfrak{g}) > n$ and let J be a subset of I such that $|J| = n$. For $m \in B_J$, there is a unique $F_{J,t}(m) \in \mathfrak{K}_{J,t}^\infty$ such that m is the unique J-dominant $\mathcal{Y}_t$-monomial of $F_{J,t}(m)$. Moreover $F_{J,t}(m) \in \tilde{D}_m$.*

For $M \in A$, the elements of $\mathfrak{K}_{J,t}^\infty \cap \tilde{D}_M$ are infinite sums $\sum_{m \in B_J \cap D_M} \lambda_m(t) F_{J,t}(m)$. In particular :

$$\mathfrak{K}_{J,t}^\infty \cap \tilde{D}_M \simeq \mathbb{Z}[t^\pm]^{B_J \cap D_M}$$

Proof : The uniqueness of $F_{J,t}(m)$ follows from lemma 5.7. Let us write $m = m_J m'$ where $m_J = \prod_{i \in J, l \in \mathbb{Z}} Y_{i,l}^{u_{i,l}(m)}$. So m_J is a dominant $\mathcal{Y}_t$-monomial of $\mathbb{Z}[Y_{i,l}^\pm]_{i \in J, l \in \mathbb{Z}}$. In particular the proposition 5.17 with the algebra $\mathcal{U}_q(\hat{\mathfrak{g}})_J$ of rank n gives $m_J \chi$ where $\chi \in$

$\mathbb{Z}[\tilde{A}_{i,l}^{\mathcal{U}_q(\hat{\mathfrak{g}})_J,-1}, t^{\pm}]_{i\in J, l\in\mathbb{Z}}$ (where for $i\in I, l\in\mathbb{Z}$, $\tilde{A}_{i,l}^{\mathcal{U}_q(\hat{\mathfrak{g}})_J,\pm} = \beta_{I,J}(\tilde{A}_{i,l}^{\pm})$ where $\beta_{I,J}(\tilde{Y}_{i,l}^{\pm}) = \delta_{i\in J}\tilde{Y}_{i,l}^{\pm}$). So we can put $F_t(m) = m\nu_{J,t}(\chi)$ where $\nu_{J,t} : \mathbb{Z}[\tilde{A}_{i,l}^{\mathcal{U}_q(\hat{\mathfrak{g}})_J,-1}, t^{\pm}]_{i\in J,l\in\mathbb{Z}} \to \mathcal{Y}_t$ is the ring homomorphism such that
$\nu_{J,t}(\tilde{A}_{i,l}^{\mathcal{U}_q(\hat{\mathfrak{g}})_J,-1}) = \tilde{A}_{i,l}^{-1}$. The last assertion is proved as in proposition 5.17. □

5.7 Step 3 : t-algorithm and end of the proof of theorem 5.11

In this section we prove the property $P(n)$ by induction on $n\geq 1$. In particular we define the t-algorithm which constructs explicitly the $F_t(m)$.

It follows from section 5.5 that $P(1)$ and $P(2)$ are true. Let be $n\geq 3$ and suppose that $P(r)$ is proved for $r<n$.

Let m_+ be in B and $m_0 = m_+ > m_1 > m_2 > ...$ the countable set D_{m_+} with a total ordering compatible with the partial ordering.

For $J\subsetneq I$ and $m\in B_J$, it follows from $P(r)$ and corollary 5.18 that there is a unique $F_{J,t}(m)\in \tilde{D}_m\cap\mathfrak{K}_{J,t}^{\infty}$ such that m is the unique J-dominant monomial of $F_{J,t}(m)$. Moreover the elements of $\tilde{D}_{m_+}\cap\mathfrak{K}_{J,t}^{\infty}$ are the infinite sums of $\mathcal{Y}_t^{\infty}$: $\sum_{m\in D_{m_+}\cap B_J}\lambda_m(t)F_{J,t}(m)$ where $\lambda_m(t)\in\mathbb{Z}[t^{\pm}]$. If $m\in A - B_J$, denote $F_{J,t}(m)=0$.

For $r,r'\geq 0$ and $J\subsetneq I$ denote by $[F_{J,t}(m_{r'})]_{m_r}\in\mathbb{Z}[t^{\pm}]$ the coefficient of m_r in $F_{J,t}(m_{r'})$.

5.7.1 Definition of the t-algorithm

Definition 5.19. *We call t-algorithm the following inductive definition of the sequences $(s(m_r)(t))_{r\geq 0}\in\mathbb{Z}[t^{\pm}]^{\mathbb{N}}$, $(s_J(m_r)(t))_{r\geq 0}\in\mathbb{Z}[t^{\pm}]^{\mathbb{N}}$ $(J\subsetneq I)$:*

$$s(m_0)(t) = 1 \ , \ s_J(m_0)(t) = 0$$

and for $r\geq 1, J\subsetneq I$:

$$s_J(m_r)(t) = \sum_{r'<r}(s(m_{r'})(t) - s_J(m_{r'})(t))[F_{J,t}(m_{r'})]_{m_r}$$

$$\textit{if } m_r\notin B_J, s(m_r)(t) = s_J(m_r)(t)$$

$$\textit{if } m_r\in B, s(m_r)(t) = 0$$

We have to prove that the t-algorithm defines the sequences in a unique way. We see that if $s(m_r), s_J(m_r)$ are defined for $r\leq R$ so are $s_J(m_{R+1})$ for $J\subsetneq I$. The $s_J(m_R)$ impose the value of $s(m_{R+1})$ and by induction the uniqueness is clear. We say that the t-algorithm is well defined to step R if there exist $s(m_r), s_J(m_r)$ such that the formulas of the t-algorithm are verified for $r\leq R$.

Lemma 5.20. *The t-algorithm is well defined to step r if and only if :*

$$\forall J_1, J_2\subsetneq I, \forall r'\leq r, m_{r'}\notin B_{J_1} \textit{ and } m_{r'}\notin B_{J_2}\Rightarrow s_{J_1}(m_{r'})(t) = s_{J_2}(m_{r'})(t)$$

Proof : If for $r' < r$ the $s(m_{r'})(t), s_J(m_{r'})(t)$ are well defined, so is $s_J(m_r)(t)$. If $m_r \in B$, $s(m_r)(t) = 0$ is well defined. If $m_r \notin B$, it is well defined if and only if $\{s_J(m_r)(t)/m_r \notin B_J\}$ has one unique element. □

5.7.2 The t-algorithm never fails

If the t-algorithm is well defined to all steps, we say that the t-algorithm never fails. If the t-algorithm is well defined to step r, for $J \subsetneq I$ we set :

$$\mu_J(m_r)(t) = s(m_r)(t) - s_J(m_r)(t)$$

$$\chi_J^r = \sum_{r' \leq r} \mu_J(m_{r'}(t)) F_{J,t}(m_{r'}) \in \mathfrak{K}_{J,t}^{\infty}$$

Lemma 5.21. *If the t-algorithm is well defined to step r, for $J \subset I$ we have :*

$$\chi_J^r \in (\sum_{r' \leq r} s(m_{r'})(t) m_{r'}) + s_J(m_{r+1})(t) m_{r+1} + \sum_{r' > r+1} \mathbb{Z}[t^{\pm}] m_{r'}$$

For $J_1 \subset J_2 \subsetneq I$, we have :

$$\chi_{J_2}^r = \chi_{J_1}^r + \sum_{r' > r} \lambda_{r'}(t) F_{J_1,t}(m_{r'})$$

where $\lambda_{r'}(t) \in \mathbb{Z}[t^{\pm}]$. In particular, if $m_{r+1} \notin B_{J_1}$, we have $s_{J_1,t}(m_{r+1}) = s_{J_2,t}(m_{r+1})$.

Proof : For $r' \leq r$ let us compute the coefficient $(\chi_J^r)_{m_{r'}} \in \mathbb{Z}[t^{\pm}]$ of $m_{r'}$ in χ_J^r :

$$\begin{aligned}(\chi_J^r)_{m_{r'}} &= \sum_{r'' \leq r'} (s(m_{r''})(t) - s_J(m_{r''})(t))[F_{J,t}(m_{r''})]_{m_{r'}} \\ &= (s(m_{r'})(t) - s_J(m_{r'})(t))[F_{J,t}(m_{r'})]_{m_{r'}} \\ &\quad + \sum_{r'' < r'} (s(m_{r''})(t) - s_J(m_{r''})(t))[F_{J,t}(m_{r''})]_{m_{r'}} \\ &= (s(m_{r'})(t) - s_J(m_{r'})(t)) + s_J(m_{r'})(t) = s(m_{r'})(t)\end{aligned}$$

Let us compute the coefficient $(\chi_J^r)_{m_{r+1}} \in \mathbb{Z}[t^{\pm}]$ of m_{r+1} in χ_J^r :

$$(\chi_J^r)_{m_{r+1}} = \sum_{r'' < r+1} (s(m_{r''})(t) - s_J(m_{r''})(t))[F_{J,t}(m_{r''})]_{m_{r+1}} = s_J(m_{r+1})$$

For the second point let $J_1 \subset J_2 \subsetneq I$. We have $\chi_{J_2}^r \in \mathfrak{K}_{J_1,t}^{\infty} \cap \tilde{D}_{m+}$ and it follows from $P(|J_1|)$ and corollary 5.18 (or section 5.5 if $|J_1| \leq 2$) that we can introduce $\lambda_{m_{r'}}(t) \in \mathbb{Z}[t^{\pm}]$ such that :

$$\chi_{J_2}^r = \sum_{r' \geq 0} \lambda_{m_{r'}}(t) F_{J_1,t}(m_{r'})$$

We show by induction on r' that for $r' \leq r$, $m_{r'} \in B_{J_1} \Rightarrow \lambda_{m_{r'}}(t) = \mu_{J_1}(m_{r'})(t)$. First we have $\lambda_{m_0}(t) = (\chi^r_{J_2})_{m_0} = s(m_0)(t) = 1 = \mu_{J_1}(m_0)$. For $r' \leq r$:

$$s(m_{r'})(t) = \lambda_{m_{r'}}(t) + \sum_{r''<r'} \lambda_{m_{r''}}(t)[F_{J_1,t}(m_{r''})]_{m_{r'}}$$

$$\lambda_{m_{r'}}(t) = s(m_{r'})(t) - \sum_{r''<r'} \mu_{J_1}(m_{r''})(t)[F_{J_1,t}(m_{r''})]_{m_{r'}}$$
$$= s(m_{r'})(t) - s_{J_1}(m_{r'})(t) = \mu_{J_1}(m_{r'})(t)$$

For the last assertion if $m_{r+1} \notin B_{J_1}$, the coefficient of m_{r+1} in $\sum_{r'>r} \mathbb{Z}[t^{\pm}]F_{J_1,t}(m_{r'})$ is 0, and $(\chi^r_{J_2})_{m_{r+1}} = (\chi^r_{J_1})_{m_{r+1}}$. It follows from the first point that $s_{J_1,t}(m_{r+1}) = s_{J_2,t}(m_{r+1})$. □

Lemma 5.22. *The t-algorithm never fails.*

Proof : Suppose that the sequence is well defined until the step $r-1$ and let $J_1, J_2 \subsetneq I$ such that $m_r \notin B_{J_1}$ and $m_r \notin B_{J_2}$. Let i be in J_1, j in J_2 such that $m_r \notin B_i$ and $m_r \notin B_j$. Consider $J = \{i,j\} \subsetneq I$. The $\chi^{r-1}_J, \chi^{r-1}_i, \chi^{r-1}_j \in \mathcal{Y}_t$ have the same coefficient $s(m_{r'})(t)$ on $m_{r'}$ for $r' \leq r-1$. Moreover :

$$s_i(m_r)(t) = (\chi^{r-1}_i)_{m_r} \ , \ s_j(m_r)(t) = (\chi^{r-1}_j)_{m_r} \ , \ s_J(m_r)(t) = (\chi^{r-1}_J)_{m_r}$$

But $m_r \notin B_J$, so :

$$\chi^{r-1}_J = \sum_{r' \leq r-1} \mu_i(m_{r'})(t)F_{i,t}(m_{r'}) + \sum_{r' \geq r+1} \lambda_{m_{r'}}(t)F_{i,t}(m_{r'})$$

So $(\chi^{r-1}_J)_{m_r} = (\chi^{r-1}_i)_{m_r}$ and we have $s_i(m_r)(t) = s_J(m_r)(t)$. In the same way we have $s_i(m_r)(t) = s_{J_1}(m_r)(t)$, $s_j(m_r)(t) = s_J(m_r)(t)$ and $s_j(m_r)(t) = s_{J_2}(m_r)(t)$. So we can conclude $s_{J_1}(m_r)(t) = s_{J_2}(m_r)(t)$. □

5.7.3 Proof of $P(n)$

It follows from lemma 5.22 that $\chi = \sum_{r \geq 0} s(m_r)(t)m_r \in \mathcal{Y}^\infty_t$ is well defined.

Lemma 5.23. *We have $\chi \in \mathfrak{K}^\infty_t \bigcap \tilde{D}_{m_+}$. Moreover the only dominant $\mathcal{Y}_t$-monomial in χ is $m_0 = m_+$.*

Proof : The defining formula of χ gives $\chi \in \tilde{D}_{m_+}$. Let i be in I and :

$$\chi_i = \sum_{r \geq 0} \mu_i(m_r)(t)F_{i,t}(m_r) \in \mathfrak{K}^\infty_{i,t}$$

Let us compute for $r \geq 0$ the coefficient of m_r in $\chi - \chi_i$:

$$(\chi - \chi_i)_{m_r} = s(m_r)(t) - \sum_{r' \leq r} \mu_i(m_{r'})(t)[F_{i,t}(m_{r'})]_{m_r}$$

$$= s(m_r)(t) - s_i(m_r)(t) - \mu_i(m_r)(t)[F_{i,t}(m_r)]_{m_r}$$

$$= (s(m_r)(t) - s_i(m_r)(t))(1 - [F_{i,t}(m_r)]_{m_r})$$

We have two cases :

if $m_r \in B_i$, we have $1 - [F_{i,t}(m_r)]_{m_r} = 0$.

if $m_r \notin B_i$, we have $s(m_r)(t) - s_i(m_r)(t) = 0$.

So $\chi = \chi_i \in \mathfrak{K}_{i,t}^\infty$, and $\chi \in \mathfrak{K}_t^\infty$. The last assertion follows from the definition of the algorithm : for $r > 0$, $m_r \in B \Rightarrow s(m_r)(t) = 0$. □

Corollary 5.24. *For $n \geq 3$, if the $P(r)$ $(r < n)$ are true, then $P(n)$ is true.*

In particular the theorem 5.11 is proved by induction on n.

6 Morphism of q,t-characters and applications

6.1 Morphism of q,t-characters

We set $\text{Rep}_t = \text{Rep} \otimes_{\mathbb{Z}} \mathbb{Z}[t^\pm] = \mathbb{Z}[X_{i,l}, t^\pm]_{i\in I, l\in\mathbb{Z}}$. We say that $M \in \text{Rep}_t$ is a Rep_t-monomial if it is of the form $M = \prod_{i\in I, l\in\mathbb{Z}} X_{i,l}^{x_{i,l}}$ $(x_{i,l} \geq 0)$. In this case denote $x_{i,l}(M) = x_{i,l}$. Recall the definition of the $E_t(m)$ (section 5.4).

Definition 6.1. *The morphism of q,t-characters is the $\mathbb{Z}[t^\pm]$-linear map $\chi_{q,t} : Rep_t \to \mathcal{Y}_t^\infty$ such that $(u_{i,l} \geq 0)$:*

$$\chi_{q,t}(\prod_{i\in I, l\in\mathbb{Z}} X_{i,l}^{u_{i,l}}) = E_t(\prod_{i\in I, l\in\mathbb{Z}} Y_{i,l}^{u_{i,l}})$$

Theorem 6.2. *We have $\pi_+(Im(\chi_{q,t})) \subset \mathcal{Y}$ and the following diagram is commutative :*

$$\begin{array}{ccc} Rep & \xrightarrow{\chi_{q,t}} & Im(\chi_{q,t}) \\ id \downarrow & & \downarrow \quad \pi_+ \\ Rep & \xrightarrow{\chi_q} & \mathcal{Y} \end{array}$$

In particular the map $\chi_{q,t}$ is injective. The $\mathbb{Z}[t^\pm]$-linear map $\chi_{q,t} : Rep_t \to \mathcal{Y}_t^\infty$ is characterized by the three following properties :

1) For a Rep_t-monomial M define $m = \pi^{-1}(\prod_{i\in I, l\in\mathbb{Z}} Y_{i,l}^{x_{i,l}(M)}) \in A$ and $\tilde{m} \in A_t$ as in section 3.5.1. Then we have :

$$\chi_{q,t}(M) = \tilde{m} + \sum_{m'<m} a_{m'}(t) m' \ (\textit{where } a_{m'}(t) \in \mathbb{Z}[t^\pm])$$

2) The image of $Im(\chi_{q,t})$ is contained in $\mathfrak{K}_t^\infty$.

3) Let M_1, M_2 be Rep_t-monomials such that

$$max\{l/\sum_{i\in I} x_{i,l}(M_1)>0\} \leq min\{l/\sum_{i\in I} x_{i,l}(M_2)>0\}$$

Then we have :

$$\chi_{q,t}(M_1M_2)=\chi_{q,t}(M_1)\chi_{q,t}(M_2)$$

Note that the properties $1,2,3$ are generalizations of the defining axioms introduced by Nakajima in [N3] for the ADE-case ; in particular in the ADE-case $\chi_{q,t}$ is the morphism of q,t-characters constructed in [N3].

Proof : $\pi_+(\mathrm{Im}(\chi_{q,t}))\subset\mathcal{Y}$ means that only a finite number of $\mathcal{Y}_t$-monomials of $E_t(m)$ have coefficient $\lambda(t)\notin(t-1)\mathbb{Z}[t^\pm]$. As $F_t(\tilde{Y}_{i,0})$ has no dominant $\mathcal{Y}_t$-monomial other than $\tilde{Y}_{i,0}$, we have the same property for $\pi_+(F_t(\tilde{Y}_{i,0}))\in\mathfrak{K}^\infty$ and $\pi_+(F_t(\tilde{Y}_{i,0}))=E(Y_{i,0})\in\mathcal{Y}$. As $\mathcal{Y}$ is a subalgebra of $\mathcal{Y}^\infty$ we get $\pi_+(E_t(m))\in\mathcal{Y}$ with the help of the defining formula.

The diagram is commutative because $\pi_+\circ s_l=s_l\circ\pi_+$ and $\pi_+(F_t(\tilde{Y}_{i,0}))=E(Y_{i,0})$. It is proved by Frenkel, Reshetikhin in [FR3] that χ_q is injective, so $\chi_{q,t}$ is injective. Let us show that $\chi_{q,t}$ verifies the three properties :

1) By definition we have $\chi_{q,t}(M)=E_t(m)$. But $s_l(F_t(\tilde{Y}_{i,0}))=F_t(\tilde{Y}_{i,l})\in\tilde{D}(\tilde{Y}_{i,l})$. In particular $s_l(F_t(\tilde{Y}_{i,0}))$ is of the form $\tilde{Y}_{i,l}+\sum\limits_{m'<Y_{i,l}}\lambda_{m'}(t)m'$ and we get the property for $E_t(m)$ by multiplication.

2) We have $s_l(F_t(\tilde{Y}_{i,0}))=E_t(\tilde{Y}_{i,l})\in\mathfrak{K}_t^\infty$ and $\mathfrak{K}_t^\infty$ is a subalgebra of $\mathcal{Y}_t^\infty$, so $\mathrm{Im}(\chi_{q,t})\subset\mathfrak{K}_t^\infty$.

3) If we set $L=\max\{l/\sum\limits_{i\in I}x_{i,l}(M_1)>0\}$, $m_1=\prod\limits_{i\in I,l\in\mathbb{Z}}Y_{i,l}^{x_{i,l}(M_1)}$ and $m_2=\prod\limits_{i\in I,l\in\mathbb{Z}}Y_{i,l}^{x_{i,l}(M_2)}$ we have :

$$E_t(m_1)=\overrightarrow{\prod_{l\leq L}}\prod_{i\in I}s_l(F_t(\tilde{Y}_{i,0}))^{x_{i,l}(M_1)}\ ,\ E_t(m_2)=\overrightarrow{\prod_{l\geq L}}\prod_{i\in I}s_l(F_t(\tilde{Y}_{i,0}))^{x_{i,l}(M_2)}$$

and in particular : $E_t(m_1m_2)=E_t(m_1)E_t(m_2)$.

Finally let $f:\mathrm{Rep}_t\to\mathcal{Y}_t^\infty$ be a $\mathbb{Z}[t^\pm]$-linear homomorphism which verifies properties 1,2,3. We saw that there is only element of $\mathfrak{K}_t^\infty$ with highest weight monomial $\tilde{Y}_{i,l}$ is $s_l(F_t(\tilde{Y}_{i,0}))$. In particular we have $f(X_{i,l})=E_t(Y_{i,l})$. Using property 3, we get for $M\in\mathrm{Rep}_t$ a monomial :

$$f(M)=\overrightarrow{\prod_{l\in\mathbb{Z}}}\prod_{i\in I}f(X_{i,l})^{u_{i,l}(m)}=\overrightarrow{\prod_{l\in\mathbb{Z}}}\prod_{i\in I}s_l(F_t(\tilde{Y}_{i,0}))^{u_{i,l}(m)}=\chi_{q,t}(M)$$

$\square$

6.2 Quantization of the Grothendieck Ring

In this section we see that $\chi_{q,t}$ allows us to define a deformed algebra structure on Rep_t generalizing the quantization of [N3]. The point is to show that $\mathrm{Im}(\chi_{q,t})$ is a subalgebra of $\mathfrak{K}_t^\infty$.

6.2.1 Construction of the quantization

Recall the definition of $\mathfrak{K}_t^{\infty,f}$ in section 5.6. For $m \in B$, all monomials of $E_t(m)$ are in $\{mA_{i_1,l_1}^{-1}...A_{i_K,l_K}^{-1}/k \geq 0, l_k \geq L\}$ where $L = \min\{l \in \mathbb{Z}, \exists i \in I, u_{i,l}(m) > 0\}$. So it follows from lemma 3.14 that $E_t(m) \in \mathfrak{K}_t^{\infty}$ has only a finite number of dominant $\mathcal{Y}_t$-monomials, that is to say $E_t(m) \in \mathfrak{K}_t^{\infty,f}$.

Proposition 6.3. *The $\mathbb{Z}[t^{\pm}]$-module $\mathfrak{K}_t^{\infty,f}$ is freely generated by the $E_t(m)$:*

$$\mathfrak{K}_t^{\infty,f} = \bigoplus_{m \in B} \mathbb{Z}[t^{\pm}]E_t(m) \simeq \mathbb{Z}[t^{\pm}]^{(B)}$$

Moreover $\mathfrak{K}_t^{\infty,f}$ is a subalgebra of $\mathfrak{K}_t^{\infty}$.

Proof : The $E_t(m)$ are $\mathbb{Z}[t^{\pm}]$-linearly independent and we saw $E_t(m) \in \mathfrak{K}_t^{\infty,f}$. It suffices to prove that the $E_t(m)$ generate the $F_t(m)$: let us look at $m_0 \in B$ and consider $L = \min\{l \in \mathbb{Z}, \exists i \in I, u_{i,l}(m_0) > 0\}$. In the proof of lemma 3.14 we saw there is only a finite dominant monomials in $\{m_0 A_{i_1,l_1}^{-v_{i_1,l_1}}...A_{i_R,l_R}^{-v_{i_R,l_R}}/R \geq 0, i_r \in I, l_r \geq L\}$. Let $m_0 > m_1 > ... > m_D \in B$ be those monomials with a total ordering compatible with the partial ordering. In particular, for $0 \leq d \leq D$ the dominant monomials of $E_t(m_d)$ are in $\{m_d, m_{d+1}, ..., m_D\}$. So there are elements $(\lambda_{d,d'}(t))_{0 \leq d,d' \leq D}$ of $\mathbb{Z}[t^{\pm}]$ such that $E_t(m_d) = \sum_{d \leq d' \leq D} \lambda_{d,d'}(t) F_t(m_{d'})$. We have $\lambda_{d,d'}(t) = 0$ if $d' < d$ and $\lambda_{d,d}(t) = 1$. We have a triangular system with 1 on the diagonal, so it is invertible in $\mathbb{Z}[t^{\pm}]$.

For the last point it suffices to prove that for $m_1, m_2 \in B$, $E_t(m_1)E_t(m_2)$ has only a finite number of dominant $\mathcal{Y}_t$-monomials. But $E_t(m_1)E_t(m_2)$ has the same monomials as $E_t(m_1 m_2)$. □

It particular $\chi_{q,t}$ is a $\mathbb{Z}[t^{\pm}]$-linear isomorphism between Rep_t and $\mathfrak{K}_t^{\infty,f}$. So we can define an associative deformed $\mathbb{Z}[t^{\pm}]$-algebra structure on Rep_t by

$$\forall \lambda_1, \lambda_2 \in \mathrm{Rep}_t, \lambda_1 * \lambda_2 = \chi_{q,t}^{-1}(\chi_{q,t}(\lambda_1)\chi_{q,t}(\lambda_2))$$

.

6.2.2 Examples : sl_2-case

We make explicit computation of the deformed multiplication in the sl_2-case :

Proposition 6.4. *In the sl_2-case, the deformed algebra structure on $Rep_t = \mathbb{Z}[X_l, t^{\pm}]_{l \in \mathbb{Z}}$ is given by :*

$$X_{l_1} * X_{l_2} * ... * X_{l_m} = X_{l_1} X_{l_2} ... X_{l_m} \text{ if } l_1 \leq l_2 \leq ... \leq l_m$$

$$X_l * X_{l'} = t^{\gamma} X_l X_{l'} = t^{\gamma} X_{l'} * X_l \text{ if } l > l' \text{ and } l \neq l' + 2$$

$$X_l * X_{l-2} = t^{-2} X_l X_{l-2} + t^{\gamma}(1 - t^{-2}) = t^{-2} X_{l-2} * X_l + (1 - t^{-2})$$

where $\gamma \in \mathbb{Z}$ is defined by $\tilde{Y}_l \tilde{Y}_{l'} = t^{\gamma} \tilde{Y}_{l'} \tilde{Y}_l$.

The proof is a straightforward computation which uses for $l \in \mathbb{Z}$ the q,t-character of $X_l : \chi_{q,t}(X_l) = \tilde{Y}_l + \tilde{Y}_{l+2}^{-1} = \tilde{Y}_l(1 + t\tilde{A}_{l+1}^{-1})$.

Note that γ were computed in section 3.5.3. We see that the new $\mathbb{Z}[t^{\pm}]$-algebra structure is not commutative and not even twisted polynomial.

6.3 An involution of the Grothendieck ring

In this section we construct an antimultiplicative involution of the Grothendieck ring Rep_t. The construction is motivated by the point view adopted in this article : it is just replacing $c_{|l|}$ by $-c_{|l|}$. In the ADE-case such an involution were introduced Nakajima [N3] with different motivations.

Lemma 6.5. *There is a unique $\mathbb{C}$-linear isomorphism of $\mathcal{H}_h$ which is antimultiplicative and such that ($m > 0, i \in I, r \in \mathbb{Z} - \{0\}$) :*

$$\overline{c_m} = -c_m \ , \ \overline{a_i[r]} = a_i[r]$$

Moreover it is an involution. The $\mathbb{Z}$-subalgebra $\mathcal{Y}_u \subset \mathcal{H}_h$ verifies $\overline{\mathcal{Y}_u} \subset \mathcal{Y}_u$.

Proof : It suffices to show that it is compatible with the defining relations of $\mathcal{H}$ ($i, j \in I, m, r \in \mathbb{Z} - \{0\}$) : $\overline{[a_i[m], a_j[r]]} = -[a_i[m], a_j[r]]$. It is an involution because $\overline{\overline{c_m}} = c_m$ and $\overline{\overline{a_i[r]}} = a_i[r]$. For the last assertion it suffices to check on the generators of $\mathcal{Y}_u$ ($R \in \mathfrak{U}, i \in I, l \in \mathbb{Z}$) :

$$\overline{t_R} = t_{-R} \ , \ \overline{\tilde{Y}_{i,l}} = t_{-\tilde{C}_{i,i}(q)(q_i - q_i^{-1})}\tilde{Y}_{i,l} \ , \ \overline{\tilde{Y}_{i,l}}^{-1} = t_{\tilde{C}_{i,i}(q)(q_i - q_i^{-1})}\tilde{Y}_{i,l}^{-1}$$

□

As for $R, R' \in \mathfrak{U}$, we have $\pi_0(R) = \pi_0(R') \Leftrightarrow \pi_0(-R) = \pi_0(-R')$, the involution of $\mathcal{Y}_u$ (resp. of $\mathcal{H}_h$) is compatible with the defining relations of $\mathcal{Y}_t$ (resp. $\mathcal{H}_t$). We get a $\mathbb{Z}$-linear involution of $\mathcal{Y}_t$ (resp. of $\mathcal{H}_t$). For $\lambda, \lambda' \in \mathcal{Y}_t, \alpha \in \mathbb{Z}$, we have :

$$\overline{\lambda.\lambda'} = \overline{\lambda'}.\overline{\lambda} \ , \ \overline{t^\alpha \lambda} = t^{-\alpha}\overline{\lambda}$$

Note that in $\mathcal{Y}_u$ for $i \in I, l \in \mathbb{Z}$, $\overline{\tilde{A}_{i,l}} = t_{(-q_i^2 + q_i^{-2})}\tilde{A}_{i,l}$. So in $\mathcal{Y}_t$ we have $\overline{\tilde{A}_{i,l}} = \tilde{A}_{i,l}$ and $\overline{\tilde{A}_{i,l}^{-1}} = \tilde{A}_{i,l}^{-1}$.

Lemma 6.6. *For $i \in I$, the $\mathcal{Y}_{i,u} \subset \mathcal{H}_h$ verifies $\overline{\mathcal{Y}_{i,u}} \subset \mathcal{Y}_{i,u}$.*

Proof : First we compute for $i \in I, l \in \mathbb{Z}$: $\overline{\tilde{S}_{i,l}} = t_{\frac{q_i + q_i^{-1}}{q_i - q_i^{-1}}}\tilde{S}_{i,l} \in \mathcal{Y}_{i,u}$. Now for $\lambda \in \mathcal{Y}_u$, we have $\overline{\lambda.\tilde{S}_{i,l}} = t_{\frac{q_i + q_i^{-1}}{q_i - q_i^{-1}}}\tilde{S}_{i,l}\overline{\lambda}$. But it is in $\mathcal{Y}_{i,u}$ because $\overline{\lambda} \in \mathcal{Y}_u$ (lemma 6.5) and $\mathcal{Y}_{i,u}$ is a $\mathcal{Y}_u$-subbimodule of $\mathcal{H}_h$ (lemma 4.6). □

In $\mathcal{H}_t$ we have $\overline{\tilde{S}_{i,l}} = t\tilde{S}_{i,l}$ because $\pi_0(\frac{q_i+q_i^{-1}}{q_i-q_i^{-1}}) = 1$. As said before we get a $\mathbb{Z}$-linear involution of $\mathcal{Y}_{i,t}$ such that :

$$\overline{\lambda\tilde{S}_{i,l}} = t\tilde{S}_{i,l}\overline{\lambda}$$

We introduced such an involution in [He1]. With this new point of view, the compatibility with the relation $\tilde{A}_{i,l-r_i}\tilde{S}_{i,l} = t^{-1}\tilde{S}_{i,l+r_i}$ is a direct consequence of lemma 4.6.

Lemma 6.7. *For $i \in I$, the subalgebra $\mathfrak{K}_{i,t} \subset \mathcal{Y}_t$ verifies $\overline{\mathfrak{K}_{i,t}} \subset \mathfrak{K}_{i,t}$.*

Proof : Suppose $\lambda \in \mathfrak{K}_{i,t}$, that is to say $S_{i,t}(\lambda) = 0$. So $\overline{(t^2-1)S_{i,t}(\lambda)} = 0$ and :

$$\sum_{l\in\mathbb{Z}}(\overline{\tilde{S}_{i,l}\lambda} - \overline{\lambda\tilde{S}_{i,l}}) = 0 \Rightarrow t\sum_{l\in\mathbb{Z}}(\overline{\lambda}\tilde{S}_{i,l} - \tilde{S}_{i,l}\overline{\lambda}) = 0$$

So $t(1-t^2)S_{i,t}(\overline{\lambda}) = 0$ and $\overline{\lambda} \in \mathfrak{K}_{i,t}$. □

Note that $\chi \in \mathcal{Y}_t$ has the same monomials as $\overline{\chi}$, that is to say if $\chi = \sum_{m\in A}\lambda(t)m$ and $\overline{\chi} = \sum_{m\in A}\mu(t)m$, we have $\lambda(t) \neq 0 \Leftrightarrow \mu(t) \neq 0$. In particular we can naturally extend our involution to an antimultiplicative involution on $\mathcal{Y}_t^\infty$. Moreover we have $\overline{\mathfrak{K}_t^\infty} \subset \mathfrak{K}_t^\infty$ and $\overline{\mathfrak{K}_t^{\infty,f}} = \overline{\mathrm{Im}(\chi_{q,t})} \subset \mathrm{Im}(\chi_{q,t})$. So we can define a $\mathbb{Z}$-linear involution of Rep_t by

$$\forall\lambda \in \mathrm{Rep}_t\ ,\ \overline{\lambda} = \chi_{q,t}^{-1}(\overline{\chi_{q,t}(\lambda)})$$

6.4 Analogues of Kazhdan-Lusztig polynomials

In this section we define analogues of Kazhdan-Lusztig polynomials (see [KL]) with the help of the antimultiplicative involution of section 6.3 in the same spirit Nakajima did for the ADE-case [N3]. Let us begin we some technical properties of the action of the involution on monomials.

6.4.1 Construction

We recall that the $\mathcal{Y}_t^A$-monomials are products of the $\tilde{A}_{i,l}^{-1}$ $(i \in I, l \in \mathbb{Z})$.

Lemma 6.8. *For M a $\mathcal{Y}_t$-monomial and m a $\mathcal{Y}_t^A$-monomial there is a unique $\alpha(M,m) \in \mathbb{Z}$ such that $\overline{t^{\alpha(M,m)}Mm} = t^{\alpha(M,m)}\overline{M}m$.*

Proof : Let $\beta \in \mathbb{Z}$ such that $\overline{m} = t^\beta m$. We have $\overline{Mm} = \overline{m}\,\overline{M} = t^{\beta+\gamma}\overline{M}m$ where $\gamma \in 2\mathbb{Z}$ (section 3.4). So it suffices to prove that $\beta \in 2\mathbb{Z}$.

Let us compute β. Let $\pi_+(m) = \prod_{i\in I,l\in\mathbb{Z}} A_{i,l}^{-v_{i,l}}$. In $\mathcal{Y}_u$ we have :

$$\pi_+(m)\pi_-(m) = t_R\pi_-(m)\pi_+(m)$$

where $\pi_0(R)=\beta$ and :

$$R(q)=\sum_{i,j\in I,r,r'\in\mathbb{Z}}v_{i,r}v_{j,r'}\sum_{l>0}q^{lr-lr'}\frac{[a_i[l],a_j[-l]]}{c_l}$$

where for $l>0$ we set $\frac{[a_i[l],a_j[-l]]}{c_l}=B_{i,j}(q^l)(q^l-q^{-l})\in\mathbb{Z}[q^{\pm}]$ which is antisymmetric. For $i=j$, we have the term :

$$\sum_{r,r'\in\mathbb{Z}}v_{i,r}v_{i,r'}\sum_{l>0}q^{lr-lr'}\frac{[a_i[l],a_i[-l]]}{c_l}$$

$$=\sum_{l>0}(\sum_{\{r,r'\}\subset\mathbb{Z},r\neq r'}v_{i,r}(m)v_{i,r'}(m)(q^{l(r-r')}+q^{l(r'-r)})+\sum_{r\in\mathbb{Z}}v_{i,r}(m)^2)\frac{[a_i[l],a_i[-l]]}{c_l}$$

It is antisymmetric, so it has no term in q^0. So $\pi_0(R)=\pi_0(R')$ where R' is the sum of the contributions for $i\neq j$:

$$\sum_{r,r'\in\mathbb{Z}}v_{i,r}(m)v_{j,r'}(m)\sum_{l>0}q^{lr-lr'}(\frac{[a_i[l],a_j[-l]]}{c_l}+\frac{[a_j[l],a_i[-l]]}{c_l})$$

$$=2\sum_{r,r'\in\mathbb{Z}}v_{i,r}(m)v_{j,r'}(m)\sum_{l>0}q^{lr-lr'}\frac{[a_i[l],a_j[-l]]}{c_l}$$

In particular $\pi_0(R')\in 2\mathbb{Z}$. □

For M a $\mathcal{Y}_t$-monomial denote $A_M^{\mathrm{inv}}=\{t^{\alpha(m,M)}Mm/m\ \mathcal{Y}_t^A\text{-monomial}\}$. In particular for $m'\in A_M^{\mathrm{inv}}$ we have $\overline{m'}m'^{-1}=\overline{M}M^{-1}$. Let $B_M^{\mathrm{inv}}=(t^{\mathbb{Z}}B)\cap A_M^{\mathrm{inv}}$.

Theorem 6.9. *For $m\in t^{\mathbb{Z}}B$ there is a unique $L_t(m)\in\mathfrak{K}_t^\infty$ such that :*

$$\overline{L_t(m)}=(\overline{m}m^{-1})L_t(m)$$

$$E_t(m)=L_t(m)+\sum_{m'<m,m'\in B_m^{inv}}P_{m',m}(t)L_t(m')$$

where $P_{m',m}(t)\in t^{-1}\mathbb{Z}[t^{-1}]$.

Those polynomials $P_{m',m}(t)$ are called analogues to Kazhdan-Lusztig polynomials and the $L_t(m)$ ($m\in B$) for a canonical basis of $\mathfrak{K}_t^{f,\infty}$. Such polynomials were introduced by Nakajima [N3] for the ADE-case.

<u>*Proof :*</u> First consider $\overline{F_t(m)}$: it is in $\mathfrak{K}_t^\infty$ and has only one dominant $\mathcal{Y}_t$-monomial $\overline{m}$, so $\overline{F_t(m)}=\overline{m}m^{-1}F_t(m)$.

Let be $m=m_L>m_{L-1}>...>m_0$ the finite set $t^{\mathbb{Z}}D(m)\cap B_m^{\mathrm{inv}}$ (see lemma 5.8) with a total ordering compatible with the partial ordering. Note that it follows from section 6.4.1 that for $L\geq l\geq 0$, we have $\overline{m_l}m_l^{-1}=\overline{m}m^{-1}$.

We have $E_t(m_0)=F_t(m_0)$ and so $\overline{E_t(m_0)}=\overline{m_0}m_0^{-1}E_t(m_0)$. As $B_{m_0}^{\mathrm{inv}}=\{m_0\}$, we have $L_t(m_0)=E_t(m_0)$. We suppose by induction that the $L_t(m_l)$ ($L-1\geq l\geq 0$) are uniquely and well defined. In particular m_l is of highest weight in $L_t(m_l)$, $\overline{L_t(m_l)} = \overline{m_l}m_l^{-1}L_t(m_l) = \overline{m}m^{-1}L_t(m_l)$, and we can write :

$$\tilde{D}_t(m_L) \cap \mathfrak{K}_t^{\infty} = \mathbb{Z}[t^{\pm}]F_t(m_L) \oplus \bigoplus_{0\leq l\leq L-1} \mathbb{Z}[t^{\pm}]L_t(m_l)$$

In particular consider $\alpha_{l,L}(t) \in \mathbb{Z}[t^{\pm}]$ such that :

$$E_t(m) = F_t(m) + \sum_{l<L} \alpha_{l,L}(t)L_t(m_l)$$

We want $L_t(m)$ of the form $L_t(m) = F_t(m) + \sum_{l<L} \beta_{l,L}(t)L_t(m_l)$. The condition $\overline{L_t(m)} = \overline{m}m^{-1}mL_t(m)$ means that the $\beta_{l,L}(t)$ are symmetric. The condition $P_{m',m}(t) \in t^{-1}\mathbb{Z}[t^{-1}]$ means $\alpha_{l,L}(t) - \beta_{l,L}(t) \in t^{-1}\mathbb{Z}[t^{-1}]$. So it suffices to prove that those two conditions uniquely define the $\beta_{l,L}(t)$: let us write $\alpha_{l,L}(t) = \alpha^+_{l,L}(t) + \alpha^0_{l,L}(t) + \alpha^-_{l,L}(t)$ (resp. $\beta_{l,L}(t) = \beta^+_{l,L}(t) + \beta^0_{l,L}(t) + \beta^-_{l,L}(t)$) where $\alpha^{\pm}_{l,L}(t) \in t^{\pm}\mathbb{Z}[t^{\pm}]$ and $\alpha^0_{l,L}(t) \in \mathbb{Z}$ (resp. for β). The condition $\alpha_{l,L}(t) - \beta_{l,L}(t) \in t^{-1}\mathbb{Z}[t^{-1}]$ means $\beta^0_{l,L}(t) = \alpha^0_{l,L}(t)$ and $\beta^-_{l,L}(t) = \alpha^-_{l,L}(t)$. The symmetry of $\beta_{l,L}(t)$ means $\beta^+_{l,L}(t) = \beta^-_{l,L}(t^{-1}) = \alpha^-_{l,L}(t^{-1})$. □

6.4.2 Examples

Proposition 6.10. *We suppose that $\mathfrak{g} = sl_2$. Let $m \in t^{\mathbb{Z}}B$ such that $\forall l \in \mathbb{Z}, u_l(m) \leq 1$. Then $L_t(m) = F_t(m)$. Moreover :*

$$E_t(m) = L_t(m) + \sum_{m'<m/m'\in B_m^{inv}} t^{-R(m')}L_t(m')$$

where $R(m') \geq 1$ is given by $\pi_+(m'm^{-1}) = A^{-1}_{i_1,l_1}...A^{-1}_{i_R,l_R}$. In particular for $m' \in B_m^{inv}$ such that $m' < m$ we have $P_{m',m}(t) = t^{-R(m')}$.

Proof : Note that a dominant monomial $m' < m$ verifies $\forall l \in \mathbb{Z}, u_l(m') \leq 1$ and appears in $E_t(m)$. We know that $\tilde{D}_m \cap \mathfrak{K}_t = \bigoplus_{m'\in t^{\mathbb{Z}}D_m\cap B_m^{\mathrm{inv}}} \mathbb{Z}[t^{\pm}]F_t(m')$. We can introduce $P_{m',m}(t) \in \mathbb{Z}[t^{\pm}]$ such that :

$$E_t(m) = F_t(m) + \sum_{m'\in t^{\mathbb{Z}}D_m\cap B^{\mathrm{inv}}-\{m\}} P_{m',m}(t)F_t(m')$$

So by induction it suffices to show that $P_{m',m}(t) \in t^{-1}\mathbb{Z}[t^{-1}]$.

$P_{m',m}(t)$ is the coefficient of m' in $E_t(m)$. A dominant $\mathcal{Y}_t$-monomial M which appears in $E_t(m)$ is of the form :

$$M = m(m_1...m_{R+1})^{-1}m_1t\tilde{A}^{-1}_{l_1}m_2t\tilde{A}^{-1}_{l_2}m_3...t\tilde{A}^{-1}_{l_R}m_{R+1}$$

where $l_1 < ... < l_R \in \mathbb{Z}$ verify $\{l_r+2, l_r-2\} \cap \{l_1, ..., l_{r-1}, l_{r+1}, ..., l_R\}$ is empty, $u_{l_r-1}(m) = u_{l_r+1}(m) = 1$ and we have set $m_r = \overrightarrow{\prod}_{l_{r-1}<l\leq l_r} \tilde{Y}_l^{u_l(m)}$. Such a monomial appears one time

in $E_t(m)$. In particular $P_{m',m}(t) = t^\alpha$ where $\alpha \in \mathbb{Z}$ is given by $M = t^\alpha m'$ that is to say $\overline{M}M^{-1} = t^{-2\alpha}m'^{-1}m' = t^{-2\alpha}m^{-1}m$. So we compute :

$$\overline{M}M^{-1} = t^{-2R}t^{4R}\tilde{A}_{l_R}^{-1}...\tilde{A}_{l_1}^{-1}\overline{m}\tilde{A}_{l_R}...\tilde{A}_{l_1}m^{-1} = t^{2R}\overline{m}m^{-1}$$

□

Let us look at another example : $\mathfrak{g} = sl_2$ and $m = \tilde{Y}_0^2\tilde{Y}_2$. We have :

$$E_t(m) = L_t(m) + t^{-2}L_t(m') \text{ , } P_{m',m}(t) = t^{-2}$$

where $m' = t\tilde{Y}_0^2\tilde{Y}_2\tilde{A}_1^{-1} \in B_m^{\text{inv}}$ and :

$$L_t(m) = F_t(\tilde{Y}_0)F_t(\tilde{Y}_0\tilde{Y}_2) = \tilde{Y}_0(1+t\tilde{A}_1^{-1})\tilde{Y}_0\tilde{Y}_2(1+t\tilde{A}_3^{-1}(1+t\tilde{A}_1^{-1}))$$

$$L_t(m') = F_t(m') = t\tilde{Y}_0^2\tilde{Y}_2\tilde{A}_1^{-1}(1+t\tilde{A}_1^{-1})$$

Let suppose that $C = B_2$ and $m = \tilde{Y}_{2,0}\tilde{Y}_{1,5}$. The formulas for $E_t(\tilde{Y}_{2,0})$ and $E_t(\tilde{Y}_{1,5})$ are given is the appendix. We have :

$$E_t(m) = L_t(m) + t^{-1}L_t(m') \text{ , } P_{m',m}(t) = t^{-1}$$

where $m' = t\tilde{Y}_{2,0}\tilde{Y}_{1,5}\tilde{A}_{2,2}^{-1}\tilde{A}_{1,4}^{-1} \in B_m^{\text{inv}}$ and :

$$L_t(m') = F_t(m') = t\tilde{Y}_{2,0}\tilde{Y}_{1,5}\tilde{A}_{2,2}^{-1}\tilde{A}_{1,4}^{-1}(1+t\tilde{A}_{1,2}^{-1}(1+t\tilde{A}_{2,4}^{-1}(1+t\tilde{A}_{1,6}^{-1})))$$

7 Questions and conjectures

7.1 Positivity of coefficients

Proposition 7.1. *If $\mathfrak{g}$ is of type A_n ($n \geq 1$), the coefficients of $\chi_{q,t}(Y_{i,0})$ are in $\mathbb{N}[t^{\pm}]$.*

Proof : We show that for all $i \in I$ the hypothesis of proposition 5.15 for $m = Y_{i,0}$ are verified ; in particular the property Q of section 5.5 will be verified.

Let i be in I. For $j \in I$, let us write $E(Y_{i,0}) = \sum_{m \in B_j} \lambda_j(m)E_j(m) \in \mathfrak{K}_j$ where $\lambda_j(m) \in \mathbb{Z}$.

Let D be the set $D = \{\text{monomials of } E_j(m) \ / j \in I, m \in B_j, \lambda_j(m) \neq 0\}$. It suffices to prove that for $j \in I$, $m \in B_j \cap D \Rightarrow u_j(m) \leq 1$ (because proposition 5.15 implies that for all $i \in I$, $F_t(\tilde{Y}_{i,0}) = \pi^{-1}(E(Y_{i,0}))$).

As $E(Y_{i,0}) = F(Y_{i,0})$, $Y_{i,0}$ is the unique dominant $\mathcal{Y}$-monomial in $E(Y_{i,0})$. So for a monomial $m \in D$ there is a finite sequence $\{m_0 = Y_{i,0}, m_1, ..., m_R = m\}$ such that for all $1 \leq r \leq R$, there is $r' < r$ and $j \in I$ such that $m_{r'} \in B_j$ and for $r' < r'' \leq r$, $m_{r''}$ is a monomial of $E_j(m_{r'})$ and $m_{r''}m_{r''-1}^{-1} \in \{A_{j,l}^{-1} / l \in \mathbb{Z}\}$. Such a sequence is said to be adapted to m. Suppose there is $j \in I$ and $m \in B_j \cap D$ such that $u_j(m) \geq 2$. So there is $m' \leq m$ in $D \cap B_j$ such that $u_j(m) = 2$. So we can consider $m_0 \in D$ such that there is $j_0 \in I$, $m_0 \in B_{j_0}$,

$u_{j_0}(m) \geq 2$ and for all $m' < m_0$ in D we have $\forall j \in I, m' \in B_j \Rightarrow u_j(m') \leq 1$. Let us write :

$$m_0 = Y_{j_0,q^l} Y_{j_0,q^m} \prod_{j \neq j_0} m_0^{(j)}$$

where for $j \neq j_0$, $m_0^{(j)} = \prod_{l \in \mathbb{Z}} Y_{j,l}^{u_{j,l}(m_0)}$. In a finite sequence adapted to m_0, a term Y_{j_0,q^l} or Y_{j_0,q^m} must come from a $E_{j_0+1}(m_1)$ or a $E_{j_0-1}(m_1)$. So for example we have $m_1 < m_0$ in D of the form $m_1 = Y_{j_0,q^m} Y_{j_0+1,q^{l-1}} \prod_{j \neq j_0, j_0+1} m_1^{(j)}$. In all cases we get a monomial $m_1 < m_0$ in D of the form :

$$m_1 = Y_{j_1,q^{m_1}} Y_{j_1+1,q^{l_1}} \prod_{j \neq j_1, j_1+1} m_1^{(j)}$$

But the term $Y_{j_1+1,q^{l-1}}$ can not come from a $E_{j_1}(m_2)$ because we would have $u_{j_1}(m_2) \geq 2$. So we have $m_2 < m_1$ in D of the form :

$$m_2 = Y_{j_2,q^{m_2}} Y_{j_2+2,q^{l_2}} \prod_{j \neq j_2, j_2+1, j_2+2} m_2^{(j)}$$

This term must come from a E_{j_2-1}, E_{j_2+3}. By induction, we get $m_N < m_0$ in D of the form :

$$m_N = Y_{1,q^{m_N}} Y_{n,q^{l_N}} \prod_{j \neq 1,..,n} m_N^{(j)} = Y_{1,q^{m-N}} Y_{n,q^{l_N}}$$

It is a dominant monomial of $D \subset D_{Y_{i,0}}$ which is not $Y_{i,0}$. It is impossible (proof of lemma 5.12). $\square$

An analog result is also geometrically proved by Nakajima for the ADE-case in [N3] (it is also algebraically for AD-cases proved in [N4]). Those results and the explicit formulas in $n = 1, 2$-cases (see the appendix) suggest :

Conjecture 7.2. *The coefficients of* $F_t(\tilde{Y}_{i,0}) = \chi_{q,t}(Y_{i,0})$ *are in* $\mathbb{N}[t^{\pm}]$.

In particular for $m \in B$, the coefficients of $E_t(m)$ would be in $\mathbb{N}[t^{\pm}]$; moreover $\chi_{q,t}(Y_{i,0})$ and $\chi_q(Y_{i,0})$ would have the same monomials, the t-algorithm would stop and $\mathrm{Im}(\chi_{q,t}) \subset \mathfrak{Y}_t$.

At the time he wrote this paper the author does not know a general proof of the conjecture[4]. However a case by case investigation seems possible : the cases G_2, B_2, C_2 are checked in the appendix and the cases F_4, B_n, C_n $(n \leq 10)$ have been checked on a computer. So a combinatorial proof for series B_n, C_n $(n \geq 2)$ analog to the proof of proposition 7.1 would complete the picture.

7.2 Decomposition in irreducible modules

The proposition 6.10 suggests :

Conjecture 7.3. *For* $m \in B$ *we have* $\pi_+(L_t(m)) = L(m)$.

[4]This conjecture is proved later in [He5].

In the ADE-case the conjecture 7.3 is proved by Nakajima with the help of geometry ([N3]). In particular this conjecture implies that the coefficients of $\pi_+(L_t(m))$ are non negative. It gives a way to compute explicitly the decomposition of a standard module in irreducible modules, because the conjecture 7.3 implies :

$$E(m) = L(m) + \sum_{m'<m} P_{m',m}(1)L(m')$$

In particular we would have $P_{m',m}(1) \geq 0$ (we have checked for numerous examples on computer that the coefficients of the $P_{m',m}(t)$ are positive).

In section 6.4.2 we have studied some examples :

-In proposition 6.10 for $\mathfrak{g} = sl_2$ and $m \in B$ such that $\forall l \in \mathbb{Z}, u_l(m) \leq 1$: we have $\pi_+(L_t(m)) = F(m) = L(m)$ and :

$$E(m) = \sum_{m' \in B / m' \leq m} L(m')$$

-For $\mathfrak{g} = sl_2$ and $m = \tilde{Y}_0^2\tilde{Y}_2$: we have $\pi_+(L_t(m)) = F(Y_0)F(Y_0Y_2) = L(m)$ and :

$$E(Y_0^2Y_2) = L(Y_0^2Y_2) + L(Y_0)$$

Note that $L(Y_0^2Y_2)$ has two dominant monomials $Y_0^2Y_2$ and Y_0 because $Y_0^2Y_2$ is irregular (lemma 4.5).

-For $C = B_2$ and $m = \tilde{Y}_{2,0}\tilde{Y}_{1,5}$. The $\pi_+(L_t(\tilde{Y}_{2,0}\tilde{Y}_{1,5}))$ has non negative coefficients and the conjecture implies $E(Y_{2,0}Y_{1,5}) = L(Y_{2,0}Y_{1,5}) + L(Y_{1,1})$.

7.3 Further applications and generalizations

We hope to address the following questions in the future :

Our presentation of deformed screening operators as commutators leads to the definition of iterated deformed screening operators, for example in order 2 :

$$\tilde{S}_{j,i,t}(m) = [\sum_{l \in \mathbb{Z}} \tilde{S}_{j,l}, S_{i,t}(m)]$$

Some generalizations of the approach used in this article would be possible :

a) the theory of q-characters at roots of unity ([FM2]) suggests a generalization to the case $q^N = 1$.

b) in this article we decided to work with $\mathcal{Y}_t$ which is a quotient of $\mathcal{Y}_u$. The same construction with $\mathcal{Y}_u$ will give characters with an infinity of parameters of deformation $t_r = \exp(\sum_{l>0} h^{2l} q^{lr} c_l)$ $(r \in \mathbb{Z})$.

c) our construction is independent of representation theory and could be established for other generalized Cartan matrices (in particular for affine cases).

8 Appendix

There are 5 types of semi-simple Lie algebra of rank 2 : $A_1 \times A_1$, A_2, C_2, B_2, G_2 (see for example [Kac]). In each case we give the formula for $E(1), E(2) \in \mathfrak{K}$ and we see that the hypothesis of proposition 5.15 is verified. In particular we have $E_t(\tilde{Y}_{1,0}) = \pi^{-1}(E(1)), E_t(\tilde{Y}_{2,0}) = \pi^{-1}(E(2)) \in \mathfrak{K}_t$.

Following [FR3], we represent the $E(1), E(2) \in \mathfrak{K}$ as a $I \times \mathbb{Z}$-oriented colored tree. For $\chi \in \mathfrak{K}$ the tree Γ_χ is defined as follows : the set of vertices is the set of $\mathcal{Y}$-monomials of χ. We draw an arrow of color (i,l) from m_1 to m_2 if $m_2 = A_{i,l}^{-1} m_1$ and if in the decomposition $\chi = \sum_{m \in B_i} \mu_m L_i(m)$ there is $M \in B_i$ such that $\mu_M \neq 0$ and m_1, m_2 appear in $L_i(M)$.

$A_1 \times A_1$-case : $C = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $r_1 = r_2 = 1$ (note that in this case the computations keep unchanged for all r_1, r_2).

$$\begin{array}{ccc} Y_{1,0} & \text{and} & Y_{2,0} \\ \downarrow{\scriptstyle 1,1} & & \downarrow{\scriptstyle 2,1} \\ Y_{1,2}^{-1} & & Y_{2,2}^{-1} \end{array}$$

A_2-case : $C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. It is symmetric, $r_1 = r_2 = 1$:

$$\begin{array}{ccc} Y_{1,0} & \text{and} & Y_{2,0} \\ \downarrow{\scriptstyle 1,1} & & \downarrow{\scriptstyle 2,1} \\ Y_{1,2}^{-1} Y_{2,1} & & Y_{2,2}^{-1} Y_{1,1} \\ \downarrow{\scriptstyle 2,2} & & \downarrow{\scriptstyle 1,2} \\ Y_{2,3}^{-1} & & Y_{1,3}^{-1} \end{array}$$

C_2, B_2-case (dual cases) : we use $C = B_2 = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$ and $r_1 = 1$, $r_2 = 2$.

$$\begin{tikzcd}
Y_{1,0} \arrow[d, "1{,}1"] & \text{and} & Y_{2,0} \arrow[d, "2{,}2"] \\
Y_{1,2}^{-1}Y_{2,1} \arrow[d, "2{,}3"] & & Y_{2,4}^{-1}Y_{1,1}Y_{1,3} \arrow[d, "1{,}4"] \\
Y_{2,5}^{-1}Y_{1,4} \arrow[d, "1{,}5"] & & Y_{1,1}Y_{1,5}^{-1} \arrow[d, "1{,}2"] \\
Y_{1,6}^{-1} & & Y_{1,3}^{-1}Y_{1,5}^{-1}Y_{2,2} \arrow[d, "2{,}4"] \\
& & Y_{2,6}^{-1}
\end{tikzcd}$$

G_2-case : $C = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$ and $r_1 = 1$, $r_2 = 3$.
First fundamental representation :

$$\begin{tikzcd}
Y_{1,0} \arrow[d, "1{,}1"] \\
Y_{1,2}^{-1}Y_{2,1} \arrow[d, "2{,}4"] \\
Y_{2,7}^{-1}Y_{1,4}Y_{1,6} \arrow[d, "1{,}7"] \\
Y_{1,4}Y_{1,8}^{-1} \arrow[d, "1{,}5"] \\
Y_{1,6}^{-1}Y_{1,8}^{-1}Y_{2,5} \arrow[d, "2{,}8"] \\
Y_{2,11}^{-1}Y_{1,10} \arrow[d, "1{,}11"] \\
Y_{1,12}^{-1}
\end{tikzcd}$$

Second fundamental representation :

$$
\begin{tikzcd}
 & Y_{2,0} \arrow[d, "{2,3}"] & \\
 & Y_{2,6}^{-1}Y_{1,5}Y_{1,3}Y_{1,1} \arrow[d, "{1,6}"] & \\
 & Y_{1,7}^{-1}Y_{1,3}Y_{1,1} \arrow[d, "{1,4}"] & \\
 & Y_{2,4}Y_{1,7}^{-1}Y_{1,5}^{-1}Y_{1,1} \arrow[dl, "{1,2}"] \arrow[dr, "{2,7}"] & \\
Y_{1,7}^{-1}Y_{1,5}^{-1}Y_{1,3}^{-1}Y_{2,4}Y_{2,2} \arrow[d, "{2,5}"] \arrow[dr, "{2,7}"] & & Y_{2,10}^{-1}Y_{1,9}Y_{1,1} \arrow[dl, "{1,2}"] \arrow[d, "{1,10}"] \\
Y_{2,4}Y_{2,8}^{-1} \arrow[d, "{2,7}"] & Y_{2,2}Y_{2,10}^{-1}Y_{1,9}Y_{1,3}^{-1} \arrow[dl, "{2,5}"] \arrow[dr, "{1,10}"] & Y_{1,11}^{-1}Y_{1,1} \arrow[d, "{1,2}"] \\
Y_{2,8}^{-1}Y_{2,10}^{-1}Y_{1,9}Y_{1,7}Y_{1,5} \arrow[dr, "{1,10}"] & & Y_{2,2}Y_{1,11}^{-1}Y_{1,3}^{-1} \arrow[dl, "{2,5}"] \\
 & Y_{2,8}^{-1}Y_{1,11}^{-1}Y_{1,7}Y_{1,5} \arrow[d, "{1,8}"] & \\
 & Y_{1,11}^{-1}Y_{1,9}^{-1}Y_{1,5} \arrow[d, "{1,6}"] & \\
 & Y_{1,11}^{-1}Y_{1,9}^{-1}Y_{1,7}^{-1}Y_{2,6} \arrow[d, "{2,9}"] & \\
 & Y_{2,12}^{-1} &
\end{tikzcd}
$$

Troisième partie

The t-analogs of q-characters at roots of unity for quantum affine algebras and beyond

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Résumé. Les q-caractères ont été construits par Frenkel et Reshetikhin [FR3] pour étudier les représentations de dimension finie des algèbres affines quantiques non tordues $\mathcal{U}_q(\hat{\mathfrak{g}})$ pour q générique. Les ϵ-caractères aux racines de l'unité ont été construits par Frenkel et Mukhin [FM2] pour étudier les représentations de dimension finie de diverses spécialisations de $\mathcal{U}_q(\hat{\mathfrak{g}})$ à $q^s = 1$. Pour les cas simplement lacés finis, Nakajima [N2, N3] définit des déformations des q-caractères appelées q,t-caractères, pour q générique ou racine de l'unité. La définition est combinatoire mais la preuve de l'existence utilise la théorie des variétés de carquois qui n'exite que dans le cas simplement lacé. Dans [He2] nous avons proposé une nouvelle approche algébrique générale (non nécessairement simplement lacée) pour les q,t-caractères avec q-générique. Dans cet article nous traitons le cas q racine de l'unité. De plus nous construisons les q-caractères et q,t-caractères pour une grande classe de matrices de Cartan généralisées (incluant les cas finis et affines exceptés $A_1^{(1)}$, $A_2^{(2)}$) en étendant l'approche de [He2]. Nous généralisons en particulier la construction des analogues des polynômes de Kazhdan-Lusztig aux racines de l'unité de [N3] à ces situations. Nous étudions également les propriétés de nombreux objets utilisés dans cet article : opérateurs d'écrantage déformés aux racines de l'unité, algèbres polynomiales t-déformées, bicaractères associés aux matrices de Cartan symmétrisables et déformation de l'algorithme de Frenkel-Mukhin.

Abstract. The q-characters were introduced by Frenkel and Reshetikhin [FR3] to study finite dimensional representations of the untwisted quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$ for q generic. The ϵ-characters at roots of unity were constructed by Frenkel and Mukhin [FM2] to study finite dimensional representations of various specializations of $\mathcal{U}_q(\hat{\mathfrak{g}})$ at $q^s = 1$. In the finite simply laced case Nakajima [N2, N3] defined deformations of q-characters called q,t-characters for q generic and also at roots of unity. The definition is combinatorial but the proof of the existence uses the geometric theory of quiver varieties which holds only in the simply laced case. In [He2] we proposed an algebraic general (non necessarily simply laced) new approach to q,t-characters for q generic. In this paper we treat the root of unity case. Moreover we construct q-characters and q,t-characters for a large class of generalized Cartan matrices (including finite and affine cases except $A_1^{(1)}$,

$A_2^{(2)}$) by extending the approach of [He2]. In particular we generalize the construction of analogs of Kazhdan-Lusztig polynomials at roots of unity of [N3] to those cases. We also study properties of various objects used in this article : deformed screening operators at roots of unity, t-deformed polynomial algebras, bicharacters arising from symmetrizable Cartan matrices, deformation of the Frenkel-Mukhin's algorithm.

1 Introduction

Drinfel'd [Dr1] and Jimbo [Jim] associated, independently, to any symmetrizable Kac-Moody algebra $\mathfrak{g}$ and any complex number $q \in \mathbb{C}^*$ a Hopf algebra $\mathcal{U}_q(\mathfrak{g})$ called quantum group or quantum Kac-Moody algebra.

First we suppose that $q \in \mathbb{C}^*$ is not a root of unity. In the case of a semi-simple Lie algebra $\mathfrak{g}$ of rank n, the structure of the Grothendieck ring $\mathrm{Rep}(\mathcal{U}_q(\mathfrak{g}))$ of finite dimensional representations of the quantum algebra of finite type $\mathcal{U}_q(\mathfrak{g})$ is well understood. It is analogous to the classical case $q = 1$. In particular we have ring isomorphisms :

$$\mathrm{Rep}(\mathcal{U}_q(\mathfrak{g})) \simeq \mathrm{Rep}(\mathfrak{g}) \simeq \mathbb{Z}[\Lambda]^W \simeq \mathbb{Z}[T_1, ..., T_n]$$

deduced from the injective homomorphism of characters χ :

$$\chi(V) = \sum_{\lambda \in \Lambda} \dim(V_\lambda)\lambda$$

where V_λ are weight spaces of a representation V and Λ is the weight lattice.

For the general case of Kac-Moody algebras the picture is less clear. The representation theory of the quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$ is of particular interest (see [CP3], [CP4]). In this case there is a crucial property of $\mathcal{U}_q(\hat{\mathfrak{g}})$: it has two realizations, the usual Drinfel'd-Jimbo realization and a new realization (see [Dr2] and [Be]) as a quantum affinization of the quantum algebra of finite type $\mathcal{U}_q(\mathfrak{g})$.

To study the finite dimensional representations of $\mathcal{U}_q(\hat{\mathfrak{g}})$ Frenkel and Reshetikhin [FR3] introduced q-characters which encode the (pseudo)-eigenvalues of some commuting elements in the Cartan subalgebra $\mathcal{U}_q(\hat{\mathfrak{h}}) \subset \mathcal{U}_q(\hat{\mathfrak{g}})$ (see also [Kn]). The morphism of q-characters is an injective ring homomorphism :

$$\chi_q : \mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}})) \to \mathbb{Z}[Y_{i,a}^{\pm}]_{i \in I, a \in \mathbb{C}^*}$$

where $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ is the Grothendieck ring of finite dimensional (type 1)-representations of $\mathcal{U}_q(\hat{\mathfrak{g}})$ and $I = \{1, ..., n\}$. In particular $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ is commutative and isomorphic to $\mathbb{Z}[X_{i,a}]_{i \in I, a \in \mathbb{C}^*}$.

The morphism of q-characters has a symmetry property analogous to the classical action of the Weyl group $\mathrm{Im}(\chi) = \mathbb{Z}[\Lambda]^W$: Frenkel and Reshetikhin [FR3] defined n screening operators S_i and showed that $\mathrm{Im}(\chi_q) = \bigcap_{i\in I}\mathrm{Ker}(S_i)$ for $\mathfrak{g} = sl_2$. The result was proved by Frenkel and Mukhin for all finite $\mathfrak{g}$ in [FM1].

In the simply laced case Nakajima [N2, N3] introduced t-analogs of q-characters. The motivations are the study of filtrations induced on representations by (pseudo)-Jordan decompositions, the study of the decomposition in irreducible modules of tensorial products and the study of cohomologies of certain quiver varieties. The morphism of q,t-characters is a $\mathbb{Z}[t^{\pm}]$-linear map

$$\chi_{q,t} : \mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}})) \to \mathbb{Z}[Y_{i,a}^{\pm}, t^{\pm}]_{i\in I, a\in\mathbb{C}^*}$$

which is a deformation of χ_q and multiplicative in a certain sense. A combinatorial axiomatic definition of q,t-characters is given. But the existence is non-trivial and is proved with the geometric theory of quiver varieties which holds only in the simply laced case.

In [He2] we defined and constructed q,t-characters in the general (non necessarily simply laced) case with a new approach motivated by the non-commutative structure of $\mathcal{U}_q(\hat{\mathfrak{h}}) \subset \mathcal{U}_q(\hat{\mathfrak{g}})$, the study of screening currents of [FR2] and of deformed screening operators $S_{i,t}$ of [He1]. In particular we have a symmetry property : the image of $\chi_{q,t}$ is a completion of $\bigcap_{i\in I}\mathrm{Ker}(S_{i,t})$.

The representation theory of the quantum affine algebras $\mathcal{U}_q(\hat{\mathfrak{g}})$ depends crucially whether q is a root of unity or not (see [CP5]). Frenkel and Mukhin [FM1] generalized q-characters at roots of unity : if ϵ is a s^{th}-primitive root of unity the morphism of ϵ-characters is :

$$\chi_\epsilon : \mathrm{Rep}(\mathcal{U}_\epsilon^{\mathrm{res}}(\hat{\mathfrak{g}})) \to \mathbb{Z}[Y_{i,a}^{\pm}]_{i\in I, a\in\mathbb{C}^*}$$

where $\mathrm{Rep}(\mathcal{U}_\epsilon^{\mathrm{res}}(\hat{\mathfrak{g}}))$ is the Grothendieck ring of finite dimensional (type 1)-representations of the restricted specialization $\mathcal{U}_\epsilon^{\mathrm{res}}(\hat{\mathfrak{g}})$ of $\mathcal{U}_q(\hat{\mathfrak{g}})$ at $q = \epsilon$. In particular $\mathrm{Rep}(\mathcal{U}_\epsilon^{\mathrm{res}}(\hat{\mathfrak{g}}))$ is commutative and isomorphic to $\mathbb{Z}[X_{i,a}]_{i\in I, a\in\mathbb{C}^*}$.

Moreover χ_ϵ can be characterized by

$$\chi_\epsilon(\prod_{i\in I, l\in\mathbb{Z}/s\mathbb{Z}} X_{i,\epsilon^l}^{x_{i,l}}) = \tau_s(\chi_q(\prod_{i\in I, 0\leq l\leq s-1} X_{i,q^l}^{x_{i,[l]}}))$$

where $\tau_s : \mathbb{Z}[Y_{i,q^l}^{\pm}]_{i\in I, l\in\mathbb{Z}} \to \mathbb{Z}[Y_{i,\epsilon^l}^{\pm}]_{i\in I, l\in\mathbb{Z}/s\mathbb{Z}}$ is the ring homomorphism such that $\tau_s(Y_{i,q^l}^{\pm}) = Y_{i,\epsilon^{[l]}}^{\pm}$ (for $l \in \mathbb{Z}$ we denote by $[l]$ its image in $\mathbb{Z}/s\mathbb{Z}$).

In the simply laced case Nakajima generalized the theory of q,t-characters at roots of unity with the help of quiver varieties [N3].

In this paper we construct q,t-characters at roots of unity in the general (non necessarily simply laced) case by extending the approach of [He2]. As an application we construct analogs of Kazhdan-Lusztig polynomials at roots of unity in the same spirit as Nakajima did for the simply laced case. We also study properties of various objects used in this paper : deformed screening operators at roots of unity, t-deformed polynomial algebras, bicharacters arising from general symmetrizable Cartan matrices, deformation of the Frenkel-Mukhin's algorithm.

The construction is also extended beyond the case of a quantum affine algebra, that is to say by replacing the finite Cartan matrix by a generalized symmetrizable Cartan matrix : the construction of q-characters as well as q,t-characters (generic and roots of unity cases) is explained in this paper for (non necessarily finite) Cartan matrices such that $i \neq j \Rightarrow C_{i,j}C_{j,i} \leq 3$ (it includes finite and affine types except $A_1^{(1)}$, $A_2^{(2)}$). The notion of a quantum affinization is more general than the construction of a quantum affine algebra from a quantum algebra of finite type : it can be extended to any general symmetrizable Cartan matrix (see [N1]). For example for an affine Cartan matrix one gets a quantum toroidal algebra (see [VV1]). In general a quantum affinization is not a quantum Kac-Moody algebra and few is known about the representation theory outside the quantum affine algebra case. However for an integrable representation one can define q-characters as Frenkel-Reshetikhin did for quantum affine algebras. So the role of the q-characters in the representation theory of the general quantum affinizations has to be studied in details. We will address further developments on this point in a separate publication.

This paper is organized as follows : after some backgrounds in section 2, we generalize in section 3 the construction of t-deformed polynomial algebras of [He2] to the root of unity case. We give a "concrete" construction using Heisenberg algebras. We show that this twisted multiplication can also be "abstractly" defined with two bicharacters d_1, d_2 as Nakajima did for the simply laced case (for which there is only one bicharacter $d_1 = d_2$).

In section 4 we remind how q,t-characters are constructed for q generic and C finite in [He2]. We extend the construction of q-characters and of q,t-characters to symmetrizable (non necessarily finite) Cartan matrices such that $i \neq j \Rightarrow C_{i,j}C_{j,i} \leq 3$, in particular for affine Cartan matrices (except $A_1^{(1)}$ and $A_2^{(2)}$). The q,t-characters can be computed by the algorithm described in [He2] which is a deformation of the algorithm of Frenkel-Mukhin [FM1].

In section 5 we construct q,t-characters at roots of unity. Let us explain the crucial technical point of this section : we can not use directly a t-deformation of the definition of Frenkel-Mukhin because there is no analog of τ_s which is an algebra homomorphism for the t-deformed structures. But we can construct $\tau_{s,t}$ which is multiplicative for some ordered products (see section 5.2.1). In particular $\tau_{s,t}$ has nice properties and we can define $\chi_{\epsilon,t}$ such that "$\chi_{\epsilon,t} = \tau_{s,t} \circ \chi_{q,t}$". We give properties of $\chi_{\epsilon,t}$ analogous to the property of χ_ϵ (theorem 4.8, 5.6, 5.11). In particular in the ADE-case we get a formula which is Axiom 4 of [N3], and so the construction coincides with the construction of [N3] for the ADE-case.

In section 6 we give some applications about Kazhdan-Lusztig polynomials and quantization of the Grothendieck ring. If C is finite the technical point in the root of unity case is to show that the algorithm produces a finite number of dominant monomials. We give a conjecture about the multiplicity of an irreducible module in a standard module at roots of unity. For the ADE-case it is a result of Nakajima [N3]. We also discuss the non finite cases.

In section 7 we give some complements : first we discuss the finiteness of the algorithm ; at $t = 1$ it stops if C is finite and it does not stop if C is affine. We relate the structure of the deformed ring in the affine $A_r^{(1)}$-case to the structure of quantum toroidal algebras. We study some combinatorial properties of the Cartan matrices which are related to the bicharacters d_1 and d_2 (propositions 7.9, 7.11, 7.12 and theorem 7.10).

In the course of writing this paper we were informed by H. Nakajima that the t-analogs of q-characters for some quantum toroidal algebras are also mentioned in the remark 6.9 of [N5]. This incited us to add the construction of analogs of Kazdhan-Lusztig polynomials at roots of unity also in the non finite cases (in section 6.1).

2 Background

2.1 Cartan matrices

A generalized Cartan matrix is $C = (C_{i,j})_{1\leq i,j\leq n}$ such that $C_{i,j} \in \mathbb{Z}$ and :

$$C_{i,i} = 2 \text{ , } i \neq j \Rightarrow C_{i,j} \leq 0 \text{ , } C_{i,j} = 0 \Leftrightarrow C_{j,i} = 0$$

Let $I = \{1, ..., n\}$.

C is said to be decomposable if it can be written in the form $C = P \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} P^{-1}$ where P is a permutation matrix, A and B are square matrices. Otherwise C is said to be indecomposable.

C is said to be symmetrizable if there is a matrix $D = \text{diag}(r_1, ..., r_n)$ ($r_i \in \mathbb{N}^*$) such that $B = DC$ is symmetric. In particular if C is symmetric then it is symmetrizable with $D = I_n$.

If C is indecomposable and symmetrizable then there is a unique choice of $r_1, ..., r_n > 0$ such that $r_1 \wedge ... \wedge r_n = 1$: indeed if $C_{j,i} \neq 0$ we have the relation $r_i = \frac{C_{j,i}}{C_{i,j}} r_j$.

In the following C is a symmetrizable and indecomposable generalized Cartan matrix. For example :

C is said to be of finite type if all its principal minors are positive (see [Bo] for a classification).

C is said to be of affine type if all its proper principal minor are positive and $\det(C) = 0$ (see [Kac] for a classification).

Let $r^\vee = \max\{r_i - 1 - C_{j,i})/i \neq j\} \cup \{1\}$. If C is finite we have $r^\vee = \max\{r_i/i \in I\} = \max\{-C_{i,j}/i \neq j\}$. In particular if C is of type ADE we have $r^\vee = 1$, if C is of type $B_lC_lF_4$ ($l \geq 2$) we have $r^\vee = 2$, if C of type G_2 we have $r^\vee = 3$.

Let z be an indeterminate and $z_i = z^{r_i}$. The matrix $C(z) = (C_{i,j}(z))_{1\leq i,j\leq n}$ with coefficients in $\mathbb{Z}[z^\pm]$ is defined by $C_{i,i}(z) = [2]_{z_i} = z_i + z_i^{-1}$ and $C_{i,j} = [C_{i,j}]_z$ for $i \neq j$ where for $l \in \mathbb{Z}$, we denote $[l]_z = \frac{z^l - z^{-l}}{z - z^{-1}} \in \mathbb{Z}[z^\pm]$.

Let $B(z) = D(z)C(z)$ where $D(z)$ is the diagonal matrix $D_{i,j}(z) = \delta_{i,j}[r_i]_z$, that is to say $B_{i,j}(z) = [r_i]_z C_{i,j}(z)$.

In particular, the coefficients of $C(z)$ and $B(z)$ are symmetric Laurent polynomials (invariant under $z \mapsto z^{-1}$).

In the following we suppose that $\det(C(z)) \neq 0$ and that $B(z)$ is symmetric. It includes finite and affine Cartan matrices (if C is of type $A_1^{(1)}$ we set $r_1 = r_2 = 2$) and also the matrices such that $i \neq j \Rightarrow C_{i,j}C_{j,i} \leq 3$ which will appear later (see lemma 6.4 and section 7.3 for complements).

2.2 Quantum affine algebras and q-characters

In the following q is a complex number $q \in \mathbb{C}^*$. If q is not a root of unity we set $s = 0$ and we say that q is generic. Otherwise $s \geq 1$ is set such that q is a s^{th} primitive root of unity.

We suppose in this section that C is finite. We refer to [FM2] for the definition of the untwisted quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$ associated to C (for q generic) and of the restricted specialization $\mathcal{U}_\epsilon^{\mathrm{res}}(\hat{\mathfrak{g}})$ of $\mathcal{U}_q(\hat{\mathfrak{g}})$ at $q = \epsilon$ (for ϵ root of unity).

We briefly describe the construction of $\mathcal{U}_\epsilon^{\mathrm{res}}(\hat{\mathfrak{g}})$ from $\mathcal{U}_q(\hat{\mathfrak{g}})$: we consider a $\mathbb{Z}[q, q^{-1}]$-subalgebra of $\mathcal{U}_q(\hat{\mathfrak{g}})$ containing the $(x_i^\pm)^{(r)} = \frac{(x_i^\pm)^s}{[r]_{q_i}!}$ (where $[r]_q! = [r]_q[r-1]_q...[1]_q$) for some generators $x_i^\pm$, and we set $q = \epsilon$.

One can define a Hopf algebra structure on $\mathcal{U}_q(\hat{\mathfrak{g}})$ and $\mathcal{U}_\epsilon^{\mathrm{res}}(\hat{\mathfrak{g}})$, and so we consider the Grothendieck ring of finite dimensional (type 1)-representations : $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ and $\mathrm{Rep}(\mathcal{U}_\epsilon^{\mathrm{res}}(\hat{\mathfrak{g}}))$.

The morphism of q-characters χ_q (Frenkel-Reshetikhin [FR3]) and the morphism of ϵ-character χ_ϵ (Frenkel-Mukhin [FM2]) are injective ring homomorphisms :

$$\chi_q : \mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}})) \rightarrow \mathbb{Z}[Y_{i,a}^\pm]_{i\in I, a\in\mathbb{C}^*} \text{ , } \chi_\epsilon : \mathrm{Rep}(\mathcal{U}_\epsilon^{\mathrm{res}}(\hat{\mathfrak{g}})) \rightarrow \mathbb{Z}[Y_{i,a}^\pm]_{i\in I, a\in\mathbb{C}^*}$$

In particular $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ and $\mathrm{Rep}(\mathcal{U}_\epsilon^{\mathrm{res}}(\hat{\mathfrak{g}}))$ are commutative and isomorphic to $\mathbb{Z}[X_{i,a}]_{i\in I, a\in\mathbb{C}^*}$.

Frenkel and Mukhin [FM1][FM2] have proven that for $i \in I$, $a \in \mathbb{C}^*$:

$$\chi_q(X_{i,a}) \in \mathbb{Z}[Y_{i,aq^m}^\pm]_{i\in I, m\in\mathbb{Z}} \text{ and } \chi_\epsilon(X_{i,a}) \in \mathbb{Z}[Y_{i,a\epsilon^m}^\pm]_{i\in I, m\in\mathbb{Z}}$$

Indeed it suffices to study (see [He2] for details) :

$$\chi_q : \mathrm{Rep} = \mathbb{Z}[X_{i,l}]_{i\in I, l\in\mathbb{Z}} \rightarrow \mathcal{Y} = \mathbb{Z}[Y_{i,l}^\pm]_{i\in I, l\in\mathbb{Z}}$$

(where $X_{i,l} = X_{i,q^l}$, $Y_{i,l}^\pm = Y_{i,q^l}^\pm$), and :

$$\chi_\epsilon^s : \mathrm{Rep}^s = \mathbb{Z}[X_{i,l}]_{i\in I, l\in\mathbb{Z}/s\mathbb{Z}} \rightarrow \mathcal{Y}^s = \mathbb{Z}[Y_{i,l}^\pm]_{i\in I, l\in\mathbb{Z}/s\mathbb{Z}}$$

(where $X_{i,l} = X_{i,\epsilon^l}$, $Y_{i,l}^\pm = Y_{i,\epsilon^l}^\pm$).

3 t-deformed polynomial algebras and screening operators

In this section we generalize at roots of unity and beyond the finite case the construction of [He2] of t-deformed polynomial algebras and screening operators. In particular the presentation with bicharacters d_1, d_2 is introduced for the non simply laced case. Technical results which will be used in the construction are also proved in this section.

3.1 The t-deformed algebra $\hat{\mathcal{Y}}_t^s$

In this section we suppose that $B(z)$ is symmetric.

Definition 3.1. *$\hat{\mathcal{H}}$ is the $\mathbb{C}$-algebra defined by generators $a_i[m], y_i[m]$ ($i \in I, m \in \mathbb{Z}-\{0\}$), central elements c_r ($r > 0$) and relations ($i, j \in I, m, r \in \mathbb{Z} - \{0\}$) :*

$$[a_i[m], a_j[r]] = \delta_{m,-r}(q^m - q^{-m})B_{i,j}(q^m)c_{|m|} \tag{6}$$

$$[a_i[m], y_j[r]] = (q^{mr_i} - q^{-r_i m})\delta_{m,-r}\delta_{i,j}c_{|m|} \tag{7}$$

$$[y_i[m], y_j[r]] = 0 \tag{8}$$

Let $\hat{\mathcal{H}}_h = \hat{\mathcal{H}}[[h]]$. For $i \in I$, $l \in \mathbb{Z}/s\mathbb{Z}$ we define $Y_{i,l}^{\pm}, A_{i,l}^{\pm}, t_l^{\pm} \in \hat{\mathcal{H}}_h$ such that :

$$Y_{i,l} = \exp(\sum_{m>0} h^m q^{lm} y_i[m])\exp(\sum_{m>0} h^m q^{-lm} y_i[-m])$$

$$A_{i,l} = \exp(\sum_{m>0} h^m q^{lm} a_i[m])\exp(\sum_{m>0} h^m q^{-lm} a_i[-m])$$

$$t_l = \exp(\sum_{m>0} h^{2m} q^{lm} c_m)$$

For $R = \sum_{l\in\mathbb{Z}} R_l z^l \in \mathbb{Z}[z^{\pm}]$ we denote $t_R = \prod_{l\in\mathbb{Z}} t_l^{R_l} = \exp(\sum_{m>0} h^{2m} R(q^m)c_m) \in \hat{\mathcal{H}}_h$.
Note that the root of unity condition, that is to say $s \geq 1$, is a periodic condition ($Y_{i,l+s} = Y_{i,l}$).

Lemma 3.2. *([He2]) We have the following relations in $\hat{\mathcal{H}}_h$:*

$$A_{i,l}Y_{j,k}A_{i,l}^{-1}Y_{j,k}^{-1} = t_{\delta_{i,j}(z^{-r_i} - z^{r_i})(-z^{(l-k)} + z^{(k-l)})}$$

$$A_{i,l}A_{j,k}A_{i,l}^{-1}A_{j,k}^{-1} = t_{B_{i,j}(z)(z^{-1} - z)(-z^{(l-k)} + z^{(k-l)})}$$

Definition 3.3. *$\hat{\mathcal{Y}}_u^s$ is the $\mathbb{Z}$-subalgebra of $\hat{\mathcal{H}}_h$ generated by the $Y_{i,l}, A_{i,l}^{-1}, t_l$ ($i \in I, l \in \mathbb{Z}/z\mathbb{Z}$).*
$\hat{\mathcal{Y}}_t^s$ is the quotient-algebra of $\hat{\mathcal{Y}}_u^s$ by relations $t_l = 1$ if $l \in \mathbb{Z}/s\mathbb{Z} - \{0\}$.

We keep the notations $Y_{i,l}, A_{i,l}^{-1}$ for their image in $\hat{\mathcal{Y}}_t^s$. We denote by t the image of $t_0 = \exp(\sum_{m>0} h^{2m} c_m)$ in $\hat{\mathcal{Y}}_t^s$. In particular the image of t_R is t^{R_0}. We denote by $\hat{\mathcal{Y}}_t = \hat{\mathcal{Y}}_t^0$ the algebra in the generic case (in the same way for a family $(M^s)_{s\geq 0}$ we will denote $M = M^0$).
Note that if $s \geq 1$, the elements $A_{i,0}^{-1}A_{i,1}^{-1}...A_{i,s-1}^{-1}$ and $Y_{i,0}Y_{i,1}...Y_{i,s-1}$ are central in $\hat{\mathcal{Y}}_t^s$.

The following theorem gives the structure of $\hat{\mathcal{Y}}_t^s$:

Theorem 3.4. *The algebra $\hat{\mathcal{Y}}_t^s$ is defined by generators $Y_{i,l}, A_{i,l}^{-1}, t^{\pm}$ ($i \in I, l \in \mathbb{Z}/s\mathbb{Z}$) and relations ($i, j \in I, k, l \in \mathbb{Z}/s\mathbb{Z}$) :*

$$Y_{i,l}Y_{j,k} = Y_{j,k}Y_{i,l} \ , \ A_{i,l}^{-1}A_{j,k}^{-1} = t^{\alpha(i,l,j,k)}A_{j,k}^{-1}A_{i,l}^{-1} \ , \ Y_{j,k}A_{i,l}^{-1} = t^{\beta(i,l,j,k)}A_{i,l}^{-1}Y_{j,k}$$

where $\alpha, \beta : (I \times \mathbb{Z}/s\mathbb{Z})^2 \to \mathbb{Z}$ are given by ($l, k \in \mathbb{Z}/s\mathbb{Z}$, $i, j \in I$) :

$$\alpha(i,l,i,k) = 2(\delta_{l-k,-2r_i} - \delta_{l-k,2r_i})$$

$$\alpha(i,l,j,k) = 2\sum_{r=C_{i,j}+1, C_{i,j}+3, ..., -C_{i,j}-1}(\delta_{l-k,r+r_i} - \delta_{l-k,r-r_i}) \ \textit{(if } i \neq j\textit{)}$$

$$\beta(i,l,j,k) = 2\delta_{i,j}(-\delta_{l-k,r_i} + \delta_{l-k,-r_i})$$

(the r_i, r are seen in $\mathbb{Z}/l\mathbb{Z}$.)

This theorem is a generalization of theorem 3.11 of [He2]. It is proved in the same way except for lemma 3.7 of [He2] whose proof is changed at roots of unity : for $N \geq 1$ we denote by $\mathbb{Z}_N[z] \subset \mathbb{Z}[z]$ the subset of polynomials of degree lower that N. The following lemma is a generalization of lemma 3.7 of [He2] at roots of unity :

Lemma 3.5. *We suppose that $s \geq 1$. Let $J = \{1, ..., r\}$ be a finite set of cardinal r and Λ be the polynomial commutative algebra $\Lambda = \mathbb{C}[\lambda_{j,m}]_{j \in J, m \geq 0}$. For $R = (R_1, ..., R_r) \in \mathbb{Z}_{s-1}[z]^r$, consider :*

$$\Lambda_R = exp(\sum_{j \in I, m > 0} h^m R_j(q^m)\lambda_{j,m}) \in \Lambda[[h]]$$

Then the $(\Lambda_R)_{R \in \mathbb{Z}_{s-1}[z]^r}$ are $\mathbb{C}$-linearly independent. In particular the $\Lambda_{j,l} = \Lambda_{(0,...,0,z^l,0,...,0)}$ ($j \in I$, $0 \leq l \leq s-1$) are $\mathbb{C}$-algebraically independent.

Proof : Suppose we have a linear combination ($\mu_R \in \mathbb{C}$, only a finite number of $\mu_R \neq 0$) :

$$\sum_{R \in \mathbb{Z}_{s-1}[z]^r} \mu_R \Lambda_R = 0$$

In the proof of lemma 3.7 of [He2] we saw that for $N \geq 0$, $j_1, ..., j_N \in J$, $l_1, ..., l_N > 0$, $\alpha_1, ..., \alpha_N \in \mathbb{C}$ we have :

$$\sum_{R \in \mathbb{Z}_{s-1}[z]^r / R_{j_1}(q^{l_1}) = \alpha_1, ..., R_{j_N}(q^{l_R}) = \alpha_N} \mu_R = 0$$

We set $N = sr$ and

$$((j_1, l_1), ..., (j_N, l_N)) = ((1,1), (1,2), ..., (1,s), (2,1), ..., (2,s), (3,1), ..., (r,s))$$

We get for all $\alpha_{j,l} \in \mathbb{C}$ ($j \in J, 1 \leq l \leq L$) :

$$\sum_{R \in \mathbb{Z}_{s-1}[z]^r / \forall j \in J, 1 \leq l \leq s, R_j(q^l) = \alpha_{j,l}} \mu_R = 0$$

It suffices to show that there is at most one term is this sum. But consider $P, Q \in \mathbb{Z}_{s-1}[z]$ such that for all $1 \leq l \leq s$, $P(q^l) = P'(q^l)$. As q is primitive the q^l are different and so $P - P' = 0$. $\square$

3.2 Bicharacters, monomials and involution

3.2.1 Presentation with bicharacters

The definition of the algebra $\hat{\mathcal{Y}}_t^s$ with the Heisenberg algebra $\hat{\mathcal{H}}$ is a "concrete" construction. It can also be defined "abstractly" with bicharacters in the same spirit as Nakajima [N3] did for the simply laced case :

For m a $\hat{\mathcal{Y}}_t^s$-monomial, $i \in I$, $l \in \mathbb{Z}/s\mathbb{Z}$ we define $y_{i,l}(m), v_{i,l}(m) \geq 0$ such that $m \in t^{\mathbb{Z}} \prod_{i \in I, l \in \mathbb{Z}/s\mathbb{Z}} Y_{i,l}^{y_{i,l}} A_{i,l}^{-v_{i,l}}$. We define also $u_{i,l}(m) \in \mathbb{Z}$ by :

$$u_{i,l}(m) = y_{i,l}(m) - v_{i,l-r_i}(m) - v_{i,l+r_i}(m) + \sum_{j \neq i, r = C_{i,j}+1, C_{i,j}+3, \dots, -C_{i,j}-1} v_{j,r}(m)$$

In particular if $C_{i,j} = 0$ we have $u_{i,l}(A_{j,k}^{-1}) = 0$ and if $C_{i,j} < 0$

$$u_{i,l}(A_{j,k}^{-1}) = \sum_{r = C_{i,j}+1 \dots -C_{i,j}-1} \delta_{l+r,k}$$

Definition 3.6. *For m_1, m_2 $\hat{\mathcal{Y}}_t^s$-monomials we define :*

$$d_1(m_1, m_2) = \sum_{i \in I, l \in \mathbb{Z}/s\mathbb{Z}} v_{i,l+r_i}(m_1) u_{i,l}(m_2) + y_{i,l+r_i}(m_1) v_{i,l}(m_2)$$

$$d_2(m_1, m_2) = \sum_{i \in I, l \in \mathbb{Z}/s\mathbb{Z}} u_{i,l+r_i}(m_1) v_{i,l}(m_2) + v_{i,l+r_i}(m_1) y_{i,l}(m_2)$$

For m a $\hat{\mathcal{Y}}_t^s$-monomial we have always $d_1(m,m) = d_2(m,m)$ (see section 7.3). In the ADE-case we have $d_1 = d_2$ and it is the bicharacter of Nakajima [N3].

By checking on generators we see that :

Proposition 3.7. *For m_1, m_2 $\hat{\mathcal{Y}}_t^s$-monomials, we have in $\hat{\mathcal{Y}}_t^s$:*

$$m_1 m_2 = t^{2d_1(m_1,m_2) - 2d_2(m_2,m_1)} m_2 m_1 = t^{2d_2(m_1,m_2) - 2d_1(m_2,m_1)} m_2 m_1$$

If $B(z)$ is not symmetric, the product is defined in section 7.3.3.

3.2.2 Involution

We consider the $\mathbb{Z}[t^{\pm}]$-antilinear antimultiplicative involution of $\hat{\mathcal{Y}}_t^s$ such that $\overline{Y_{i,l}} = Y_{i,l}$, $\overline{A_{i,l}^{-1}} = A_{i,l}^{-1}$, $\overline{t} = t^{-1}$ (in [He2] we gave a "concrete" construction of this involution for the generic case : in $\hat{\mathcal{Y}}_u$ the involution is defined by $c_m \mapsto -c_m$).

A computation analog to lemma 6.8 of [He2] gives :

Lemma 3.8. *There is a $\mathbb{Z}[t^\pm]$-basis A_t^s of $\hat{\mathcal{Y}}_t^s$ such that all $m \in A_t^s$ is a $\hat{\mathcal{Y}}_t^s$-monomial and :*

$$\overline{m} = t^{2d_1(m,m)}m = t^{2d_2(m,m)}m$$

Moreover for $m_1, m_2 \in A_t^s$ we have $m_1 m_2 t^{-d_1(m_1,m_2)-d_2(m_1,m_2)} \in A_t^s$.

For example we have $Y_{i,l} \in A_t^s$ (because $d_1(Y_{i,l}, Y_{i,l}) = 0$) and if $s = 0$ or $s > 2r_i$ we have $tA_{i,l}^{-1} \in A_t^s$ (because $d_1(A_{i,l}^{-1}, A_{i,l}^{-1}) = -1$).

For $m_1, m_2 \in A_t^s$ we set $m_1.m_2 = m_1 m_2 t^{-d_1(m_1,m_2)-d_2(m_1,m_2)} \in A_t^s$. We have $m_1.m_2 = m_2.m_1$. The non commutative multiplication can be defined from . by setting ($m_1, m_2 \in A_t^s$) :

$$m_1 m_2 = t^{d_1(m_1,m_2)+d_2(m_1,m_2)} m_1.m_2$$

In the ADE-case it is the point of view adopted in [N3]. In particular if $s = 0$ or $s > 2r_i$, $Y_{i,l}$ (resp. $A_{i,l}^{-1}$) is denoted by $W_{i,l}$ (resp. $t^{-1}V_{i,l}$) in [N3].

For $s \geq 0$ there is a surjective map $p_s : A_t \to A_t^s$ such that for $m \in A_t$, $p_s(m)$ is the unique element of A_t^s such that for $i \in I, l \in \mathbb{Z}/s\mathbb{Z}$:

$$y_{i,l}(p_s(m)) = \sum_{l' \in \mathbb{Z}/[l']=l} y_{i,l}(m) \ , \ v_{i,l}(p_s(m)) = \sum_{l' \in \mathbb{Z}/[l']=l} v_{i,l}(m)$$

In particular it gives a $\mathbb{Z}[t^\pm]$-linear map $p_s : \hat{\mathcal{Y}}_t \to \hat{\mathcal{Y}}_t^s$.

3.2.3 Dominant monomials

A $\hat{\mathcal{Y}}_t^s$-monomial is said to be i-dominant (resp. i-antidominant) if $\forall l \in \mathbb{Z}/s\mathbb{Z}$, $u_{i,l}(m) \geq 0$ (resp. $u_{i,l}(m) \leq 0$). We denote by $B_{i,t}^s$ the set of i-dominant monomials m such that $m \in A_t^s$.

A $\hat{\mathcal{Y}}_t^s$-monomial is said to be dominant (resp. antidominant) if $\forall l \in \mathbb{Z}/s\mathbb{Z}, \forall i \in I$, $u_{i,l}(m) \geq 0$ (resp. $u_{i,l}(m) \leq 0$). We denote by B_t^s the set of dominant monomials m such that $m \in A_t^s$.

We define $\hat{\Pi} : \hat{\mathcal{Y}}_t^s \to \mathcal{Y}^s = \mathbb{Z}[Y_{i,l}^\pm]_{i\in I, l\in\mathbb{Z}/s\mathbb{Z}}$ as the ring morphism such that for m a $\hat{\mathcal{Y}}_t$-monomial $\hat{\Pi}(m) = \prod_{i\in I, l\in\mathbb{Z}/s\mathbb{Z}} Y_{i,l}^{u_{i,l}(m)}$ ($\mathcal{Y}^s$ is commutative).

In particular for $i \in I$, $l \in \mathbb{Z}/s\mathbb{Z}$, we have :

$$\hat{\Pi}(A_{i,l}^{-1}) = Y_{i,l-r_i}^{-1} Y_{i,l+r_i}^{-1} \prod_{j/C_{j,i}<0} \prod_{k=C_{j,i}+1, C_{j,i}+3, \dots, -C_{j,i}-1} Y_{j,l+2k}$$

and we denote this term by $A_{i,l}^{-1} = \hat{\Pi}(A_{i,l}^{-1})$. Let $A^s = \hat{\Pi}(A_t^s)$ the set of $\mathcal{Y}^s$-monomials, $B_i^s = \hat{\Pi}(B_{i,t}^s)$, $B^s = \bigcap_{i\in I} B_i^s = \hat{\Pi}(B_t^s)$.

We recall that there is a partial ordering on the $\hat{\mathcal{Y}}_t^s$-monomials such that
$m = t^\alpha A_{i_1,l_1}^{-1} \dots A_{i_k,l_k}^{-1} m' \Leftrightarrow m < m'$.

We have a generalization of lemma 3.14 of [He2] :

Lemma 3.9. *Let $s \geq 0$ and M be in A_t^s. Then :*

i) If C is finite then there is at most a finite number of $m' \in A_t^s$ such that $m' \leq M$ and m' is dominant.

ii) If C is finite or $det(C(q)) \neq 0$ then for $m \in A^s$ there is at most a finite number of $m' \in A_t^s$ such that $m' \leq M$ and $\hat{\Pi}(m') = m$.

iii) We suppose that there are $(a_i)_{i \in I} \in \mathbb{Z}^I$ such that $a_i > 0$ and for $i \in I$ $\sum_{j \in I} a_j C_{j,i} = 0$. Then for $M \in A^s$ there are at most a finite number of dominant monomials $m \in B^s$ of the form $m = MA_{i_1,l_1}^{-1}A_{i_2,l_3}^{-1}...A_{i_k,l_k}^{-1}$.

In particular an affine Cartan matrix verifies the property of (iii) (see [Kac] for the coefficients a_j).

Proof : i) If $s = 0$ see lemma 3.14 of [He2]. Suppose that $s \geq 1$ and let m' be in A_t^s with $m' = t^\alpha M \prod_{i \in I, l \in \mathbb{Z}/s\mathbb{Z}} A_{i,l}^{-v_{i,l}}$ and the $v_{i,l} \geq 0$. It suffices to show that the condition m' dominant implies that the $v_i = \sum_{l' \in \mathbb{Z}/s\mathbb{Z}} v_{i,l}$ are bounded (because $\mathbb{Z}/s\mathbb{Z}$ is finite). This condition implies :

$$u_i(m') = -2v_i + \sum_{j \neq i} (-C_{i,j}v_j) + u_i(M) \geq 0$$

Let U be the column vector with coefficients $(u_1(M), ..., u_n(M))$ and V the column vector with coefficients $(v_1, ..., v_n)$. So we have $U - CV \geq 0$. As C is finite, the theorem 4.3 of [Kac] implies that $C^{-1}U - V \geq 0$ and so the v_i are bounded.

ii) If $\det(C(q)) \neq 0$ then the $A_{i,l}^{-1}$ are algebraically independent (see [He2]) and the result is clear. If C is finite and $s \geq 1$, we use the proof of (i) with the condition :

$$u_i(m') = -2v_i + \sum_{j \neq i} (-C_{i,j}v_j) + u_i(M) = u_i(m)$$

iii) Consider $m' = \prod_{i \in I, l \in \mathbb{Z}/s\mathbb{Z}} A_{i,l}^{-v_{i,l}}$ and $m = Mm'$. For $i \in I$ let $v_i = \sum_{l \in \mathbb{Z}/s\mathbb{Z}} v_{i,l} \geq 0$. We have :

$$\sum_{i \in I} a_i u_i(m') = \sum_{i \in I} a_i \sum_{j \in I} (-C_{i,j})v_j = -\sum_{j \in I} v_j (\sum_{i \in I} a_i C_{i,j}) = 0$$

We suppose that m is dominant, in particular $u_{i,l}(m') \geq -u_{i,l}(M)$. So :

$$u_{i,l}(m') = u_i(m') - \sum_{l' \in \mathbb{Z}/s\mathbb{Z}, l' \neq l} u_{i,l'}(m') \leq u_i(m') + \sum_{l' \in \mathbb{Z}/s\mathbb{Z}, l' \neq l} u_{i,l'}(M)$$

$$\leq \frac{1}{a_i} \sum_{j \neq i} a_j(-u_j(m')) + \sum_{l' \in \mathbb{Z}/s\mathbb{Z}, l' \neq l} u_{i,l'}(M) \leq \frac{1}{a_i} \sum_{j \neq i} a_j u_j(M) + \sum_{l' \in \mathbb{Z}/s\mathbb{Z}, l' \neq l} u_{i,l'}(M)$$

So the $u_{i,l}(m')$ $(i \in I, l \in \mathbb{Z}/s\mathbb{Z})$ are bounded and there is at most a finite number of m' such that m is dominant. $\square$

In (ii) the condition $\det(C(q))\neq 0$ is essential : if we suppose that C is of type $A_2^{(1)}$ and $s=3$ (so $\det(C(q))=q^{-3}(q^3-1)^2=0$). Then for all $L\geq 0$, we have $\hat{\Pi}(Y_{1,0}A_{1,1}^{-L}A_{2,2}^{-L}A_{3,3}^{-L})=Y_{1,0}$ and $\hat{\Pi}^{-1}(Y_{1,0})$ is infinite.

3.3 Screening operators

Classical screening operators were introduced in [FR3] and t-deformed screening operators were introduced in [He1] for C finite and q-generic. We define classical and deformed screening operators in the general case :

Definition 3.10. *$\hat{\mathfrak{Y}}_{i,u}^s$ is the $\hat{\mathfrak{Y}}_u^s$-bimodule defined by generators $S_{i,l}$ ($i\in I, l\in\mathbb{Z}/s\mathbb{Z}$) and relations :*

$$S_{i,l}A_{j,k}^{-1}=t_{-C_{i,j}(z)(z^{(k-l)}+z^{(l-k)})}A_{j,k}^{-1}S_{i,l}\ ,\ S_{i,l}Y_{j,k}=t_{\delta_{i,j}(z^{(k-l)}+z^{(l-k)})}Y_{j,k}S_{i,l}$$

$$S_{i,l}t=tS_{i,l}\ ,\ S_{i,l-r_i}=t_{-q^{-2r_i}-1}A_{i,l}^{-1}S_{i,l+r_i}$$

$\hat{\mathfrak{Y}}_{i,t}^s$ is the $\hat{\mathfrak{Y}}_t^s$-bimodule $\hat{\mathfrak{Y}}_t^s\otimes_{\hat{\mathfrak{Y}}_u^s}\hat{\mathfrak{Y}}_{i,u}^s\otimes_{\hat{\mathfrak{Y}}_u^s}\hat{\mathfrak{Y}}_t^s$.

(Note that $\hat{\mathfrak{Y}}_t^s$ is a $\hat{\mathfrak{Y}}_u^s$-bimodule using the projection $\hat{\mathfrak{Y}}_u^s\to\hat{\mathfrak{Y}}_t^s$). For $s\geq 1$ we have a periodic condition $S_{i,l+s}=S_{i,l}$.

For m a $\hat{\mathfrak{Y}}_t^s$-monomial only a finite number of $[S_{i,l},m]=(t^2-1)t^{u_{i,l}(m)-1}[u_{i,l}(m)]_tS_{i,l}\in(t^2-1)\hat{\mathfrak{Y}}_{i,t}$ are not equal to 0, so we can define :

Definition 3.11. *The i^{th}-deformed screening operator is the map $S_{i,t}^s:\hat{\mathfrak{Y}}_t^s\to\hat{\mathfrak{Y}}_{i,t}^s$ defined by ($\lambda\in\hat{\mathfrak{Y}}_t^s$) :*

$$S_{i,t}^s(\lambda)=\frac{1}{t^2-1}\sum_{l\in\mathbb{Z}/s\mathbb{Z}}[S_{i,l},\lambda]\in\hat{\mathfrak{Y}}_{i,t}^s$$

At $t=1$ we define the classical screening operators :

$$S_i^s:\mathfrak{Y}^s\to\mathfrak{Y}_i^s=\bigoplus_{l\in\mathbb{Z}/s\mathbb{Z}}\mathfrak{Y}^sS_{i,l}/\sum_{l\in\mathbb{Z}/s\mathbb{Z}}\mathfrak{Y}^s.(S_{i,l-r_i}-A_{i,l}^{-1}S_{i,l+r_i})$$

such that for $m\in A^s$, $S_i^s(m)=m\sum_{l\in\mathbb{Z}/s\mathbb{Z}}u_{i,l}(m)S_{i,l}$. For $\lambda\in\hat{\mathfrak{Y}}_t^s$ we have $S_i(\hat{\Pi}(\lambda))=\hat{\Pi}(S_{i,t}(\lambda))$ where $\hat{\Pi}:\hat{\mathfrak{Y}}_{i,t}^s\to\mathfrak{Y}_i^s$ is defined by $\hat{\Pi}(mS_{i,l})=\hat{\Pi}(m)S_{i,l}$.

The $S_{i,t}^s,S_i^s$ are derivations. Denote $\hat{\mathfrak{K}}_{i,t}^s=\mathrm{Ker}(S_{i,t}^s)$ and $\hat{\mathfrak{K}}_t^s=\bigcap_{i\in I}\hat{\mathfrak{K}}_{i,t}^s$ (resp. $\mathfrak{K}_i^s=\mathrm{Ker}(S_i^s)$ and $\mathfrak{K}^s=\bigcap_{i\in I}\mathfrak{K}_i^s$) be the subalgebras of $\hat{\mathfrak{Y}}_t^s$ (resp. of $\mathfrak{Y}^s$).

4 q, t-characters in the generic case

In [He2] we defined q, t-characters for all finite Cartan matrices in the generic case. In this section we define q and q, t-characters for all symmetrizable (non necessarily finite) Cartan matrix such that $i \neq j \Rightarrow C_{i,j}C_{j,i} \leq 3$, in particular for Cartan matrices of affine type (except $A_1^{(1)}$, $A_2^{(2)}$). We suppose $s = 0$, that is to say q is generic. The root of unity case will be studied in section 5.

4.1 Reminder on the algorithm of Frenkel-Mukhin and on the deformed algorithm

4.1.1 Kernel of deformed screening operators

In the following a product $\overrightarrow{\prod\limits_{l\in\mathbb{Z}}} M_l$ (resp. $\overleftarrow{\prod\limits_{l\in\mathbb{Z}}} M_l$) is the ordered product $...M_{-2}M_{-1}M_0M_1...$ (resp. $...M_2M_1M_0M_{-1}...$).

Definition 4.1. *For $M \in \hat{\mathcal{Y}}_t$ a i-dominant monomial we define :*

$$\overleftarrow{E}_{i,t}(M) = M(\prod_{l\in\mathbb{Z}} Y_{i,l}^{u_{i,l}(M)})^{-1}\overleftarrow{\prod_{l\in\mathbb{Z}}}(Y_{i,l}(1+tA_{i,l+r_i}^{-1}))^{u_{i,l}(M)} \in \hat{\mathcal{Y}}_t$$

Theorem 4.2. *([He1]) For all Cartan matrix C, the kernel $\hat{\mathfrak{K}}_{i,t}$ of $S_{i,t}$ is the $\mathbb{Z}[t^\pm]$-subalgebra of $\hat{\mathcal{Y}}_t$ generated by the ($l \in \mathbb{Z}, j \neq i$) :*

$$Y_{i,l}(1+tA_{i,l+r_i}^{-1})\ ,\ A_{i,l}^{-1}Y_{i,l+r_i}Y_{i,l-r_i}\ ,\ Y_{j,l}\ ,\ \overleftarrow{E}_{i,t}(A_{j,l}^{-1})$$

For M a i-dominant monomial we have $\overleftarrow{E}_{i,t}(M) \in \hat{\mathfrak{K}}_{i,t}$, and $\hat{\mathfrak{K}}_{i,t} = \bigoplus\limits_{M\in B_{i,t}} \mathbb{Z}[t^\pm]\overleftarrow{E}_{i,t}(M)$.

Note that the proof of [He1] works also if C is not finite : the point of this proof is that an element $\chi \in \hat{\mathfrak{K}}_{i,t} - \{0\}$ has at least one i-dominant monomial, which is shown as in the sl_2-case.

At $t = 1$ it is a classical result of [FR3].

Note that in the ADE-case the identification (see section 3.2.2) between the $tA_{i,l}^{-1}$ and the $V_{i,l}$ shows that the notation $\hat{\mathfrak{K}}_{i,t}$ coincides with the notation of [N3].

4.1.2 Completed algebras

We define a $\mathbb{Z}^-$-gradation of $\hat{\mathcal{Y}}_t$ by putting $\deg(A_{i,l}^{-1}) = -1$, $\deg(Y_{i,l}) = 0$. Note that $m < m' \Rightarrow \deg(m) < \deg(m')$. We define the algebra $\hat{\mathcal{Y}}_t^\infty \supset \hat{\mathcal{Y}}_t$ as the completion for this gradation. In particular the elements of $\hat{\mathcal{Y}}_t^\infty$ are (infinite) sums $\sum\limits_{k\leq 0}\lambda_k$ such that λ_k is homogeneous of degree k.

In the same way we define $\hat{\mathfrak{K}}_{i,t}^{\infty}$ such that $\hat{\mathcal{Y}}_t^{\infty} \supset \hat{\mathfrak{K}}_{i,t}^{\infty} \supset \hat{\mathfrak{K}}_{i,t}$, that is to say $\chi \in \hat{\mathcal{Y}}_t^{\infty}$ is in $\hat{\mathfrak{K}}_{i,t}^{\infty}$ if and only if it is of the form $\chi = \sum\limits_{k \leq 0} \chi_k$ where :

$$\chi_k \in \bigoplus_{M \in B_{i,t}/\deg(M)=k} \mathbb{Z}[t^{\pm}]\overleftarrow{E}_{i,t}(M)$$

Let $\hat{\mathfrak{K}}_t^{\infty} = \bigcap\limits_{i \in I} \hat{\mathfrak{K}}_{i,t}^{\infty}$. In particular :

Lemma 4.3. *An element of $\hat{\mathfrak{K}}_t^{\infty} - \{0\}$ has at least one dominant monomial. An element of $\hat{\mathfrak{K}}_t - \{0\}$ has at least one dominant monomial and one antidominant monomial.*

In the same way for $t = 1$ we define $\mathcal{Y}^{\infty} \supset \mathcal{Y}$, $\mathfrak{K}^{\infty} \supset \mathfrak{K}$. They are well defined because in $\mathcal{Y}$ the $A_{i,l}^{-1}$ are algebraically independent (see section 3.2.3) and $\hat{\Pi}$ preserves the degree. In particular the maps $\hat{\Pi}$ can be extended to a map $\hat{\Pi} : \hat{\mathcal{Y}}_t^{\infty} \to \mathcal{Y}^{\infty}$.

For m a $\hat{\mathcal{Y}}_t$-monomial let $u(m) = \max\{l \in \mathbb{Z}/\forall k < l, \forall i \in I, u_{i,k}(m) = 0\}$. We define the subset $C(m) \subset A_t$

$$C(m) = \{t^{\mathbb{Z}} m A_{i_1,l_1}^{-1} ... A_{i_N,l_N}^{-1}/N \geq 0, l_1, ..., l_N \geq u(m)\} \cap A_t$$

We define the $\mathbb{Z}[t^{\pm}]$-submodule of $\hat{\mathcal{Y}}_t^{\infty}$:

$$\tilde{C}(m) = \{\chi \in \hat{\mathcal{Y}}_t^{\infty}/\chi = \sum_{m' \in C(m)} \lambda_{m'}(t) m'\}$$

4.1.3 Algorithms

In [He2] we defined a deformed algorithm to compute q, t-characters for C finite. We had to show that this algorithm is well defined, that is to say that at each step the different ways to compute each term give the same result.

The formulas of [He2] gives also a (non necessarily well defined) deformed algorithm for all Cartan matrices, that is to say :

Let $m \in B_t$. If the deformed algorithm beginning with m is well defined, it gives an element $F_t(m) \in \hat{\mathfrak{K}}_t^{\infty}$ such that m is the unique dominant monomial of $F_t(m)$.

An algorithm was also used by Nakajima in the ADE-case in [N2]. If we set $t = 1$ and apply $\hat{\Pi}$ we get a classical algorithm (it is analogous to the algorithm constructed by Frenkel and Mukhin in [FM1]). So :

Let $m \in B$. If the classical algorithm beginning with m is well defined, it gives an element $F(m) \in \mathfrak{K}^{\infty}$ such that m is the unique dominant monomial of $F(m)$.

We say that the classical algorithm (resp. the deformed algorithm) is well defined if for all $m \in B$ (resp. all $m \in B_t$) the classical algorithm (resp. deformed algorithm) beginning with m is well defined.

Lemma 4.4. *If the deformed algorithm is well defined then the classical algorithm is well defined.*

Proof : If the deformed algorithm beginning with m is well defined then the classical algorithm beginning with $\hat{\Pi}(m)$ is well defined and $F(\hat{\Pi}(m)) = \hat{\Pi}(F_t(m))$. $\square$

The following results are known :

If C is finite then the classical algorithm is well defined ([FM1]).

If C is finite and symmetric then the deformed algorithm is well defined ([N3]).

If C is finite then the deformed algorithm is well defined ([He2]).

In this section (theorem 4.6) we show that the classical and the deformed algorithms are well defined for a (non necessarily finite) Cartan matrix such that $i \neq j \Rightarrow C_{i,j}C_{j,i} \leq 3$.

4.2 Morphism of q, t-characters beyond the finite case

The construction of [He2] is based on the fact that we can compute explicitly q, t-characters for the submatrices of format 2 of the Cartan matrix. So :

Proposition 4.5. *We suppose that C is a Cartan matrix of rank 2. The following properties are equivalent :*

i) For all $m \in B$, $F(m) \in \mathfrak{K}$

ii) C is finite

iii) $C_{1,2}C_{2,1} \leq 3$

iv) For $i = 1$ or 2, $\hat{\mathfrak{K}}_t \cap \tilde{C}(Y_{i,0}) \neq \{0\}$

v) For $i = 1$ or 2, $C(Y_{i,0})$ has an antidominant monomial

Proof : The Cartan matrices of rank 2 such that $C_{1,2}C_{2,1} \leq 3$ are matrices of type $A_1 \times A_1$, A_2, B_2, C_2, G_2 or G_2^t. Those are finite Cartan matrices of rank 2, so $(ii) \Leftrightarrow (iii)$. Moreover if C is finite, the classical theory of q-characters shows $(ii) \Rightarrow (i)$.

We have seen in [He2] that $(ii) \Rightarrow (iv)$. It follows from lemma 4.3 that $(iv) \Rightarrow (v)$ and $(i) \Rightarrow (v)$.

So it suffices to show that $(v) \Rightarrow (iii)$. We suppose there is an antidominant monomial $m \in C(Y_{1,0})$. We can suppose $C_{1,2} < 0$ and $C_{2,1} < 0$. m verifies :

$$\hat{\Pi}(m) = Y_{1,0}A_{1,l_1}^{-1}...A_{1,l_L}^{-1}A_{2,l_1}^{-1}...A_{2,l_M}^{-1}$$

where $L, M \geq 0$. In particular we have :

$$u_1(m) = 1 - 2L - MC_{1,2} \text{ and } u_2(m) = -2M - LC_{2,1}$$

As m is antidominant, we have $u_1(m), u_2(m) \leq 0$.

if $M = 0$, we have $u_2(m) = -LC_{2,1} \leq 0 \Rightarrow L = 0$ and $u_1(m) = 1 > 0$, impossible.

if $M > 0$, we have $\frac{L}{M} > \frac{-C_{1,2}}{2}$ and $\frac{L}{M} \leq \frac{2}{-C_{2,1}}$ so $\frac{-C_{1,2}}{2} < \frac{2}{-C_{2,1}} \Rightarrow C_{1,2}C_{2,1} \leq 3$. $\square$

Let us look at the general case :

Theorem 4.6. *If $i \neq j \Rightarrow C_{i,j}C_{j,i} \leq 3$, then the classical and the deformed algorithms are well defined.*

Proof : It suffices to show that the deformed algorithm is well defined (lemma 4.4).We follow the proof of theorem 5.11 of [He2] : it suffices to construct $F_t(m)$ for $m = Y_{i,0}$ ($i \in I$) and it suffices to see the property for the matrices $\begin{pmatrix} 2 & C_{i,j} \\ C_{j,i} & 2 \end{pmatrix}$. If $r_i \wedge r_j = 1$ this follows from proposition 4.5. If $r_i \wedge r_j > 1$ it suffices to replace r_i, r_j with $\frac{r_i}{r_i \wedge r_j}, \frac{r_j}{r_i \wedge r_j}$ (in fact it means that we replace q by $q^{r_i \wedge r_j}$). $\square$

In the following we suppose that $i \neq j \Rightarrow C_{i,j}C_{j,i} \leq 3$. For example C could be of finite or affine type (except $A_1^{(1)}$, $A_2^{(2)}$).

We conjecture that for C of type $A_1^{(1)}$ (with $r_1 = r_2 = 2$) and of type $A_2^{(2)}$ the algorithms are well defined (note that for C of type $A_1^{(1)}$ and $r_1 = r_2 = 1$ the classical algorithm is not well defined).

We verify as in [He2] that $F_t(Y_{i,l})F_t(Y_{j,l}) = F_t(Y_{j,l})F_t(Y_{i,l})$. Let Rep $= \mathbb{Z}[X_{i,l}]_{i\in I, l\in\mathbb{Z}}$ as in section 2.2 and a Rep-monomials is a product of the $X_{i,l}$.

Definition 4.7. *The morphism of q,t-characters $\chi_{q,t} : Rep \rightarrow \hat{\mathfrak{K}}_t^\infty$ is the $\mathbb{Z}$-linear map such that :*

$$\chi_{q,t}(\prod_{i\in I, l\in\mathbb{Z}} X_{i,l}^{x_{i,l}}) = \overrightarrow{\prod_{l\in\mathbb{Z}}} \prod_{i\in I} F_t(Y_{i,l})^{x_{i,l}}$$

The morphism of q-characters $\chi_q : Rep \rightarrow \mathfrak{K}^\infty$ is defined by $\chi_q = \hat{\Pi} \circ \hat{\chi}_{q,t}$.

We show as in [He2] :

Theorem 4.8. *The $\mathbb{Z}$-linear map $\chi_{q,t} : Rep \rightarrow \hat{\mathcal{Y}}_t^\infty$ is injective and is characterized by the three following properties :*

1) For M a Rep-monomial define $m = \prod_{i\in I, l\in\mathbb{Z}} Y_{i,l}^{x_{i,l}(M)} \in B_t$. Then we have :

$$\chi_{q,t}(M) = m + \sum_{m'<m} a_{m'}(t)m' \text{ (where } a_{m'}(t) \in \mathbb{Z}[t^\pm])$$

2) The image $Im(\chi_{q,t})$ is contained in $\hat{\mathfrak{K}}_t^\infty$.

3) Let M_1, M_2 be Rep-monomials such that

$$max\{l / \sum_{i\in I} x_{i,l}(M_1) > 0\} \leq min\{l / \sum_{i\in I} x_{i,l}(M_2) > 0\}$$

We have :

$$\chi_{q,t}(M_1M_2) = \chi_{q,t}(M_1)\chi_{q,t}(M_2)$$

Remark : Those properties are generalizations of Nakajima's axioms [N3] and of properties of [He2]. In particular if C is finite then we have $\hat{\Pi}(\mathrm{Im}(\chi_{q,t})) \subset \mathcal{Y}$, $\chi_q : \mathrm{Rep} \rightarrow \mathcal{Y}$ is the classical morphism of q-characters and $\chi_{q,t}$ is the morphism of [He2]. If C is of type ADE then $\chi_{q,t}$ is the morphism of q,t-characters of [N3].

5 ϵ, t-characters in the root of unity case

In this section we define and study ϵ, t-characters at roots of unity : let $\epsilon \in \mathbb{C}^*$ be a s^{th}-primitive root of unity. We suppose that $s > 2r^\vee$.

The case $t = 1$ was studied in [FM2]. The t-deformations were studied in the ADE-case by Nakajima in [N3] using quiver varieties. In this section we suppose that $i \neq j \Rightarrow C_{i,j}C_{j,i} \leq 3$ and $B(z)$ is symmetric. In particular C can be of finite type or of affine type (except $A_1^{(1)}$, $A_2^{(2)}$, see section 7.3.2). The deformed algorithm is well defined and $\chi_{q,t}$ exists (theorem 4.6). We define the completed algebra $\hat{\mathcal{Y}}_t^{s,\infty}$, $\hat{\mathfrak{K}}_{i,t}^{s,\infty}$, $\hat{\mathfrak{K}}_t^{s,\infty}$ as we did for the generic case (section 4.1.2).

5.1 Reminder : classical ϵ-characters at roots of unity

We define $\tau_s : \mathcal{Y} \rightarrow \mathcal{Y}_s$ as the ring homomorphism such that $\tau_s(Y_{i,l}) = Y_{i,[l]}$ where for $l \in \mathbb{Z}$ we denote by $[l]$ its image in $\mathbb{Z}/s\mathbb{Z}$.

If C is finite the morphism of ϵ-characters $\chi_\epsilon : \mathrm{Rep}^s \rightarrow \mathcal{Y}^s$ is defined by Frenkel and Mukhin (see section 2.2). We have the following characterization :

Theorem 5.1. *([FM2]) If C is finite, the morphism of ϵ-characters $\chi_\epsilon : Rep^s \rightarrow \mathcal{Y}^s$ is the linear map such that ($l_0 \in \mathbb{Z}$) :*

$$\chi_\epsilon(\prod_{i\in I, l\in\mathbb{Z}/s\mathbb{Z}} X_{i,l}^{x_{i,l}}) = \tau_s(\chi_q(\prod_{i\in I, l_0\leq l\leq l_0+s-1} X_{i,l}^{x_{i,[l]}}))$$

If C is not finite, we can consider $\hat{\mathcal{Y}}^s = \mathbb{Z}[Y_{i,l}, A_{i,l}^{-1}]_{i\in I, l\in\mathbb{Z}/s\mathbb{Z}}$ and the completion $\hat{\mathcal{Y}}_t^{s,\infty}$ as in the generic case. We define $\hat{\chi}_\epsilon : \mathrm{Rep}^s \rightarrow \hat{\mathcal{Y}}^{s,\infty}$ with the formula of the theorem 5.1.

In the following we give an analogous construction in the deformed case $t \neq 1$.

5.2 Construction of $\chi_{\epsilon,t}$

The point for the t-deformation is that we can not define a natural t-analog of τ_s which is a ring homomorphism. However we construct an analog $\tau_{s,t}$ of τ_s which has nice properties.

5.2.1 Definition of $\tau_{s,t}$

First let us briefly explain how $\tau_{s,t}$ is constructed. The main property is a compatibility with some ordered products : suppose that $l_1 > l_2$ ($l_1, l_2 \in \mathbb{Z}$), that $m_1 \in \hat{\mathcal{Y}}_t$ involves only the $Y_{i,l_1}, A_{i,l_1}^{-1}$ and that m_2 involves only the $Y_{i,l_2}, A_{i,l_2}^{-1}$. Then $\tau_{s,t}$ is defined such that $\tau_{s,t}(m_1 m_2) = \tau_{s,t}(m_1)\tau_{s,t}(m_2)$. Let us now write it in a formal way :

For m a $\hat{\mathcal{Y}}_t$-monomial and $l \in \mathbb{Z}$, let :

$$\pi_l(m) = (\prod_{i\in I} Y_{i,l}^{y_{i,l}(m)})(\prod_{i\in I} A_{i,l}^{-v_{i,l}(m)})$$

It is well defined because for $i, j \in I$ and $l \in \mathbb{Z}$ we have $Y_{i,l}Y_{j,l} = Y_{j,l}Y_{i,l}$, $A_{i,l}^{-1}A_{j,l}^{-1} = A_{j,l}^{-1}A_{i,l}^{-1}$ and for $i \neq j$, $A_{i,l}^{-1}Y_{j,l} = Y_{j,l}A_{i,l}^{-1}$ (theorem 3.4).

Let $\overrightarrow{m} = \overrightarrow{\prod_{l\in\mathbb{Z}}}\pi_l(m)$, $\overleftarrow{m} = \overleftarrow{\prod_{l\in\mathbb{Z}}}\pi_l(m)$, and the $\mathbb{Z}[t^\pm]$-basis of $\hat{\mathcal{Y}}_t$:

$$\overrightarrow{A} = \{\overrightarrow{m}/\ m\ \hat{\mathcal{Y}}_t\text{-monomial}\} \text{ and } \overleftarrow{A} = \{\overleftarrow{m}/\ m\ \hat{\mathcal{Y}}_t\text{-monomial}\}$$

Definition 5.2. *We define $\tau_{s,t} : \hat{\mathcal{Y}}_t \to \hat{\mathcal{Y}}_t^s$ as the $\mathbb{Z}[t^\pm]$-linear map such that for $m \in \overleftarrow{A}$:*

$$\tau_{s,t}(m) = \overleftarrow{\prod_{l\in\mathbb{Z}}}(\prod_{j\in I} A_{j,[l]}^{-v_{j,l}(m)})(\prod_{j\in I} Y_{j,[l]}^{y_{j,l}(m)})$$

Note that $\tau_{s,t}$ is not a ring homomorphism and is not injective; the motivation of the construction are the results of theorem 5.6 and 5.11.

5.2.2 Definition of $\chi_{\epsilon,t}$

$\tau_{s,t}$ is compatible with the gradations of $\hat{\mathcal{Y}}_t$ and $\hat{\mathcal{Y}}_t^s$ and is extended to a map $\tau_{s,t}$: $\hat{\mathcal{Y}}_t^\infty \to \hat{\mathcal{Y}}_t^{s,\infty}$.

Definition 5.3. *The morphism of q,t-characters at the s^{th}-primitive roots of unity $\chi_{\epsilon,t}$: $Rep^s \to \hat{\mathcal{Y}}_t^{s,\infty}$ is the $\mathbb{Z}$-linear map such that :*

$$\chi_{\epsilon,t}(\prod_{i\in I, l\in\mathbb{Z}/s\mathbb{Z}} X_{i,l}^{x_{i,l}}) = \tau_{s,t}(\chi_{q,t}(\prod_{i\in I, 0\leq l\leq s-1} X_{i,l}^{x_{i,[l]}}))$$

It follows from theorem 4.8 and 5.1 :

Proposition 5.4. *The morphism $\chi_{\epsilon,t}$ verifies the following properties :*

1) If C is finite we have $\hat{\Pi} \circ \chi_{\epsilon,t} = \chi_\epsilon$.

2) The map $\chi_{\epsilon,t}$ is injective.

3) For a Rep-monomial M define $m = \prod_{i\in I, l\in\mathbb{Z}/s\mathbb{Z}} Y_{i,l}^{x_{i,l}(M)} \in B_t^s$. Then we have :

$$\chi_{q,t}(M) = m + \sum_{m'<m} a_{m'}(t)m' \ \textit{(where } a_{m'}(t) \in \mathbb{Z}[t^\pm]\textit{)}$$

Note that 1) means that in the finite case we get at $t=1$ the map of [FM1].

Remark that in general we have an analog of 1, that is to say $\Pi\circ\chi_{\epsilon,t}=\hat{\chi}_\epsilon$ where $\Pi:\hat{\mathcal{Y}}_t^{s,\infty}\rightarrow\hat{\mathcal{Y}}^{\infty,s}$ is defined in an obvious way.

In the following we show other fundamental properties of $\chi_{\epsilon,t}$ (theorem 5.6 and theorem 5.11).

5.3 The image of $\chi_{\epsilon,t}$

In this section we show an analog of the property 2 of theorem 4.8 at roots of unity (theorem 5.6).

5.3.1 Classical case $t=1$

The map $\tau_s:\mathcal{Y}\rightarrow\mathcal{Y}^s$ is a ring homomorphism. In particular we can define a $\mathbb{Z}$-linear map $\tau_s:\mathcal{Y}_i\rightarrow\mathcal{Y}_i^s$ such that $\tau_s(mS_{i,l})=\tau_s(m)S_{i,[l]}$. Indeed it suffices to see it agrees with the defining relations of $\mathcal{Y}_i$:

$$\tau_s(mA_{i,l+r_i}^{-1}S_{i,l+2r_i})=\tau_s(mA_{i,l+r_i}^{-1})S_{i,[l+2r_i]}=\tau_s(m)A_{i,[l+r_i]}^{-1}S_{i,[l+2r_i]}=\tau_s(m)S_{i,[l]}=\tau_s(mS_{i,l})$$

Lemma 5.5. *We have $\tau_s\circ S_i=S_i^s\circ\tau_s$. In particular $\tau_s(\mathfrak{K})\subset\mathfrak{K}_i^s$.*

Proof : It suffices to see for m a $\mathcal{Y}$-monomial :

$$\tau_s(S_i(m))=\sum_{l\in\mathbb{Z}}u_{i,l}(m)\tau_s(mS_{i,l})=\tau_s(m)\sum_{l\in\mathbb{Z}}u_{i,l}(m)S_{i,[l]}=\tau_s(m)\sum_{0\leq l\leq s-1}(\sum_{r\in\mathbb{Z}}u_{i,l+rs}(m))S_{i,[l]}$$

$$=\tau_s(m)\sum_{0\leq l\leq s-1}u_{i,[l]}(\tau_s(m))S_{i,[l]}=S_i^s(\tau_s(m))$$

□

Remark : $\tau_s(\mathfrak{K}_i)$ is a subalgebra of $\mathfrak{K}_i^s$ and $\tau_s(\mathfrak{K}_i)=\bigoplus_{m\in B_i}E_i(m)$ where for $m\in B_i^s$, we set $E_i(m)=m\prod_{l\in\mathbb{Z}/s\mathbb{Z}}(1+A_{i,l+r_i}^{-1})^{u_{i,l}(m)}$. In particular if $\chi\in\tau_s(\mathfrak{K}_i)$ has no i-dominant monomial then $\chi=0$.

5.3.2 Deformed case

Theorem 5.6. *The image of $\chi_{\epsilon,t}$ is contained in $\hat{\mathfrak{K}}_t^{s,\infty}$.*

In this section we prove this theorem. With the help of theorem 4.8 it suffices to show that $\tau_{s,t}(\hat{\mathfrak{K}}_{i,t})\subset\hat{\mathfrak{K}}_{i,t}^s$ which will be done in proposition 5.10.

For m a $\hat{\mathcal{Y}}_t$-monomial and $k\in\mathbb{Z}$ let :

$$m[k]=m(\overleftarrow{m})^{-1}\overleftarrow{\prod_{l\in\mathbb{Z}}}(\prod_{j\in I}Y_{j,l}^{y_{j,l+k}(m)})(\prod_{j\in I}A_{j,l}^{-v_{j,l+k}(m)})$$

Note that $\tau_{s,t}(m[ks]) = \tau_{s,t}(m)$ and for $m \in \overleftarrow{A}$, $k \in \mathbb{Z}$ we have $m[k] \in \overleftarrow{A}$.

For m_1, m_2 $\hat{\mathcal{Y}}_t$-monomials, and $k \in \mathbb{Z}$ we have :

$$d_1(m_1, m_2[k]) = d_1(m_1[-k], m_2) \text{ and } d_2(m_1, m_2[k]) = d_2(m_1[-k], m_2)$$

Lemma 5.7. *For m_1, m_2 $\hat{\mathcal{Y}}_t$-monomials we have :*

$$d_1(\tau_{s,t}(m_1), \tau_{s,t}(m_2)) = \sum_{r\in\mathbb{Z}} d_1(m_1, m_2[rs]) = \sum_{r\in\mathbb{Z}} d_1(m_1[rs], m_2)$$

$$d_2(\tau_{s,t}(m_1), \tau_{s,t}(m_2)) = \sum_{r\in\mathbb{Z}} d_2(m_1, m_2[rs]) = \sum_{r\in\mathbb{Z}} d_2(m_1[rs], m_2)$$

Proof : For example for d_1 we compute :

$$\begin{aligned} &d_1(\tau_{s,t}(m_1), \tau_{s,t}(m_2)) \\ &= \sum_{i\in I, l\in\mathbb{Z}/s\mathbb{Z}} v_{i,l+r_i}(\tau_{s,t}(m_1)) u_{i,l}(\tau_{s,t}(m_2)) + w_{i,l+r_i}(\tau_{s,t}(m_1)) v_{i,l}(\tau_{s,t}(m_2)) \\ &= \sum_{i\in I, 0\leq l\leq s-1, r\in\mathbb{Z}, r'\in\mathbb{Z}} v_{i,l+r_i+rs}(m_1) u_{i,l+r's}(m_2) + w_{i,l+r_i+rs}(m_1) v_{i,l+r's}(m_2) \\ &= \sum_{i\in I, l\in\mathbb{Z}, r\in\mathbb{Z}} v_{i,l+r_i}(m_1) u_{i,l+rs}(m_2) + w_{i,l+r_i}(m_1) v_{i,l+rs}(m_2) = \sum_{r\in\mathbb{Z}} d_1(m_1, m_2[rs]) \end{aligned}$$

□

Lemma 5.8. *Let m be a $\hat{\mathcal{Y}}_t$-monomial of the form $m = Z_1 Z_2 ... Z_K$ where $Z_k = Y_{i_k,l_k}$ or $Z_k = A^{-1}_{i_k,l_k}$. We suppose that $k > k'$ implies $l_k \leq l_{k'} + r^\vee$ and $(Z_k, Z_{k'}) \notin \{(A^{-1}_{i,l}, A^{-1}_{i,l'})/i \in I, l' < l\}$. Then we have :*

$$\tau_{s,t}(m) = \tau_{s,t}(Z_1)\tau_{s,t}(Z_2)...\tau_{s,t}(Z_K)$$

Proof : First we order the factors $m = t^{2\sum_{k<k'/l_k<l_{k'}} d_1(Z_k, Z_{k'}) - d_2(Z_{k'}, Z_k)} \overleftarrow{m}$. So we can apply $\tau_{s,t}$:

$$\tau_{s,t}(m) = t^{2\sum_{k<k'/l_k<l_{k'}} d_1(Z_k, Z_{k'}) - d_2(Z_{k'}, Z_k)} \tau_{s,t}(\overleftarrow{m}) \text{ , } \tau_{s,t}(\overleftarrow{m}) = \overleftarrow{\prod_{l\in\mathbb{Z}}}(\prod_{j\in I} Y_{j,[l]}^{y_{j,l}(m)})(\prod_{j\in I} A_{j,[l]}^{-v_{j,l}(m)})$$

If we order the factors of $\tau_{s,t}(Z_1)\tau_{s,t}(Z_2)...\tau_{s,t}(Z_k)$, we get :

$$\tau_{s,t}(Z_1)\tau_{s,t}(Z_2)...\tau_{s,t}(Z_k) = t^{2\sum_{k<k'/l_k<l_{k'}} (d_1(\tau_{s,t}(Z_k), \tau_{s,t}(Z_{k'})) - d_2(\tau_{s,t}(Z_{k'}), \tau_{s,t}(Z_k)))} \tau_{s,t}(\overleftarrow{m})$$

So it suffices to show that $k < k'$ and $l_k < l_{k'}$ implies $d_1(Z_k, Z_{k'}) - d_2(Z_{k'}, Z_k) = d_1(\tau_{s,t}(Z_k), \tau_{s,t}(Z_{k'})) - d_2(\tau_{s,t}(Z_{k'}), \tau_{s,t}(Z_k))$. But we have $0 < l_{k'} - l_k \leq r^\vee$ and $s > 2r^\vee$. So for $p \in \mathbb{Z}$ such that $p \neq 0$ we have $|l_k - l_{k'} + ps| > r^\vee$. But in general for k_1, k_2, we have :

$$[Z_{k_1}, Z_{k_2}] \neq 0 \Rightarrow (Z_{k_1}, Z_{k_2}) = (A^{-1}_{i_k,l_k}, A^{-1}_{i_k,l_k\pm 2r_{i_k}}) \text{ or } |l_{k_1} - l_{k_2}| \leq r^\vee$$

So in our situation we have $d_1(Z_k, Z_{k'}[ps]) = d_2(Z_{k'}, Z_k[ps]) = 0$. In particular :

$$d_1(\tau_{s,t}(Z_k), \tau_{s,t}(Z_{k'})) - d_2(\tau_{s,t}(Z_{k'}), \tau_{s,t}(Z_k))$$

$$= d_1(Z_k, Z_{k'}) - d_2(Z_{k'}, Z_k) + \sum_{p\neq 0}(d_1(Z_k, Z_{k'}[ps]) - d_2(Z_{k'}, Z_k[ps])) = d_1(Z_k, Z_{k'}) - d_2(Z_{k'}, Z_k)$$

□

Lemma 5.9. *Let m be a $\hat{\mathcal{Y}}_t^s$-monomial and $l, l' \in \mathbb{Z}$.*

$$l' \geq l + s - r_i \Rightarrow u_{i,l'}(\pi_l(m)) = 0$$

$$l' \leq l + r_i - s + 1 \Rightarrow u_{i,l'}(\pi_l(m)\pi_{l-1}(m)...) = u_{i,l'}(m)$$

Proof : First notice that for $l, l' \in \mathbb{Z}$, we have :

$$u_{i,l'}(Y_{i,l}) \neq 0 \Rightarrow l' = l \ , \ u_{i,l'}(A_{i,l}) \neq 0 \Rightarrow l' = l \pm r_i$$

$$i \neq j \ , \ u_{i,l'}(A_{j,l}) \neq 0 \Rightarrow |l' - l| \leq -C_{j,i} - 1 \leq r^\vee - 1$$

As $r_i \leq r^\vee$ we have : $u_{i,l'}(\pi_l(m)) \neq 0 \Rightarrow l - r^\vee \leq l' \leq l + r^\vee$.

If we suppose $l' \geq l + s - r_i \geq l + 2r^\vee + 1 - r_i \geq l + r^\vee + 1$ we have $u_{i,l'}(\pi_l(m)) = 0$ and this gives the first point.

We suppose that $l' \leq l + r_i - s + 1$. If $k \geq l + 1 \geq l' + s - r_i \geq l' + r^\vee + 1$ we have $u_{i,l'}(\pi_k(m)) = 0$. So : $u_{i,l'}(\pi_l(m)\pi_{l-1}(m)...) = u_{i,l'}(m) - \sum_{k>l} u_{i,l'}(\pi_k(m)) = u_{i,l'}(m)$. $\square$

Proposition 5.10. *We have $\tau_{s,t}(\hat{\mathfrak{K}}_{i,t}) \subset \hat{\mathfrak{K}}_{i,t}^s$. Moreover for m a i-dominant monomial :*

$$\tau_{s,t}(\overleftarrow{E}_{i,t}(m)) = \tau_{s,t}(m)\tau_{s,t}(\hat{m}^i)^{-1}\overleftarrow{\prod_{l\in\mathbb{Z}}}(Y_{i,[l]}(1 + tA_{i,[l+r_i]}^{-1}))^{u_{i,l}(m)}$$

where $\hat{m}^i = \prod_{l\in\mathbb{Z}} Y_{i,l}^{u_{i,l}(m)} \in B_{i,t}$.

Proof : We have to show that for m a i-dominant monomial, $\tau_{s,t}(\overleftarrow{E}_{i,t}(m)) \in \hat{\mathfrak{K}}_{i,t}^s$. The proof has three steps :

1) First we suppose that $m = Y_{i,l}$ where $l \in \mathbb{Z}$. We have $\overleftarrow{E}_{i,t}(Y_{i,l}) = Y_{i,l}(1 + tA_{i,l+r_i}^{-1})$, and :

$$\tau_{s,t}(\overleftarrow{E}_{i,t}(Y_{i,l})) = \tau_{s,t}((1 + t^{-1}A_{i,l+r_i}^{-1})Y_{i,l}) = 1 + t^{-1}A_{i,[l+r_i]}^{-1})Y_{i,[l]} = Y_{i,[l]}(1 + tA_{i,[l+r_i]}^{-1})$$

and so :

$$S_{i,t}^s(\tau_{s,t}(\overleftarrow{E}_{i,t}(Y_{i,l})) = Y_{i,[l]}S_{i,[l]} - t^{-2}tY_{i,[l]}A_{i,[l+r_i]}^{-1}S_{i,[l+2r_i]} = Y_{i,[l]}(S_{i,[l]} - t^{-1}A_{i,[l+r_i]}^{-1}S_{i,[l+2r_i]}) = 0$$

2) Next we suppose that $m = \prod_{l\in\mathbb{Z}} Y_{i,l}^{u_{i,l}}$. We have $\overleftarrow{E}_{i,t}(m) = \overleftarrow{\prod_{l\in\mathbb{Z}}}(\overleftarrow{E}_{i,t}(Y_{i,l}))^{u_{i,l}}$. But $r_i \leq r^\vee$, and in $(\overleftarrow{E}_{i,t}(Y_{i,l}))^{u_{i,l}}$ there are only $Y_{i,l}$ and $A_{i,l+r_i}^{-1}$. So we are in the situation of the lemma 5.8, and :

$$\tau_{s,t}(\overleftarrow{E}_{i,t}(m)) = \overleftarrow{\prod_{l\in\mathbb{Z}}}(\tau_{s,t}(\overleftarrow{E}_{i,t}(Y_{i,l})))^{u_{i,l}}$$

As $\hat{\mathfrak{K}}_{i,t}^s$ is a subalgebra of $\hat{\mathcal{Y}}_t^s$, it follows from the first step that $\tau_{s,t}(\overleftarrow{E}_{i,t}(m)) \in \hat{\mathfrak{K}}_{i,t}^s$.

3) Finally let $m \in B_{i,t}$ be an i-dominant monomial. As for all $l \in \mathbb{Z}$, $u_{i,l}(m) = u_{i,l}(\hat{m}^i)$, we have :

$$(\tau_{s,t}(m))^{-1} S_{i,t}^s(\tau_{s,t}(m)) = (\tau_{s,t}(\hat{m}^i))^{-1} S_{i,t}^s(\tau_{s,t}(\hat{m}^i))$$

It follows from the second point that $\tau_{s,t}(\overleftarrow{E}_{i,t}(\hat{m}^i)) \in \hat{\mathfrak{K}}_{i,t}^s$. Let $\hat{\chi} = \tau_{s,t}(\hat{m}^i)^{-1}\tau_{s,t}(\overleftarrow{E}_{i,t}(\hat{m}^i)) \in \hat{\mathcal{Y}}_t^s$. We have $\tau_{s,t}(m)\hat{\chi} \in \hat{\mathfrak{K}}_{i,t}^s$, because :

$S_{i,t}^s(\tau_{s,t}(m)\hat{\chi}) = S_{i,t}^s(\tau_{s,t}(m))\hat{\chi} + \tau_{s,t}(m) S_{i,t}^s(\hat{\chi})$
$= \tau_{s,t}(m)(\tau_{s,t}(\hat{m}^i))^{-1}(S_{i,t}^s(\tau_{s,t}(\hat{m}^i))\hat{\chi} + \tau_{s,t}(\hat{m}^i) S_{i,t}^s(\hat{\chi}))$
$= \tau_{s,t}(m)(\tau_{s,t}(\hat{m}^i))^{-1} S_{i,t}^s(\tau_{s,t}(\hat{m}^i)\hat{\chi})$
$= S_{i,t}^s(\tau_{s,t}(\overleftarrow{E}_{i,t}(m))) = 0$

So it suffices to show that $\tau_{s,t}(\overleftarrow{E}_{i,t}(m)) = \tau_{s,t}(m)\hat{\chi}$.

Let $\chi = (\hat{m}^i)^{-1}\overleftarrow{E}_{i,t}(\hat{m}^i) \in \hat{\mathcal{Y}}_t$. By definition of $\overleftarrow{E}_{i,t}(m)$, we have in $\hat{\mathcal{Y}}_t$: $\overleftarrow{E}_{i,t}(m) = m\chi$. In particular we want to show that $\tau_{s,t}(m\chi) = \tau_{s,t}(m)\tau_{s,t}(\hat{m}^i)^{-1}\tau_{s,t}(\hat{m}^i\chi)$. Let $\lambda_{m'}(t)$ be in $\mathbb{Z}[t^{\pm}]$ such that :

$$\chi = \sum_{m' \in A_t} \lambda_{m'}(t) m'$$

If $\lambda_{m'}(t) \neq 0$ then m' is of the form $m' = A_{i,l_1}^{-1}...A_{i,l_k}^{-1}$. As $\tau_{s,t}$ is $\mathbb{Z}[t^{\pm}]$-linear, it suffices to show that for all m' of this form, we have :

$$\tau_{s,t}(m)\tau_{s,t}(\hat{m}^i)^{-1}\tau_{s,t}(\hat{m}^i m') = \tau_{s,t}(mm')$$

That is to say $\alpha = \beta$ where $\alpha, \beta \in \mathbb{Z}$ are defined by :

$$\tau_{s,t}(mm') = t^{\alpha}\tau_{s,t}(m)\tau_{s,t}(m') \text{ and } \tau_{s,t}(\hat{m}^i m') = t^{\beta}\tau_{s,t}(\hat{m}^i)\tau_{s,t}(m')$$

We can suppose without loss of generality that $m \in \overleftarrow{A}$ and $m' \in \overleftarrow{A}$ (because $\tau_{s,t}$ is $\mathbb{Z}[t^{\pm}]$-linear). Let us compute α. First we have in $\hat{\mathcal{Y}}_t$:

$$mm' = t^{2\sum_{l'>l} d_2(\pi_l(m), \pi_{l'}(m')) - d_1(\pi_{l'}(m'), \pi_l(m))} \overleftarrow{\prod_{l \in \mathbb{Z}}} \pi_l(m)\pi_l(m')$$

We are in the situation of lemma 5.8, so :

$$\tau_{s,t}(mm') = t^{2\sum_{l'>l} d_2(\pi_l(m), \pi_{l'}(m')) - d_1(\pi_{l'}(m'), \pi_l(m))} \overleftarrow{\prod_{l \in \mathbb{Z}}} \tau_{s,t}(\pi_l(m))\tau_{s,t}(\pi_l(m'))$$

But we have in $\hat{\mathcal{Y}}_t^s$ (lemma 5.7) :

$$\tau_{s,t}(m)\tau_{s,t}(m') = t^{2\sum_{l'>l} d_2(\tau_{s,t}(\pi_l(m)), \tau_{s,t}(\pi_{l'}(m'))) - d_1(\tau_{s,t}(\pi_{l'}(m')), \tau_{s,t}(\pi_l(m)))} \overleftarrow{\prod_{l \in \mathbb{Z}}} \tau_{s,t}(\pi_l(m))\tau_{s,t}(\pi_l(m'))$$

And we get :

$$\alpha = 2\sum_{l'>l} d_2(\pi_l(m), \pi_{l'}(m')) - d_1(\pi_{l'}(m'), \pi_l(m))$$

$$-2\sum_{l'>l} d_2(\tau_{s,t}(\pi_l(m)),\tau_{s,t}(\pi_{l'}(m'))) - d_1(\tau_{s,t}(\pi_{l'}(m')),\tau_{s,t}(\pi_l(m)))$$

And so we have from lemma 5.7 :

$\alpha = 2\sum_{l'>l}(d_2(\pi_l(m),\pi_{l'}(m')) - d_1(\pi_{l'}(m'),\pi_l(m)))$
$-\, 2\sum_{l'>l, r\in\mathbb{Z}}(d_2(\pi_l(m)[rs],\pi_{l'}(m')) - d_1(\pi_{l'}(m'),\pi_l(m)[rs]))$
$= -2\sum_{l'>l, r\neq 0}(d_2(\pi_l(m)[rs],\pi_{l'}(m')) - d_1(\pi_{l'}(m'),\pi_l(m)[rs]))$

But we have $\pi_{l'}(m')$ of the form $A_{i,l'}^{-v_{i,l'}}$, and so :

$\alpha = -2\sum_{l'>l, r\neq 0} v_{i,l'}(m')(u_{i,l'+r_i}(\pi_l(m)[rs]) - u_{i,l'-r_i}(\pi_l(m)[rs]))$
$= -2\sum_{l'\in\mathbb{Z}} v_{i,l'}(m')\sum_{l<l', r\neq 0}(u_{i,l'+r_i-rs} - u_{i,l'-r_i-rs})(\pi_l(m))$
$= -2\sum_{l'\in\mathbb{Z}} v_{i,l'}(m')\sum_{r\neq 0}(u_{i,l'+r_i-rs} - u_{i,l'-r_i-rs})(\pi_{l'-1}(m)\pi_{l'-2}(m)...)$

We use lemma 5.9 :

$\alpha = -2\sum_{l'\in\mathbb{Z}} v_{i,l'}(m')\sum_{r>0}(u_{i,l'+r_i-rs} - u_{i,l'-r_i-rs})(\pi_{l'-1}(m)\pi_{l'-2}(m)...)$
$= -2\sum_{l'\in\mathbb{Z}} v_{i,l'}(m')\sum_{r>0}(u_{i,l'+r_i-rs} - u_{i,l'-r_i-rs})(m)$

It depends only of the $u_{i,l}(m)$, so with the same computation we get :

$$\beta = -2\sum_{l'\in\mathbb{Z}} v_{i,l'}(m')\sum_{r>0}(u_{i,l'+r_i-rs} - u_{i,l'-r_i-rs})(\hat{m}^i)$$

and we can conclude $\alpha = \beta$ because for all $l \in \mathbb{Z}$, $u_{i,l}(m) = u_{i,l}(\hat{m}^i)$. □

5.4 Description of $\chi_{\epsilon,t}$

Recall the map p_s of section 3.2.2 :

Theorem 5.11. *If* $\chi_{q,t}(\prod_{i\in I, 0\leq l\leq s-1} X_{i,l}^{x_{i,[l]}}) = \sum_{m\in A_t}\lambda_m(t)m$, *then :*

$$\chi_{\epsilon,t}(\prod_{i\in I, 0\leq l\leq s-1} X_{i,[l]}^{x_{i,l}}) = \sum_{m\in A_t}\lambda_m(t)t^{D_1^-(m)+D_2^-(m)}p_s(m)$$

where for m a $\hat{\mathfrak{Y}}_t$-monomial :

$$D_1^-(m) = \sum_{k<0} d_1(m, m[ks]) \ ,\ D_2^-(m) = \sum_{k<0} d_2(m, m[ks])$$

In this section we prove this theorem (consequence of lemma 5.12 and proposition 5.13). Note that this result is a generalization of the axiom 4 of [N3] to the non necessarily finite simply laced case. In particular our construction fits with [N3] in the ADE-case (where $D_1^- = D_2^-$).

Lemma 5.12. *For m a $\hat{\mathcal{Y}}_t$-monomial we have $t^{\gamma}\overrightarrow{m} \in A_t$ and $t^{-\gamma-2d_1(m,m)}\overleftarrow{m} \in A_t$ where :*

$$\gamma = \sum_{l\in\mathbb{Z}}(\sum_{i\in I}v_{i,l}^2(m) - \sum_{i,j/C_{i,j}+r_i=-1}v_{i,l}(m)v_{j,l}(m)$$

$$-\sum_{i,j/C_{i,j}=-3 \text{ and } r_i=1}v_{i,l}(m)(v_{j,l+1}(m)+v_{j,l-1}(m)))$$

Proof : We have $\overline{\overrightarrow{m}} = \overleftarrow{m} = t^{2\beta}\overrightarrow{m}$ where :

$$\beta = \sum_{l>l'}d_1(\pi_l(m),\pi_{l'}(m)) - d_2(\pi_{l'}(m),\pi_l(m))$$

$$= d_1(m,m) - \sum_{l\in\mathbb{Z}}d_1(\pi_l(m),\pi_l(m)) - \sum_{l<l'}d_1(\pi_l(m),\pi_{l'}(m)) + d_2(\pi_l(m),\pi_{l'}(m))$$

So $\overline{t^{\gamma}\overrightarrow{m}} = t^{2d_1(m,m)}t^{\gamma}\overrightarrow{m}$ where :

$$\gamma = -\sum_{l\in\mathbb{Z}}d_1(\pi_l(m),\pi_l(m)) - \sum_{l<l'}d_1(\pi_l(m),\pi_{l'}(m)) + d_2(\pi_l(m),\pi_{l'}(m))$$

But for $l\in\mathbb{Z}$, $d_1(\pi_l(m),\pi_l(m))$ is equal to :

$$-\sum_{i\in I}v_{i,l}^2(m) + \sum_{i,j/C_{i,j}=-2 \text{ and } r_i=-1}v_{i,l}(m)v_{j,l}(m) + \sum_{i,j/C_{i,j}=-3 \text{ and } r_i=2}v_{i,l}(m)v_{j,l}(m)$$

$$= -\sum_{i\in I}v_{i,l}^2(m) + \sum_{i,j/C_{i,j}+r_i=-1}v_{i,l}(m)v_{j,l}(m)$$

For $l<l'$ we have :

$$d_1(\pi_l(m),\pi_{l'}(m)) = \delta_{l'=l+1}\sum_{i,j/C_{i,j}=-3 \text{ and } r_i=1}v_{i,l}(m)v_{j,l+1}(m)$$

$$d_2(\pi_l(m),\pi_{l'}(m)) = \delta_{l'=l+1}\sum_{i,j/C_{i,j}=-3 \text{ and } r_i=1}v_{i,l+1}(m)v_{j,l}(m)$$

and we get for γ the annonced value.
For the second point we show that $t^{-\gamma-2d_1(m,m)}\overleftarrow{m}\in A_t$:

$$\overline{t^{-\gamma-2d_1(m,m)}\overleftarrow{m}} = t^{\gamma+2d_1(m,m)}\overline{\overleftarrow{m}} = t^{\gamma+2d_1(m,m)-2\beta}\overleftarrow{m} = t^{-\gamma}\overleftarrow{m} = t^{2d_1(m,m)}(t^{-\gamma-2d_1(m,m)}\overleftarrow{m})$$

□

Proposition 5.13. *For $m\in A_t$ we have :*

$$\tau_{s,t}(m) = t^{D_1^-(m)+D_2^-(m)}p_s(m)$$

Proof : Using lemma 5.12 we can write $m = t^{-\gamma - 2d_1(m,m)}\overleftarrow{m}$. So we have :

$$\tau_{s,t}(m) = t^{-\gamma-2d_1(m,m)}\overleftarrow{\prod_{l\in\mathbb{Z}}}\tau_{s,t}(\pi_l)$$

where $\pi_l = \pi_l(m)$. So we have $\overline{\tau_{s,t}(m)} = t^{2\alpha}\tau_{s,t}(m)$ where :

$$\alpha = \gamma + 2d_1(m,m) + \sum_{l<l'} d_1(\tau_{s,t}(\pi_l), \tau_{s,t}(\pi_{l'})) - d_2(\tau_{s,t}(\pi_{l'}), \tau_{s,t}(\pi_l))$$

$$= \gamma + 2d_1(m,m) + \sum_{l<l'} d_1(\tau_{s,t}(\pi_l), \tau_{s,t}(\pi_{l'})) - d_2(\tau_{s,t}(\pi_{l'}), \tau_{s,t}(\pi_l))$$

So it suffices to show that $\alpha = -D_1^-(m) - D_2^-(m) + d_1(p_s(m), p_s(m))$. But we have :

$$d_1(p_s(m), p_s(m)) = \sum_{l<l'} d_1(\tau_{s,t}(\pi_l), \tau_{s,t}(\pi_{l'})) + \sum_{l\geq l'} d_1(\tau_{s,t}(\pi_l), \tau_{s,t}(\pi_{l'}))$$

So we want to show :

$$-d_2(\tau_{s,t}(\pi_{l'}), \tau_{s,t}(\pi_l))$$
$$= \sum_{l\geq l'} d_1(\tau_{s,t}(\pi_l), \tau_{s,t}(\pi_{l'})) - (d_1(m,m) + D_1^-(m)) - (d_2(m,m) + D_2^-(m)) - \gamma$$

The second term is :

$$\sum_{l\geq l', r\in\mathbb{Z}} d_1(\pi_l, \pi_{l'}[rs]) - \sum_{l,l'\in\mathbb{Z}, r\leq 0} (d_1(\pi_l, \pi_{l'}[rs]) + d_2(\pi_l, \pi_{l'}[rs]))$$

$$+ \sum_{l\in\mathbb{Z}} d_2(\pi_l, \pi_l) + \sum_{l<l'} (d_1(\pi_l, \pi_{l'}) + d_2(\pi_l, \pi_{l'}))$$

But for $l < l'$ and $r < 0$ (resp. $l \geq l'$ and $r > 0$) we have $d_1(\pi_l, \pi_{l'}[rs]) = d_2(\pi_l, \pi_{l'}[rs]) = 0$. So this term is :

$$\sum_{l\geq l', r\leq 0} d_1(\pi_l, \pi_{l'}[rs]) - \sum_{l\geq l', r\leq 0} (d_1(\pi_l, \pi_{l'}[rs]) + d_2(\pi_l, \pi_{l'}[rs])) + \sum_{l\in\mathbb{Z}} d_2(\pi_l, \pi_l)$$

$$= -\sum_{l>l', r\leq 0} d_2(\pi_l, \pi_{l'}[rs]) = -\sum_{l>l', r\in\mathbb{Z}} d_2(\pi_l, \pi_{l'}[rs]) = -\sum_{l>l'} d_2(\tau_{s,t}(\pi_l), \tau_{s,t}(\pi_{l'}))$$

□

6 Applications

In this section we see how we can generalize at roots of unity results of [He2] about Kazhdan-Lusztig polynomials and quantization of the Grothendieck ring. We suppose that $i \neq j \Rightarrow C_{i,j}C_{j,i} \leq 3$ (C is not necessarily finite) and that $s = 0$ or ($s > 2r^\vee$ and $B(z)$ is symmetric). Such constructions were made by Nakajima [N3] in the finite simply laced case.

6.1 Analogs of Kazhdan-Lusztig polynomials

The involution of $\hat{\mathcal{Y}}^s_t$ is extended to an involution of $\hat{\mathcal{Y}}^{s,\infty}_t$. For $m \in B_t$ a dominant $\hat{\mathcal{Y}}_t$-monomial we set :

$$\overrightarrow{E}_t(m) = m(\overrightarrow{\prod_{l\in\mathbb{Z}}\prod_{i\in I}} Y_{i,l}^{u_{i,l}(m)})^{-1}\overrightarrow{\prod_{l\in\mathbb{Z}}\prod_{i\in I}} F_t(Y_{i,l})^{u_{i,l}(m)}$$

For $m \in B^s_t$ we define $\overrightarrow{E}_t(m) = m(\tau_{s,t}(M))^{-1}\tau_{s,t}(\overrightarrow{E}_t(M))$ where $M = \prod_{i\in I, l=0\dots s} Y_{i,l}^{u_{i,[l]}(m)}$. It follows from the proposition 5.10 that $\overrightarrow{E}_t(m) \in \hat{\mathfrak{K}}^{s,\infty}_t$.

Let $A^{s,\mathrm{inv}}_t = \{t^{\alpha(m)}m/m \in A^s_t\}$ and $B^{s,\mathrm{inv}}_t = \{t^{\alpha(m)}m/m \in B^s_t\}$ where $\alpha(m)$ is defined by $\overline{t^\alpha m} = t^{\alpha(m)}m$ (see the proof of lemma 6.8 of [He2]).

The following theorem was first given in [N3] for the ADE-case and in [He2] for the generic finite case.

Theorem 6.1. *For $m \in B^{s,inv}_t$ there is a unique $L^s_t(m) \in \hat{\mathfrak{K}}^{s,\infty}_t$ of the form $L^s_t(m) = m + \sum_{m'<m} \mu_{m',m}(t)m'$ such that $\overline{L^s_t(m)} = L^s_t(m)$ and :*

$$\overrightarrow{E}_t(m) = L^s_t(m) + \sum_{m'<m} P^s_{m',m}(t) L^s_t(m')$$

where $P^s_{m',m}(t) \in t^{-1}\mathbb{Z}[t^{-1}]$. Moreover we have :

$$\hat{\Pi}(m) = \hat{\Pi}(m') \Rightarrow m^{-1}L^s_t(m) = m'^{-1}L^s_t(m')$$

In this section we prove this result and study possible interpretation. In finite cases the proof is essentially in [He2]. In the cases $s \neq 1$ and C non finite we have to give another proof.

6.1.1 Proof of theorem 6.1

i) If C is finite : it follows from the lemma 3.9 (i) that for $m \in B^s_t$, the set $\{m' \leq m\} \cap B^s_t$ is finite. In particular $\overrightarrow{E}_t(m)$ and a product $\overrightarrow{E}_t(m)\overrightarrow{E}_t(m')$ have a finite number of dominant monomial. Moreover we see as in theorem 3.8 of [N3] that $\sum_{m\in B^s_t} \mathbb{Z}[t^\pm]\overrightarrow{E}_t(m)$ is stable by the multiplication and the involution. So we can apply the proof of theorem 6.9 of [He2].

But in general (C non necessarily finite) an infinite number of dominant monomials can appear in the q,t-character : let us briefly explain it for the example of section 3.2.3. We consider the case C of type $A^{(1)}_2$ and $s = 3$. We have the following subgraph in the q-character given by the classical algorithm :

$$Y_{1,0} \to Y^{-1}_{1,2}Y_{2,1}Y_{3,1} \to Y_{3,2}Y_{3,1}Y^{-1}_{2,3} \to Y^{-1}_{3,4}Y_{3,1}Y_{1,0}$$

But at $s=3$ we have $Y_{3,4}^{-1}Y_{3,1}Y_{1,0}\simeq Y_{1,0}$. So we have a periodic chain and an infinity of dominant monomials in $\tau_{s,t}(F_t(Y_{1,0}))$. So we have to give another proof :

ii) General case : for $m\in B_t^{s,\text{inv}}$ and $k\geq 0$ we denote by $B_{t,k}^s(m)\subset B_t^{s,\text{inv}}$ the set of dominant monomials of the form $m'=t^\alpha mA_{i_1,l_1}^{-1}...A_{i_k,l_k}^{-1}$. We set also $B^s(m)=\bigcup_{k\geq 0}B_k^s(m)$.

It will be useful to construct the element $F_t^s(m)\in\hat{\mathfrak{K}}_t^{s,\infty}$ with a unique dominant monomial m : we denote by $m_0=m>m_1>m_2>...$ the dominant monomials appearing in $\overrightarrow{E}_t(m)$ with a total ordering compatible with the partial ordering and the degree (the set is countable because there is a finite number of monomials of degree $-k$ in $\overrightarrow{E}_t(m)$). We define $\lambda_k(t)\in\mathbb{Z}[t^\pm]$ inductively as the coefficient of m_k in $\overrightarrow{E}_t(m)-\sum_{1\leq l\leq k-1}\lambda_l(t)\overrightarrow{E}_t(m_l)$. We define :

$$F_t^s(m)=\overrightarrow{E}_t(m)-\sum_{l\geq 1}\lambda_l(t)\overrightarrow{E}_t(m_l)\in\hat{\mathfrak{K}}_t^{s,\infty}$$

(this infinite sum is allowed in $\hat{\mathfrak{K}}_t^{s,\infty}$). The unique dominant monomial of $F_t^s(m)$ is m. In particular $\overline{F_t^s(m)}=F_t^s(m)$ (because as for the finite case the set of (infinite) sums of $\overrightarrow{E}_t(m)$ in $\hat{\mathfrak{K}}_t^{s,\infty}$ is stable by the involution).

We aim at defining in a unique way the $\mu_{m'm}(t)\in\mathbb{Z}[t^\pm]$ such that :

$$L_t^s(m)=\sum_{m'\in B_t^s(m)}\mu_{m',m}(t)F_t(m')$$

The condition $\overline{L_t^s(m)}=L_t^s(m)$ means that $\mu_{m',m}(t^{-1})=\mu_{m',m}(t)$.

We define by induction on $k\geq 0$, for $m'\in B_{t,k}^s(m)$ the $P_{m',m}^s(t)$ and the $\mu_{m',m}(t)$ such that :

$$\hat{E}_t(m)-\sum_{k\geq l\geq 0,m'\in B_{t,l}^s(m)}P_{m',m}^s(t)\sum_{k\geq r\geq 0,m''\in B_{t,r}^s(m')}\mu_{m'',m'}(t)F_t(m'')$$

$$\in\sum_{m'\in B_{t,k+1}^s(m)}(\mu_{m',m}(t)+P_{m',m}^s(t))F_t(m')+\sum_{l>k+1,m'\in B_{t,l}^s(m')}\mathbb{Z}[t^\pm]F_t(m')$$

For $k=0$ we have $P_{m,m}^s(t)=\mu_{m,m}(t)=1$. And the the equation determines uniquely $P_{m',m}^s(t)\in t^{-1}\mathbb{Z}[t^{-1}]$ and $\mu_{m',m}(t)\in\mathbb{Z}[t^\pm]$ such that $\mu_{m',m}(t)=\mu_{m',m}(t^{-1})$.

For the last point of the theorem we see also by induction on k that for $m_1,m_2\in B_t^{s,\text{inv}}$ such that $\hat{\Pi}(m_1)=\hat{\Pi}(m_2)$ and $m_1'\in B_t^s(m_1)$, $m_2'\in B_t^s(m_2)$ such that $m_1^{-1}m_1'\in t^{\mathbb{Z}}m_2^{-1}m_2'$ we have :

$$\mu_{m_1',m_1}(t)=\mu_{m_2',m_2}(t)\ ,\ P_{m_1',m_1}^s(t)=P_{m_2',m_2}^s(t)$$

6.1.2 Conjecture in finite case

In the following example we suppose that we are in the sl_2-case and we study the decomposition with $m_0=Y_0Y_1Y_2$.

If $s=0$, let $m_1 = t^2 Y_0 A_1^{-1} Y_1 Y_2$. We have :

$$\overrightarrow{E}_t(m_0) = L_t(m_0) + t^{-1} L_t(m_1)$$

$$L_t(m_0) = m_0(1 + tA_3^{-1}(1 + tA_1^{-1}))(1 + tA_2^{-1}) \ , \ L_t(m_1) = m_1(1 + tA_2^{-1})$$

If $s=3$, let $m_1 = t^2 Y_0 A_1^{-1} Y_1 Y_2$, $m_2 = Y_0 Y_1 A_2^{-1} Y_2$, $m_3 = Y_0 Y_1 Y_2 A_3^{-1}$. We have :

$$\overrightarrow{E}_t(m_0) = L_t^s(m_0) + t^{-1} L_t^s(m_1) + t^{-1} L_t^s(m_2) + t^{-1} L_t^s(m_3)$$
$$L_t^s(m_0) = m_0 + A_1^{-1} Y_2 A_3^{-1} Y_4 A_5^{-1}$$
$$L_t^s(m_1) = m_1(1 + A_3^{-1}) \ , \ L_t^s(m_2) = m_2(1 + tA_2^{-1}) \ , \ L_t^s(m_3) = m_3(1 + tA_2^{-1})$$

In particular we see in this example that the decomposition of $\overrightarrow{E}_t(m)$ in general is not necessarily the same if $s=0$ or $s \neq 0$.

We recall that irreducible representations of $\mathcal{U}_q(\hat{\mathfrak{g}})$ (resp. $\mathcal{U}_\epsilon^{\mathrm{res}}(\hat{\mathfrak{g}})$) are classified by dominant monomials of $\mathcal{Y}$ (resp. $\mathcal{Y}^s$) or by Drinfel'd polynomials (see [CP3], [CP5], [FR3], [FM2]).

For $m \in B$ (resp. $m \in B^s$) we denote by $V_m^0 = V_m \in \mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ (resp. $V_m^s \in \mathrm{Rep}(\mathcal{U}_\epsilon^{\mathrm{res}}(\hat{\mathfrak{g}}))$) the irreducible module of highest weight m. In particular for $i \in I, l \in \mathbb{Z}/s\mathbb{Z}$ let $V_{i,l}^s = V_{Y_{i,l}}$. The simple modules $V_{i,l}^s$ are called fundamental representations. In the ring Rep^s it is denoted by $X_{i,l}$.

For $m \in B$ (resp. $m \in B^s$) we denote by $M_m^s \in \mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ (resp. $M_m^s \in \mathrm{Rep}(\mathcal{U}_\epsilon^{\mathrm{res}}(\hat{\mathfrak{g}}))$) the module $M_m^s = \bigotimes_{i \in I, l \in \mathbb{Z}/s\mathbb{Z}} V_{i,l}^{s, \otimes u_{i,l}(m)}$. It is called a standard module and in Rep^s it is denoted by $\prod_{i \in I, l \in \mathbb{Z}/s\mathbb{Z}} X_{i,l}^{u_{i,l}(m)}$.

The irreducible $\mathcal{U}_q(\hat{sl}_2)$-representation with highest weight m is $V_m = V_{Y_0 Y_2} \otimes V_{Y_1}$ (see [CP3] or [FR3]). In particular $\dim(V_m) = 6$, that is to say the number of monomials of $L_t(m)$.

For ϵ such that $s=3$, the irreducible $\mathcal{U}_\epsilon^{\mathrm{res}}(\hat{\mathfrak{g}})$-representation with highest weight m is V_m^s the pull back by the Frobenius morphism of the $\overline{\mathcal{U}}(\hat{sl}_2)$-module $\overline{V}$ of Drinfel'd polynomial $(1-u)$ (see [CP5] or [FM2]). In particular $\dim(V_m^s) = 2$, that is to say the number of monomials of $L_t^s(m)$.

Those observations would be explained by the following conjecture which is a generalization of the conjecture 7.3 of [He2] to the root of unity case. We know from [N3] that the result is true in the simply laced case (in particular in the last example).

For $m = \prod_{i \in I, l \in \mathbb{Z}/s\mathbb{Z}} Y_{i,l}^{u_{i,l}}$ a dominant $\mathcal{Y}^s$-monomial let $M = \prod_{i \in I, l \in \mathbb{Z}/s\mathbb{Z}} Y_{i,l}^{u_{i,l}} \in \hat{\mathcal{Y}}_t^s$. We suppose that C is finite.

Conjecture 6.2. *For m a dominant $\mathcal{Y}_s$-monomial, $\hat{\Pi}(L_t^s(M))$ is the ϵ-character of the irreducible $\mathcal{U}_\epsilon^{res}(\hat{\mathfrak{g}})$-representation V_m^s associated to m. In particular for m' another dominant $\mathcal{Y}_s$-monomials the multiplicity of $V_{m'}^s$ in the standard module M_m^s associated to m*

is :

$$\sum_{m''\in B_t^{s,inv}/\hat{\Pi}(m'')=m'} P^s_{m'',M}(1)$$

Let us look at an application of the conjecture in the non-simply laced case : we suppose that $C=\begin{pmatrix}2 & -2\\ -1 & 2\end{pmatrix}$ and $m=Y_{1,0}Y_{1,1}$.

First we suppose that $s=0$. The formulas for $F_t(Y_{1,0})$ and $F_t(Y_{1,1})$ are given in [He2] :

$$F_t(Y_{1,0})=Y_{1,0}(1+tA_{1,1}^{-1}(1+tA_{2,3}^{-1}(1+tA_{1,5}^{-1})))$$

$$F_t(Y_{1,1})=Y_{1,1}(1+tA_{1,2}^{-1}(1+tA_{2,4}^{-1}(1+tA_{1,6}^{-1})))$$

The product $F_t(Y_{1,0})F_t(Y_{1,1})$ has a unique dominant monomial $Y_{1,0}Y_{2,0}$, so :

$$\overrightarrow{E}_t(Y_{1,0}Y_{2,0})=F_t(Y_{1,0}Y_{2,0})=L_t(Y_{1,0}Y_{2,0})=F_t(Y_{1,0})F_t(Y_{1,1})$$

In particular the $V_{1,0}\otimes V_{1,1}$ is irreducible (also a consequence of classical theory of q-characters).

We suppose now that $s=5>4=2r^\vee$.

$$\tau_{s,t}(\overrightarrow{E}_t(Y_{1,0}Y_{1,1}))=L_t^s(Y_{1,0}Y_{1,1})+t^{-1}L_t(1)$$

where $L_t(1)=1$. So if the conjecture is true, at $s=5$ the $V^s_{1,0}\otimes V^s_{1,1}$ is not irreducible and contains the trivial representation with multiplicity one.

6.1.3 Non finite cases

Let us look at an example : we suppose that C is of type $A_2^{(1)}$. In the generic case, the classical algorithm gives the q-characters beginning with $Y_{1,0}$, and the first terms are :

$$\begin{tikzcd}
 & Y_{1,0} \arrow[d, "{1,1}"] & \\
 & Y_{1,2}^{-1}Y_{3,1}Y_{2,1} \arrow[dl, "{3,2}"'] \arrow[dr, "{2,2}"] & \\
Y_{2,0}^{-1}Y_{3,1}Y_{3,2} & & Y_{3,3}^{-1}Y_{2,1}Y_{2,2}
\end{tikzcd}$$

The deformed algorithm gives :

$$\overrightarrow{E}_t(Y_{1,0})=Y_{1,0}(1+tA_{1,1}^{-1}(1+tA_{2,2}^{-1}+tA_{3,2}^{-1}))+\text{terms of lower degree}$$

We suppose now that $s=3$. First let $m=Y_{1,0}Y_{1,2}$, $m'=t^2Y_{1,0}A_{1,1}^{-1}Y_{1,2}$. We have :

$$\overrightarrow{E}_t(m)=F_t(m)+t^{-1}F_t(m')+...$$

In particular $P^s_{m',m}(t)=t^{-1}$.

Let $m=Y_{2,1}Y_{3,1}$, $m'=tY_{2,1}Y_{3,1}A^{-1}_{3,2}A^{-1}_{2,3}$, $m''=tY_{2,1}Y_{3,1}A^{-1}_{2,2}A^{-1}_{3,3}$. We have :

$$\overrightarrow{E}_t(m)=F_t(m)+t^{-1}F_t(m')+t^{-1}F_t(m'')+...$$

In particular $P^s_{m',m}(t)=t^{-1}$ and $P^s_{m'',m}(t)=t^{-1}$.

Let us go back to general case : for $m,m'\in B^s$ we want to define $P^s_{m',m}(t)$. We can not set as in the finite case $P^s_{m',m}(t)=\sum_{M'\in B^s_t(M)/\hat{\Pi}(M')=m'}P_{M',M}(t)$ (where $M\in B^{s,\text{inv}}_t$ verifies $\hat{\Pi}(M)=m$) because this sum is not finite in general.

However we propose the following construction. For $m,m'\in B^s$, we define $k(m,m')\geq 0$ such that for $M\in\hat{\Pi}^{-1}(m)$ we have $k(m,m')=\min\{k\geq 0/\exists M'\in B^s_{t,k}(M),\hat{\Pi}(M')=m'\}$.

Definition 6.3. *For $m,m'\in B^s$ we define $P^s_{m',m}(t)\in\mathbb{Z}[t^{\pm}]$ by :*

$$P^s_{m',m}(t)=\sum_{M'\in B^s_t(M)/\hat{\Pi}(M')=m' \text{ and } deg(M')=deg(M)-k(m,m')}P_{M',M}(t)$$

where M an element of $B^{s,inv}_t\cap\hat{\Pi}^{-1}(m)$.

Note that if C affine it follows from lemma 3.9 (iii) that for each $m\in B^s$, there is a finite number of $m'\in B^s$ such that $P^s_{m',m}(t)\neq 0$. In particular in this situation the proof of the theorem gives an algorithm to compute the polynomials with a finite number of steps (although there could be an infinite number of monomials in the ϵ,t-character).

For example if C is of type $A^{(1)}_2$ and $s=3$ we have :

$$P_{Y_{3,1}Y_{2,1},Y_{1,0}Y_{1,2}}(t)=t^{-1}\ ,\ P_{Y_{1,0},Y_{1,2},Y_{3,1}Y_{2,1}}=2t^{-1}$$

6.2 Quantization of the Grothendieck ring

6.2.1 General quantization

We set $\text{Rep}^s_t=\text{Rep}^s\otimes\mathbb{Z}[t^{\pm}]=\mathbb{Z}[X_{i,l},t^{\pm}]_{i\in I,l\in\mathbb{Z}/s\mathbb{Z}}$ and we extend $\chi_{\epsilon,t}$ to a $\mathbb{Z}[t^{\pm}]$-linear injective map $\chi_{\epsilon,t}:\text{Rep}^s_t\rightarrow\hat{\mathfrak{K}}^{\infty,s}_t$. We set $B^s=\{m=\prod_{i\in I,l\in\mathbb{Z}/s\mathbb{Z}}Y^{u_{i,l}(m)}_{i,l}\}\subset B^s_t$. We have a map $\pi:B^s_t\rightarrow B^s$ defined by $\pi(m)=\prod_{i\in I,l\in\mathbb{Z}/s\mathbb{Z}}Y^{u_{i,l}(m)}_{i,l}$.

We have :

$$\text{Im}(\chi_{\epsilon,t})=\bigoplus_{m\in B^s}\mathbb{Z}[t^{\pm}]\overrightarrow{E}_t(m)\subset\hat{\mathfrak{K}}^{s,\infty}_t$$

But in general $\text{Im}(\chi_{\epsilon,t})$ is not a subalgebra of $\hat{\mathfrak{K}}^{s,\infty}_t$.

If $s = 0$ or C is finite we have $\mathrm{Im}(\chi_{\epsilon,t}) \subset \bigoplus_{m \in B_t^s} \mathbb{Z}[t^{\pm}]\overrightarrow{E}_t(m)$ (subalgebra of $\hat{\mathfrak{K}}_t^{s,\infty}$) and we have a $\mathbb{Z}[t^{\pm}]$-linear map $\pi : \bigoplus_{m \in B_t^s} \mathbb{Z}[t^{\pm}]\overrightarrow{E}_t(m) \to \mathrm{Im}(\chi_{\epsilon,t})$ such that for $m \in B_t^s$:

$$\pi(\overrightarrow{E}_t(m)) = \overrightarrow{E}_t(\pi(m))$$

If $s > 2r^{\vee}$ and C verifies the property of (ii) lemma 3.9 (for example C is affine) then there is a $\mathbb{Z}[t^{\pm}]$-linear map $\pi : \mathrm{Im}(\chi_{\epsilon,t})\mathrm{Im}(\chi_{\epsilon,t}) \to \mathrm{Im}(\chi_{\epsilon,t})$ such that for $m \in B_t^s$ of the form $m = Mm'$ where $M \in B^s$ and $m' = t^{\alpha}A_{i_1,l_1}^{-1}...A_{i_k,l_k}^{-1}$ (see the definition of $k(m_1, m_2) \in \mathbb{Z}$ in section 6.1.3) :

$$\pi(\overrightarrow{E}_t(m)) = \overrightarrow{E}_t(\pi(m)) \text{ if } k = k(\hat{\Pi}(m), \hat{\Pi}(M))$$
$$\pi(\overrightarrow{E}_t(m)) = 0 \text{ if } k > k(\hat{\Pi}(m), \hat{\Pi}(M))$$

Indeed the elements of $\mathrm{Im}(\chi_{\epsilon,t})\mathrm{Im}(\chi_{\epsilon,t})$ are infinite sums of $\overrightarrow{E}_t(m)$.

In both cases, as $\chi_{\epsilon,t}$ is injective, we can define a $\mathbb{Z}[t^{\pm}]$-bilinear map $*$ such that for $\alpha, \beta \in \mathrm{Rep}_t^s$:

$$\alpha * \beta = \chi_{\epsilon,t}^{-1}(\pi(\chi_{\epsilon,t}(\alpha)\chi_{\epsilon,t}(\beta)))$$

This is a deformed multiplication on Rep_t^s. But in general this multiplication is not associative.

6.2.2 Associative quantization

In some cases it is possible to define an associative quantization (see [VV3], [N3], [He2]). The point is to use a t-deformed algebra $\mathcal{Y}_t = \mathbb{Z}[Y_{i,l}^{\pm}, t^{\pm}]_{i \in I, l \in \mathbb{Z}}$ instead of $\hat{\mathcal{Y}}_t$: in this case $\mathrm{Im}(\chi_{q,t})$ is an algebra and we have an associative quantization of the Grothendieck ring (see [He2] for details). In this section we see how this construction can be generalized to other Cartan matrices. We suppose that $s = 0$ and that q is transcendental.

Lemma 6.4. *Let C be a Cartan matrix such that :*

$$C_{i,j} < -1 \Rightarrow -C_{j,i} \leq r_i$$

Then :

$$det(C(z)) = z^{-R} + \alpha_{-R+1}z^{-R+1} + ... + \alpha_{R-1}z^{R-1} + z^R$$

where $R = \sum_{i=1...n} r_i$ and $\alpha(-l) = \alpha(l) \in \mathbb{Z}$.

In particular finite and affine Cartan matrices ($A_1^{(1)}$ with $r_1 = r_2 = 2$) verify the property of lemma 6.4. Note that the condition $C_{i,j} < 0 \Rightarrow C_{i,j} = -1$ or $C_{j,i} = -1$ is sufficient ; in particular Cartan matrices such that $i \neq j \Rightarrow C_{i,j}C_{j,i} \leq 3$ verify the property.

Proof : For $\sigma \in S_n$ let us look at the term $\det_{\sigma} = \prod_{i \in I} C_{i,\sigma(i)}(z)$ of $\det(C(z))$. If $\sigma = \mathrm{Id}$ then the degree $\deg(\det_{\mathrm{Id}})$ is $\sum_{i \in I} r_i$. So it suffices to show that for $\sigma \neq \mathrm{Id}$ we have $\deg(\det_{\sigma}) < \sum_{i \in I} r_i$. If $i \neq \sigma(i)$, we have the following cases :

if $C_{i,\sigma(i)} = 0$ or -1, $\deg([C_{i,\sigma(i)}]_z) \leq 0 < r_{\sigma(i)}$

if $C_{i,\sigma(i)} < -1$, we have $C_{\sigma(i),i} = -1$ and so $r_i C_{i,\sigma(i)} = -r_{\sigma(i)}$ and so

$$\deg([C_{i,\sigma(i)}]_z) = -C_{i,\sigma(i)} - 1 = -\frac{r_{\sigma(i)} C_{\sigma(i),i}}{r_i} - 1 \leq r_{\sigma(i)} - 1 < r_{\sigma(i)}$$

So if $\sigma \neq \mathrm{Id}$ we have :

$$\deg(\det_\sigma) = \sum_{i \in I / i = \sigma(i)} r_i + \sum_{i \in I / i \neq \sigma(i)} \deg([C_{i,\sigma(i)}]_{z_i}) < \sum_{i \in I / i = \sigma(i)} r_i + \sum_{i \in I / i \neq \sigma(i)} r_{\sigma(i)} = \sum_{i \in I} r_i$$

For the last point $\det(C(z))$ is symmetric polynomial because the coefficients of $C(z)$ are symmetric. □

We suppose in this section that C verifies the property of lemma 6.4.

In particular $\det(C(z)) \neq 0$ and $C(z)$ has an inverse $\tilde{C}(z)$ with coefficients of the form $\frac{P(z)}{Q(z^{-1})}$ where $P(z) \in \mathbb{Z}[z^{\pm}]$, $Q(z) \in \mathbb{Z}[z]$, $Q(0) = \pm 1$ and the dominant coefficient of Q is ± 1. We denote by $\mathfrak{V} \subset \mathbb{Z}((z^{-1}))$ the set of rational fractions of this form. Note that $\mathfrak{V}$ is a subring of $\mathbb{Q}(z)$, and for $R(z) \in \mathfrak{V}, m \in \mathbb{Z}$ we have $R(z^m) \in \mathfrak{V}$. In particular for $m \in \mathbb{Z} - \{0\}$, $\tilde{C}(q^m)$ makes sense.

We denote by $\mathbb{Z}((z^{-1}))$ the ring of series of the form $P = \sum_{r \leq R_P} P_r z^r$ where $R_P \in \mathbb{Z}$ and the coefficients $P_r \in \mathbb{Z}$. We have an embedding $\mathfrak{V} \subset \mathbb{Z}((z^{-1}))$ by expanding $\frac{1}{Q(z^{-1})}$ in $\mathbb{Z}[[z^{-1}]]$ for $Q(z) \in \mathbb{Z}[z]$ such that $Q(0) = 1$. So we can introduce maps $(\pi_r,\, r \in \mathbb{Z})$:

$$\pi_r : \mathfrak{V} \to \mathbb{Z} \ , \ P = \sum_{r \leq R_P} P_r z^r \mapsto P_r$$

We denote by $\mathcal{H}$ the algebra with generators $a_i[m], y_i[m], c_r$, relations 6, 7 (of definition 3.1) and $(j \in I, m \neq 0)$:

$$y_j[m] = \sum_{i \in I} \tilde{C}_{i,j}(q^m) a_i[m] \tag{9}$$

Note that the relations 9 are compatible with the relations 7.

We define $\mathcal{Y}_u$ as the subalgebra of $\mathcal{H}[[h]]$ generated by the $Y_{i,l}^{\pm}$, $A_{i,l}^{\pm}$ $(i \in I, l \in \mathbb{Z})$, t_R $(R \in \mathfrak{V})$ and $\mathcal{Y}_t$ in analogy to the definition of $\hat{\mathcal{Y}}_t$.

The following theorem is a generalization of theorem 3.11 of [He2] :

Theorem 6.5. *The algebra $\mathcal{Y}_t$ is defined by generators $Y_{i,l}^{\pm}$, $(i \in I, l \in \mathbb{Z})$ central elements $t^{\pm}$ and relations $(i, j \in I, k, l \in \mathbb{Z})$:*

$$Y_{i,l} Y_{j,k} = t^{\gamma(i,l,j,k)} Y_{j,k} Y_{i,l}$$

where $\gamma : (I \times \mathbb{Z})^2 \to \mathbb{Z}$ is given by :

$$\gamma(i,l,j,k) = \sum_{r \in \mathbb{Z}} \pi_r(\tilde{C}_{j,i}(z))(-\delta_{l-k,-r_j-r} - \delta_{l-k,r-r_j} + \delta_{l-k,r_j-r} + \delta_{l-k,r_j+r})$$

7 Complements

7.1 Finiteness of algorithms

In the construction of q,t and ϵ,t-character we deal with completed algebras $\hat{\mathcal{Y}}_t^{s,\infty}$, so the algorithms can produce an infinite number of monomials. In some cases we can say when this number is finite :

Definition 7.1. *We say that the classical algorithm stops if the classical algorithm is well defined and for all $m \in B$, $F(m) \in \mathfrak{K}$.*

It follows from the classical theory of q-characters that if C is finite then the classical algorithm stops.

Proposition 7.2. *We suppose that there are $(a_i)_{i\in I} \in \mathbb{Z}^I$ such that $a_i > 0$ and $\sum\limits_{j\in I} a_j C_{j,i} = 0$. Then the classical algorithm does not stop.*

In particular if C is an affine Cartan matrix then the classical algorithm does not stop.

Proof : It follows from lemma 4.3 at $t = 1$ that it suffices to show that there is no antidominant monomial in $C(Y_{1,0})$. So let $m = Y_{1,0} \prod\limits_{i\in I, l\in\mathbb{Z}} A_{i,l}^{-v_{i,l}}$ be in $C(Y_{1,0})$. We see as in lemma 3.9 (iii) that $u_i(Y_{1,0}^{-1}m) = 0$. In particular $u_1(m) = 1$ and m is not antidominant. $\square$

Proposition 7.3. *The following properties are equivalent :*

i) For all $i \in I$, $F_t(Y_{i,0}) \in \hat{\mathfrak{K}}_t$.

ii) For all $m \in B_t$, $F_t(m) \in \hat{\mathfrak{K}}_t$.

iii) $Im(\chi_{q,t}) \subset \hat{\mathfrak{K}}_t$.

Definition 7.4. *If the properties of the proposition 7.3 are verified we say that the deformed algorithm stops.*

Let us give some examples :

-If C is of type ADE then the deformed algorithm stops : [N3] (geometric proof) and [N4] (algebraic proof in AD cases)

-If C is of rank 2 ($A_1 \times A_1$, A_2, B_2, C_2, G_2) then the deformed algorithm stops : [He2] (algebraic proof)

-In [He2] we give an alternative algebraic proof for Cartan matrices of type A_n ($n \geq 1$) and we conjecture that for all finite Cartan matrices the deformed algorithm stops. The cases F_4, B_n, C_n ($n \leq 10$) have been checked on a computer (with the help of T. Schedler).

Lemma 7.5. *If the deformed algorithm stops then the classical algorithm stops.*

Proof : This is a consequence of the formula $F(\hat{\Pi}(m)) = \hat{\Pi}(F_t(m))$ (see section 4.1.3). $\square$

In particular if C is affine then the deformed algorithm does not stop.

Let C be a Cartan matrix such that $i \neq j \Rightarrow C_{i,j}C_{j,i} \leq 3$. We conjecture[5] that the deformed algorithm stops if and only if the classical algorithm stops.

7.2 q,t-characters of affine type and quantum toroidal algebras

We have seen in [He2] that if C is finite then the defining relations of $\hat{\mathcal{H}}$:

$$[a_i[m], a_j[r]] = \delta_{m,-r}(q^m - q^{-m})B_{i,j}(q^m)c_{|m|}$$

appear in the $\mathbb{C}$-subalgebra $\mathcal{U}_q(\hat{\mathfrak{h}})$ of $\mathcal{U}_q(\hat{\mathfrak{g}})$ generated by the $h_{i,m}, c^{\pm}$ ($i \in I, m \in \mathbb{Z} - \{0\}$) : it suffices to send $a_i[m]$ to $(q-q^{-1})h_{i,m}$ and c_r to $\frac{c^r - c^{-r}}{r}$.

In this section we see that in the affine case $A_n^{(1)}$ ($n \geq 2$) the relations of $\hat{\mathcal{H}}$ appear in the structure of the quantum toroidal algebra. In particular we hope that q,t-characters will play a role in representation theory of quantum toroidal algebras (see the introduction).

Let be $d \in \mathbb{C}^*$ and $n \geq 3$. In the quantum toroidal algebra of type sl_n there is a subalgebra $\mathcal{Z}$ generated by the $k_i^{\pm}, h_{i,l}$ ($i \in \{1, ..., n\}$, $l \in \mathbb{Z} - \{0\}$) with relations :

$$k_i k_i^{-1} = cc^{-1} = 1 \ , \ [k_{\pm,i}(z), k_{\pm,j}(w)] = 0$$

$$\theta_{-a_{i,j}}(c^2 d^{-m_{i,j}} w z^{-1}) k_{+,i}(z) k_{-,j}(w) = \theta_{-a_{i,j}}(c^{-2} d^{-m_{i,j}} w z^{-1}) k_{-,j} k_{+,i}(z) \qquad (10)$$

(see [VV1] for the definition of the $k_{\pm,i}(z) \in \mathcal{Z}[[z]]$, $\theta_m(z) \in \mathbb{C}[[z]]$, the matrices A and M). A computation gives :

Lemma 7.6. *The relation (10) are consequences of :*

$$[h_{i,l}, h_{j,m}] = \delta_{l,-m} \frac{q^{la_{i,j}} - q^{-la_{i,j}}}{(q-q^{-1})^2} d^{-|l| m_{i,j}} \frac{c^{2l} - c^{-2l}}{l}$$

In particular for $d = 1$, $a_i[m] = \frac{h_{i,m}}{q-q^{-1}}$ and $c_m = \frac{c^{2m} - c^{-2m}}{m}$, we get the defining relation of the Heisenberg algebra $\hat{\mathcal{H}}$ of section 3 in the affine case $A_{n-1}^{(1)}$:

$$[a_i[m], a_j[r]] = \delta_{m,-r}(q^m - q^{-m})[B_{i,j}]_{q^m} c_{|m|}$$

In the case $d \neq 1$ we have to extend the former construction : let us study the case $d \neq 1$: we suppose that q, d are indeterminate and we construct a t-deformation of $\mathbb{Z}[A_{i,l,k}^{\pm}]_{i \in I, l, k \in \mathbb{Z}}$.

We define the $\mathbb{C}[q^{\pm}, d^{\pm}]$-algebra $\hat{\mathcal{H}}_d$ by generators $a_i[m]$ ($i \in I = \{1, ..., n\}, m \in \mathbb{Z}$) and relations :

$$[a_i[m], a_j[r]] = \delta_{m,-r}(q^m - q^{-m})[A_{i,j}]_{q^m} d^{-|m| m_{i,j}} c_{|m|}$$

[5]This conjecture is proved later in [He5].

For $i \in I, l, k \in \mathbb{Z}$ and $R(q,d) \in \mathbb{Z}[q^{\pm}, d^{\pm}]$ we define :

$$A_{i,l,k} = \exp(\sum_{m>0} h^m q^{lm} d^{km} a_i[m]) \exp(\sum_{m>0} h^m q^{-lm} d^{-km} a_i[-m]) \in \hat{\mathcal{H}}_d[[h]]$$

$$t_{R(q,d)} = \exp(\sum_{m>0} h^m R(q^m, d^m) c_m) \in \hat{\mathcal{H}}_d[[h]]$$

A computation analogous to the proof of lemma 3.2 gives :

$$A_{i,l,p} A_{j,k,r} A_{i,l,p}^{-1} A_{j,k,r}^{-1} = t_{(q-q^{-1})[A_{i,j}]_q (-q^{l-k} d^{p-r} + q^{k-l} d^{r-p}) d^{-m_{i,j}}}$$

In particular, in the quotient of $\hat{\mathcal{H}}_d[[h]]$ by relations $t_R = 1$ if $R \neq 0$, we have :

$$A_{i,l,p}^{-1} A_{j,k,r}^{-1} = t^{\alpha(i,j,k,l,p,r)} A_{j,k,r}^{-1} A_{i,l,p}^{-1}$$

where $\alpha : (I \times \mathbb{Z} \times \mathbb{Z})^2 \to \mathbb{Z}$ is given by $(l, k \in \mathbb{Z},\ i, j \in I)$:

$$\alpha(i,i,l,k,p,r) = 2(\delta_{l-k,2r_i} - \delta_{l-k,-2r_i})\delta_{r,p}$$

$$\alpha(i,j,l,k,p,r) = \sum_{r=r_iC_{i,j}+1, r_iC_{i,j}+3, \ldots, -r_iC_{i,j}-1} (-\delta_{l-k,r+r_i}\delta_{p-r,m_{i,j}} + \delta_{l-k,r-r_i}\delta_{r-p,m_{i,j}}) \text{ (if } i \neq j)$$

In particular this would lead to the construction of q,t-characters with variables $Y_{i,l,p}$, $A_{i,l,p}^{-1}$ associated to quantum toroidal algebras. But we shall leave further discussion of this point to another place.

7.3 Combinatorics of bicharacters and Cartan matrices

In this section $C = (C_{i,j})_{1 \leq i,j \leq n}$ is an indecomposable generalized (non necessarily symmetrizable) Cartan matrix and $(r_1, \ldots, r_n)$ are positive integers. Let $D = \mathrm{diag}(r_1, \ldots, r_n)$ and $B = DC$ (which is non necessarily symmetric).

We show that the quantization of $\hat{\mathfrak{Y}}^s \otimes \mathbb{Z}[t^{\pm}] = \mathbb{Z}[Y_{i,l}, V_{i,l}, t^{\pm}]_{i \in I, l \in \mathbb{Z}/s\mathbb{Z}}$ is linked to fundamental combinatorial properties of C and $(r_1, \ldots, r_n)$ (propositions 7.9, 7.11, 7.12 and theorem 7.10). Let us begin with some general background about twisted multiplication defined by bicharacters.

7.3.1 Bicharacters and twisted multiplication

Let Λ be a set, Y be the commutative polynomial ring :

$$Y = \mathbb{Z}[X_\alpha, t^{\pm}]_{\alpha \in \Lambda}$$

and A the set of monomials of the form $m = \prod_{\alpha \in \Lambda} X_\alpha^{x_\alpha(m)} \in Y$. The usual commutative multiplication of Y is denoted by . in the following.

Definition 7.7. *A bicharacter on A is a map $d : A\times A \to \mathbb{Z}$ such that $(m_1, m_2, m_3 \in A)$:*

$$d(m_1.m_2, m_3) = d(m_1, m_3) + d(m_2, m_3)\ ,\ d(m_1, m_2.m_3) = d(m_1, m_2) + d(m_1, m_3)$$

The symmetric bicharacter $\mathfrak{S}d$ and the antisymmetric bicharacter $\mathfrak{A}d$ of d are defined by :

$$\mathfrak{S}d(m_1, m_2) = \frac{1}{2}(d(m_1, m_2) + d(m_2, m_1))\ ,\ \mathfrak{A}d(m_1, m_2) = \frac{1}{2}(d(m_1, m_2) - d(m_2, m_1))$$

and we have $d = \mathfrak{A}d + \mathfrak{S}d$.

Let be d be a bicharacter on A. One can define a $\mathbb{Z}[t^{\pm}]$-bilinear map $* : Y \times Y \to Y$ such that :

$$m_1 * m_2 = t^{d(m_1,m_2)} m_1.m_2$$

This map is associative[6] and we get a $\mathbb{Z}[t^{\pm}]$-algebra structure on Y. We say that the new multiplication is the twisted multiplication associated to the bicharacter d, and it is given by formulas :

$$m_1 * m_2 = t^{d(m_1,m_2)-d(m_2,m_1)} m_2 * m_1 = t^{2\mathfrak{A}d(m_1,m_2)} m_2 * m_1$$

Lemma 7.8. *Let d_1, d_2 be two bicharacters. One can define a multiplication on Y such that $(m_1, m_2 \in A)$:*

$$m_1 * m_2 = t^{2d_1(m_1,m_2)-2d_2(m_2,m_1)} m_2 * m_1$$

if and only if $\mathfrak{S}d_1 = \mathfrak{S}d_2$.

In this case, the multiplication is the twisted multiplication associated to the bicharacter $d = d_1 + d_2$:

$$m_1 * m_2 = t^{d_1(m_1,m_2)+d_2(m_1,m_2)} m_1.m_2$$

Proof : It follows immediately from the definition of $*$:

$$m_1 * m_2 = t^{2d_1(m_1,m_2)-2d_2(m_2,m_1)} m_2 * m_1 = t^{4(\mathfrak{S}d_1-\mathfrak{S}d_2)(m_1,m_2)} m_1 * m_2$$

If $\mathfrak{S}d_1 = \mathfrak{S}d_2$, let $*$ be the twisted multiplication associated with the bicharacter $d = d_1 + d_2$. We have :

$$m_1 * m_2 = t^{d_1(m_1,m_2)+d_2(m_1,m_2)-d_1(m_2,m_1)-d_2(m_2,m_1)} m_2 * m_1 = t^{2d_1(m_1,m_2)-2d_2(m_2,m_1)} m_2 * m_1$$

□

7.3.2 Bicharacters and symmetrizable Cartan matrices

Let $s \geq 0$ and A_t^s be the set of monomials of the form $m = \prod_{(i,l)\in I\times\mathbb{Z}/s\mathbb{Z}} Y_{i,l}^{y_{i,l}(m)} V_{i,l}^{v_{i,l}(m)}$.

Let $D(z) = \mathrm{diag}([r_1]_z, ..., [r_n]_z)$.

For $(i,l) \in I\times\mathbb{Z}/s\mathbb{Z}$, we define a character [7] $u_{i,l}$ and bicharacters d_1, d_2 on A_t^s as in section 3.2.1.

[6] In fact it suffices that $-d(m_2, m_3) + d(m_1m_2, m_3) - d(m_1, m_2m_3) + d(m_1, m_2) = 0$.

[7] ie. $u_\alpha(m_1.m_2) = u_\alpha(m_1) + u_\alpha(m_2)$

Proposition 7.9. *The following properties are equivalent :*

i) For $s \geq 0$, $d_1 = d_2$

ii) For $s \geq 0$, $\forall (i,l),(j,k) \in I \times \mathbb{Z}/s\mathbb{Z}, u_{i,l}(V_{j,k}) = u_{j,k+r_j}(V_{i,l+r_i})$

iii) C is symmetric and $\forall i,j \in I$, $r_i = r_j$.

Proof : Only $(ii) \Leftrightarrow (iii)$ needs a proof : for $i,j \in I$ and $l,k \in \mathbb{Z}/s\mathbb{Z}$ we have :

$$u_{i,l}(V_{j,k}) = \sum_{r=C_{i,j}+1...-C_{i,j}-1} \delta_{l+r,k} = \sum_{r=C_{i,j}+1...-C_{i,j}-1} \delta_{l-k,r}$$

$$u_{j,k+r_j}(V_{i,l+r_i}) = \sum_{r=C_{j,i}+1...-C_{j,i}-1} \delta_{k+r+r_j,l+r_i} = \sum_{r=C_{j,i}+1...-C_{j,i}-1} \delta_{l-k,r_j-r_i+r}$$

If $s = 0$, those terms are equal for all $l,k \in \mathbb{Z}$ if and only if $C_{i,j} \neq 0$ implies $C_{i,j} = C_{j,i}$ and $r_i = r_j$. So as C is indecomposable we have $(ii) \Leftrightarrow (iii)$.

If $s \geq 0$ and (iii) is verified we see in the same way that those terms are equal, so $(iii) \Rightarrow (ii)$. $\square$

In particular if C is of type ADE, we get the bicharacter of [N3] and $d_1 = d_2$ is the equation ([N3], 2.1).

We have seen in lemma 7.8 that we can define a twisted multiplication if and only if $\mathfrak{S}d_1 = \mathfrak{S}d_2$, so we investigate those cases :

Theorem 7.10. *The following properties are equivalent :*

i) For $s \geq 0$, we have $\mathfrak{S}d_1 = \mathfrak{S}d_2$

ii) For $s \geq 0$, $\forall (i,l),(j,k) \in I \times \mathbb{Z}/s\mathbb{Z}, u_{i,l}(V_{j,k+r_j}) - u_{i,l+2r_i}(V_{j,k+r_j}) = u_{j,k+2r_j}(V_{i,l+r_i}) - u_{j,k}(V_{i,l+r_i})$

iii) For $s \geq 0$ and $m \in A_t^s$, $d_1(m,m) = d_2(m,m)$

iv) $B(z)$ is symmetric

v) B is symmetric and $C_{i,j} \neq C_{j,i} \Longrightarrow (r_i = -C_{j,i}$ and $r_j = -C_{i,j})$

Proof : Only $(ii) \Leftarrow (v)$ needs a proof : the equation (ii) means :

$$\sum_{r=C_{i,j}+1...-C_{i,j}-1} \delta_{l-k,r_j-r} - \delta_{l-k,r_j-2r_i-r} = \sum_{r=C_{j,i}+1...-C_{j,i}-1} \delta_{l-k,2r_j+r-r_i} - \delta_{l-k,r-r_i}$$

At $s = 0$, the formula holds for all $l,k \in \mathbb{Z}$, if and only the coefficients of Kronecker's functions are equal, that is to say in $\mathbb{Z}[X^{\pm}]$:

$$\sum_{r=C_{i,j}+1...-C_{i,j}-1} X^{r_j-r} - X^{r_j-2r_i-r} = \sum_{r=C_{j,i}+1...-C_{j,i}-1} X^{2r_j+r-r_i} - X^{r-r_i}$$

$$X^{r_j-2r_i+C_{i,j}}(1 - X^{-2C_{i,j}})(1 - X^{2r_i}) = X^{-r_i+C_{j,i}}(1 - X^{-2C_{j,i}})(1 - X^{2r_j})$$

$$(C_{i,j} = C_{j,i} = 0) \text{ or } (r_i = r_j \text{ and } C_{i,j} = C_{j,i} \neq 0) \text{ or } (r_j = -C_{i,j} \text{ and } r_i = -C_{j,i})$$

and so $(ii) \Rightarrow (v)$. $\square$

7.3.3 Bicharacters and q-symmetrizable Cartan matrices

There is a way to define a deformation multiplication if $B(z)$ is non necessarily symmetric. First we define the matrix $C'_{i,j}(z) = [C_{i,j}]_{z_i}$ and the characters :

$$u'_{i,l}(m) = y_{i,l}(m) - v_{i,l-r_i}(m) - v_{i,l+r_i}(m) + \sum_{j\in I, r=C_{i,j}+1, C_{i,j}+3,\dots,-C_{i,j}-1} v_{i,l+r_i r}(m)$$

We define the bicharacters d'_1 and d'_2 from $\tilde{u}_{i,l}$ in the same way d_1 and d_2 were defined from $u_{i,l}$ (section 7.3.2).

We also define $B'_{i,j}(z) = [B_{i,j}]_z$. Note that we have always $B'(z) = D(z)C'(z)$.

Proposition 7.11. *The following properties are equivalent :*

i) For $s \geq 0$, $\mathfrak{S}d'_1 = \mathfrak{S}d'_2$

ii) For $s \geq 0$, $\forall (i,l),(j,k) \in I \times \mathbb{Z}/s\mathbb{Z}$, $u'_{i,l}(V_{j,k+r_j} - u'_{i,l+2r_i}(V_{j,k+r_j}) = u'_{j,k+2r_j}(V_{i,l+r_i}) - u'_{j,k}(V_{i,l+r_i})$

iii) $B = DC$ *is symmetric*

iv) $B'(z)$ *is symmetric*

In particular if C is symmetrizable we can define the deformed structure for all $s \geq 0$.

Proof : Only $(ii) \Rightarrow (iii)$ needs a proof : let us write the equation (ii) :

$$u'_{i,l}(V_{j,k+r_j}) - u'_{i,l+2r_i}(V_{j,k+r_j}) = u'_{j,k+2r_j}(V_{i,l+r_i}) - u'_{j,k}(V_{i,l+r_i})$$

If $i = j$, we are in the symmetric case, and it follows from proposition 7.9 that this equation is verified. In the case $i \neq j$, if $C_{i,j} = 0$ then all is equal to 0. In the cases $C_{i,j} < 0$ the equation reads :

$$\sum_{r=C_{i,j}+1\dots-C_{i,j}-1} \delta_{l+r_i r,k+r_j} - \delta_{l+2r_i+rr_i,k+r_j} = \sum_{l=C_{j,i}+1\dots-C_{j,i}-1} \delta_{k+2r_j+lr_j,l+r_i} - \delta_{k+rr_j,l+r_i}$$

$$\sum_{r=C_{i,j}+1\dots-C_{i,j}-1} \delta_{l-k,r_j-rr_i} - \delta_{l-k,r_j-2r_i-r_i r} = \sum_{r=C_{j,i}+1\dots-C_{j,i}-1} \delta_{l-k,2r_j+rr_j-r_i} - \delta_{l-k,rr_j-r_i}$$

$$\delta_{l-k,r_j-r_i-r_iC_{i,j}} - \delta_{l-k,r_j-r_i+r_iC_{i,j}} = \delta_{l-k,r_j-r_i-r_jC_{j,i}} - \delta_{l-k,r_j-r_i+r_jC_{j,i}}$$

That is to say :

$$(2r_iC_{i,j} \in s\mathbb{Z} \text{ and } 2r_jC_{j,i} \in s\mathbb{Z}) \text{ or } r_iC_{i,j} - r_jC_{j,i} \in s\mathbb{Z}$$

If $s = 0$, the equation means $r_iC_{i,j} = r_jC_{j,i}$ that is to say $B = DC$ symmetric. So $(ii) \Leftrightarrow (iii)$. □

In some situations the two constructions are the same :

Proposition 7.12. *The following properties are equivalent :*

i) For $s \geq 0$, $u' = u$

ii) For $s \geq 0$, $d_1' = d_1$

iii) For $s \geq 0$, $d_2' = d_2$

iv) $C'(z) = C(z)$

v) $B'(z) = B(z)$

vi) $i \neq j \Rightarrow (r_i = 1$ or $C_{i,j} = -1$ or $C_{i,j} = 0)$

Proof : Only $(v) \Leftrightarrow (vi)$ needs a proof : we have always :

$$B_{i,i}(z) = \frac{z^{r_i} - z^{-r_i}}{z - z^{-1}}(z^{r_i} + z^{-r_i}) = \frac{z^{2r_i} - z^{-2r_i}}{z - z^{-1}} = [2r_i]_z = [B_{i,i}]_z$$

If $i \neq j$, the equality $B_{i,j}(z) = B'_{i,j}(z)$ means :

$$z^{r_i + C_{i,j}} + z^{-r_i - C_{i,j}} - z^{C_{i,j} - r_i} - z^{r_i - C_{i,j}} = z^{r_i C_{i,j} + 1} + z^{-1 - r_i C_{i,j}} - z^{r_i C_{i,j} - 1} - z^{1 - r_i C_{i,j}}$$

If $r_i = 1$ or $C_{i,j} = -1$ or $C_{i,j} = 0$ the equality is clear and so $(vi) \Rightarrow (v)$. Suppose that (v) is true and let be $i \neq j$. We have to study different cases :

$r_i + C_{i,j} = C_{i,j} - r_i \Rightarrow r_i = 0$ (impossible)

$r_i + C_{i,j} = r_i - C_{i,j} \Rightarrow C_{i,j} = 0$

$r_i + C_{i,j} = r_i C_{i,j} + 1$ and $r_i C_{i,j} - 1 = C_{i,j} - r_i \Rightarrow r_i = 1$

$r_i + C_{i,j} = r_i C_{i,j} + 1$ and $-r_i C_{i,j} + 1 = C_{i,j} - r_i \Rightarrow C_{i,j} = 1$ (impossible)

$r_i + C_{i,j} = -r_i C_{i,j} - 1$ and $r_i C_{i,j} - 1 = C_{i,j} - r_i \Rightarrow C_{i,j} = -1$

$r_i + C_{i,j} = -r_i C_{i,j} - 1$ and $-r_i C_{i,j} + 1 = C_{i,j} - r_i \Rightarrow r_i = -1$ (impossible)

and so we get (vi). □

Lemma 7.13. *If the properties of the proposition 7.12 are verified and $B = DC$ is symmetric then the properties of the proposition 7.11 are verified.*

Proof : We verify the property (iv) of proposition 7.11 : we suppose that $C_{i,j} \neq C_{j,i}$. So $C_{i,j} \neq 0$, $C_{j,i} \neq 0$ and we do not have $C_{i,j} = C_{j,i} = -1$. As $r_i C_{i,j} = r_j C_{j,i}$, we do not have $r_i = r_j = 1$. So we have (property (vi) of proposition 7.12) $r_i = -C_{j,i} = 1$ or $r_j = -C_{i,j} = 1$. For example in the first case, $r_i C_{i,j} = r_j C_{j,i}$ leads to $C_{i,j} = -r_j$. □

Definition 7.14. *We say that C is q-symmetrizable if $B = DC$ is symmetric and :*

$$i \neq j \Rightarrow (r_i = 1 \text{ or } C_{i,j} = -1 \text{ or } C_{i,j} = 0)$$

In particular C q-symmetrizable verifies the properties of proposition 7.11, 7.12 and of theorem 7.10.

7.3.4 Examples

If C is symmetric then for all $i \in I$ we have $r_i = 1$ and so C is q-symmetrizable.

Lemma 7.15. *The Cartan matrices of finite or affine type (except $A_1^{(1)}$, $A_{2l}^{(2)}$ case, $l \geq 2$) are q-symmetrizable. The affine Cartan matrices $A_1^{(1)}$, $A_{2l}^{(2)}$ with $l \geq 2$ are not q-symmetrizable.*

In particular if C is finite then $u = \tilde{u}$ and the presentation adopted in this paper fits with former articles, in particular in the non symmetric cases ([FR3], [FM1], [FM2], [He2]).

Proof : As those matrices are symmetrizable, it suffices to check the property (vi) of proposition 7.12 :

the finite Cartan matrices A_l ($l \geq 1$), D_l ($l \geq 4$), E_6, E_7, E_8 and the affine Cartan matrices $A_l^{(1)}$ ($l \geq 1$), $D_l^{(1)}$ ($l \geq 4$), $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$ are symmetric and so q-symmetrizable.

the finite Cartan matrices B_l ($l \geq 2$), G_2 and the affine Cartan matrices $B_l^{(1)}$ ($l \geq 3$), $G_2^{(1)}$ verify $r_n = 1$ and for $i \neq j : i \leq n-1 \Rightarrow C_{i,j} = -1$ or 0.

the finite Cartan matrices C_l ($l \geq 2$), the affine Cartan matrices $A_{2l-1}^{(2)}$ ($l \geq 3$), $D_4^{(3)}$ verify $r_1 = ... = r_{n-1} = 1$, $C_{n,1} = ... = C_{n,n-2} = 0$ and $C_{n,n-1} = -1$.

the affine Cartan matrices $C_l^{(1)}$ ($l \geq 2$) verify $r_2 = ... = r_{n-1} = 1$ and $C_{1,3} = ... = C_{1,n} = 0$, $C_{1,2} = -1$, $C_{n,1} = ... = C_{n,n-2} = 0$, $C_{n,n-1} = -1$.

the affine Cartan matrices $D_{l+1}^{(2)}$ ($l \geq 2$) verify $r_1 = r_n = 1$ and for $i \neq j : 2 \leq i \leq n-1 \Rightarrow C_{i,j} = -1$ or 0.

The other particular cases ($F_4, F_4^{(1)}, A_2^{(2)}, E_6^{(2)}$) are studied case by case.

Finally the affine Cartan matrices $A_1^{(1)}$ and $A_{2l}^{(2)}$ ($l \geq 2$) are not q-symmetrizable because $C_{n-1,n} = -2$ and $r_{n-1} = 2$. $\square$

One can understand "intuitively" the fact that $A_{2l}^{(2)}$ ($l \geq 2$) is not q-symmetrizable : in the Dynkin diagram there is an oriented path without loop with two arrows in the same direction.

There are q-symmetrizable Cartan matrices which are not finite and not affine : here is an example such that for all $i, j \in I$, $C_{i,j} \geq -2$:

$$C = \begin{pmatrix} 2 & -2 & -2 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -2 & -2 & 2 \end{pmatrix}$$

$$(r_1, r_2, r_3, r_4) = (1, 2, 2, 1)$$

Quatrième partie

Representations of quantum affinizations and fusion product

À paraître dans *Tranformation Groups*

Prépublication arXiv : math.QA/0312336

Résumé. Dans cet article nous étudions les affinisées quantiques générales $\mathcal{U}_q(\hat{\mathfrak{g}})$ des algèbres de Kac-Moody symmétrisables quantiques et nous développons leur théorie des représentations. Nous en démontrons une décomposition triangulaire et donnons une classification des représentations intégrables de plus haut poids (de type 1) à la Drinfel'd-Chari-Pressley. Une généralisation du morphisme de q-caractères que Frenkel-Reshetikhin ont défini pour les algèbres affines quantiques se révèle être un outil puissant dans cette étude. Pour une grande classe d'affinisées quantiques (incluant les algèbres affines et toroïdales quantiques) la combinatoire des q-caractères donne une structure d'anneau $*$ sur le groupe de Grothendieck $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ des représentations intégrables que nous avons classifiées. Nous proposons une nouvelle construction de produits tensoriels dans une catégorie plus large en utilisant une déformation du nouveau coproduit de Drinfel'd (qui ne peut pas être directement utilisé pour $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ car il fait intervenir des sommes infinies). Nous démontrons en particulier que $*$ est un produit de fusion (un produit de représentations est une représentation).

Abstract. In this paper we study general quantum affinizations $\mathcal{U}_q(\hat{\mathfrak{g}})$ of symmetrizable quantum Kac-Moody algebras and we develop their representation theory. We prove a triangular decomposition and we give a classication of (type 1) highest weight simple integrable representations analog to Drinfel'd-Chari-Pressley one. A generalization of the q-characters morphism, introduced by Frenkel-Reshetikhin for quantum affine algebras, appears to be a powerful tool for this investigation. For a large class of quantum affinizations (including quantum affine algebras and quantum toroidal algebras), the combinatorics of q-characters give a ring structure $*$ on the Grothendieck group $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ of the integrable representations that we classified. We propose a new construction of tensor products in a larger category by using a deformation of the Drinfel'd new coproduct (it can not directly be used for $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ because it involves infinite sums). In particular we prove that $*$ is a fusion product (a product of representations is a representation).

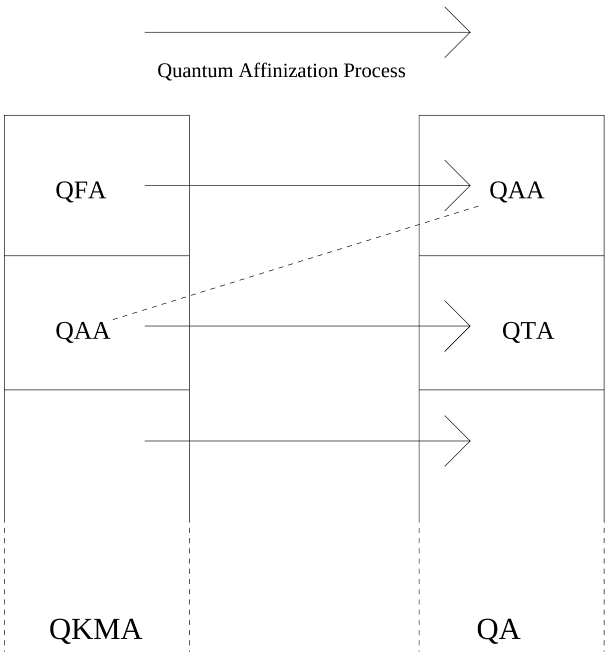
1 Introduction

In this paper $q \in \mathbb{C}^*$ is not a root of unity.

Drinfel'd [Dr1] and Jimbo [Jim] associated, independently, to any symmetrizable Kac-Moody algebra $\mathfrak{g}$ and $q \in \mathbb{C}^*$ a Hopf algebra $\mathcal{U}_q(\mathfrak{g})$ called quantum Kac-Moody algebra. The structure of the Grothendieck ring of integrable representations is well understood : it is analogous to the classical case $q = 1$.

The quantum algebras of finite type $\mathcal{U}_q(\mathfrak{g})$ ($\mathfrak{g}$ of finite type) have been intensively studied (see for example [CP4, L, R] and references therein). The quantum affine algebras $\mathcal{U}_q(\hat{\mathfrak{g}})$ ($\hat{\mathfrak{g}}$ affine algebra) are also of particular interest : they have two realizations, the usual Drinfel'd-Jimbo realization and a new realization (see [Dr2, Be]) as a quantum affinization of a quantum algebra of finite type $\mathcal{U}_q(\mathfrak{g})$. The finite dimensional representations of quantum affine algebras are the subject of intense research (see among others [AK, CP1, CP3, CP4, EM, FR3, FM1, N1, N3, VV2] and references therein). In particular they were classified by Chari-Pressley [CP3, CP4], and Frenkel-Reshetikhin [FR3] introduced the q-characters morphism which is a powerful tool for the study of these representations (see also [Kn, FM1]).

The quantum affinization process (that Drinfel'd [Dr2] described for constructing the second realization of a quantum affine algebra) can be extended to all symmetrizable quantum Kac-Moody algebras $\mathcal{U}_q(\mathfrak{g})$ (see [Jin, N1]). One obtains a new class of algebras called quantum affinizations : the quantum affinization of $\mathcal{U}_q(\mathfrak{g})$ is denoted by $\mathcal{U}_q(\hat{\mathfrak{g}})$. The quantum affine algebras are the simplest examples and are very special because they are also quantum Kac-Moody algebras. When C is affine, the quantum affinization $\mathcal{U}_q(\hat{\mathfrak{g}})$ is called a quantum toroidal algebra. It is known not to be a quantum Kac-Moody algebra but it is also of particular interest (see for example [GKV, M1, M2, N1, N5, Sa, Sc, STU, TU, VV1] and references therein). This setting is summed up in this picture :



(QKMA : Quantum Kac-Moody Algebras, QFA : Quantum Algebras of Finite type, QAA : Quantum Affine Algebras, QTA : Quantum Toroidal Algebras, QA : Quantum Affinizations ; the line between the two QAA symbolizes the Drinfel'd-Beck correspondence.)

In [N1] Nakajima gave a classification of (type 1) simple integrable highest weight modules of $\mathcal{U}_q(\hat{\mathfrak{g}})$ when $\mathfrak{g}$ is symmetric. The case C of type $A_n^{(1)}$ (toroidal $\hat{sl}_n$-case) was also studied by Miki in [M1] (a coproduct is also used with an approach specific to the $A_n^{(1)}$-case ; but it is technically different from the general construction proposed in this paper). In [He3] we proposed a combinatorial construction of q-characters (and also of their t-deformations) for generalized Cartan matrix C such that $i \neq j \Rightarrow C_{i,j}C_{j,i} \leq 3$ (it includes finite and affine types except $A_1^{(1)}$, $A_2^{(2)}$) ; we conjectured that they were linked with a general representation theory. But in general little is known about the representation theory outside the case of quantum affine algebras.

In this paper we study general quantum affinizations and we develop their representation theory. First we prove a triangular decomposition of $\mathcal{U}_q(\hat{\mathfrak{g}})$. We classify the (type 1) simple highest weight integrable representations, we define and study a generalization of the morphism of q-characters χ_q which appears to be a natural tool for this investigation (the approach is different from [He3] because q-characters are obtained from the representation theory and not from purely combinatorial constructions). If the quantized Cartan matrix $C(z)$ is invertible (it includes all quantum affine algebras and quantum toroidal algebras), a symmetry property of those q-characters with respect to the action of screening operators is proved (analog of the invariance for the action of the Weyl group in classical finite cases ; the result is proved in [FM1] for quantum affine algebras) ; in particular those q-characters are the combinatorial objects considered in [He3]. Moreover we get that the image of χ_q is a ring and we can define a formal ring structure on the Grothendieck group. Although quantum affine algebras are Hopf algebras, in general no coproduct has been defined for quantum affinizations (this point was also raised by Nakajima in [N5]). Drinfel'd gave formulas for a new coproduct which can be written for all quantum affinizations. They can not directly be used to define a tensor product of representations because they involve infinite sums. We propose a new construction of tensor products in a larger category with a generalization of the new Drinfel'd coproduct. We define a specialization process which allows us to interpret the ring structure that we defined on the Grothendieck group : we prove that it is a fusion product, that is to say that a product of representations is a representation (see [F] for generalities on fusion rings and physical motivations).

In more details, this paper is organized as follows :

in section 2 we recall backgrounds on quantum Kac-Moody algebras. In section 3 we recall the definition of quantum affinizations and we prove a triangular decomposition (theorem 3.2). Some computations are needed to prove the compatibility with affine quantum Serre relations (section 3.3) ; note that we get a new proof of a combinatorial identity discovered by Jing (consequence of lemma 3.10). The triangular decomposition is used in section 4.2.1 to define the Verma modules of $\mathcal{U}_q(\hat{\mathfrak{g}})$.

In section 4 we recall the classification of (type 1) simple integrable highest weight representations of quantum Kac-Moody algebras, and we prove such a classification for quantum affinizations (theorem 4.9 ; the proof is analog to the proof given by Chari-Pressley for quantum affine algebras). The point is to give an adapted definition of a weight which we call a l-weight : we need a more precise definition than in the case of quantum affine algebras (a l-weight must be characterized by the action of $\mathcal{U}_q(\hat{\mathfrak{h}}) \subset \mathcal{U}_q(\hat{\mathfrak{g}})$ on a l-weight space). We also give the definition of the category $\mathcal{O}(\mathcal{U}_q(\hat{\mathfrak{g}}))$.

In section 5 we construct q-characters of integrable modules in the category $\mathcal{O}(\mathcal{U}_q(\hat{\mathfrak{g}}))$. New technical points are to be considered (in comparison to quantum affine algebra cases) : we have to add terms of the form k_λ (λ coweight of $\mathcal{U}_q(\mathfrak{g})$) for the well-definedness in the general case. The original definition of q-characters ([FR3]) was based on an explicit formula for the universal $\mathcal{R}$-matrix. In general no universal $\mathcal{R}$-matrix has been defined for a quantum affinization. However q-characters can be obtained with a piece of the formula of a "$\mathcal{R}$-matrix" in the same spirit as the original approach (theorem 5.7). In section 5.5 we prove that the image of χ_q is the intersection of the kernels of screening operators (theorem 5.15) in the same spirit as Frenkel-Mukhin [FM1] did for quantum affine algebras ; new technical points are involved because of the k_λ (we suppose that the quantized Cartan matrix $C(z)$ is invertible). In particular it unifies this approach with [He3] and enables us to prove some results announced in [He3]. We prove that the image of χ_q is a ring. As χ_q is injective, we get an induced ring structure $*$ on the Grothendieck group.

In section 6 we prove that $*$ is a fusion product (theorem 6.2), that is to say that there is a product of modules. We use the new Drinfel'd coproduct (proposition 6.3) ; as it involves infinite sums, we have to work in a larger category where the tensor product is well-defined (theorem 6.7). To conclude the proof of theorem 6.2 we define specializations of certain forms which allow us to go from the larger category to $\mathcal{O}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ (section 6.5.2). We also give some concrete examples of explicit computations in section 6.6.

2 Background

2.1 Cartan matrix

In this section we give some general backgrounds about Cartan matrices (for more details see [Kac]). A generalized Cartan matrix is $C = (C_{i,j})_{1\leq i,j\leq n}$ such that $C_{i,j} \in \mathbb{Z}$, $C_{i,i} = 2$, $i \neq j \Rightarrow C_{i,j} \leq 0$, $C_{i,j} = 0 \Leftrightarrow C_{j,i} = 0$. We denote $I = \{1, ..., n\}$ and $l = \text{rank}(C)$.

In the following we suppose that C is symmetrizable, that is to say there is a matrix $D = \text{diag}(r_1, ..., r_n)$ ($r_i \in \mathbb{N}^*$) such that $B = DC$ is symmetric. In particular if C is symmetric then it is symmetrizable with $D = I_n$. For example :

C is said to be of finite type if all its principal minors are in $\mathbb{N}^*$ (see [Bo] for a classification).

C is said to be of affine type if all its proper principal minor are in $\mathbb{N}^*$ and $\det(C) = 0$ (see [Kac] for a classification).

Let z be an indeterminate. We put $z_i = z^{r_i}$ and for $l \in \mathbb{Z}$, $[l]_z = \frac{z^l - z^{-l}}{z - z^{-1}} \in \mathbb{Z}[z^{\pm}]$. For $l \geq 0$ we put $[l]_z! = [l]_z[l-1]_z...[1]_z$ and for $0 \leq r \leq m$: $\begin{bmatrix} m \\ r \end{bmatrix}_z = [m]_z!/[r]_z![m-r]_z$.

Let $C(z)$ be the quantized Cartan matrix defined by ($i \neq j \in I$) :

$$C_{i,i}(z) = z_i + z_i^{-1} \text{ , } C_{i,j}(z) = [C_{i,j}]_z$$

In sections 5.5 and 6 we suppose that $C(z)$ is invertible. We have seen in lemma 6.4 of [He3] that the condition ($C_{i,j} < -1 \Rightarrow -C_{j,i} \leq r_i$) implies that $\det(C(z)) \neq 0$. In particular finite and affine Cartan matrices (where we impose $r_1 = r_2 = 2$ for $A_1^{(1)}$) satisfy this condition and that quantum affine algebras and quantum toroidal algebra are included in our study. We denote by $\tilde{C}(z)$ the inverse matrix of $C(z)$ and $D(z)$ the diagonal matrix such that for $i, j \in I$, $D_{i,j}(z) = \delta_{i,j}[r_i]_z$.

We consider a realization $(\mathfrak{h}, \Pi, \Pi^\vee)$ of C (see [Kac]) : $\mathfrak{h}$ is a $2n - l$ dimensional $\mathbb{Q}$-vector space, $\Pi = \{\alpha_1, ..., \alpha_n\} \subset \mathfrak{h}^*$ (set of the simple roots), $\Pi^\vee = \{\alpha_1^\vee, ..., \alpha_n^\vee\} \subset \mathfrak{h}$ (set of simple coroots) and for $1 \leq i, j \leq n$:

$$\alpha_j(\alpha_i^\vee) = C_{i,j}$$

Denote by $\Lambda_1, ..., \Lambda_n \in \mathfrak{h}^*$ (resp. the $\Lambda_1^\vee, ..., \Lambda_n^\vee \in \mathfrak{h}$) the set of fundamental weights (resp. coweights) : we have $\alpha_i(\Lambda_j^\vee) = \Lambda_i(\alpha_j^\vee) = \delta_{i,j}$.

Consider a symmetric bilinear form $(,) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{Q}$ such that for $i \in I$, $h \in \mathfrak{h}^*$: $(\alpha_i, h) = h(r_i\alpha_i^\vee)$. It is non degenerate and gives an isomorphism $\nu : \mathfrak{h}^* \to \mathfrak{h}$. In particular for $i \in I$ we have $\nu(\alpha_i) = r_i\alpha_i^\vee$ and for $\lambda, \mu \in \mathfrak{h}^*$, $\lambda(\nu(\mu)) = \mu(\nu(\lambda))$.

Denote $P = \{\lambda \in \mathfrak{h}^* / \forall i \in I, \lambda(\alpha_i^\vee) \in \mathbb{Z}\}$ the set of weights and $P^+ = \{\lambda \in P / \forall i \in I, \lambda(\alpha_i^\vee) \geq 0\}$ the set of dominant weights. For example we have $\alpha_1, ..., \alpha_n \in P$ and $\Lambda_1, ..., \Lambda_n \in P^+$. Denote $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i \subset P$ the root lattice and $Q^+ = \sum_{i \in I} \mathbb{N}\alpha_i \subset Q$. For $\lambda, \mu \in \mathfrak{h}^*$, write $\lambda \geq \mu$ if $\lambda - \mu \in Q^+$.

If C is finite we have $n = l = \dim(\mathfrak{h})$ and for $\lambda \in \mathfrak{h}^*$, $\lambda = \sum_{i \in I} \alpha_i^\vee(\lambda)\Lambda_i$. In particular $\alpha_i = \sum_{j \in I} C_{j,i}\Lambda_j$. In general the simple roots can not be expressed in terms of the fundamental weights.

2.2 Quantum Kac-Moody algebra

Definition 2.1. *The quantum Kac-Moody algebra $\mathcal{U}_q(\mathfrak{g})$ is the $\mathbb{C}$-algebra with generators k_h ($h \in \mathfrak{h}$), $x_i^\pm$ ($i \in I$) and relations :*

$$k_h k_{h'} = k_{h+h'} \ , \ k_0 = 1 \tag{11}$$

$$k_h x_j^\pm k_{-h} = q^{\pm\alpha_j(h)} x_j^\pm \tag{12}$$

$$[x_i^+, x_j^-] = \delta_{i,j} \frac{k_{r_i\alpha_i^\vee} - k_{-r_i\alpha_i^\vee}}{q_i - q_i^{-1}} \tag{13}$$

$$\sum_{r=0...1-C_{i,j}} (-1)^r \begin{bmatrix} 1 - C_{i,j} \\ r \end{bmatrix}_{q_i} (x_i^\pm)^{1-C_{i,j}-r} x_j^\pm (x_i^\pm)^r = 0 \text{ (for } i \neq j) \tag{14}$$

This algebra was introduced independently by Jimbo [Jim] and Drinfel'd [Dr1] and is also called a quantum group. It is remarkable that one can define a Hopf algebra structure on $\mathcal{U}_q(\mathfrak{g})$ by setting :

$$\Delta(k_h) = k_h \otimes k_h$$

$$\Delta(x_i^+) = x_i^+ \otimes 1 + k_i^+ \otimes x_i^+ \ ,\ \Delta(x_i^-) = x_i^- \otimes k_i^- + 1 \otimes x_i^-$$
$$S(k_h) = k_{-h} \ ,\ S(x_i^+) = -x_i^+ k_i^{-1} \ ,\ S(x_i^-) = -k_i^+ x_i^-$$
$$\epsilon(k_h) = 1 \ ,\ \epsilon(x_i^+) = \epsilon(x_i^-) = 0$$

where we use the notation $k_i^{\pm} = k_{\pm r_i \alpha_i^\vee}$.

For $i \in I$ let U_i be the subalgebra of $\mathcal{U}_q(\mathfrak{g})$ generated by the $x_i^{\pm}, k_{p\alpha_i^\vee}$ ($p \in \mathbb{Q}$). Then U_i is isomorphic to $\mathcal{U}_{q_i}(sl_2)$, and so a $\mathcal{U}_q(\mathfrak{g})$-module has also a structure of $\mathcal{U}_{q_i}(sl_2)$-module.

Definition 2.2. *A triangular decomposition of an algebra A is the data of three subalgebras (A^-, H, A^+) of A such that the multiplication $x^- \otimes h \otimes x^+ \mapsto x^- h x^+$ defines an isomorphism of $\mathbb{C}$-vector space $A^- \otimes H \otimes A^+ \simeq A$.*

Let $\mathcal{U}_q(\mathfrak{g})^+$ (resp. $\mathcal{U}_q(\mathfrak{g})^-$, $\mathcal{U}_q(\mathfrak{h})$) be the subalgebra of $\mathcal{U}_q(\mathfrak{g})$ generated by the x_i^+ (resp. the x_i^-, resp. the k_h). We have (see [L]) :

Theorem 2.3. *$(\mathcal{U}_q(\mathfrak{g})^-, \mathcal{U}_q(\mathfrak{h}), \mathcal{U}_q(\mathfrak{g})^+)$ is a triangular decomposition of $\mathcal{U}_q(\mathfrak{g})$. Moreover $\mathcal{U}_q(\mathfrak{h})$ (resp. $\mathcal{U}_q(\mathfrak{g})^+$, $\mathcal{U}_q(\mathfrak{g})^-$) is isomorphic to the algebra with generators k_h (resp x_i^+, x_i^-) and relations (11) (resp. relations (14) with $+$, relations (14) with $-$).*

3 Quantum affinization $\mathcal{U}_q(\hat{\mathfrak{g}})$ and triangular decomposition

In this section we define general quantum affinizations (without central charge), we give the relations between the currents (section 3.2) and we prove a triangular decomposition (theorem 3.2).

3.1 Definition

Definition 3.1. *The quantum affinization of $\mathcal{U}_q(\mathfrak{g})$ is the $\mathbb{C}$-algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$ with generators $x_{i,r}^{\pm}$ ($i \in I, r \in \mathbb{Z}$), k_h ($h \in \mathfrak{h}$), $h_{i,m}$ ($i \in I, m \in \mathbb{Z} - \{0\}$) and the following relations ($i, j \in I, r, r' \in \mathbb{Z}, m \in \mathbb{Z} - \{0\}$) :*

$$k_h k_{h'} = k_{h+h'} \ ,\ k_0 = 1 \ ,\ [k_h, h_{j,m}] = 0 \ ,\ [h_{i,m}, h_{j,m'}] = 0 \tag{15}$$

$$k_h x_{j,r}^{\pm} k_{-h} = q^{\pm \alpha_j(h)} x_{j,r}^{\pm} \tag{16}$$

$$[h_{i,m}, x_{j,r}^{\pm}] = \pm \frac{1}{m} [m B_{i,j}]_q x_{j,m+r}^{\pm} \tag{17}$$

$$[x_{i,r}^+, x_{j,r'}^-] = \delta_{ij} \frac{\phi_{i,r+r'}^+ - \phi_{i,r+r'}^-}{q_i - q_i^{-1}} \tag{18}$$

$$x_{i,r+1}^{\pm} x_{j,r'}^{\pm} - q^{\pm B_{ij}} x_{j,r'}^{\pm} x_{i,r+1}^{\pm} = q^{\pm B_{ij}} x_{i,r}^{\pm} x_{j,r'+1}^{\pm} - x_{j,r'+1}^{\pm} x_{i,r}^{\pm} \tag{19}$$

$$\sum_{\pi \in \Sigma_s} \sum_{k=0..s} (-1)^k \begin{bmatrix} s \\ k \end{bmatrix}_{q_i} x_{i,r_{\pi(1)}}^{\pm} ... x_{i,r_{\pi(k)}}^{\pm} x_{j,r'}^{\pm} x_{i,r_{\pi(k+1)}}^{\pm} ... x_{i,r_{\pi(s)}}^{\pm} = 0 \tag{20}$$

where the last relation holds for all $i \neq j$, $s = 1 - C_{ij}$, all sequences of integers $r_1, ..., r_s$. Σ_s is the symmetric group on s letters. For $i \in I$ and $m \in \mathbb{Z}$, $\phi^{\pm}_{i,m} \in \mathcal{U}_q(\hat{\mathfrak{g}})$ is determined by the formal power series in $\mathcal{U}_q(\hat{\mathfrak{g}})[[z]]$ (resp. in $\mathcal{U}_q(\hat{\mathfrak{g}})[[z^{-1}]]$) :

$$\sum_{m\geq 0}\phi^{\pm}_{i,\pm m}z^{\pm m} = k_{\pm r_i\alpha_i^\vee}\, exp(\pm(q-q^{-1})\sum_{m'\geq 1}h_{i,\pm m'}z^{\pm m'})$$

and $\phi^+_{i,m} = 0$ for $m < 0$, $\phi^-_{i,m} = 0$ for $m > 0$.

The relations (20) are called affine quantum Serre relations. The notation $k_i^{\pm} = k_{\pm r_i\alpha_i^\vee}$ is also used. We have $k_ik_i^{-1} = k_i^{-1}k_i = 1$, $k_ix^{\pm}_{j,m}k_i^{-1} = q^{\pm B_{ij}}x^{\pm}_{j,m}$.

There is an algebra morphism $\mathcal{U}_q(\mathfrak{g}) \rightarrow \mathcal{U}_q(\hat{\mathfrak{g}})$ defined by $(h \in \mathfrak{h}, i \in I)$ $k_h \mapsto k_h$, $x_i^{\pm} \mapsto x^{\pm}_{i,0}$. In particular a $\mathcal{U}_q(\hat{\mathfrak{g}})$-module has also a structure of a $\mathcal{U}_q(\mathfrak{g})$-module.

3.2 Relations between the currents

For $i \in I$, consider the series (also called currents) :

$$x_i^{\pm}(w) = \sum_{r\in\mathbb{Z}}x^{\pm}_{i,r}w^r \ ,\ \phi_i^+(z) = \sum_{m\geq 0}\phi^+_{i,m}z^m \ ,\ \phi_i^-(z) = \sum_{m\geq 0}\phi^-_{i,-m}z^{-m}$$

The defining relations of $\mathcal{U}_q(\hat{\mathfrak{g}})$ can be written with currents $(h, h' \in \mathfrak{h},\ i, j \in I)$:

$$k_hk_{h'} = k_{h+h'} \ ,\ k_0 = 1 \ ,\ k_h\phi_i^{\pm}(z) = \phi_i^{\pm}(z)k_h \tag{21}$$

$$k_hx_j^{\pm}(z) = q^{\pm\alpha_j(h)}x_j^{\pm}(z)k_h \tag{22}$$

$$\phi_i^+(z)x_j^{\pm}(w) = \frac{q^{\pm B_{i,j}}w - z}{w - q^{\pm B_{i,j}}z}x_j^{\pm}(w)\phi_i^+(z) \tag{23}$$

$$\phi_i^-(z)x_j^{\pm}(w) = \frac{q^{\pm B_{i,j}}w - z}{w - q^{\pm B_{i,j}}z}x_j^{\pm}(w)\phi_i^-(z) \tag{24}$$

$$[x_i^+(z), x_j^-(w)] = \frac{\delta_{i,j}}{q_i - q_i^{-1}}[\delta(\frac{w}{z})\phi_i^+(w) - \delta(\frac{z}{w})\phi_i^-(z)] \tag{25}$$

$$(w - q^{\pm B_{i,j}}z)x_i^{\pm}(z)x_j^{\pm}(w) = (q^{\pm B_{i,j}}w - z)x_j^{\pm}(w)x_i^{\pm}(z) \tag{26}$$

$$\sum_{\pi\in\Sigma_s}\sum_{k=0...s}(-1)^k\begin{bmatrix}s\\k\end{bmatrix}_{q_i}x_i^{\pm}(w_{\pi(1)})...x_i^{\pm}(w_{\pi(k)})x_j^{\pm}(z)x_i^{\pm}(w_{\pi(k+1)})...x_i^{\pm}(w_{\pi(s)}) = 0 \tag{27}$$

where $\delta(z) = \sum_{r\in\mathbb{Z}}z^r$. The equation (23) (resp. equation (24)) is expanded for $|z| < |w|$ (resp. $|w| < |z|$).

Remark : in the relations (26), the terms can not be divided by $w - q^{\pm B_{i,j}}z$: it would involve infinite sums and make no sense.

The following equivalences are clear : (relations (15) $\Leftrightarrow$ relations (21)) ; (relations (16) $\Leftrightarrow$ relations (22)) ; (relations (19) $\Leftrightarrow$ relations (26)) ; (relations (18) $\Leftrightarrow$ relations (25)) ; (relations (20) $\Leftrightarrow$ relations (27)).

We suppose that the relations (16) are verified and we prove the equivalence (relations (17) with $m \geq 1 \Leftrightarrow$ relations (23)) ((relations (17) with $m \leq -1 \Leftrightarrow$ relations (24)) is proved in an similar way) : consider $h_i^+(z) = \sum_{m\geq 1} m h_{i,m} z^{m-1}$. The relation (17) with $m \geq 1$ are equivalent to (expanded for $|z| < |w|$) :

$$[h_i^+(z), x_j^\pm(w)] = \pm [B_{i,j}]_q \frac{w^{-1} x_j^\pm(w)}{(1-\frac{z}{w}q^{B_{i,j}})(1-\frac{z}{w}q^{-B_{i,j}})}$$

It is equivalent to the data of a $\alpha_\pm(z,w) \in (\mathbb{C}[w,w^{-1}])[[z]]$ such that

$$\phi_i^+(z) x_j^\pm(w) = \alpha_\pm(z,w) x_j^\pm(w) \phi_i^+(z)$$

So it suffices to prove that this term is the $\frac{q^{\pm B_{i,j}} w - z}{w - q^{\pm B_{i,j}} z}$ of relation (23). Let us compute this term : we have $\frac{\partial \phi_i^+(z)}{\partial z} = (q-q^{-1}) h_i^+(z) \phi_i^+(z)$ and so the relations (17) imply :

$$(q-q^{-1}) \phi_i^+(z) [h_i^+(z), x_j^\pm(w)] = \frac{\partial \alpha_\pm(z,w)}{\partial z} x_j^\pm(w) \phi_i^+(z)$$

$$(\pm [B_{i,j}]_q \frac{w^{-1}}{(1-\frac{z}{w}q^{B_{i,j}})(1-\frac{z}{w}q^{-B_{i,j}})} \alpha_\pm(z,w) - \frac{1}{q-q^{-1}} \frac{\partial \alpha_\pm(z,w)}{\partial z}) x_j^\pm(w) \phi_i^+(z) = 0$$

$$\frac{\partial \alpha_\pm(z,w)}{\partial z} = \pm (q^{B_{i,j}} - q^{-B_{i,j}}) \frac{w^{-1}}{(1-\frac{z}{w}q^{B_{i,j}})(1-\frac{z}{w}q^{-B_{i,j}})} \alpha_\pm(z,w)$$

As $\frac{q^{\pm B_{i,j}} w - z}{w - q^{\pm B_{i,j}} z}$ is a solution, we have $\alpha_\pm(z,w) = \lambda(w) \frac{q^{\pm B_{i,j}} w - z}{w - q^{\pm B_{i,j}} z}$. But at $z = 0$ we know $\alpha_\pm(0,w) = q^{\pm B_{i,j}}$ (relations (16)) and so $\lambda(w) = 1$.

3.3 Triangular decomposition

3.3.1 Statement

Let $\mathcal{U}_q(\hat{\mathfrak{g}})^+$ (resp. $\mathcal{U}_q(\hat{\mathfrak{g}})^-$, $\mathcal{U}_q(\hat{\mathfrak{h}})$) be the subalgebra of $\mathcal{U}_q(\hat{\mathfrak{g}})$ generated by the $x_{i,r}^+$ (resp. the $x_{i,r}^-$, resp. the k_h, $h_{i,r}$).

Theorem 3.2. *$(\mathcal{U}_q(\hat{\mathfrak{g}})^-, \mathcal{U}_q(\hat{\mathfrak{h}}), \mathcal{U}_q(\hat{\mathfrak{g}})^+)$ is a triangular decomposition of $\mathcal{U}_q(\hat{\mathfrak{g}})$. Moreover $\mathcal{U}_q(\hat{\mathfrak{h}})$ (resp. $\mathcal{U}_q(\hat{\mathfrak{g}})^+$, $\mathcal{U}_q(\hat{\mathfrak{g}})^-$) is isomorphic to the algebra with generators $k_h, h_{i,m}$ (resp $x_{i,r}^+$, $x_{i,r}^-$) and relations (15) (resp. relations (19), (20) with $+$, relations (19), (20) with $-$).*

For a quantum affine algebra (C finite) it is proved in [Be].

In this section 3.3 we prove this theorem in general. We will use the algebras $\mathcal{U}_q^l(\hat{\mathfrak{g}})$, $\tilde{\mathcal{U}}_q(\hat{\mathfrak{g}})$ defined by :

Definition 3.3. *$\mathcal{U}_q^l(\hat{\mathfrak{g}})$ is the $\mathbb{C}$-algebra with generators $x_{i,r}^\pm$, $h_{i,m}$, k_h ($i \in I$, $r \in \mathbb{Z}$, $m \in \mathbb{Z} - \{0\}$, $h \in \mathfrak{h}$) and relations (15), (16), (17), (18) (or relations (21), (22), (23), (24), (25)).*

$\tilde{\mathcal{U}}_q(\hat{\mathfrak{g}})$ is the quotient of $\mathcal{U}_q^l(\hat{\mathfrak{g}})$ by relations (19) (or relations (26)).

Note that $\mathcal{U}_q(\hat{\mathfrak{g}})$ is a quotient of $\mathcal{U}_q^l(\hat{\mathfrak{g}})$ and that $(\mathcal{U}_q^{l,-}(\hat{\mathfrak{g}}),\mathcal{U}_q(\hat{\mathfrak{h}}),\mathcal{U}_q^{l,+}(\hat{\mathfrak{g}}))$ is a triangular decomposition of $\mathcal{U}_q^l(\hat{\mathfrak{g}})$ where $\mathcal{U}_q^{l,\pm}(\hat{\mathfrak{g}})$ is generated by the $x_{i,r}^{\pm}$ without relations. In the sl_2-case we have $\tilde{\mathcal{U}}_q(\hat{sl_2})=\mathcal{U}_q(\hat{sl_2})$.

Let us sketch the proof of theorem 3.2. We use a method analog to the proof for classic cases or quantum Kac-Moody algebras (see for example the chapter 4 of [Ja]) : we have to check a compatibility condition between the relations and the product as explained in section 3.3.2. After some preliminary technical lemmas about polynomials in section 3.3.3, the heart of the proof is given in section 3.3.4 : properties of $\mathcal{U}_q^l(\hat{\mathfrak{g}})$ (lemma 3.9) lead to a triangular decomposition of $\tilde{\mathcal{U}}_q(\hat{\mathfrak{g}})$. Properties of $\tilde{\mathcal{U}}_q(\hat{\mathfrak{g}})$ proved in lemmas 3.10, 3.11 imply theorem 3.2. Note that the intermediate algebra $\tilde{\mathcal{U}}_q(\hat{\mathfrak{g}})$ is also studied because it will be used in the last section of this paper.

Remark : lemma 3.10 gives a new proof of a combinatorial identity discovered by Jing.

The theorem 3.2 is used in section 4.2.1 to define the Verma modules of $\mathcal{U}_q(\hat{\mathfrak{g}})$. Let us give another consequence of theorem 3.2 : for $i\in I$, let $\hat{U}_i$ be the subalgebra of $\mathcal{U}_q(\hat{\mathfrak{g}})$ generated by the $x_{i,r}^{\pm}, k_{p\alpha_i^\vee}, h_{i,m}$ ($r\in\mathbb{Z}$, $m\in\mathbb{Z}-\{0\}$, $p\in\mathbb{Q}$). We have a morphism $\mathcal{U}_{q_i}(\hat{sl_2})\rightarrow\hat{U}_i$ (in particular any $\mathcal{U}_q(\hat{\mathfrak{g}})$-module also has a structure of $\mathcal{U}_{q_i}(\hat{sl_2})$-module). Moreover theorem 3.2 implies :

Corollary 3.4. *$\hat{U}_i$ is isomorphic to $\mathcal{U}_{q_i}(\hat{sl_2})$.*

3.3.2 General proof of triangular decompositions

Let A be an algebra with a triangular decomposition (A^-,H,A^+). Let B^+ (resp. B^-) be a two-sided ideal of A^+ (resp. A^-). Let $C=A/(A.(B^++B^-).A)$ and denote by $C^{\pm}$ the image of $B^{\pm}$ in C.

Lemma 3.5. *If $B^+.A\subset A.B^+$ and $A.B^-\subset B^-.A$ then (C^-,H,C^+) is a triangular decomposition of C and the algebra $C^{\pm}$ is isomorphic to $A^{\pm}/B^{\pm}$.*

Proof : We use the proof of section 4.21 in [Ja] : indeed the product gives an isomorphism of $\mathbb{C}$-vector space $A.(B^++B^-).A\simeq B^+\otimes H\otimes A^-+A^+\otimes H\otimes B^-$. □

3.3.3 Technical lemmas

Let $i\neq j$ and $s=1-C_{i,j}$. Define $P_{\pm}(w_1,...,w_s,z)\in\mathbb{C}[w_1,...,w_s,z]$ by the formula :

$$\sum_{k=0...s}(-1)^k\begin{bmatrix}s\\k\end{bmatrix}_{q_i}(w_1-q^{\pm B_{i,j}}z)...(w_k-q^{\pm B_{i,j}}z)(w_{k+1}q^{\pm B_{i,j}}-z)...(w_sq^{\pm B_{i,j}}-z)$$

Lemma 3.6. *There are polynomials $(f_{\pm,r})_{r=1,...,s-1}$ of $s-1$ variables such that :*

$$P_{\pm}(w_1,...,w_s,z)=\sum_{1\leq r\leq s-1}(w_{r+1}-q_i^{\pm 2}w_r)f_{\pm,r}(w_1,...,w_{r-1},w_{r+2},...,w_s,z)$$

Proof : It suffices to prove it for P_+ (because P_- is obtained from P_+ by $q \mapsto q^{-1}$). First we prove that $P_+(q_i^{-2(s-1)}w, q_i^{-2(s-2)}w, ..., q_i^{-2}w, w, z) = 0$. Indeed it is equal to :

$$z^s \sum_{k=0...s} (-1)^k \begin{bmatrix} s \\ k \end{bmatrix}_{q_i} q_i^{k(1-s)}(q_i^{-2(s-1)}q_i^{s-1} - \frac{z}{w})$$

$$...(q_i^{-2(s-k)}q_i^{s-1} - \frac{z}{w})(q_i^{-2(s-k-1)}q_i^{1-s} - \frac{z}{w})...(q_i^{1-s} - \frac{z}{w})$$

$$= z^s(q_i^{1-s} - \frac{z}{w})q^{-3s+3}M_{q_i}(\frac{z}{w}q^{3s-3})$$

where :

$$M_q(u) = \sum_{k=0...s} (-1)^k \begin{bmatrix} s \\ k \end{bmatrix}_q q^{k(1-s)}(q^{2k} - u)(q^{2(k+1)} - u)...(q^{2(k+s-2)} - u)$$

Let $\alpha_0(q), ..., \alpha_{s-1}(q) \in \mathbb{Z}[q]$ such that $(a-u)(a-uq^2)...(a-uq^{2(s-2)}) = u^{s-1}\alpha_{s-1}(q) + u^{s-2}a\alpha_{s-2}(q) + ... + a^{s-1}\alpha_0(q)$. So :

$$M_q(u) = \sum_{p=0...s-1} \alpha_{s-p}(q)u^{s-p} \sum_{k=0...s} (-1)^k \begin{bmatrix} s \\ k \end{bmatrix}_q q^{k(1-s+2p)}$$

And so $M_q(u) = 0$ because of the q-binomial identity for $p' = 1-s, 3-s, ..., s-1$ (see [L]) :

$$\sum_{k=0...s} (-1)^k \begin{bmatrix} s \\ k \end{bmatrix}_q q^{rp'} = 0$$

As a consequence P_+ is in the kernel of the projection

$$\phi : \mathbb{C}[w_1, ..., w_s, z] \to \mathbb{C}[w_1, ..., w_s, z]/((w_2 - q_i^2 w_1), ..., (w_s - q_i^2 w_{s-1}))$$

that is to say $P_+(w_1, ..., w_s, z) = \sum\limits_{1 \leq r \leq s-1} (w_{r+1} - q^{B_{i,j}}w_r)f_r(w_1, ..., w_s, z)$ where the $f_r \in \mathbb{C}[w_1, ..., w_s, z]$.

Let us prove that we can choose the $(f_r)_{1 \leq r \leq s-1}$ so that for all $1 \leq s \leq r-1$, f_r does not depend of w_r, w_{r+1}. Let $\mathcal{A} \subset \mathrm{Ker}(\phi)$ be the subspace of polynomials which are degree at most of 1 in each variable $w_1, ..., w_s$. In particular $P \in \mathcal{A}$. We can decompose in a unique way $P = \alpha + w_2\beta + w_1\gamma$ where $\alpha, \gamma \in \mathbb{C}[w_3, ..., w_s, z]$, $\beta \in \mathbb{C}[w_1, w_3, ..., w_s, z]$. Consider $\lambda^{(1)} = -q_i^{-2}\gamma(w_2 - q_i^2 w_1) \in \mathcal{A}$ and $P^{(1)} = P - \lambda^{(1)} \in \mathcal{A}$. We have in particular $P^{(1)} = \mu_3^{(1)} + w_2\mu_2^{(1)} + w_2w_1\mu_1^{(1)}$ where $\mu_1^{(1)}, \mu_2^{(1)}, \mu_3^{(1)} \in \mathbb{C}[w_3, ..., w_s, z]$. In the same way we define by induction on r ($1 \leq r \leq s-1$) the $\lambda^{(r)} \in \mathcal{A}$ such that $P^{(r)} = P^{(r-1)} - \lambda^{(r)} \in \mathcal{A}$ is of the form :

$$P^{(r)} = \mu_{r+2}^{(r)} + w_{r+1}\mu_{r+1}^{(r)} + w_{r+1}w_r\mu_r^{(r)} + ... + w_{r+1}w_r...w_1\mu_1^{(r)}$$

where for $1 \leq r' \leq r+2$, $\mu_{r'}^{(r)} \in \mathbb{C}[w_{r+2}, ..., w_s, z]$. Indeed in the part of $P^{(r)}$ without w_{r+2} we can change the terms $w_{r+1}\lambda(w_{r+3}, ..., w_s, z)$ to $q_i^{-2}w_{r+2}\lambda(w_{r+3}, ..., w_s, z)$ by adding $q_i^{-2}(w_{r+2} - q_i^2 w_{r+1})\lambda \in \mathcal{A}$, we can change the terms $w_{r+1}w_r\lambda'(w_{r+3}, ..., w_s, z)$ to

$q_i^{-4}w_{r+2}w_{r+1}\lambda'(w_{r+3},...,w_s,z)$ by adding $q_i^{-4}(w_{r+2}-q_i^2w_{r+1})\lambda+q_i^{-2}(w_{r+2}-q_i^2w_{r+1})\lambda\in\mathcal{A}$, and so on. In particular for $r=s-1$:

$$P^{(s-1)}=\mu_{s+1}^{(s-1)}+\mu_s^{(s-1)}w_s+\mu_{s-1}^{(s-1)}w_sw_{s-1}+...+\mu_1^{(s-1)}w_sw_{s-1}...w_1$$

where $\mu_{s+1}^{(s-1)},...,\mu_1^{(s-1)}\in\mathbb{C}[z]$. But :

$$0=\phi(P^{(s-1)})=\mu_{s+1}^{(s-1)}+\mu_s^{(s-1)}w_s+\mu_{s-1}^{(s-1)}q_i^{-2}w_s^2+...+\mu_1^{(s-1)}q_i^{-2-4-...-2(s-1)}w_s^s$$

So for all $1\leq r'\leq s+1$, $\mu_{r'}^{(s-1)}=0$, and so $P^{(s-1)}=0$. In particular $P=\lambda^{(1)}+\lambda^{(2)}+...+\lambda^{(s-1)}$. □

For $1\leq k\leq s$ consider $P_\pm^{(k)}(w_1,w_2,...,w_s,z)\in\mathbb{C}[w_1,...,w_s,z]$ defined by :

$$(-1)^k\begin{bmatrix}s\\k\end{bmatrix}_{q_i}\sum_{k'=1...k}(zq_i^{\pm(1-s)}-w_1)(w_2-q_i^{\pm2}w_1)...(w_{k'}-q_i^{\pm2}w_1)(w_{k'+1}q_i^{\pm2}-w_1)...(w_sq_i^{\pm2}-w_1)$$

$$+(-1)^{k-1}\begin{bmatrix}s\\k-1\end{bmatrix}_{q_i}\sum_{k'=k...s}(z-w_1q_i^{\pm(1-s)})(w_2-q_i^{\pm2}w_1)$$
$$...(w_{k'}-q_i^{\pm2}w_1)(w_{k'+1}q_i^{\pm2}-w_1)...(w_sq_i^{\pm2}-w_1)$$

Lemma 3.7. *i) For $2\leq k\leq s-1$ there are polynomials $(f_{\pm,r}^{(k)})_{r=1,...,s-1}$ of $s-1$ variables, of degree at most 1 in each variable, such that $P_\pm^{(k)}(w_1,...,w_s,z)$ is equal to :*

$$(z-q_i^{\pm(1-s)}w_k)f_{\pm,k-1}^{(k)}(w_1,...,w_{k-1},w_{k+1},...,w_s,z)$$
$$+(w_{k+1}-q_i^{\pm(1-s)}z)f_{\pm,s-1}^{(k)}(w_1,...,w_k,w_{k+2},...,w_s,z)$$
$$+\sum_{1\leq r\leq s-2,r\neq k-1}(w_{r+2}-q_i^{\pm2}w_{r+1})f_{\pm,r}^{(k)}(w_1,...,w_{r-1},w_{r+2},...,w_s,z)$$

ii) There are polynomials $(f_{\pm,r}^{(1)})_{r=1,...,s-1}$ of $s-1$ variables, of degree at most 1 in each variable, such that $P_\pm^{(1)}(w_1,...,w_s,z)$ is equal to :

$$(w_2-q_i^{\pm(1-s)}z)f_{\pm,s}^{(1)}(w_3,...,w_s,z)+\sum_{1\leq r\leq s-2}(w_{r+2}-q_i^{\pm2}w_{r+1})f_{\pm,r}^{(k)}(w_1,...,w_{r-1},w_{r+2},...,w_s,z)$$

iii) There are polynomials $(f_{\pm,r}^{(s)})_{r=1,...,s-1}$ of $s-1$ variables, of degree at most 1 in each variable, such that $P_\pm^{(s)}(w_1,...,w_s,z)$ is equal to :

$$(z-q_i^{\pm(1-s)}w_s)f_{\pm,s-1}^{(s)}(w_1,...,w_{s-1},z)$$
$$+\sum_{1\leq r\leq s-2}(w_{r+2}-q_i^{\pm2}w_{r+1})f_{\pm,r}^{(s)}(w_1,...,w_{r-1},w_{r+2},...,w_s,z)$$

Proof : It suffices to prove it for $P_+^{(k)}$ (because $P_-^{(k)}$ is obtained from $P_+^{(k)}$ by $q \mapsto q^{-1}$).

For i) : we see as in lemma 3.6 that it suffices to check that $P_+^{(k)}(w_1, ..., w_s, z) = 0$ if $w_3 = q_i^2 w_2$, ... , $w_k = q_i^2 w_{k-1}$, $w_{k+2} = q_i^2 w_{k+1}$, ... , $w_s = q_i^2 w_{s-1}$, $z = q_i^{1-s} w_k$ and $w_{k+1} = q_i^{1-s} z$. It means $w_3 = q_i^2 w_2$, ..., $w_k = q_i^{2(k-2)} w_2$, $w_{k+1} = q^{2k-2-2s} w_2$, ... ,$w_s = q^{-4} w_2$, $z = q^{2k-3-s} w_2$. So if we set $u = w_1/w_2$ we find for $P_+^{(k)} w_2^{-s}$:

$(-1)^k \begin{bmatrix} s \\ k \end{bmatrix}_{q_i} \sum_{k'=1...k} q_i^{2(k'-1)}(q_i^{2k-2-2s} - u)(q_i^{2k-2s} - u)...(q_i^{2k'-6} - u)(q_i^{2k'} - u)...(q_i^{2k-2} - u)(q_i^{-2} - u)$

$+ (-1)^{k-1} \begin{bmatrix} s \\ k-1 \end{bmatrix}_{q_i} \sum_{k'=k...s} q_i^{2k'-s-1}(q_i^{2k-2s-4} - u)...(q_i^{2k'-2s-6} - u)(q_i^{2k'-2s} - u)...(q_i^{2k-4} - u)(q_i^{-2} - u)$

It is a multiple of :

$\frac{[s-k+1]_{q_i}}{q_i^{2k-2s-4} - u}[\sum_{k'=1...k} \frac{q_i^{2k'-1}}{(q_i^{2k'-2}-u)(q_i^{2k'-4}-u)}] - \frac{[k]_{q_i}}{q_i^{2k-2}-u} q_i^s [\sum_{k'=k...s} \frac{q_i^{2k'-1-s}}{(q_i^{2k'-2s-2}-u)(q_i^{2k'-2s-4}-u)}]$

$= \frac{q_i^2[s-k+1]_{q_i}}{(1-q_i^2)(q_i^{2k-2s-4}-u)}[\sum_{k'=1...k} \frac{1}{q_i^{2k'-2}-u} - \frac{1}{q_i^{2k'-4}-u}] - \frac{q_i^2[k]_{q_i}}{(1-q_i^2)q_i^{2k-2}-u} q_i^s [\sum_{k'=k...s} \frac{1}{q_i^{2k'-2s-2}-u} - \frac{1}{q_i^{2k'-2s-4}-u}]$

$= \frac{q_i^2[s-k+1]_{q_i}}{(1-q_i^2)(q_i^{2k-2s-4}-u)}[\frac{1}{q_i^{2k-2}-u} - \frac{1}{q_i^{-2}-u}] - \frac{q_i^2[k]_{q_i}}{(1-q_i^2)(q_i^{2k-2}-u)} q_i^s [\frac{1}{q_i^{-2}-u} - \frac{1}{q_i^{2k-2s-4}-u}] = 0$

For ii) : as for i) we check that $P_+^{(1)}(w_1, ..., w_s, z) = 0$ if $w_3 = q_i^2 w_2$, ... , $w_s = q_i^2 w_{s-1}$, $z = q_i^{s-1} w_2$. It means $w_{k'} = q_i^{2(k'-2)} w_2$ for $2 \leq k' \leq s$. So it we set $u = w_1/w_2$ we find for $P_+^{(1)} w_2^{-s}$:

$-[s]_{q_i}(1-u)(q_i^2 - u)...(q_i^{2s-2} - u) + q_i^{1-s} \sum_{k'=1...s} q_i^{2k'-2}(q_i^{2s-2} - u)(q_i^{-2} - u)...(q_i^{2k'-6} - u)(q_i^{2k'} - u)...(q_i^{2s-2} - u)$

It is a multiple of : $-\frac{q_i^2[s]_{q_i}}{q_i^{2s}-1}(\frac{1}{q_i^{-2}-u} - \frac{1}{q_i^{2s-2}-u}) + \frac{q_i^{1-s}}{1-q_i^{-2}}(\frac{1}{q_i^{-2}-u} - \frac{1}{q_i^{2s-2}-u}) = 0.$

For iii) : as for i) we check that $P_+^{(k)}(w_1, ..., w_s, z) = 0$ if $w_3 = q_i^2 w_2$, ... , $w_s = q_i^2 w_{s-1}$, $z = q_i^{1-s} w_s$. It means $w_{k'} = q_i^{2(k'-2)} w_2$ for $2 \leq k' \leq s$ and $z = q_i^{s-3} w_2$. The computation is analogous to i). □

Lemma 3.8. *For all choices of polynomials $(f_{\pm,r}^{(k')})_{1 \leq k' \leq s, 1 \leq r \leq s-1}$ in lemma 3.7 and each $2 \leq k \leq s$ there are polynomials $(g_{\pm,r}^{(k)})_{r=1,...,s-2}$ of $s-1$ variables such that :*

$$f_{\pm,k-1}^{(k)} - f_{\pm,s-1}^{(k-1)} = \sum_{1 \leq r \leq s-2} (w_{r+2} - q_i^{\pm 2} w_{r+1}) g_{\pm,r}^{(k)}(w_1, ..., w_{r-1}, w_{r+2}, ..., w_s, z)$$

Proof : We see as in lemma 3.6 that it suffices to check that $f_{+,k-1}^{(k)} + f_{+,s-1}^{(k-1)} = 0$ if $w_3 = q_i^2 w_2$, ... , $w_s = q_i^2 w_{s-1}$. So we suppose that $w_{k'} = q_i^{2(k'-2)}$ for all $2 \leq k' \leq s$. Let $Q = w_1^{s-1}(w_2 q_i^{-2} - 1)(w_2 - 1)...(w_2 q_i^{2s-2})/(q_i^2 - 1))$. It suffices to prove that for $2 \leq k \leq s$, we have :

$$(q_i^2 - q_i^{2-2s}) f_{+,k-1}^{(k)} (Q(-1)^k \begin{bmatrix} s \\ k \end{bmatrix}_{q_i} [k]_{q_i}(q_i - q_i^{-1}))^{-1}$$

$$\frac{q_i^{k+1+s}+q_i^{-s-k+3}-q_i^{-s+k+1}-q_i^{3-k+s}}{(vq_i^{-2}-1)(vq_i^{2k-4}-1)(vq_i^{2s-2}-1)} \tag{28}$$

and

$$(q_i^2-q_i^{2-2s})f_{+,s-1}^{(k-1)}(Q(-1)^{k-1}\begin{bmatrix} s \\ k-1\end{bmatrix}_{q_i}[k-1]_{q_i}(q_i-q_i^{-1}))^{-1}$$
$$=\frac{q_i^{k+1}+q_i^{-k+3}-q_i^{-2s+k+1}-q_i^{3-k+2s}}{(vq_i^{-2}-1)(vq_i^{2k-4}-1)(vq_i^{2s-2}-1)} \tag{29}$$

because we have the relation :

$$\begin{bmatrix} s \\ k\end{bmatrix}_{q_i}[k]_{q_i}(q_i^{k+1+s}+q_i^{-s-k+3}-q_i^{-s+k+1}-q_i^{3-k+s})$$
$$=-\begin{bmatrix} s \\ k-1\end{bmatrix}_{q_i}[k-1]_{q_i}(q_i^{k+1}+q_i^{-k+3}-q_i^{-2s+k+1}-q_i^{3-k+2s})$$

First suppose that $3\leq k\leq s-1$. We have $P_+^{(k)}=(z-q_i^{1-s}w_k)f_{+,k-1}^{(k)}+(q_i^2w_k-q_i^{1-s}z)f_{+,s-1}^{(k)}$. So for α_k,β_k such that $P_+^{(k)}=z\alpha_k+w_k\beta_k$, we have $f_{+,k-1}^{(k)}=\frac{q_i^2\alpha_k+q_i^{1-s}\beta_k}{q_i^2-q_i^{2-2s}}$ and $f_{+,s-1}^{(k)}=\frac{q_i^{1-s}\alpha_k+\beta_k}{q_i^2-q_i^{2-2s}}$. But we have $P_+^{(k)}=z(q_i^{1-s}\lambda_k+\mu_k)-w_1(\lambda_k+q_i^{1-s}\mu_k)$ where (we put $v=w_2/w_1$) :

$\lambda_k=(-1)^kw_1^{s-1}\begin{bmatrix} s \\ k\end{bmatrix}_{q_i}\sum_{k'=1...k}(v-q_i^2)(vq_i^2-q_i^2)...(vq_i^{2(k'-2)}-q_i^2)(vq_i^{2k'}-1)...(vq_i^{2(s-2)+2}-1)$
$=Q(-1)^k\begin{bmatrix} s \\ k\end{bmatrix}_{q_i}[\frac{1}{vq_i^{-2}-1}-\frac{q_i^{2k}}{vq_i^{2k-2}-1}]$

$\mu_k=(-1)^{k-1}w_1^{s-1}\begin{bmatrix} s \\ k-1\end{bmatrix}_{q_i}\sum_{k'=k...s}(v-q_i^2)(vq_i^2-q_i^2)...(vq_i^{2(k'-2)}-q_i^2)(vq_i^{2k'}-1)...(vq_i^{2(s-2)+2}-1)$
$=Q(-1)^{k-1}\begin{bmatrix} s \\ k-1\end{bmatrix}_{q_i}[\frac{q_i^{2k-2}}{vq_i^{2k-4}-1}-\frac{q_i^{2s}}{vq_i^{2s-2}-1}]$

As $\alpha_k=q^{1-s}\lambda_k+\mu_k$ and $\beta_k=-(\lambda_k+q_i^{1-s})/(q_i^{k-2}w_2)$, we have :

$$\alpha_k=Q(-1)^k\begin{bmatrix} s \\ k\end{bmatrix}_{q_i}[k]_{q_i}(q_i-q_i^{-1})$$
$$.\frac{((q_i^{k+1-s}-q_i^{s+k-1})+v(q_i^{s+k-3}+q_i^{s+3k-3}-q_i^{3k-3-s}-q_i^{s+k-1}))}{(vq_i^{-2}-1)(vq_i^{2k-2}-1)(vq_i^{2k-4}-1)(vq_i^{2s-2}-1)}$$

$$\beta_k=Q\frac{(-1)^k\begin{bmatrix} s \\ k\end{bmatrix}_{q_i}[k]_{q_i}(q_i-q_i^{-1})((q_i^k+q_i^{2s-k+2}-q_i^{-k+2}-q_i^{k+2})+v(-q_i^{k+2s-2}+q_i^k))}{(vq_i^{-2}-1)(vq_i^{2k-2}-1)(vq_i^{2k-4}-1)(vq_i^{2s-2}-1)}$$

In particular $(q_i^2-q_i^{2-2s})f_{+,k-1}^{(k)}(Q(-1)^k\begin{bmatrix} s \\ k\end{bmatrix}_{q_i}[k]_{q_i}(q_i-q_i^{-1}))^{-1}$ is :

$$\frac{(q_i^{k+1-s}+q_i^{s-k+3}-q_i^{s+k+1}-q_i^{3-k-s})+v(q_i^{s+3k-1}+q_i^{k+1-s}-q_i^{3k-1-s}-q_i^{s+k+1})}{(vq_i^{-2}-1)(vq_i^{2k-2}-1)(vq_i^{2k-4}-1)(vq_i^{2s-2}-1)}$$

and we get formula 28 for k. Moreover $(q_i^2 - q_i^{2-2s})f^{(k)}_{+,s-1}(Q(-1)^k \begin{bmatrix} s \\ k \end{bmatrix}_{q_i} [k]_{q_i}(q_i - q_i^{-1}))^{-1}$ is :

$$\frac{(q_i^{k+2-2s} + q_i^{2s-k+2} - q_i^{-k+2} - q_i^{k+2}) + v(q_i^{k-2} + q_i^{3k-2} - q_i^{3k-2s-2} - q_i^{k+2s-2})}{(vq_i^{-2} - 1)(vq_i^{2k-2} - 1)(vq_i^{2k-4} - 1)(vq_i^{2s-2} - 1)}$$

and we get formula 29 for $k+1$.

So it remains to prove formula 29 with $k=2$ and formula 28 with $k=s$.

$$P^{(1)}_+ = (w_2 - q_i^{(1-s)}z)f^{(1)}_{+,s-1} = -[s]_{q_i}(zq_i^{1-s} - w_1)(q_i^2w_2 - w_1)...(q_i^{2s-2}w_2 - w_1)$$

$$+(z - w_1q_i^{1-s})\sum_{k'=1...s} q_i^{2k'-2}(q_i^{-2}w_2 - w_1)...(q_i^{2k'-6}w_2 - w_1)(q_i^{2k'}w_2 - w_1)...(q_i^{2s-2}w_2 - w_1)$$

$$\Rightarrow f^{(1)}_{+,s-1} = -q_i^{1-s}Q[\frac{-[s]_{q_i}q_i^{1-s}(q_i^2 - 1)}{(vq_i^{-2} - 1)(v - 1)} + \sum_{k'=1...s}\frac{q_i^{2k'-2}}{(q_i^{2k'-4}v - 1)(q_i^{2k'-2}v - 1)}]$$

And so we have for $f^{(1)}_{+,s-1}(q_i^2 - q_i^{2-2s})(-Q[s]_{q_i}(q_i - q_i^{-1}))^{-1}$:

$$\frac{q_i + q_i^3 - q_i^{2s+1} - q_i{-2s+3}}{(vq_i^{-2} - 1)(v - 1)(vq_i^{2s-2}v - 1)}$$

that it to say the formula 29 with $k=2$.

$P^{(s)}_+$ is equal to :

$$(z - q_i^{(1-s)}q_i^{2(s-2)}w_2)f^{(s)}_{+,s-1} = (-1)^{s-1}[s]_{q_i}(z - w_1q_i^{1-s})q_i^{2(s-1)}(q_i^{-2}w_2 - w_1)...(q_i^{2s-6}w_2 - w_1)$$

$$+(-1)^s(zq_i^{1-s} - w_1)\sum_{k'=1...s} q_i^{2k'-2}(q_i^{-2}w_2 - w_1)...(q_i^{2k'-6}w_2 - w_1)(q_i^{2k'}w_2 - w_1)...(q_i^{2s-2}w_2 - w_1)$$

$$\Rightarrow f^{(s)}_{+,s-1} = Q[\frac{(-1)^{s-1}[s]_{q_i}q_i^{2(s-1)}(q_i^2 - 1)}{(vq_i^{2s-4} - 1)(vq_i^{2s-2} - 1)} + (-1)^sq_i^{1-s}\sum_{k'=1...s}\frac{q_i^{2k'-2}}{(q_i^{2k'-4}v - 1)(q_i^{2k'-2}v - 1)}]$$

And so we have for $f^{(s)}_{+,s-1}(q_i^2 - q_i^{2-2s})((-1)^sQ[s]_{q_i}(q_i - q_i^{-1}))^{-1}$:

$$\frac{q_i^{2s+1} + q_i^{3-2s} - q_i - q_i^3}{(vq_i^{-2} - 1)(vq_i^{2s-4} - 1)(vq_i^{2s-2}v - 1)}$$

that it to say the formula 28 with $k=s$. □

3.3.4 Proof of theorem 3.2

The algebras $\mathcal{U}_q^l(\hat{\mathfrak{g}}), \tilde{\mathcal{U}}_q(\hat{\mathfrak{g}}), \mathcal{U}_q^{l,\pm}(\hat{\mathfrak{g}})$ are defined in section 3.3. Let $\tilde{\mathcal{U}}_q^\pm(\hat{\mathfrak{g}}) \subset \tilde{\mathcal{U}}_q(\hat{\mathfrak{g}})$ be the subalgebra generated by the $x^\pm_{i,r}$. Let $\tau_\pm$ be the two-sided ideal of $\mathcal{U}_q^{l,\pm}(\hat{\mathfrak{g}})$ generated by the left terms of relations (19) (with the $x^\pm_{i,r}$).

Lemma 3.9. *We have $\tau_+\mathcal{U}_q^l(\hat{\mathfrak{g}}) \subset \mathcal{U}_q^l(\hat{\mathfrak{g}})\tau_+$ and $\mathcal{U}_q^l(\hat{\mathfrak{g}})\tau_- \subset \tau_-\mathcal{U}_q^l(\hat{\mathfrak{g}})$.*
In particular $(\tilde{\mathcal{U}}_q^-(\hat{\mathfrak{g}}), \mathcal{U}_q(\hat{\mathfrak{h}}), \tilde{\mathcal{U}}_q^+(\hat{\mathfrak{g}}))$ is a triangular decomposition of $\tilde{\mathcal{U}}_q(\hat{\mathfrak{g}})$.

Proof : First $\tau_+\mathcal{U}_q(\hat{\mathfrak{h}}) \subset \mathcal{U}_q(\hat{\mathfrak{h}})\tau_+$, $\mathcal{U}_q(\hat{\mathfrak{h}})\tau_- \subset \tau_-\mathcal{U}_q(\hat{\mathfrak{h}})$ are direct consequences of relations (22), (23), (24). We have also (we use relations (25) and (23), (24)) :

$$[(w-q^{\pm B_{i,j}}z)x_i^\pm(z)x_j^\pm(w)-(q^{\pm B_{i,j}}w-z)x_j^\pm(w)x_i^\pm(z),x_k^\mp(u)]$$

$$=(w-q^{\pm B_{i,j}}z)x_i^\pm(z)[x_j^\pm(w),x_k^\mp(u)]-(q^{\pm B_{i,j}}w-z)[x_j^\pm(w),x_k^\mp(u)]x_i^\pm(z)$$
$$-(q^{\pm B_{i,j}}w-z)x_j^\pm(w)[x_i^\pm(z),x_k^\mp(u)]+(w-q^{\pm B_{i,j}}z)[x_i^\pm(z),x_k^\mp(u)]x_j^\pm(w)=0$$

and so $\tau_+\mathcal{U}_q^{l,-}(\hat{\mathfrak{g}}) \subset \mathcal{U}_q^l(\hat{\mathfrak{g}})\tau_+$, $\mathcal{U}_q^{l,+}(\hat{\mathfrak{g}})\tau_- \subset \tau_-\mathcal{U}_q^l(\hat{\mathfrak{g}})$.

The last point follows from $\tilde{\mathcal{U}}_q(\hat{\mathfrak{g}})=\mathcal{U}_q^l(\hat{\mathfrak{g}})/(\mathcal{U}_q^l(\hat{\mathfrak{g}}).(\tau_++\tau_-).\mathcal{U}_q^l(\hat{\mathfrak{g}}))$, the triangular decomposition of $\mathcal{U}_q^l(\hat{\mathfrak{g}})$ and lemma 3.5. □

Lemma 3.10. *Let $i\neq j$, $s=1-C_{i,j}$ $\mu=1$ or $\mu=-1$. We have in $\tilde{\mathcal{U}}_q(\hat{\mathfrak{g}})$:*

$$\sum_{\pi\in\Sigma_s}\sum_{k=0..s}(-1)^k\begin{bmatrix}s\\k\end{bmatrix}_{q_i}x_i^\pm(w_{\pi(1)})...x_i^\pm(w_{\pi(k)})\phi_j^\mu(z)x_i^\pm(w_{\pi(k+1)})...x_i^\pm(w_{\pi(s)})=0 \tag{30}$$

$$\sum_{\pi\in\Sigma_s}\sum_{k=0..s}(-1)^k\begin{bmatrix}s\\k\end{bmatrix}_{q_i}\xi_i(w_{\pi(1)})...\xi_i(w_{\pi(k)})x_j^\pm(z)\xi_i(w_{\pi(k+1)})...\xi_i(w_{\pi(s)})=0 \tag{31}$$

where $\xi_i(w_p)=x_i^\pm(w_p)$ if $p\neq 1$ and $\xi_i(w_1)=\phi_i^\mu(w_1)$.

Remark : in particular if we multiply the equation (30) by $(\prod_{r=1...s}(w_r-q_i^{s-1}z))(\prod_{1\leq r'<r\leq s}(w_r-q_i^2w_{r'}))$ and we project it on $x_i^+(w_1)...x_i^+(w_s)\phi_j^+(z)$ (we can use the relations (26) thanks to the multiplied polynomial), we get the combinatorial identity discovered by Jing in [Jin], which was also proved in a combinatorial way in [DJ] : for $\pi\in\Sigma_s$ denote by $\epsilon(\pi)\in\{1,-1\}$ the signature of π (we have replaced $z\mapsto z^{-1}$, $w_{k'}\mapsto w_{k'}^{-1}$ to get the formula in the same form as in [Jin]) :

$$0=\sum_{\pi\in\Sigma_s}\epsilon(\pi)\sum_{k=0...s}\begin{bmatrix}s\\k\end{bmatrix}_q(z-q^{s-1}w_{\pi(1)})...(z-q^{s-1}w_{\pi(k)})$$
$$(w_{\pi(k+1)}-q^{s-1}z)...(w_{\pi(s)}-q^{s-1}z)\prod_{1\leq r<r'\leq s}(w_{\pi(r)}-q^2w_{\pi(r')})$$

Proof : First we prove the equation (30) with $\mu=1$ ($\mu=-1$ is analog). The left term is (relations (23)) :

$$\frac{\phi_j^+(z)}{(w_1q^{\pm B_{i,j}}-z)...(w_sq^{\pm B_{i,j}}-z)}\sum_{\pi\in\Sigma_s}P_\pm(w_{\pi(1)},...,w_{\pi(s)},z)x_i^\pm(w_{\pi(1)})...x_i^\pm(w_{\pi(s)})$$

that is to say (see lemma 3.6) it is equal to :

$$\sum_{\pi\in\Sigma_s}\sum_{1\leq r\leq s-1} f_{r,\pm}(w_{\pi(1)},...,w_{\pi(r-1)},w_{\pi(r+2)},...,w_{\pi(s)},z)$$

$$(w_{\pi(r+1)}-q_i^{\pm 2}w_{\pi(r)})x_i^{\pm}(w_{\pi(1)})...x_i^{\pm}(w_{\pi(s)})$$

For each r, we put together the $\pi,\pi'\in\Sigma_s$ such that $\pi(r)=\pi'(r+1)$, $\pi(r+1)=\pi'(r)$, and $\pi(r'')=\pi'(r'')$ for all $r''\neq r,r+1$. So we get a sum of terms :

$$f_{r,\pm}(w_{\pi(1)},...,w_{\pi(r-1)},w_{\pi(r+2)},...,w_{\pi(s)},z)x_i^{\pm}(w_{\pi(1)})...x_i^{\pm}(w_{\pi(r-1)})$$

$$A^{\pm}_{\{\pi(r),\pi(r+1)\}}x_i^{\pm}(w_{\pi(r+2)}...x_i^{\pm}(w_{\pi(s)})$$

where $A^{\pm}_{\{k,k'\}}=(w_k-q_i^{\pm 2}w_{k'})x_i^{\pm}(w_{k'})x_i^{\pm}(w_k)+(w_{k'}-q_i^{\pm 2}w_k)x_i^{\pm}(w_k)x_i^{\pm}(w_{k'})$

But $A^{\pm}_{\{k,k'\}}=0$ in $\tilde{\mathcal{U}}_q(\mathfrak{g})$.

Let us prove the equation (31) with $\mu=1$ ($\mu=-1$ is analog). The left term is :

$$\frac{\phi_i^+(w_1)}{(w_2q_i^{\pm 2}-w_1)...(w_sq_i^{\pm 2}-w_1)(zq_i^{\pm(1-s)}-w_1)}$$

$$\sum_{\pi\in\Sigma_{s-1},k=1...s} P_{\pm}^{(k)}(w_1,w_{\pi(2)},...,w_{\pi(s)},z)x_i^{\pm}(w_{\pi(2)})...x_i^{\pm}(w_{\pi(k)})x_j^{\pm}(z)x_i^{\pm}(w_{\pi(k+1)})...x_i^{\pm}(w_{\pi(s)})$$

where Σ_{s-1} acts on $\{2,...,s\}$. With the help of lemma 3.7 and in analogy to the previous case, for each $1\leq k\leq s$ each $r\neq k$, we put together the $\pi,\pi'\in\Sigma_s$ such that $\pi(r)=\pi'(r+1)$, $\pi(r+1)=\pi'(r)$, and $\pi(r'')=\pi'(r'')$ for all $r''\neq r,r+1$. So the terms with polynomials $f^{(k)}_{\pm,k'}$ with $k'\neq s,k-1$ are erased. We get : $\frac{\phi_i^+(w_1)}{(w_2q_i^{\pm 2}-w_1)...(w_sq_i^{\pm 2}-w_1)(zq_i^{\pm(1-s)}-w_1)}$
$\sum_{\pi\in\Sigma_{s-1},k=1...s}((z-q_i^{\pm(1-s)}w_{\pi(k)})f^{(k)}_{\pm,k-1}+(w_{\pi(k+1)}-q_i^{\pm(1-s)}z)f^{(k)}_{\pm,s-1})$
$x_i^{\pm}(w_{\pi(2)})...x_i^{\pm}(w_{\pi(k)})x_j^{\pm}(z)x_i^{\pm}(w_{\pi(k+1)})...x_i^{\pm}(w_{\pi(s)})$

But this last sum is equal to :

$$\sum_{\pi\in\Sigma_{s-1},k=2...s}(z-q_i^{\pm(1-s)}w_{\pi(k)})(f^{(k)}_{\pm,k-1}-f^{(k-1)}_{\pm,s-1})$$

$$x_i^{\pm}(w_{\pi(2)})...x_i^{\pm}(w_{\pi(k)})x_j^{\pm}(z)x_i^{\pm}(w_{\pi(k+1)})...x_i^{\pm}(w_{\pi(s)})$$

where we can replace $(z-q_i^{\pm(1-s)}w_{\pi(k)})x_i^{\pm}(w_{\pi(k)})x_j^{\pm}(z)$ by
$(-w_{\pi(k)}+q_i^{\pm(1-s)}z)x_j^{\pm}(z)x_i^{\pm}(w_{\pi(k)})$ (relations (26) in $\tilde{\mathcal{U}}_q(\hat{\mathfrak{g}})$). As in the previous cases it follows from lemma 3.8 that this term is equal to 0. □

Let $\tilde{\tau}_{\pm}$ be the two-sided ideal of $\tilde{\mathcal{U}}_q^{\pm}(\hat{\mathfrak{g}})$ generated by the left terms of relations (20) with the $x^{\pm}_{i,r}$.

Lemma 3.11. *We have $\tilde{\tau}_+\tilde{\mathcal{U}}_q(\hat{\mathfrak{g}})\subset\tilde{\mathcal{U}}_q(\hat{\mathfrak{g}})\tilde{\tau}_+$ and $\tilde{\mathcal{U}}_q(\hat{\mathfrak{g}})\tilde{\tau}_-\subset\tilde{\tau}_-\tilde{\mathcal{U}}_q(\hat{\mathfrak{g}})$.*

In particular as $\mathcal{U}_q(\hat{\mathfrak{g}}) = \tilde{\mathcal{U}}_q^l(\hat{\mathfrak{g}})/(\tilde{\mathcal{U}}_q^l(\hat{\mathfrak{g}}).(\tilde{\tau}_+ + \tilde{\tau}_-).\tilde{\mathcal{U}}_q^l(\hat{\mathfrak{g}}))$ the result of theorem 3.2 follows from lemma 3.5 and the triangular decomposition of $\tilde{\mathcal{U}}_q(\hat{\mathfrak{g}})$ proved in lemma 3.9.

Proof : First $\tilde{\tau}_+\mathcal{U}_q(\hat{\mathfrak{h}}) \subset \mathcal{U}_q(\hat{\mathfrak{h}})\tau_+$, $\mathcal{U}_q(\hat{\mathfrak{h}})\tilde{\tau}_- \subset \tilde{\tau}_-\mathcal{U}_q(\hat{\mathfrak{h}})$ are direct consequences of relations (22), (23), (24). Let us show that :

$$[\sum_{\pi\in\Sigma_s}\sum_{k=0..s}(-1)^k \begin{bmatrix} s \\ k \end{bmatrix}_{q_i} x_i^\pm(w_{\pi(1)})...x_i^\pm(w_{\pi(k)})x_j^\pm(z)x_i^\pm(w_{\pi(k+1)})...x_i^\pm(w_{\pi(s)}), x_l^\mp(u)] = 0 \quad (32)$$

where $i,j,l \in I$, $i \neq j$. If $l \neq j$ and $l \neq i$ the equation (32) follows from relations (25)). If $l = j$, the equation (32) follows from the identity (30) of lemma 3.10 because the left term is :

$$\sum_{\pi\in\Sigma_s}\sum_{k=0..s}(-1)^k \begin{bmatrix} s \\ k \end{bmatrix}_{q_i} x_i^\pm(w_{\pi(1)})...x_i^\pm(w_{\pi(k)})(\delta(\frac{z}{u})\phi_j^\pm(z) - \delta(\frac{z}{u})\phi_j^\mp(z))x_i^\pm(w_{\pi(k+1)})...x_i^\pm(w_{\pi(s)})$$

If $l = i$, the equation (32) follows from the identity (31) of lemma 3.10 because the left term is :

$$\begin{aligned}\sum_{\pi\in\Sigma_s}\sum_{k=0..s}(-1)^k \begin{bmatrix} s \\ k \end{bmatrix}_{q_i} (&\sum_{k'=1...k} x_i^\pm(w_{\pi(1)})...x_i^\pm(w_{\pi(k'-1)})\delta(\frac{w_{k'}}{u})(\phi_i^\pm(w_{k'}) - \phi_i^\mp(w_{k'})) \\ &x_i^\pm(w_{\pi(k'+1)})...x_i^\pm(w_{\pi(k)})x_j^\pm(z)x_i^\pm(w_{\pi(k+1)})...x_i^\pm(w_{\pi(s)}) \\ &+\sum_{k'=k+1...s} x_i^\pm(w_{\pi(1)})...x_i^\pm(w_{\pi(k)})x_j^\pm(z)x_i^\pm(w_{\pi(k+1)})...x_i^\pm(w_{\pi(k'-1)}) \\ &\delta(\frac{w_{k'}}{u})(\phi_i^\pm(w_{k'}) - \phi_i^\mp(w_{k'}))x_i^\pm(w_{\pi(k'+1)})...x_i^\pm(w_{\pi(s)}))\end{aligned}$$

So we have proved the equation (32) and in particular $\tilde{\tau}_+\tilde{\mathcal{U}}_q^-(\hat{\mathfrak{g}}) \subset \tilde{\mathcal{U}}_q(\hat{\mathfrak{g}})\tilde{\tau}_+$, $\tilde{\mathcal{U}}_q^+(\hat{\mathfrak{g}})\tilde{\tau}_- \subset \tilde{\tau}_-\tilde{\mathcal{U}}_q(\hat{\mathfrak{g}})$. □

4 Integrable representations and category $\mathcal{O}(\mathcal{U}_q(\hat{\mathfrak{g}}))$

In this section we study highest weight representations of $\mathcal{U}_q(\hat{\mathfrak{g}})$. In particular the theorem 4.9 is a generalization of a result of Chari-Pressley about integrable representations.

4.1 Integrable representations of quantum Kac-Moody algebras (reminder)

In this section we review some known properties of integrable representations of $\mathcal{U}_q(\mathfrak{g})$.
For V a $\mathcal{U}_q(\mathfrak{h})$-module and $\omega \in \mathfrak{h}^*$ we denote by V_ω the weight space of weight ω :

$$V_\omega = \{v \in V / \forall h \in \mathfrak{h}, k_h.v = q^{\omega(h)}v\}$$

In particular for $v \in V_\omega$ we have $k_i.v = q_i^{\omega(\alpha_i^\vee)}v$ and for $i \in I$ we have $x_i^\pm.V_\omega \subset V_{\omega\pm\alpha_i}$.
We say that V is $\mathcal{U}_q(\mathfrak{h})$-diagonalizable if $V = \bigoplus_{\omega\in\mathfrak{h}^*} V_\omega$ (in particular V is of type 1).

Definition 4.1. *A $\mathcal{U}_q(\mathfrak{g})$-module V is said to be integrable if V is $\mathcal{U}_q(\mathfrak{h})$-diagonalizable, $\forall\omega \in \mathfrak{h}^*$, V_ω is finite dimensional, and for $\mu \in \mathfrak{h}^*$, $i \in I$ there is $R \geq 0$ such that $r \geq R \Rightarrow V_{\mu\pm r\alpha_i} = \{0\}$.*

In particular for all $v \in V$ there is $m_v \geq 0$ such that for all $i \in I$, $m \geq m_v$, $(x_i^+)^m.v = (x_i^-)^m.v = 0$, and $U_i.v$ is finite dimensional.

Definition 4.2. *A $\mathcal{U}_q(\mathfrak{g})$-module V is said to be of highest weight $\omega \in \mathfrak{h}^*$ if there is $v \in V_\omega$ such that V is generated by v and $\forall i \in I$, $x_i^+.v = 0$.*

In particular $V = \mathcal{U}_q(\mathfrak{g})^-.v$ (theorem 2.3), V is $\mathcal{U}_q(\mathfrak{h})$-diagonalizable, and $V = \bigoplus\limits_{\lambda\leq\omega} V_\lambda$. We have (see [L]) :

Theorem 4.3. *For any $\omega \in \mathfrak{h}^*$ there is a unique up to isomorphism simple highest weight module $L(\omega)$ of highest weight ω. The highest weight module $L(\omega)$ is integrable if and only $\omega \in P^+$.*

4.2 Integrable representations of quantum affinizations

In this section we generalize results of Chari-Pressley [CP3, CP4] to all quantum affinizations.

4.2.1 l-highest weight modules

We give the following notion of l-weight :

Definition 4.4. *A couple (λ, Ψ) such that $\lambda \in \mathfrak{h}^*$, $\Psi = (\Psi_{i,\pm m}^\pm)_{i\in I, m\geq 0}$, $\Psi_{i,\pm m}^\pm \in \mathbb{C}$, $\Psi_{i,0}^\pm = q_i^{\pm\lambda(\alpha_i^\vee)}$ is called a l-weight.*

The condition $\Psi_{i,0}^\pm = q_i^{\pm\lambda(\alpha_i^\vee)}$ is a compatibility condition which comes from $\phi_{i,0}^\pm = k_i^\pm$. We denote by P_l the set of l-weights. Note that in the finite case λ is uniquely determined by Ψ because $\lambda = \sum\limits_{i\in I} \lambda(\alpha_i^\vee)\Lambda_i$. Analogs of those l-weights were also used in [M1] for toroidal $\hat{sl}_n$-cases.

Definition 4.5. *A $\mathcal{U}_q(\hat{\mathfrak{g}})$-module V is said to be of l-highest weight $(\lambda, \Psi) \in P_l$ if there is $v \in V$ such that ($i \in I, r \in \mathbb{Z}, m \geq 0, h \in \mathfrak{h}$) :*

$$x_{i,r}^+.v = 0 \ , \ V = \mathcal{U}_q(\hat{\mathfrak{g}}).v \ , \ \phi_{i,\pm m}^\pm.v = \Psi_{i,\pm m}^\pm v \ , \ k_h.v = q^{\lambda(h)}.v$$

In particular $\mathcal{U}_q(\hat{\mathfrak{g}})^-.v = V$ (theorem 3.2), V is $\mathcal{U}_q(\mathfrak{h})$-diagonalizable and $V = \bigoplus\limits_{\lambda\leq\omega} V_\lambda$. Note that the l-weight $(\lambda, \Psi) \in P_l$ is uniquely determined by V. It is called the l-highest weight of V.

The notion of l-highest weight is different from the notion of highest weight for quantum affine algebras. The term “pseudo highest weight” is also used in the literature.

Example : for any $(\lambda, \Psi) \in P_l$, define the Verma module $M(\lambda, \Psi)$ as the quotient of $\mathcal{U}_q(\hat{\mathfrak{g}})$ by the left ideal generated by the $x_{i,r}^+$ $(i \in I, r \in \mathbb{Z})$, $k_h - q^{\lambda(h)}$ $(h \in \mathfrak{h})$, $\phi^{\pm}_{i,\pm m} - \Psi^{\pm}_{i,\pm m}$ $(i \in I, m \geq 0)$. It follows from theorem 3.2 that $M(\lambda, \Psi)$ is a free $\mathcal{U}_q^-(\hat{\mathfrak{g}})$-module of rank 1. In particular it is non trivial and it is a l-highest weight module of highest weight (λ, Ψ). Moreover it has a unique proper submodule (mimic the classical argument in [Kac]), and :

Proposition 4.6. *For any $(\lambda, \Psi) \in P_l$ there is a unique up to isomorphism simple l-highest weight module $L(\lambda, \Psi)$ of l-highest weight (λ, Ψ).*

4.2.2 Integrable $\mathcal{U}_q(\hat{\mathfrak{g}})$-modules

Definition 4.7. *A $\mathcal{U}_q(\hat{\mathfrak{g}})$-module V is said to be integrable if V is integrable as a $\mathcal{U}_q(\mathfrak{g})$-module.*

Note that in the case of a quantum affine algebra, the two notions of integrability do not coincide. Throughout the paper only the notion of integrability of definition 4.7 is used.

For $i \in I, r \in \mathbb{Z}$ and $\omega \in \mathfrak{h}^*$ we have $x^{\pm}_{i,r}.V_\omega \subset V_{\omega \pm \alpha_i}$. So if V is integrable, for all $v \in V$, $\hat{U}_i.v$ is finite dimensional and there is $m_0 \geq 1$ such that for all $i \in I$, $r \in \mathbb{Z}$, $m \geq m_0 \Rightarrow (x^+_{i,r})^m.v = (x^-_{i,r})^m.v = 0$.

Definition 4.8. *The set P_l^+ of dominant l-weights is the set of $(\lambda, \Psi) \in P_l$ such that there exist (Drinfel'd)-polynomials $P_i(z) \in \mathbb{C}[z]$ $(i \in I)$ of constant term 1 such that in $\mathbb{C}[[z]]$ (resp. in $\mathbb{C}[[z^{-1}]]$) :*

$$\sum_{m \geq 0} \Psi^{\pm}_{i,\pm m} z^{\pm m} = q_i^{deg(P_i)} \frac{P_i(zq_i^{-1})}{P_i(zq_i)}$$

In particular for all $i \in I$, $\lambda(\alpha_i^\vee) = \deg(P_i) \geq 0$ and so $\lambda \in P^+$ is a dominant weight.

Theorem 4.9. *For $(\lambda, \Psi) \in P_l$, $L(\lambda, \Psi)$ is integrable if and only $(\lambda, \Psi) \in P_l^+$.*

If $\mathfrak{g}$ is finite (case of a quantum affine algebra) it is a result of Chari-Pressley in [CP3] (if part) and in [CP4] (only if part). Moreover in this case the integrable $L(\lambda, \Psi)$ are finite dimensional. If $\mathfrak{g}$ is symmetric the result is geometrically proved by Nakajima in [N1]. If C is of type $A_n^{(1)}$ (toroidal $\hat{sl}_n$-case) the result is algebraically proved by Miki in [M1].

For the general case we propose a proof similar to the proof given by Chari-Pressley in the finite case. For $\lambda \in \mathfrak{h}^*$ denote $D(\lambda) = \{\omega \in \mathfrak{h}^* / \omega \leq \lambda\}$.

Proof : The proof uses the result for $\mathcal{U}_q(\hat{sl}_2)$ which is proved in [CP1, CP3].

First suppose that $L = L(\lambda, \Psi)$ is integrable and for $i \in I$ let L_i be the $\hat{U}_i$-submodule of L generated by the highest weight vector v. It is a l-highest weight $\mathcal{U}_{q_i}(\hat{sl}_2)$-module of highest weight $(\lambda(\alpha_i^\vee),\Psi_i^\pm)$. As L is integrable, L_i is finite dimensional. So the result for $\mathcal{U}_{q_i}(\hat{sl}_2)$ gives $P_i(z)\in\mathbb{C}[z]$ such that :

$$\sum_{m\geq 0}\Psi_{i,\pm m}^{\pm}z^{\pm m}=q_i^{\deg(P_i)}\frac{P_i(zq_i^{-1})}{P_i(zq_i)}\text{ , }\lambda(\alpha_i^\vee)=\deg(P_i)\geq 0$$

Now we prove that $L=L(\lambda,\Psi)=\mathcal{U}_q(\hat{\mathfrak{g}}).v$ is integrable where $(\lambda,\Psi)\in P_l^+$. It suffices to prove that :

(1) For all $\mu\leq\lambda$, if $L_\mu\neq\{0\}$ then there exists $M>0$ such that $m>M\Rightarrow L_{\mu-m\alpha_i}=L_{\mu+m\alpha_i}=0$ for all $i\in I$.

(2) For all $\mu\leq\lambda$, $\dim(L_\mu)<\infty$.

The proof goes roughly as in section 5 of [CP3], with the following modifications :

For (1) : the existence of M for $L_{\mu+m\alpha_i}=0$ is clear because the weights of L are in $D(\lambda)$. Put $r^\vee=\max\{-C_{i,j}/i\neq j\}$. In particular if C is finite, we have $r^\vee\leq 3$. First we prove that for $m>0$, the space $L_{\mu-m\alpha_i}$ is spanned by vectors of the form $X_1^-x_{i_1,k_1}^-...X_h^-x_{i_h,k_h}^-X_{h+1}^-.v$ where $\lambda-\mu=\alpha_{i_1}+...+\alpha_{i_h}$, $k_1,...,k_h\in\mathbb{Z}$, X_p^- is of the form $X_p^-=x_{i,l_{1,p}}^-...x_{i,l_{m_p,p}}^-$ where $m_1+...+m_{h+1}=m$ and $m_1,...,m_h\leq r^\vee$ (which is the crucial condition). It is proved by induction on h (see [CP3] section 5, (e)) with the help of the relations (20). Note that in [CP3] $r^\vee=3$. Now it suffices to prove that $\hat{U}_i.v$ is finite dimensional : indeed if $m>r^\vee h+\dim(\hat{U}_i.v)$ we have $m_{h+1}>\dim(\hat{U}_i.v)$ and $X_{h+1}^-.v=0$. It is shown exactly as in lemma 2.3 of [CP2] that $\hat{U}_i.v$ is irreducible as $\hat{U}_i$-module, and so is finite dimensional.

For (2) : let us write $\lambda-\mu=\alpha_{i_1}+...+\alpha_{i_h}$. The result is proved by induction on h. We have seen that $\hat{U}_i.v$ is finite dimensional. The induction is shown exactly as in [CP3] (section 5. (b)) by considering the $L_{\lambda-\mu+\alpha_{i_j}}$ and with the help of relation (19). ☐

4.3 Category $\mathcal{O}(\mathcal{U}_q(\hat{\mathfrak{g}}))$

In the following by subcategory we mean full subcategory.

Definition 4.10. *A $\mathcal{U}_q(\mathfrak{h})$-module V is said to be in the category $\mathcal{O}(\mathcal{U}_q(\mathfrak{h}))$ if :*

i) V is $\mathcal{U}_q(\mathfrak{h})$-diagonalizable

ii) for all $\omega\in\mathfrak{h}^*$, dim$(V_\omega)<\infty$

iii) there is a finite number of element $\lambda_1,...,\lambda_s\in\mathfrak{h}^*$ such that the weights of V are in $\bigcup_{j=1...s}D(\lambda_j)$

A $\mathcal{U}_q(\mathfrak{g})$-module (resp. a $\mathcal{U}_q(\hat{\mathfrak{g}})$-module) is said to be in the category $\mathcal{O}(\mathcal{U}_q(\mathfrak{g}))$ (resp. $\mathcal{O}(\mathcal{U}_q(\hat{\mathfrak{g}}))$) if it is in the category $\mathcal{O}(\mathcal{U}_q(\mathfrak{h}))$ as a $\mathcal{U}_q(\mathfrak{h})$-module.

In particular we have a restriction functor res : $\mathcal{O}(\mathcal{U}_q(\hat{\mathfrak{g}}))\to\mathcal{O}(\mathcal{U}_q(\mathfrak{g}))$.

For example a highest weight $\mathcal{U}_q(\mathfrak{g})$-module is in the category $\mathcal{O}(\mathcal{U}_q(\mathfrak{g}))$ and the product $\otimes$ is well-defined on $\mathcal{O}(\mathcal{U}_q(\mathfrak{g}))$. An integrable l-highest weight module is in the category

$\mathcal{O}(\mathcal{U}_q(\hat{\mathfrak{g}}))$. But in general a l-highest weight module is not in the category $\mathcal{O}(\mathcal{U}_q(\hat{\mathfrak{g}}))$, indeed ($\mathbb{C}_r[z]$ is the space of polynomials of degree lower that r) :

Lemma 4.11. *Consider a l-weight $(\omega, \Psi) \in P_l$ and $i \in I$. If $dim(L(\omega,\Psi)_{\omega-\alpha_i}) = r \in \mathbb{N}$ then there is $P(z) \in \mathbb{C}_r[z]$ such that $P(z)\Psi_i(z) = 0$ where $\Psi_i(z) = \sum_{r\geq 0}(\Psi_{i,r}^+ z^r - \Psi_{i,-r}^- z^{-r})$.*

In particular the existence of a $P(z) \in \mathbb{C}[z]$ such that $P(z)\Psi_i(z) = 0$ for all $i \in I$ is a necessary condition for $L(\omega,\Psi) \in \mathcal{O}(\mathcal{U}_q(\hat{\mathfrak{g}}))$.

Proof : Let $v_0, v_1, ..., v_r \in L(\omega,\Psi)$ such that :

$$L(\omega,\Psi)_\omega = \mathbb{C}v_0 \text{ , } L(\omega,\Psi)_{\omega-\alpha_i} = \mathbb{C}v_1 \oplus ... \oplus \mathbb{C}v_r$$

For $m \in \mathbb{Z}$ let $\Psi_{i,m} = \Psi_{i,m}^+ - \Psi_{i,m}^-$. As $x_{i.m}^+.v_0 = 0$, we have :

$$x_{i,m}^+ x_{i,m'}^-.v_0 = \frac{1}{q_i - q_i^{-1}}\Psi_{i,m+m'}v_0$$

As $x_{i,m}^-.v_0 \in L(\omega,\Psi)_{\omega-\alpha_i}$ and $x_{i,m}^+.v_j \in L(\omega,\Psi)_\omega$, there are $\lambda_m^j, \mu_m^j \in \mathbb{C}$ ($m \in \mathbb{Z}, 1 \leq j \leq r$) such that :

$$x_{i,m}^-.v_0 = \lambda_m^1 v_1 + ... + \lambda_m^r v_r \text{ , } x_{i,m}^+.v_j = \mu_m^j v_0$$

In particular we have : $\Psi_{i,m+m'} = (q_i - q_i^{-1})\sum_{j=1...r}\lambda_{m'}^j \mu_m^j$. We set $\lambda^j(z) = \sum_{m'\in\mathbb{Z}}\lambda_{m'}^j z^{m'}$, $\Psi_i(z) = \sum_{r\geq 0}\Psi_{i,r}^+ z^r - \Psi_{i,-r}^- z^{-r}$ and we have :

$$z^{-m}\Psi_i(z) = (q_i - q_i^{-1})\sum_{j=1...r}\mu_m^j \lambda^j(z)$$

So the $\{\Psi_i(z), z\Psi_i(z), ..., z^r\Psi_i(z)\}$ are not linearly independent. □

5 q-characters

For a quantum Kac-Moody algebra, one can define a character morphism as in the classical case. For quantum affine algebras a more precise morphism, called morphism of q-characters, was introduced by Frenkel-Reshetikhin [FR3] (in particular to distinguish finite dimensional representations). In this section we generalize the construction of q-characters to quantum affinizations. The technical point is to add terms k_λ ($\lambda \in \mathfrak{h}^*$) to make it well-defined in the general case. We prove a symmetry property of q-characters that generalizes a result of Frenkel-Mukhin : the image of χ_q is the intersection of the kernels of screening operators (theorem 5.15).

5.1 Reminder : classical character

Let $\mathcal{U}_q(\mathfrak{g})$ be a quantum Kac-Moody algebra. Let $\mathcal{E} \subset (\mathfrak{h}^*)^{\mathbb{Z}}$ be the subset of $c : \mathfrak{h}^* \to \mathbb{Z}$ such that $c(\lambda) = 0$ for λ outside the union of a finite number of sets of the form $D(\mu)$. For

$\lambda \in \mathfrak{h}^*$ denote $e(\lambda) \in \mathcal{E}$ such that $e(\lambda)(\mu) = \delta_{\lambda,\mu}$. $\mathcal{E}$ has a natural structure of commutative $\mathbb{Z}$-algebra such that $e(\lambda)e(\mu) = e(\lambda + \mu)$ (see [Kac]).

The classical character is the map $\mathrm{ch} : \mathcal{O}(\mathcal{U}_q(\mathfrak{g})) \to \mathcal{E}$ such that for $V \in \mathcal{O}(\mathcal{U}_q(\mathfrak{g}))$:

$$\mathrm{ch}(V) = \sum_{\omega \in \mathfrak{h}^*} \dim(V_\omega) e(\omega)$$

ch is a ring morphism and $\mathrm{ch}(L(\omega_1)) = \mathrm{ch}(L(\omega_2)) \Rightarrow \omega_1 = \omega_2$.

5.2 Formal character

Let $\mathcal{U}_q(\hat{\mathfrak{g}})$ be a quantum affinization. In general the map $\mathrm{ch} \circ \mathrm{res}$ does not distinguish the simple integrable representations in $\mathcal{O}(\mathcal{U}_q(\hat{\mathfrak{g}}))$. That is why Frenkel-Reshetikhin [FR3] introduced the theory of q-characters for quantum affine algebras. We generalize the construction for quantum affinizations.

Let V be in $\mathcal{O}(\mathcal{U}_q(\hat{\mathfrak{g}}))$. For $\omega \in \mathfrak{h}^*$, the subspace $V_\omega \subset V$ is stable by the operators $\phi^\pm_{i,\pm m}$ ($i \in I$, $m \geq 0$). Moreover they commute and $[\phi^\pm_{i,m}, k_h] = 0$, so we have a pseudo-weight space decomposition :

$$V_\omega = \bigoplus_{\gamma/(\omega,\gamma) \in P_l} V_{\omega,\gamma}$$

where $V_{\omega,\gamma}$ is a simultaneous generalized eigenspace :

$$V_{\omega,\gamma} = \{x \in V_\omega / \exists p \in \mathbb{N}, \forall i \in \{1, ..., n\}, \forall m \geq 0, (\phi^\pm_{i,\pm m} - \gamma^\pm_{i,\pm m})^p.x = 0\}$$

As V_ω is finite dimensional the $V_{\omega,\gamma}$ are finite dimensional.

Let $\mathcal{E}_l \subset P_l^{\mathbb{Z}}$ be the ring of maps $c : P_l \to \mathbb{Z}$ such that $c(\lambda, \Psi) = 0$ for λ outside the union of a finite number of sets of the form $D(\mu)$.

Definition 5.1. *The formal character of a module V in the category $\mathcal{O}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ is $ch_q(V) \in \mathcal{E}_l$ defined by :*

$$ch_q(V) = \sum_{(\mu,\Gamma) \in P_l} dim(V_{\mu,\Gamma}) e(\mu, \Gamma)$$

We have the following commutative diagram :

$$\begin{array}{ccc} \mathcal{O}(\mathcal{U}_q(\hat{\mathfrak{g}})) & \xrightarrow{\mathrm{ch}_q} & \mathcal{E}_l \\ \downarrow \mathrm{res} & & \downarrow \beta \\ \mathcal{O}(\mathcal{U}_q(\mathfrak{g})) & \xrightarrow{\mathrm{ch}} & \mathcal{E} \end{array}$$

where $\beta : \mathcal{E}_l \to \mathcal{E}$ is constructed from the first projection $\pi_1 : P_l \to P$.

5.3 Morphism of q-characters

The combinatorics of formal characters can be studied with a morphism of q-characters χ_q which is defined on a category $\mathcal{O}_{\mathrm{int}}(\mathcal{U}_q(\hat{\mathfrak{g}}))$:

5.3.1 The category $\mathcal{O}_{\mathbf{int}}(\mathcal{U}_q(\hat{\mathfrak{g}}))$

Denote by $\mathcal{O}_{\text{int}}(\mathcal{U}_q(\mathfrak{g}))$ (resp. $\mathcal{O}_{\text{int}}(\mathcal{U}_q(\hat{\mathfrak{g}}))$) the category of integrable representations in the category $\mathcal{O}(\mathcal{U}_q(\mathfrak{g}))$ (resp. $\mathcal{O}(\mathcal{U}_q(\hat{\mathfrak{g}}))$). For example a simple integrable l-highest weight $\mathcal{U}_q(\hat{\mathfrak{g}})$-modules is in $\mathcal{O}_{\text{int}}(\mathcal{U}_q(\hat{\mathfrak{g}}))$. Moreover :

Proposition 5.2. *For V a module in $\mathcal{O}_{int}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ there are $P_{(\lambda,\Psi)} \geq 0$ ($(\lambda, \Psi) \in P_l^+$) such that :*

$$ch_q(V) = \sum_{(\lambda,\Psi)\in P_l^+} P_{(\lambda,\Psi)} ch_q(L(\lambda, \Psi))$$

Proof : We have two preliminary points :

1) a submodule, a quotient of an integrable module is integrable.

2) for $V \in \mathcal{O}_{\text{int}}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ a module and μ a maximal weight of V, then there is $v \in V_\mu$ such that $\mathcal{U}_q(\hat{\mathfrak{g}}).v$ is a l-highest weight module : indeed for $(\mu, \gamma) \in P_l$ such that $V_{\mu,\gamma} \neq \{0\}$ there is $v \in V_{\mu,\gamma} - \{0\}$ such that $\forall i \in I, r \geq 0$, $\phi^{\pm}_{i,\pm r}.v = \gamma^{\pm}_{i,\pm r} v$ (because for all $i \in I, r \geq 0$, $\text{Ker}(\phi^{\pm}_{i,\pm r} - \gamma^{\pm}_{i,\pm r}) \cap V_{\mu,\gamma} \neq \{0\}$).

The end of the proof is essentially made in [Kac] (proposition 9.7) : first we prove that for $\lambda \in \mathfrak{h}^*$ there exists a filtration by a sequence of submodules in $\mathcal{O}_{\text{int}}(\mathcal{U}_q(\hat{\mathfrak{g}}))$: $V = V_t \supset V_{t-1} \supset ... \supset V_1 \supset V_0 = 0$ and $J \subset \{1, ..., t\}$ such that :

(i) if $j \in J$, then $V_j/V_{j-1} \simeq L(\lambda_j, \Psi_j)$ for some $(\lambda_j, \Psi_j) \in P_l^+$ such that $\lambda_j \geq \lambda$

(ii) if $j \notin J$, then $(V_j/V_{j-1})_\mu = 0$ for every $\mu \geq \lambda$

(see the lemma 9.6 of [Kac]). Next for $(\mu, \Psi) \in P_l^+$, fix λ such that $\mu \geq \lambda$ and introduce $P_{(\mu,\Psi)}$ the number of times (μ, Ψ) appears among the (λ_j, Ψ_j) (it is independent of the choice of the filtration and of μ). We conclude as in proposition 9.7 of [Kac]. □

Definition 5.3. *QP_l^+ is the set of $(\mu, \gamma) \in P_l$ such that there exist polynomials $Q_i(z), R_i(z) \in \mathbb{C}[z]$ ($i \in I$) of constant term 1 such that in $\mathbb{C}[[z]]$ (resp. in $\mathbb{C}[[z^{-1}]]$) :*

$$\sum_{m\geq 0} \gamma^{\pm}_{i,\pm m} z^{\pm m} = q_i^{deg(Q_i)-deg(R_i)} \frac{Q_i(zq_i^{-1})R_i(zq_i)}{Q_i(zq_i)R_i(zq_i^{-1})}$$

and such that there exit $\omega \in P^+$, $\alpha \in Q^+$ satisfying $\mu = \omega - \alpha$.

In particular $P_l^+ \subset QP_l^+$.

Proposition 5.4. *Let V be a module in $\mathcal{O}_{int}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ and $(\mu, \gamma) \in P_l$. If $dim(V_{\mu,\gamma}) > 0$ then $(\mu, \gamma) \in QP_l^+$.*

Proof : The existence of the polynomials is shown as in [FR3] (proposition 1) : it reduces to the sl_2-case because for $v \in V$, $\hat{U}_i.v$ is finite dimensional. The existence of $\omega \in P$ and $\alpha \in Q^+$ is a consequence of proposition 5.2 and theorem 4.9. □

5.3.2 Construction of q-characters

Consider formal variables $Y_{i,a}^{\pm}$ ($i \in I, a \in \mathbb{C}^*$) and k_ω ($\omega \in \mathfrak{h}$). Let $\tilde{A}$ be the commutative group of monomials of the form $m = \prod_{i\in I, a\in\mathbb{C}^*} Y_{i,a}^{u_{i,a}(m)} k_{\omega(m)}$ ($k_0 = 1$) where only a finite number of $u_{i,a}(m) \in \mathbb{Z}$ are non zero, $\omega(m) \in \mathfrak{h}$ (the coweight of m), and such that for $i \in I$:

$$\alpha_i(\omega(m)) = r_i u_i(m) = r_i \sum_{a\in\mathbb{C}^*} u_{i,a}(m)$$

The product is given by $u_{i,a}(m_1 m_2) = u_{i,a}(m_1) + u_{i,a}(m_2)$ and $\omega(m_1 m_2) = \omega(m_1) + \omega(m_2)$. For example for $i \in I, a \in \mathbb{C}^*$, we have $k_{\nu(\Lambda_i)} Y_{i,a} \in \tilde{A}$ because for $j \in I$, $\alpha_j(\nu(\Lambda_i)) = \Lambda_i(\nu(\alpha_j)) = r_j \Lambda_i(\alpha_j^\vee) = r_j \delta_{i,j}$. For $(\mu, \Gamma) \in QP_l^+$ we define $Y_{\mu,\Gamma} \in \tilde{A}$ by :

$$Y_{\mu,\Gamma} = k_{\nu(\mu)} \prod_{i\in I, a\in\mathbb{C}^*} Y_{i,a}^{\beta_{i,a}-\gamma_{i,a}}$$

where $\beta_{i,a}, \gamma_{i,a} \in \mathbb{Z}$ are defined by $Q_i(u) = \prod_{a\in\mathbb{C}^*} (1-ua)^{\beta_{i,a}}$, $R_i(u) = \prod_{a\in\mathbb{C}^*} (1-ua)^{\gamma_{i,a}}$. We have $Y_{\mu,\Gamma} \in \tilde{A}$ because for $i \in I$:

$$\alpha_i(\nu(\mu)) = \mu(\nu(\alpha_i)) = r_i \mu(\alpha_i^\vee) = r_i(\deg(Q_i) - \deg(R_i)) = r_i u_i(Y_{\mu,\Gamma})$$

For $\chi \in \tilde{A}^{\mathbb{Z}}$ we say $\chi \in \mathcal{Y}$ if there is a finite number of element $\lambda_1, ..., \lambda_s \in \mathfrak{h}^*$ such that the coweights of monomials of χ are in $\bigcup_{j=1...s} \nu(D(\lambda_j))$. In particular $\mathcal{Y}$ has a structure of $\mathfrak{h}$-graded $\mathbb{Z}$-algebra.

Definition 5.5. *The q-character of a module $V \in \mathcal{O}_{int}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ is :*

$$\chi_q(V) = \sum_{(\mu,\Gamma)\in QP_l^+} d(\mu,\Gamma) Y_{\mu,\Gamma} \in \mathcal{Y}$$

where $d(\mu,\Gamma) \in \mathbb{Z}$ is defined by $ch_q(V) = \sum_{(\mu,\Gamma)\in QP_l^+} d(\mu,\Gamma) e(\mu,\Gamma)$.

We have a commutative diagram :

$$\begin{array}{ccc} \mathcal{O}_{\text{int}}(\mathcal{U}_q(\hat{\mathfrak{g}})) & \xrightarrow{\chi_q} & \mathcal{Y} \\ \downarrow \text{res} & & \downarrow \beta \\ \mathcal{O}_{\text{int}}(\mathcal{U}_q(\mathfrak{g})) & \xrightarrow{\text{ch}} & \mathcal{E} \end{array}$$

where for $m \in \tilde{A}$, $\beta(m) = e(\omega(m))$.

If C is of finite type then the weight of a monomial $m \in \mathcal{Y}$ is $\omega(m) = \sum_{i\in I} u_i(m) \nu(\Lambda_i)$. So we can forget the k_h, and we get the q-characters defined in [FR3]. In this case the integrable simple modules are finite dimensional.

Note that in the same way one can define the q-character of a finite dimensional $\mathcal{U}_q(\hat{\mathfrak{h}})$-module.

5.3.3 Morphism of q-characters

Denote by $\mathrm{Rep}(\mathcal{U}_q(\mathfrak{g}))$ (resp. $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$) the Grothendieck group generated by the modules V in $\mathcal{O}_{\mathrm{int}}(\mathcal{U}_q(\mathfrak{g}))$ (resp. $\mathcal{O}_{\mathrm{int}}(\mathcal{U}_q(\hat{\mathfrak{g}}))$) which have a composition series (a sequence of modules $V \supset V_1 \supset V_2 \supset ...$ such that V_i/V_{i+1} is irreducible).

The tensor product defines a ring structure on $\mathrm{Rep}(\mathcal{U}_q(\mathfrak{g}))$ and ch gives a ring morphism $\chi : \mathrm{Rep}(\mathcal{U}_q(\mathfrak{g})) \to \mathcal{E}$.

The q-characters are compatible with exact sequences and so we get a group morphism $\chi_q : \mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}})) \to \mathcal{Y}$ which is called morphism of q-characters.

Proposition 5.6. *The morphism χ_q is injective and the following diagram is commutative :*

$$\begin{array}{ccc} Rep(\mathcal{U}_q(\hat{\mathfrak{g}})) & \xrightarrow{\chi_q} & \mathcal{Y} \\ \downarrow res & & \downarrow \beta \\ Rep(\mathcal{U}_q(\mathfrak{g})) & \xrightarrow{\chi} & \mathcal{E} \end{array}$$

The commutativity of the diagram follows from the definition. To see that χ_q is injective, let us give some definitions :

A monomial $m \in \tilde{A}$ is said to be dominant if $u_{i,a}(m) \geq 0$ for all $i \in I, a \in \mathbb{C}^*$. If a l-weight (ω, Ψ) belongs to P_l^+ then $Y_{(\omega,\Psi)} \in \tilde{A}$ is dominant. Moreover the map $(\omega, \Psi) \mapsto Y_{(\omega,\Psi)}$ defines a bijection between P_l^+ and dominant monomials. For $m \in \tilde{A}$ a dominant monomial we denote by $L(m) \in \mathcal{Y}$ the q-character of $L(\omega, \Psi)$ where (ω, Ψ) is the corresponding dominant l-weight. In particular $L(m) = m +$ monomials of lower weight (in the sense of the ordering on P), and so the $L(m)$ are linearly independent.

A module with composition series in determined in the Grothendieck group by the multiplicity of the simple modules, and we have seen that the $\chi_q(L(\lambda, \Psi))$ ($(\lambda, \Psi) \in P_l^+$) are linearly independent in $\mathcal{Y}$. So χ_q is injective.

5.4 q-characters and universal $\mathcal{R}$-matrix

The original definition of q-characters ([FR3]) was based on an explicit formula for the universal $\mathcal{R}$-matrix established in [KT, LSS, Da]. In general no universal $\mathcal{R}$-matrix has been defined for a quantum affinization. However q-characters can be obtained with a piece of the formula of a "$\mathcal{R}$-matrix" in the same spirit as the original approach :

We refer to the chapter 3 of [Gu] for general background on h-formal deformations. Consider $\mathcal{U}_h(\hat{\mathfrak{g}})$ the $\mathbb{C}[[h]]$-algebra which is h-topologically generated by $\mathfrak{h}$ and the $x_{i,r}^{\pm}$ ($i \in I, r \in \mathbb{Z}$), $h_{i,m}$ ($i \in I, m \in \mathbb{Z} - \{0\}$) and with the relations of $\mathcal{U}_q(\hat{\mathfrak{g}})$ (where we set for $\omega \in \mathfrak{h}$, $k_\omega = \exp(h\omega)$). The subalgebra $\mathcal{U}_h(\hat{\mathfrak{h}}) \subset \mathcal{U}_h(\hat{\mathfrak{g}})$ is h-topologically generated by $\mathfrak{h}$ and the $h_{i,m}$ ($i \in I, m \in \mathbb{Z} - \{0\}$).

If V is a $\mathcal{U}_q(\hat{\mathfrak{g}})$-module (resp. $\mathcal{U}_q(\hat{\mathfrak{h}})$-module) which is $\mathcal{U}_q(\mathfrak{h})$-diagonalizable then we have an algebra morphism $\pi_V(h) : \mathcal{U}_h(\hat{\mathfrak{g}}) \to \mathrm{End}(V)[[h]]$ (resp. $\pi_V(h) : \mathcal{U}_h(\hat{\mathfrak{h}}) \to \mathrm{End}(V)[[h]]$) (Remark : for $\lambda \in \mathfrak{h}^*$, $\omega \in \mathfrak{h}$, $v \in V_\lambda$ we set $\omega.v = \lambda(\omega)v$).

Define $\mathcal{R}^0$ and T in $\mathcal{U}_h(\hat{\mathfrak{h}})\hat{\otimes}\mathcal{U}_h(\hat{\mathfrak{h}}) \subset \mathcal{U}_h(\hat{\mathfrak{g}})\hat{\otimes}\mathcal{U}_h(\hat{\mathfrak{g}})$ (h-topological completion of the tensor product) by the formula :

$$\mathcal{R}^0 = \exp(-(q-q^{-1})\sum_{i,j\in I, m>0}\frac{m}{[m]_q}\tilde{B}_{i,j}(q^m)h^m h_{i,m}\otimes h_{j,-m})$$

$$T = \exp(-h\sum_{1\leq i\leq 2n-l}\Lambda_i^\vee \otimes \nu(\alpha_i))$$

Remark : we have the usual property of T (see [FR3]) : for $\lambda,\mu\in\mathfrak{h}^*$, $x\in V_\lambda$, $y\in V_\mu$, we have $T.(x\otimes y) = q^{-(\lambda,\mu)}(x\otimes y)$. Indeed :

$$\sum_{1\leq i\leq 2n-l}\lambda(\Lambda_i^\vee)\mu(\nu(\alpha_i)) = (\mu, \sum_{1\leq i\leq 2n-l}\lambda(\Lambda_i^\vee)\alpha_i) = (\mu,\lambda)$$

For $i\in I, m\in\mathbb{Z}-\{0\}$ denote $\tilde{h}_{i,m} = \sum_{j\in I}\tilde{C}_{j,i}(q^m)h_{j,m}$. We have an inclusion $\tilde{A}\subset\mathcal{U}_h(\hat{\mathfrak{h}})$ because the elements $Y_{i,a}^{\pm} = k_{\mp\nu(\Lambda_i)}\exp(\mp(q-q^{-1})\sum_{m\geq 1}h^m a^{-m}\tilde{h}_{i,m})\in\mathcal{U}_h(\hat{\mathfrak{g}})$ $(i\in I, a\in\mathbb{C}^*)$ are algebraically independent.

Theorem 5.7. *For V a finite dimensional $\mathcal{U}_q(\hat{\mathfrak{h}})$-module, $((Tr_V\circ\pi_V(h))\otimes Id)(\mathcal{R}^0T))\in\mathcal{U}_h(\hat{\mathfrak{h}})$ is equal to $\chi_q(V)$.*

Proof : For $(\lambda,\Psi)\in P_l$ consider $V_{(\lambda,\Psi)}$ and $((\mathrm{Tr}_{V_{(\lambda,\Psi)}}\circ\pi_{V_{(\lambda,\Psi)}}(h))\otimes\mathrm{Id})(\mathcal{R}^0T)$. First we see as in [FR3] that the term $\mathcal{R}^0$ gives $\prod_{i\in I,a\in\mathbb{C}^*}Y_{i,a}^{u_{i,a}(Y_{\lambda,\Psi})}$. But we have :

$$\sum_{1\leq i\leq 2n-l}\lambda(\Lambda_i^\vee)\nu(\alpha_i) = \nu(\sum_{1\leq i\leq 2n-l}\lambda(\Lambda_i^\vee)\alpha_i) = \nu(\lambda)$$

and so T gives $k_{-\nu(\lambda)}$. $\square$

In general for $V\in\mathcal{O}_{\mathrm{int}}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ we can consider a filtration $(V_r)_{r\geq 0}$ of finite dimensional sub $\mathcal{U}_q(\hat{\mathfrak{h}})$-modules of V such that $\bigcup_{r\geq 0}V_r = V$; so $\chi_q(V)$ is the "limit" of the $((\mathrm{Tr}_{V_r}\circ\pi_{V_r}(h))\otimes\mathrm{Id})(\mathcal{R}^0T)$ in $\mathcal{Y}$.

5.5 Combinatorics of q-characters

In this section we prove a symmetry property of general q-characters : the image of χ_q is the intersection of the kernels of screening operators (theorem 5.15). Our proof is analog to the proof used by Frenkel-Mukhin [FM1] for quantum affine algebras ; however new technical points are involved because of the k_λ and infinite sums. In particular it shows that those q-characters are the combinatorial objects considered in [He3] (which were constructed in the kernel of screening operators).

In sections 5.5 and 6 we suppose that $C(z)$ is invertible (it includes the cases of quantum affine algebras and quantum toroidal algebras, see section 2). We write $\tilde{C}(z) = \frac{\tilde{C}'(z)}{d(z)}$ where $d(z), \tilde{C}'_{i,j}(z)\in\mathbb{Z}[z^{\pm}]$. For $r\in\mathbb{Z}$ let $p_{i,j}(r) = [(D(z)\tilde{C}'(z))_{i,j}]_r$ where for a Laurent polynomial $P(z)\in\mathbb{Z}[z^{\pm}]$ we put $P(z) = \sum_{r\in\mathbb{Z}}[P(z)]_r z^r$.

5.5.1 Construction of screening operators

Let $\mathcal{Y}^{\mathrm{int}} \subset \mathcal{Y}$ be the subset consisting of those $\chi \in \mathcal{Y}$ satisfying the following property : if λ is the coweight of a monomial of χ there is $K \geq 0$ such that $k \geq K$ implies that for all $i \in I$, $\lambda \pm k r_i \alpha_i^\vee$ is not the coweight of a monomial of χ.

Lemma 5.8. *$\mathcal{Y}^{int}$ is a subalgebra of $\mathcal{Y}$ and $Im(\chi_q) \subset \mathcal{Y}^{int}$.*

Consider the free $\mathcal{Y}^{\mathrm{int}}$-module $\tilde{\mathcal{Y}}_i = \prod_{a \in \mathbb{C}^*} \mathcal{Y}^{\mathrm{int}} S_{i,a}$ and the linear map $\tilde{S}_i : \mathcal{Y}^{\mathrm{int}} \to \tilde{\mathcal{Y}}_i$ such that, for a monomial m :

$$\tilde{S}_i(m) = m \sum_{a \in \mathbb{C}^*} u_{i,a}(m) S_{i,a}$$

In particular $\tilde{S}_i$ is a derivation. Let us choose a representative a for each class of $\mathbb{C}^*/q_i^{2\mathbb{Z}}$ and consider :

$$\mathcal{Y}_i = \prod_{a \in \mathbb{C}^*/q_i^{2\mathbb{Z}}} \mathcal{Y}^{\mathrm{int}} S_{i,a}$$

For $i \in I$ and $a \in \mathbb{C}^*$ we set :

$$A_{i,a} = k_i Y_{i,aq_i^{-1}} Y_{i,aq_i} \prod_{j/C_{j,i}<0\ ,\ r=C_{j,i}+1, C_{j,i}+3, \ldots, -C_{j,i}-1} Y_{j,aq^r}^{-1} \in \tilde{A}$$

We have $A_{i,a} \in \tilde{A}$ because for $j \in I : \alpha_j(r_i \alpha_i^\vee) = r_i C_{i,j} = r_j C_{j,i} = r_j u_j(A_{i,a})$.

We would like to see $\mathcal{Y}_i$ as a quotient of $\tilde{\mathcal{Y}}_i$ by the relations $S_{i,aq_i} = A_{i,a} S_{i,aq_i^{-1}}$. But the projection is not defined for all elements of $\tilde{\mathcal{Y}}_i$ because there are infinite sums. However if $\chi \in \mathcal{Y}^{\mathrm{int}}$ and m is a monomial of χ there is a finite number of monomials in χ of the form $mA_{i,aq_i}^{-1} A_{i,aq_i^3}^{-1} \ldots A_{i,aq_i^r}^{-1}$ or of the form $mA_{i,aq_i^{-1}} A_{i,aq_i^{-3}}^{-1} \ldots A_{i,aq_i^{-r}}^{-1}$. So the projection on $\mathcal{Y}_i$ is well defined on $\tilde{S}_i(\mathcal{Y}^{\mathrm{int}}) \subset \tilde{\mathcal{Y}}_i$. In particular we can define by projection of $\tilde{S}_i$ the i^{th} screening operator $S_i : \mathcal{Y}^{\mathrm{int}} \to \mathcal{Y}_i$.

The original definition for the finite case is in [FR3].

5.5.2 The morphism τ_i

Some operators τ_i $(i \in I)$ were defined for the finite case in [FM1]. We generalize the construction and the properties of the operators τ_i (lemma 5.9 and 5.10).

Let $i \in I$. Denote $\mathfrak{h}_i^\perp = \{\omega \in \mathfrak{h} / \alpha_i(\omega) = 0\}$.

Consider formal variables $k_r^{(i)}$ $(r \in \mathbb{Z})$, k_ω $(\omega \in \mathfrak{h})$, $Y_{i,a}^{\pm}$ $(a \in \mathbb{C}^*)$, $Z_{j,c}$ $(j \in I - \{i\}, c \in \mathbb{C}^*)$. Let $\tilde{A}^{(i)}$ be the commutative group of monomials :

$$m = k_{r(m)}^{(i)} k_{\omega(m)} \prod_{a \in \mathbb{C}^*} Y_{i,a}^{u_{i,a}(m)} \prod_{j \in I, j \neq i, c \in \mathbb{C}^*} Z_{j,c}^{z_{j,c}(m)}$$

where a finite number of $u_{i,a}(m), z_{j,c}(m), r(m) \in \mathbb{Z}$ are non zero, $\omega(m) \in \mathfrak{h}_i^\perp$ and such that $r(m) = r_i u_i(m) = r_i \sum_{a \in \mathbb{C}^*} u_{i,a}(m)$. The product is defined as for $\tilde{A}$. We call $(r(m), \omega(m)) \in \mathbb{Z} \times \mathfrak{h}_i^\perp$ the coweight of the monomial m.

Let $\tau_i : \tilde{A} \to \tilde{A}^{(i)}$ be the group morphism defined by ($j \in I$, $a \in \mathbb{C}^*$, $\lambda \in \mathfrak{h}$) :

$$\tau_i(Y_{j,a}) = Y_{i,a}^{\delta_{i,j}} \prod_{k\neq i\ ,\ r\in\mathbb{Z}} Z_{k,aq^r}^{p_{j,k}(r)}\ ,\ \tau_i(k_\lambda) = k^{(i)}_{\alpha_i(\lambda)} k_{\lambda-\alpha_i(\lambda)\frac{\alpha_i^\vee}{2}}$$

(note that it is a formal definition because $Y_{j,a}k_{\nu(\Lambda_j)} \in \tilde{A}$ but $Y_{j,a} \notin \tilde{A}$). It is well defined because for $m \in \tilde{A}$, $\alpha_i(\omega(m)) = r_i u_i(m)$ and $\alpha_i(\omega(m) - \alpha_i(\omega(m))\frac{\alpha_i^\vee}{2}) = 0$.

Lemma 5.9. *The morphism τ_i is injective and for $a \in \mathbb{C}^*$ we have :*

$$\tau_i(A_{i,a}) = k^{(i)}_{2r_i} Y_{i,aq_i^{-1}} Y_{i,aq_i}$$

Proof : Let $m \in \tilde{A}$ such that $\tau_i(m) = 1$. For $a \in \mathbb{C}^*$ we have $u_{i,a}(m) = u_{i,a}(\tau_i(m)) = 0$. For $k \in I$, $a \in \mathbb{C}^*$ denote $u_{k,a}(m)(z) = \sum\limits_{r\in\mathbb{Z}} u_{k,aq^r}(m)z^r \in \mathbb{Z}[z^\pm]$. For $j \in I - \{i\}$, we have :

$$0 = z_{j,aq^R}(\tau_i(m)) = \sum_{k\in I, r+r'=R} p_{k,j}(r')u_{k,aq^r}(m) = [\sum_{k\in I} \tilde{C}'_{k,j}(z)u_{k,a}(m)(z)]_R$$

As $\tilde{C}(z)$ is invertible we get $u_{k,a}(m) = 0$ for all $a \in \mathbb{C}^*$. In particular for $j \in I$ we have $\alpha_j(\omega(m)) = r_j u_j(m) = 0$. But $\omega(m) - \alpha_i(\omega(m))\frac{\alpha_i^\vee}{2} = 0 = \omega(m)$ and so $m = 1$.
For the second point let $M = \tau_i(A_{i,a})$. First for $b \in \mathbb{C}^*$, $u_{i,b}(M) = u_{i,b}(A_{i,a}) = \delta_{a/b,q_i} + \delta_{a/b,q_i^{-1}}$. For $R \in \mathbb{Z}$ and $j \neq i$ we have :

$$z_{j,aq^R}(M) = [(\tilde{C}'(z)C(z))_{i,j}]_R = [(d(z)D(z))_{i,j}]_R = 0$$

Finally we have $r(M) = r_i\alpha_i(\alpha_i^\vee) = -2r_i$ and $\omega(M) = r_i\alpha_i^\vee - r_i\alpha_i(\alpha_i^\vee)\frac{\alpha_i^\vee}{2} = 0$. □

Formally we have $\tau_i(k_i) = k^{(i)}_{2r_i}$ and for $j \in I - \{i\}$ $\tau_i(k_j) = k^{(i)}_{B_{j,i}} k_{\alpha_j^{(i)}}$ where $\alpha_j^{(i)} = r_j\alpha_j^\vee - \frac{B_{j,i}}{2}\alpha_i^\vee$. This motivates the following definition : for $(r,\omega) \in \mathbb{Z}\times\mathfrak{h}_i^\perp$ we set :

$$D(r,\omega) = \{(r',\omega') \in \mathbb{Z}\times\mathfrak{h}_i^\perp / \omega' = \omega - \sum_{j\in I, j\neq i} m_j\alpha_j^{(i)}\ ,\ r' = r - \sum_{j\in I, j\neq i} B_{j,i}m_j - 2r_ik/m_j, k \geq 0\}$$

Denote $\mathfrak{Y}^{\mathrm{int},(i)} \subset (\tilde{A}^{(i)})^{\mathbb{Z}}$ as the set of χ such that :

i) there is a finite number of elements $(r_1,\omega_1),...,(r_s,\omega_s) \in \mathbb{Z}\times\mathfrak{h}_i^\perp$ such that the coweights of monomials of χ are in $\bigcup\limits_{j=1...s} D(r_j,\omega_j)$.

ii) for (r,λ) the coweight of a monomial of χ there is $K \geq 0$ such that $k \geq K$ implies that for all $j \in I$, $j \neq i$, $(r \pm B_{j,i}k, \lambda \pm k\alpha_j^{(i)})$ and $(r \pm 2kr_i, \lambda)$ are not the coweight of a monomial of χ.

In particular $\mathfrak{Y}^{\mathrm{int},(i)}$ has a structure of $\mathbb{Z}\times\mathfrak{h}_i^\perp$-graded $\mathbb{Z}$-algebra.

The morphism τ_i can be extended to a unique morphism of $\mathbb{Z}$-algebra $\tau_i : \mathfrak{Y}^{\mathrm{int}} \to \mathfrak{Y}^{\mathrm{int},(i)}$. Denote by χ_q^i the morphism of q-characters for the algebra $\mathcal{U}_{q_i}(\hat{sl}_2)$.

Lemma 5.10. *Consider $V \in \mathcal{O}_{int}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ and a decomposition $\tau_i(\chi_q(V)) = \sum_k P_k Q_k$ where $P_k \in \mathbb{Z}[Y_{i,a}^{\pm} k_{\pm r_i}^{(i)}]_{a\in\mathbb{C}^*}$, Q_k is a monomial in $\mathbb{Z}[Z_{j,c}^{\pm}, k_h]_{j\neq i, a\in\mathbb{C}^*, h\in\mathfrak{h}_i^{\perp}}$ and all monomials Q_k are distinct. Then there exists a $\hat{U}_i$-module $\bigoplus_k V_k$ isomorphic to the restriction of V to $\hat{U}_i$ and such that $\chi_q^i(V_k) = P_k$.*

Proof : Let $\mathcal{U}_q(\hat{\mathfrak{h}})_i^{\perp}$ the subalgebra of $\mathcal{U}_q(\hat{\mathfrak{g}})$ generated by the k_h ($h \in \mathfrak{h}_i^{\perp}$), $h_{j,m}$ ($j \neq i$, $m \in \mathbb{Z}-\{0\}$). We can apply the proof of lemma 3.4 of [FM1] with $\hat{U}_i$ and $\mathcal{U}_q(\hat{\mathfrak{h}})_i^{\perp}$ because :

i) $\hat{U}_i$ and $\mathcal{U}_q(\hat{\mathfrak{h}})_i^{\perp}$ commute in $\mathcal{U}_q(\hat{\mathfrak{g}})$

ii) The image $\omega - \alpha_i(\omega)\frac{\alpha_i^{\vee}}{2}$ in $\mathfrak{h}_i^{\perp}$ of $\omega \in \mathfrak{h}$ suffices to encode the action of the k_h ($h \in \mathfrak{h}_i^{\perp}$) on a vector of weight $\nu^{-1}(\omega) = \lambda$. Indeed for $h \in \mathfrak{h}_i^{\perp}$, we have :

$$\lambda(h) = (\nu^{-1}(h), \nu^{-1}(\omega)) = \nu^{-1}(h)(\omega) = \nu^{-1}(h)(\omega - \omega(\alpha_i)\frac{\alpha_i^{\vee}}{2})$$

because $\alpha_i(h) = 0 \Rightarrow \nu^{-1}(\alpha_i^{\vee}) = 0$. □

5.5.3 τ_i and screening operators

In this section we prove that $\mathrm{Im}(\chi_q) \subset \mathrm{Ker}(S_i)$ (proposition 5.12) with a generalization of the proof of Frenkel-Mukhin [FM1].

Consider the $\mathcal{Y}^{\mathrm{int},(i)}$-module $\tilde{\mathcal{Y}}_i^{(i)} = \prod_{a\in\mathbb{C}^*} \mathcal{Y}^{\mathrm{int},(i)} S_{i,a}$ and the linear map $\overline{S}_i : \mathcal{Y}^{\mathrm{int},(i)} \to \tilde{\mathcal{Y}}_i^{(i)}$ such that, for a monomial m :

$$\overline{S}_i(m) = m \sum_{a\in\mathbb{C}^*} u_{i,a}(m) S_{i,a}$$

In particular $\overline{S}_i$ is a derivation. Consider $\mathcal{Y}_i^{(i)} = \prod_{a\in\mathbb{C}^*/q_i^{2\mathbb{Z}}} \mathcal{Y}^{\mathrm{int},(i)} S_{i,a}$. The $\overline{S}_i(\mathcal{Y}^{\mathrm{int},(i)}) \subset \tilde{\mathcal{Y}}_i^{(i)}$ can be projected in $\mathcal{Y}_i^{(i)}$ by the relations :

$$S_{i,aq_i} = Y_{i,aq_i} Y_{i,aq_i^{-1}} k_{2r_i}^{(i)} S_{i,aq_i^{-1}}$$

and we get a derivation that we denote also by $\overline{S}_i : \mathcal{Y}^{\mathrm{int},(i)} \to \mathcal{Y}_i^{(i)}$.

We also define a map $\tau_i : \mathcal{Y}_i \to \mathcal{Y}_i^{(i)}$ in an obvious way (with the help of lemma 5.9). We see as in lemma 5.4 of [FM1] that :

Lemma 5.11. *We have a commutative diagram :*

$$\begin{array}{ccc} \mathcal{Y}^{int} & \xrightarrow{S_i} & \mathcal{Y}_i \\ \downarrow \tau_i & & \downarrow \tau_i \\ \mathcal{Y}^{int,(i)} & \xrightarrow{\overline{S}_i} & \mathcal{Y}_i^{(i)} \end{array}$$

With the help of lemma 5.9, 5.10 and 5.11 we see as in corollary 5.5 of [FM1] :

Lemma 5.12. *We have* $Im(\chi_q) \subset \bigcap_{i \in I} Ker(S_i)$.

In the following we denote $\mathfrak{K}_i = \mathrm{Ker}(S_i)$ and $\mathfrak{K} = \bigcap_{i \in I} \mathfrak{K}_i$.

Lemma 5.13. *An element* $\chi \in \mathcal{Y}^{int}$ *is in* $\mathfrak{K}_i$ *if and only if it can be written in the form* $\chi = \sum_k P_k Q_k$ *where* $P_k \in \mathbb{Z}[k_{\nu(\Lambda_i)} Y_{i,a}(1 + A_{i,aq_i}^{-1})]_{a \in \mathbb{C}^*}$*,* $Q_k \in \mathbb{Z}[Y_{j,a}^{\pm}, k_h]_{j \neq i, a \in \mathbb{C}^*, h \in P_i^{*,\perp}}$ *and all* Q_k *are distinct monomials.*

Proof : We use the result for the sl_2-case which is proved in [FR3]. First an element of this form is in $\mathfrak{K}_i$. Consider $\chi \in \mathfrak{K}_i$ and write $\tau_i(\chi) = \sum_k P_k' Q_k'$ as in lemma 5.10. From lemma 5.11 we have $0 = \overline{S}_i(\chi) = \sum_k \overline{S}_i(P_k') Q_k$. So all $\overline{S}_i(P_k') = 0$ and it follows from the sl_2-case that $P_k' \in \mathbb{Z}[Y_{i,a} k_{r_i}^{(i)} + Y_{i,aq_i^2}^{-1} k_{-r_i}^{(i)}]_{a \in \mathbb{C}^*}$. The lemma 5.9 lead us to the conclusion. $\square$

5.5.4 Description of Im(χ_q)

Dominant monomials are defined in section 5.3.3. We have :

Lemma 5.14. *An element* $\chi \in \mathfrak{K}$ *has at least one dominant monomial.*

With the help of lemma 5.13 we can use the proof of lemma 5.6 of [FM1] (see also the proof of theorem 4.1 in [He1] at $t = 1$).

Theorem 5.15. *We have* $Im(\chi_q) = \mathfrak{K}$*. Moreover the elements of* $\mathfrak{K}$ *are the sums :*

$$\sum_{m \text{ dominant}} \lambda_m L(m)$$

where $\lambda_m = 0$ *for* $\omega(m)$ *outside the union of a finite number of sets of the form* $D(\mu)$*.*

Proof : The inclusion $\mathrm{Im}(\chi_q) \subset \mathfrak{K}$ is proved in lemma 5.12. For the other one, consider $\chi \in \mathfrak{K}$. We can suppose that the weights of χ are in a set $D(\lambda)$ (because the weights of each $L(m)$ are in a set $D(\mu)$). We define by induction $L^{(k)}(m) \in \mathrm{Im}(\chi_q)$ ($k \geq 0$) in the following way : we set $L^{(0)} = \sum_{\omega(m) = \lambda} [\chi]_m L(m)$. If $L^{(k)}$ is defined, we consider the set $\tilde{A}_{k+1}$ of monomials m' which appear in $\chi - L^{(k)}$ such that $\lambda - \omega(m') = m_1 r_1 \alpha_1^\vee + ... + m_n r_n \alpha_n^\vee$ where $m_1, ..., m_n \geq 0$ and $m_1 + ... + m_n = k$. We set :

$$L^{(k+1)} = L^{(k)} + \sum_{m' \in \tilde{A}_{k+1}} [\chi - L^{(k)}]_{m'} L(m')$$

Then we set $L^\infty = \sum_{k \geq 0 / m \in \tilde{A}_k} [L^{(k)}]_m L(m) \in \mathrm{Im}(\chi_q)$ and it follows from lemma 5.14 that $L^\infty = \chi$. $\square$

Note that proposition 5.2 gives that for $\chi_q(V)$ (V module in $\mathcal{O}_{\mathrm{int}}(\mathcal{U}_q(\hat{\mathfrak{g}}))$) the λ_m are non-negative.

Remark : for $m \in \tilde{A}$ a dominant monomial we prove in the same way that there is a unique $F(m) \in \mathfrak{K}$ such that m has coefficient 1 in $F(m)$ and m is the unique dominant monomial in $F(m)$. In the finite case an algorithm was given by Frenkel-Mukhin [FM1] to compute the $F(m)$. In [He3] we extended the definition of the algorithm for generalized Cartan matrix and showed that it is well-defined if $i \neq j \Rightarrow C_{i,j}C_{j,i} \leq 3$ (see also [He2] for the detailed description of this algorithm at $t = 1$). Theorem 5.15 allows us to prove two results announced in [He3] : the algorithm is well defined for

$A_1^{(1)}$ (with $r_1 = r_2 = 2$) because $\det(C(z)) = z^4 - z^2 - z^{-2} + z^{-4} \neq 0$

$A_2^{(2)}$ (with $r_1 = 4$, $r_2 = 1$) because $\det(C(z)) = z^5 - z - z^{-1} + z^{-5} \neq 0$

But for $A_1^{(1)}$ (with $r_1 = r_2 = 1$) we have $\det(C(z)) = 0$; we observed in [He3] that the algorithm is not well-defined in this case.

6 Drinfel'd new coproduct and fusion product

Our study of combinatorics of q-characters gives a ring structure on $\mathrm{Im}(\chi_q)$ (corollary 6.1). As χ_q is injective we get an induced ring structure on the Grothendieck group. In this section we prove that it is a fusion product (theorem 6.2), that is to say that the product of two modules is a module. We use the Drinfel'd new coproduct (proposition 6.3) ; as it involves infinite sums, we have to work in a larger category where the tensor product is well-defined (theorem 6.7). To end the proof of theorem 6.2 we define specializations of certain forms which allow us to go from the larger category to $\mathcal{O}(\mathcal{U}_q(\hat{\mathfrak{g}}))$. Note that in our construction we do not use that $C(z)$ is invertible.

6.1 Fusion product

As the S_i are derivations, theorem 5.15 gives :

Corollary 6.1. *$Im(\chi_q)$ is a subring of $\mathcal{Y}$.*

Since χ_q is injective on $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$, the product of $\mathcal{Y}$ gives an induced commutative product $*$ on $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$. For $(\lambda, \Psi), (\lambda', \Psi'), (\mu, \Phi) \in P_l^+$ there are $Q_{\lambda,\Psi,\lambda',\Psi'}(\mu, \Phi) \in \mathbb{Z}$ such that :

$$L(\lambda, \Psi) * L(\lambda', \Psi') = L(\lambda + \lambda', \Psi\Psi') + \sum_{(\mu,\Phi)\in P_l^+/\mu<\lambda+\lambda'} Q_{\lambda,\Psi,\lambda',\Psi'}(\mu, \Phi)L(\mu, \Phi)$$

We will interpret this product as a fusion product related to the basis of simple modules : that is to say we will show that a product of modules is a module (see [F] for generalities on fusion rings and physical motivations). Let us explain it in more details : consider :

$$\mathrm{Rep}^+(\mathcal{U}_q(\hat{\mathfrak{g}})) = \bigoplus_{(\lambda,\Psi)\in P_l^+} \mathbb{N}.L(\lambda, \Psi) \subset \mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}})) = \bigoplus_{(\lambda,\Psi)\in P_l^+} \mathbb{Z}.L(\lambda, \Psi)$$

Theorem 6.2. *The subset $Rep^+(\mathcal{U}_q(\hat{\mathfrak{g}})) \subset Rep(\mathcal{U}_q(\hat{\mathfrak{g}}))$ is stable by *.*

In this section 6 we prove this theorem by interpreting $*$ with the help of a generalization of the new Drinfel'd coproduct. Note that theorem 6.2 means that for $(\lambda, \Psi), (\lambda', \Psi') \in P_l^+$ we have $Q_{\lambda,\Psi,\lambda',\Psi'}(\mu, \Phi) \geq 0$.

6.2 Coproduct

6.2.1 Reminder : case of a quantum affine algebra and Drinfel'd-Jimbo coproduct

As said before the case of a quantum affine algebra is a very special one because there are two realizations (if we add a central charge) ; in particular there is a coproduct on $\mathcal{U}_q(\hat{\mathfrak{g}})$, a tensor product on $\mathcal{O}_{\text{int}}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ and $\text{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ is a ring. It is the product * because it is shown in [FR3] that χ_q is a ring morphism. In particular the tensor product is commutative. So theorem 6.2 is proved in this case.

6.2.2 General case : new Drinfel'd coproduct

In general we have a coproduct $\Delta_{\hat{\mathfrak{h}}} : \mathcal{U}_q(\hat{\mathfrak{h}}) \to \mathcal{U}_q(\hat{\mathfrak{h}}) \otimes \mathcal{U}_q(\hat{\mathfrak{h}})$ for the commutative algebra $\mathcal{U}_q(\hat{\mathfrak{h}})$ defined by ($h \in P^*$, $i \in I$, $m \neq 0$) :

$$\Delta_{\hat{\mathfrak{h}}}(k_h) = k_h \otimes k_h \ , \ \Delta_{\hat{\mathfrak{h}}}(h_{i,m}) = 1 \otimes h_{i,m} + h_{i,m} \otimes 1$$

In particular we have $(i \in I, m \geq 0)$: $\Delta_{\hat{\mathfrak{h}}}(\phi^{\pm}_{i,\pm m}) = \sum\limits_{0 \leq l \leq m} \phi^{\pm}_{i,\pm(m-l)} \otimes \phi^{\pm}_{i,\pm l}$.

No coproduct has been defined for the entire $\mathcal{U}_q(\hat{\mathfrak{g}})$. However Drinfel'd (unpublished note, see also [DI, DF]) defined for $\mathcal{U}_q(\hat{sl}_n)$ a map which behaves as a new coproduct adapted to the affinization realization. In this section we use those formulas for general quantum affinizations ; as infinite sums are involved, we use a formal parameter u so that it makes sense.

Let $\mathcal{C} = \mathbb{C}((u))$ be the field of Laurent series $\sum\limits_{r \geq R} \lambda_r u^r$ ($R \in \mathbb{Z}$, $\lambda_r \in \mathbb{C}$). The algebra $\tilde{\mathcal{U}}_q(\hat{\mathfrak{g}})$ is defined in section 3.3.1. Consider the $\mathcal{C}$-algebra $\tilde{\mathcal{U}}'_q(\hat{\mathfrak{g}}) = \mathcal{C} \otimes \tilde{\mathcal{U}}_q(\hat{\mathfrak{g}})$ (resp. $\mathcal{U}'_q(\hat{\mathfrak{g}}) = \mathcal{C} \otimes \mathcal{U}_q(\hat{\mathfrak{g}})$). Let $\tilde{\mathcal{U}}'_q(\hat{\mathfrak{g}}) \hat{\otimes} \tilde{\mathcal{U}}'_q(\hat{\mathfrak{g}}) = (\tilde{\mathcal{U}}_q(\hat{\mathfrak{g}}) \otimes_{\mathbb{C}} \tilde{\mathcal{U}}_q(\hat{\mathfrak{g}}))((u))$ be the u-topological completion of $\tilde{\mathcal{U}}'_q(\hat{\mathfrak{g}}) \otimes_{\mathcal{C}} \tilde{\mathcal{U}}'_q(\hat{\mathfrak{g}})$. It is also a $\mathcal{C}$-algebra.

Proposition 6.3. *There is a unique morphism of $\mathcal{C}$-algebra $\Delta_u : \tilde{\mathcal{U}}'_q(\hat{\mathfrak{g}}) \to \tilde{\mathcal{U}}'_q(\hat{\mathfrak{g}}) \hat{\otimes} \tilde{\mathcal{U}}'_q(\hat{\mathfrak{g}})$ such that for $i \in I$, $r \in \mathbb{Z}$, $m \geq 0$, $h \in \mathfrak{h}$:*

$$\Delta_u(x^+_{i,r}) = x^+_{i,r} \otimes 1 + \sum_{l \geq 0} u^{r+l} (\phi^-_{i,-l} \otimes x^+_{i,r+l})$$

$$\Delta_u(x^-_{i,r}) = u^r (1 \otimes x^-_{i,r}) + \sum_{l \geq 0} u^l (x^-_{i,r-l} \otimes \phi^+_{i,l})$$

$$\Delta_u(\phi^\pm_{i,\pm m}) = \sum_{0\leq l\leq m} u^{\pm l}(\phi^\pm_{i,\pm(m-l)} \otimes \phi^\pm_{i,\pm l}) \ , \ \Delta_u(k_h) = k_h \otimes k_h$$

Proof : We can easily check the compatibility with relations (21), (22), (23), (24), (25), (26) because Δ_u can also be given in terms of the currents of section 3.2 : we have in $(\tilde{\mathcal{U}}'_q(\hat{\mathfrak{g}}) \otimes_{\mathcal{C}} \tilde{\mathcal{U}}'_q(\hat{\mathfrak{g}}))[[z, z^{-1}]]$:

$$\Delta_u(x_i^+(z)) = x_i^+(z) \otimes 1 + \phi_i^-(z) \otimes x_i^+(zu) \ , \ \Delta_u(x_i^-(z)) = 1 \otimes x_i^-(zu) + x_i^-(z) \otimes \phi_i^+(zu)$$

$$\Delta_u(\phi_i^\pm(z)) = \phi_i^\pm(z) \otimes \phi_i^\pm(zu)$$

$\square$

Remark 1 : If C is finite or simply laced then Δ_u is compatible with the affine quantum Serre relations (relations (20)) and can be defined for $\mathcal{U}'_q(\hat{\mathfrak{g}})$ (see [DI] for symmetric cases and [E, Gr] for other finite cases). We conjecture that it is also true for general C, but we do not need it for our purposes.

Remark 2 : let $T : \tilde{\mathcal{U}}_q(\hat{\mathfrak{g}}) \to \tilde{\mathcal{U}}'_q(\hat{\mathfrak{g}})$ be the $\mathbb{Z}$-gradation morphism defined by $T(x^\pm_{i,r}) = u^r x^\pm_{i,r}$, $T(\phi^\pm_{i,m}) = u^m \phi^\pm_{i,m}$, $T(k_h) = k_h$. The power of u in Δ_u is put in such that $\Delta_u = (\mathrm{Id} \otimes T) \circ \Delta$ where Δ is the usual new Drinfel'd coproduct (without u).

Remark 3 : The map Δ_u is not coassociative , indeed in $(\tilde{\mathcal{U}}'_q(\hat{\mathfrak{g}}) \otimes_{\mathcal{C}} \tilde{\mathcal{U}}'_q(\hat{\mathfrak{g}}) \otimes_{\mathcal{C}} \tilde{\mathcal{U}}'_q(\hat{\mathfrak{g}}))[z]$:

$$((\Delta_u \otimes \mathrm{Id}) \circ \Delta_u)(\phi_i^+(z)) = \phi_i^+(z) \otimes \phi_i^+(uz) \otimes \phi_i^+(uz)$$

$$((\mathrm{Id} \otimes \Delta_u) \circ \Delta_u)(\phi_i^+(z)) = \phi_i^+(z) \otimes \phi_i^+(uz) \otimes \phi_i^+(u^2 z)$$

Remark 4 : Although is is not defined in a strict sense, the "limit" of Δ_u at $u = 1$ is coassociative. On $\mathcal{U}_q(\hat{\mathfrak{h}})$ the limit at $u = 1$ makes sense and is $\Delta_{\hat{\mathfrak{h}}}$.

6.3 Tensor products of representations of $\tilde{\mathcal{U}}'_q(\hat{\mathfrak{g}})$

As the coproduct involves infinite sums, we have to introduce a category larger than $\mathcal{O}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ in order to define tensor products :

6.3.1 The category $\mathcal{O}(\tilde{\mathcal{U}}'_q(\hat{\mathfrak{g}}))$

Definition 6.4. *The set of l, u-weights $P_{l,u}$ is the set of couple $(\lambda, \Psi(u))$ such that $\lambda \in \mathfrak{h}^*$, $\Psi(u) = (\Psi^\pm_{i,\pm m}(u))_{i\in I, m\geq 0}$, $\Psi^\pm_{i,\pm m}(u) \in \mathbb{C}[u, u^{-1}]$ and $\Psi^\pm_{i,0}(u) = q_i^{\pm\lambda(\alpha_i^\vee)}$.*

Definition 6.5. *An object V of the category $\mathcal{O}(\tilde{\mathcal{U}}'_q(\hat{\mathfrak{g}}))$ is a $\mathcal{C}$-vector space with a structure of $\tilde{\mathcal{U}}'_q(\hat{\mathfrak{g}})$-module such that :*

i) V is $\mathcal{U}_q(\mathfrak{h})$-diagonalizable

ii) For all $\lambda \in \mathfrak{h}^*$, the sub $\mathcal{C}$-vector space $V_\lambda \subset V$ is finite dimensional

iii) there are a finite number of element $\lambda_1, ..., \lambda_s \in \mathfrak{h}^*$ such that the weights of V are in $\bigcup_{j=1...s} D(\lambda_j)$

iv) for $\lambda \in \mathfrak{h}^*$, $V_\lambda = \bigoplus_{(\lambda,\Psi(u))\in P_{l,u}} V_{(\lambda,\Psi(u))}$ where :

$$V_{\lambda,\Psi(u)} = \{x \in V_\lambda / \exists p \in \mathbb{N}, \forall i \in \{1,...,n\}, \forall r \geq 0, (\phi^{\pm}_{i,\pm r} - \Psi^{\pm}_{i,\pm r}(u))^p.x = 0\}$$

The property iv) is added because $\mathcal{C}$ is not algebraically closed.

The scalar extension and the projection $\tilde{\mathcal{U}}_q(\hat{\mathfrak{g}}) \to \mathcal{U}_q(\hat{\mathfrak{g}})$ gives an injection $i : \mathcal{O}(\mathcal{U}_q(\hat{\mathfrak{g}})) \to \mathcal{O}(\tilde{\mathcal{U}}'_q(\hat{\mathfrak{g}}))$ such that for $V \in \mathcal{O}(\mathcal{U}_q(\hat{\mathfrak{g}}))$, $i(V) = V \otimes \mathcal{C}$.

Let $\mathcal{E}_{l,u} \subset P^{\mathbb{Z}}_{l,u}$ be defined as $\mathcal{E}_l$. The formal character of a module V in the category $\mathcal{O}(\tilde{\mathcal{U}}'_q(\hat{\mathfrak{g}}))$ is :

$$\mathrm{ch}_{q,u}(V) = \sum_{(\mu,\Gamma(u))\in P_{l,u}} \dim_{\mathcal{C}}(V_{\mu,\Gamma(u)}) e(\mu,\Gamma(u)) \in \mathcal{E}_{l,u}$$

We have a map $i_{\mathcal{E}} : \mathcal{E}_l \to \mathcal{E}_{l,u}$ such that $i_{\mathcal{E}}((\lambda,\Psi)) = (\lambda, (\Psi^{\pm m}_{i,\pm m}))$ and a commutative diagram :

$$\begin{array}{ccc} \mathcal{O}(\mathcal{U}_q(\hat{\mathfrak{g}})) & \stackrel{\mathrm{ch}_q}{\longrightarrow} & \mathcal{E}_l \\ \downarrow \mathrm{i} & & \downarrow i_{\mathcal{E}} \\ \mathcal{O}(\tilde{\mathcal{U}}'_q(\hat{\mathfrak{g}})) & \stackrel{\mathrm{ch}_{q,u}}{\longrightarrow} & \mathcal{E}_{l,u} \end{array}$$

In an analogous way one defines the category $\mathcal{O}(\mathcal{U}'_q(\hat{\mathfrak{g}}))$ and a formal character $\mathrm{ch}_{q,u}$ on $\mathcal{O}(\mathcal{U}'_q(\hat{\mathfrak{g}}))$.

6.3.2 Tensor products

We consider subcategories of $\mathcal{O}(\tilde{\mathcal{U}}'_q(\hat{\mathfrak{g}}))$. Let $R \in \mathbb{Z}$, $R \geq 0$:

Definition 6.6. *$\mathcal{O}^R(\tilde{\mathcal{U}}'_q(\hat{\mathfrak{g}}))$ is the category of modules $V \in \mathcal{O}(\tilde{\mathcal{U}}'_q(\hat{\mathfrak{g}}))$ such that for all $\lambda \in \mathfrak{h}^*$, there is a $\mathcal{C}$-basis $(v^\lambda_\alpha)_\alpha$ of V_λ satisfying :*

i) for all $m \in \mathbb{Z}$, α, β, the coefficient of $x^+_{i,m}.v^\lambda_\alpha$ on $v^{\lambda+\alpha_i}_\beta$ (resp. of $x^-_{i,m}.v^\lambda_\alpha$ on $v^{\lambda-\alpha_i}_\beta$) is in $\mathbb{C}[[u]]$ if $m \geq 0$, in $u^{Rm}\mathbb{C}[[u]]$ if $m \leq 0$.

ii) for all $m \geq 0$, α, β the coefficient of $\phi^-_{i,-m}.v^\lambda_\alpha$ on v^λ_β is in $u^{-mR}\mathbb{C}[[u]]$

iii) for all $m \geq 0$, α, β the coefficient of $\phi^+_{i,m}.v^\lambda_\alpha$ on v^λ_β is in $\mathbb{C}[[u]]$

Example : For $V \in \mathcal{O}(\mathcal{U}_q(\hat{\mathfrak{g}}))$, we have $i(V) \in \mathcal{O}^0(\tilde{\mathcal{U}}_q(\hat{\mathfrak{g}}))$.

Theorem 6.7. *Let $V_1 \in \mathcal{O}(\tilde{\mathcal{U}}_q(\hat{\mathfrak{g}}))$ and $V_2 \in \mathcal{O}^R(\tilde{\mathcal{U}}'_q(\hat{\mathfrak{g}}))$. Then Δ_u defines a structure of $\tilde{\mathcal{U}}'_q(\hat{\mathfrak{g}})$-module on $i(V_1) \otimes_{\mathcal{C}} V_2$ which is in $\mathcal{O}^{R+1}(\tilde{\mathcal{U}}'_q(\hat{\mathfrak{g}}))$. Moreover the l,u-weights of $i(V_1) \otimes_{\mathcal{C}} V_2$ are of the form $(\lambda_1 + \lambda_2, \gamma_1(z)\gamma_2(uz))$ where (λ_1, γ_1) is a l-weight of V_1 and (λ_2, γ_2) is a l,u-weight of V_2.*

Remark : $\gamma(u)(z) = \gamma_1(z)\gamma_2(uz)$ means that for $i \in I, m \geq 0$:

$$\gamma^{\pm}_{i,\pm m}(u) = \sum_{0 \leq l \leq m} (\gamma_1)_{i,\pm l}(u)(\gamma_2)_{i,\pm(m-l)}(u)u^{\pm(m-l)}$$

Proof : As the definition of Δ_u involves infinite sums, we have to prove that the action formally defined by Δ_u makes sense on $V_1' \otimes_{\mathcal{C}} V_2$ where we denote $V_1' = i(V_1)$. Indeed the weight spaces of V_1' and V_2 are finite dimensional and for $\lambda, \mu \in \mathfrak{h}^*$ we can use a $\mathbb{C}$-base $(v_\alpha^{1,\lambda})_\alpha$ of $(V_1)_\lambda$ as a $\mathcal{C}$-base of $(V_1')_\lambda$ and the $\mathcal{C}$-basis $(v_{\alpha'}^{2,\mu})$ of $(V_2)_\mu$ given by the definition of $\mathcal{O}^R(\tilde{\mathcal{U}}_q'(\hat{\mathfrak{g}}))$. So consider $\lambda, \mu \in \mathfrak{h}^*$, $i \in I$ and let us investigate the coefficients $(r \in \mathbb{Z}, m \geq 0)$:

we have $x_{i,r}^+.((V_1')_\lambda \otimes (V_2)_\mu) \subset (V_1')_{\lambda+\alpha_i} \otimes (V_2)_\mu \oplus (V_1')_\lambda \otimes (V_2)_{\mu+\alpha_i}$.

on $(V_1')_\lambda \otimes (V_2)_{\mu+\alpha_i}$: the coefficient of $x_{i,m}^+.(v_\alpha^{1,\lambda} \otimes v_{\alpha'}^{2,\mu})$ on $v_\beta^{1,\lambda} \otimes v_{\beta'}^{2,\mu+\alpha_i}$ is in $\sum_{l\geq 0} u^{r+l}\mathbb{C}[[u]] \subset \mathbb{C}[[u]]$ if $r \geq 0$, in $\sum_{l\geq 0} u^{r+l}u^{R(r+l)}\mathbb{C}[[u]] \subset u^{(R+1)r}\mathbb{C}[[u]]$ if $r \leq 0$.

on $(V_1')_{\lambda+\alpha_i} \otimes (V_2)_\mu$: the coefficient of $x_{i,r}^+.(v_\alpha^{1,\lambda} \otimes v_{\alpha'}^{2,\lambda})$ on $v_\beta^{1,\lambda+\alpha_i} \otimes v_{\beta'}^{2,\mu}$ is in $\mathbb{C}$.

we have $x_{i,r}^-.((V_1')_\lambda \otimes (V_2)_\mu) \subset (V_1')_{\lambda-\alpha_i} \otimes (V_2)_\mu \oplus (V_1')_\lambda \otimes (V_2)_{\mu-\alpha_i}$.

on $(V_1')_\lambda \otimes (V_2)_{\mu-\alpha_i}$: the coefficient of $x_{i,r}^-.(v_\alpha^{1,\lambda} \otimes v_{\alpha'}^{2,\mu})$ on $v_\beta^{1,\lambda} \otimes v_{\beta'}^{2,\mu-\alpha_i}$ is in $u^r\mathbb{C}[[u]] \subset \mathbb{C}[[u]]$ if $r \geq 0$, in $u^r u^{Rr}\mathbb{C}[[u]]$ if $r \leq 0$.

on $(V_1')_{\lambda-\alpha_i} \otimes (V_2)_\mu$: the coefficient of $x_{i,r}^-.(v_\alpha^{1,\lambda} \otimes v_{\alpha'}^{2,\mu})$ on $v_\beta^{1,\lambda-\alpha_i} \otimes v_{\beta''}^{2,\mu}$ is in $\sum_{l\geq 0} u^l\mathbb{C}[[u]] \subset \mathbb{C}[[u]]$.

we have $\phi_{i,\pm m}^\pm.((V_1')_\lambda \otimes (V_2)_\mu) \subset ((V_1')_\lambda \otimes (V_2)_\mu)$.

the coefficient of $\phi_{i,m}^+.(v_\alpha^{1,\lambda} \otimes v_{\alpha'}^{2,\mu})$ on $v_\beta^{1,\lambda} \otimes v_{\beta'}^{2,\mu}$ is in $\sum_{0\leq l\leq m} u^l\mathbb{C}[[u]] \subset \mathbb{C}[[u]]$.

the coefficient of $\phi_{i,-m}^-.(v_\alpha^{1,\lambda} \otimes v_{\alpha'}^{2,\mu})$ on $v_\beta^{1,\lambda} \otimes v_{\beta'}^{2,\mu}$ is in $\sum_{0\leq l\leq m} u^{-l}u^{-lR}\mathbb{C}[[u]]$ $\subset u^{-m(R+1)}\mathbb{C}[[u]]$.

So we have a structure of $\tilde{\mathcal{U}}_q'(\hat{\mathfrak{g}})$-module on $V_1' \otimes_{\mathcal{C}} V_2$. Let us prove that it is in $\mathcal{O}(\tilde{\mathcal{U}}_q'(\hat{\mathfrak{g}}))$. We verify the properties of definition 6.5 : i) ii) iii) are clear because the restriction of Δ_u to $\mathcal{U}_q(\hat{\mathfrak{h}})$ is the restriction of $\Delta_{\hat{\mathfrak{h}}}$. For iv) we note that for $(\lambda_1, \gamma_1), (\lambda_2, \gamma_2) \in P_{l,u}$, the $(V_1')_{\lambda_1,\gamma_1} \otimes (V_2)_{\lambda_2,\gamma_2}$ is in the pseudo weight space of l, u-weight $(\lambda_1 + \lambda_2, \gamma_1(z)\gamma_2(zu))$ because $\Delta_u(\phi_i^\pm(z)) = \phi_i^\pm(z) \otimes \phi_i^\pm(zu)$ (it also proves the last point of the proposition).

Finally we see in the above computations that the coefficients verify the property of $\mathcal{O}^{R+1}(\tilde{\mathcal{U}}_q'(\hat{\mathfrak{g}}))$, so $V_1' \otimes_{\mathcal{C}} V_2$ is in $\mathcal{O}^{R+1}(\tilde{\mathcal{U}}_q'(\hat{\mathfrak{g}}))$. □

Definition 6.8. *For $R \geq 0$, we denote $\otimes_R : \mathcal{O}(\mathcal{U}_q(\hat{\mathfrak{g}})) \times \mathcal{O}^R(\tilde{\mathcal{U}}_q'(\hat{\mathfrak{g}})) \to \mathcal{O}^{R+1}(\tilde{\mathcal{U}}_q'(\hat{\mathfrak{g}}))$ the bilinear map constructed in theorem 6.7.*

See section 6.6 for explicit examples. For $R \geq 2$ and $V_1, V_2, ..., V_R \in \mathcal{O}(\mathcal{U}_q(\hat{\mathfrak{g}}))$, one can define the iterated tensor product $V_1 \otimes_{R-2} (V_2 \otimes_{R-3} (... \otimes_0 V_R))...)$ which is in $\mathcal{O}^{R-1}(\tilde{\mathcal{U}}_q'(\hat{\mathfrak{g}}))$.

6.4 Simple modules of $\tilde{\mathcal{U}}_q'(\hat{\mathfrak{g}})$

6.4.1 l, u-highest weight modules

For $(\lambda, \Psi(u)) \in P_{l,u}$, let $\tilde{M}(\lambda, \Psi(u))$ be the Verma $\tilde{\mathcal{U}}_q'(\hat{\mathfrak{g}})$ module of highest weight $(\lambda, \Psi(u))$ (it is non trivial thanks to the triangular decomposition of $\tilde{\mathcal{U}}_q(\hat{\mathfrak{g}})$ in lemma 3.10). So we have an analog of proposition 4.6 : for $(\lambda, \Psi(u)) \in P_{l,u}$, there is a unique up to isomorphism simple $\tilde{\mathcal{U}}_q'(\hat{\mathfrak{g}})$-module $\tilde{L}(\lambda, \Psi(u))$ of l, u-highest weight $(\lambda, \Psi(u))$ that is to say that there is $v \in \tilde{L}(\lambda, \Psi(u))$ such that $(i \in I, r \in \mathbb{Z}, m \geq 0, h \in \mathfrak{h})$:

$$x_{i,r}^+.v = 0 \ , \ \tilde{L}(\lambda, \Psi(u)) = \tilde{\mathcal{U}}_q'(\hat{\mathfrak{g}}).v \ , \ \phi_{i,\pm m}^{\pm}.v = \Psi_{i,\pm m}^{\pm}(u)v \ , \ k_h.v = q^{\lambda(h)}.v$$

In a similar way one define the simple $\mathcal{U}_q'(\hat{\mathfrak{g}})$-module $L(\lambda, \Psi(u))$ of l, u-highest weight $(\lambda, \Psi(u))$ (it is non trivial thanks to theorem 3.2).

Lemma 6.9. *For $(\lambda, \Psi(u)) \in P_{l,u}$ we have an isomorphism of $\mathcal{U}_q(\hat{\mathfrak{h}})$-modules $\tilde{L}(\lambda, \Psi(u)) \simeq L(\lambda, \Psi(u))$.*

Proof : Let $\tilde{M}'(\lambda, \Psi(u)) \subset \tilde{M}(\lambda, \Psi(u))$ be the maximal proper $\tilde{\mathcal{U}}_q'(\hat{\mathfrak{g}})$-submodule of $\tilde{M}'(\lambda, \Psi(u))$. It suffices to prove that $\tilde{\tau}_-.1$ is included in $\tilde{M}'(\lambda, \Psi(u))$ (see section 3.3.4 ; it implies that $\tilde{L}(\lambda, \Psi(u))$ is also a $\mathcal{U}_q'(\hat{\mathfrak{g}})$-modules). It is a consequence of lemma 3.11. □

In particular $\tilde{L}(\lambda, \Psi(u)) \in \mathcal{O}(\tilde{\mathcal{U}}_q'(\hat{\mathfrak{g}})) \Leftrightarrow L(\lambda, \Psi(u)) \in \mathcal{O}(\mathcal{U}_q'(\hat{\mathfrak{g}}))$ and in this case $\mathrm{ch}_{q,u}(\tilde{L}(\lambda, \Psi(u))) = \mathrm{ch}_{q,u}(L(\lambda, \Psi(u)))$.

6.4.2 The category $\mathcal{O}_{\mathbf{int}}(\tilde{\mathcal{U}}_q'(\hat{\mathfrak{g}}))$

Definition 6.10. *$QP_{l,u}^+$ is the set of $(\lambda, \Psi(u)) \in P_{l,u}$ satisfying the following conditions :*

i) for $i \in I$ there exist polynomials $Q_{i,u}(z) = (1 - za_{i,1}u^{b_{i,1}})...(1 - za_{i,N_i}u^{b_{i,N_i}})$, $R_{i,u}(z) = (1 - zc_{i,1}u^{d_{i,1}})...(1 - zc_{i,N_i'}u^{d_{i,N_i'}})$ ($a_{i,j}, c_{i,j} \in \mathbb{C}^*$, $b_{i,j}, d_{i,j} \geq 0$) such that in $\mathbb{C}[u, u^{-1}][[z]]$ (resp. in $\mathbb{C}[u, u^{-1}][[z^{-1}]]$) :

$$\sum_{r \geq 0} \Psi_{i,\pm r}^{\pm}(u) z^{\pm r} = q_i^{deg(Q_{i,u}) - deg(R_{i,u})} \frac{Q_{i,u}(zq_i^{-1}) R_{i,u}(zq_i)}{Q_{i,u}(zq_i) R_{i,u}(zq_i^{-1})}$$

ii) there exist $\omega \in P^+$, $\alpha \in Q^+$ satisfying $\lambda = \omega - \alpha$.

$P_{l,u}^+$ is the set of $(\lambda, \Psi(u)) \in QP_{l,u}^+$ such that one can choose $R_{i,u} = 1$ (in this case we denote $P_{i,u} = Q_{i,u}$).

Lemma 6.11. *If $(\lambda, \Psi(u)) \in P_{l,u}^+$ then $\tilde{L}(\lambda, \Psi(u)) \in \mathcal{O}(\tilde{\mathcal{U}}_q'(\hat{\mathfrak{g}}))$. Moreover for $(\mu, \gamma(u)) \in P_{l,u}$ we have $dim(\tilde{L}(\lambda, \Psi(u))_{\mu,\gamma(u)}) \neq 0 \Rightarrow (\mu, \gamma(u)) \in QP_{l,u}^+$.*

Remark : it follows from lemma 6.9 that we have the same results for $L(\lambda, \Psi(u)) \in \mathcal{O}(\mathcal{U}_q'(\hat{\mathfrak{g}}))$.

Proof : Let $(\lambda, \Psi(u)) \in P_{l,u}^+$ and decompose $P_{i,u}(z)$ in the form :

$$P_{i,u}(z) = P_i^{(0)}(z)P_i^{(1)}(uz)...P_i^{(R)}(u^R z)$$

where $R \geq 0$, $P_i^{(k)}(z) \in \mathbb{C}[z]$, $P_i^{(k)}(0) = 1$ for $0 \leq k \leq R$ (R can be taken large enough so that we have this form for all $i \in I$). For $0 \leq k \leq R$, set $\Psi_i^{(k)}(z) = q_i^{\deg(P_i^{(k)})} \frac{P_i^{(k)}(zq_i^{-1})}{P_i(zq_i)}$. For $1 \leq k \leq R$ define $\lambda_k = \sum_{i \in I} \deg(P_i^{(k)})\Lambda_i \in \mathfrak{h}^*$. Set $\lambda_0 = \lambda - \sum_{k=1...R} \lambda_k$. Then for $0 \leq k \leq R$ the $(\lambda_k, \Psi^{(k)}) \in P_l^+$ and we can consider $L(\lambda_k, \Psi^{(k)}) \in \mathcal{O}(\mathcal{U}_q(\hat{\mathfrak{g}}))$. Let $V \in \mathcal{O}^R(\tilde{\mathcal{U}}_q'(\hat{\mathfrak{g}}))$ be defined by :

$$V = i(L(\lambda_0, \Psi^{(0)})) \otimes_{R-1} (i(L(\lambda_1, \Psi^{(1)})) \otimes_{R-2} ...(i(L(\lambda_{R-1}, \Psi^{(R-1)})) \otimes_0 i(L(\lambda_R, \Psi^{(R)})))...)$$

Consider the $\tilde{\mathcal{U}}_q'(\hat{\mathfrak{g}})$ submodule L of V generated by the tensor product of the highest weight vectors. It is a highest weight module of highest weight $(\lambda, \Psi(u))$. So $\tilde{L}(\lambda, \Psi(u))$ is a quotient of L and so is in $\mathcal{O}(\tilde{\mathcal{U}}_q'(\hat{\mathfrak{g}}))$.

For the second point it follows from proposition 5.4 that the l, u-weight of $i(L(\lambda_k, \Psi^{(k)}))$ are in $QP_{l,u}^+$. So with the help of the last point of theorem 6.7 we see that the l, u-weights of V are in $QP_{l,u}^+$ and we have the property for $\tilde{L}(\lambda, \Psi(u))$. □

Definition 6.12. *Let $\mathcal{O}_{int}(\tilde{\mathcal{U}}_q'(\hat{\mathfrak{g}}))$ be the subcategory of modules $V \in \mathcal{O}(\tilde{\mathcal{U}}_q'(\hat{\mathfrak{g}}))$ whose l, u-weights are in $QP_{l,u}^+$.*

Lemma 6.13. *For a module $V \in \mathcal{O}_{int}(\tilde{\mathcal{U}}_q'(\hat{\mathfrak{g}}))$ there are $P_{(\lambda,\Psi(u))} \geq 0$ ($(\lambda, \Psi(u)) \in P_{l,u}^+$) such that :*

$$ch_{q,u}(V) = \sum_{(\lambda,\Psi(u)) \in P_{l,u}^+} P_{(\lambda,\Psi(u))} ch_{q,u}(\tilde{L}(\lambda, \Psi(u))) = \sum_{(\lambda,\Psi(u)) \in P_{l,u}^+} P_{(\lambda,\Psi(u))} ch_{q,u}(L(\lambda, \Psi(u)))$$

Proof : Analogous to the proof of proposition 5.2 (the second identity follows from lemma 6.9). □

6.5 $\mathbb{C}[u^{\pm}]$-forms and specialization

6.5.1 $\mathbb{C}[u^{\pm}]$-forms

Let $\mathcal{U}_q^u(\hat{\mathfrak{g}}) = \mathcal{U}_q(\hat{\mathfrak{g}}) \otimes_{\mathbb{C}} \mathbb{C}[u^{\pm}] \subset \mathcal{U}_q'(\hat{\mathfrak{g}})$.

Definition 6.14. *A $\mathbb{C}[u^{\pm}]$-form of a $\mathcal{U}_q'(\hat{\mathfrak{g}})$-module V is a sub-$\mathcal{U}_q^u(\hat{\mathfrak{g}})$-module L of V such that the map $\mathcal{C} \otimes_{\mathbb{C}[u^{\pm}]} L \to V$ is an isomorphism of $\mathcal{U}_q'(\hat{\mathfrak{g}})$-module.*

Note that it means that L generates V as $\mathcal{C}$-vector space and that some vectors which are $\mathbb{C}[u^{\pm}]$-linearly independent in L are $\mathcal{C}$-linearly independent in V.

Let us look at some examples :

Proposition 6.15. *For $(\lambda, \Psi(u)) \in P_{l,u}$ and v a highest weight vector of the Verma module $M(\lambda, \Psi(u))$ (resp. the simple module $L(\lambda, \Psi(u))$), the $\mathcal{U}_q^u(\hat{\mathfrak{g}})$-module $\mathcal{U}_q^u(\hat{\mathfrak{g}}).v$ is a $\mathbb{C}[u^{\pm}]$-form of $M(\lambda, \Psi(u))$ (resp. of $L(\lambda, \Psi(u))$) which is isomorphic to the Verma (resp. the simple) $\mathcal{U}_q^u(\hat{\mathfrak{g}})$-module of l,u-highest weight $(\lambda, \Psi(u))$.*

Proof : As $(\lambda, \Psi(u))$ is fixed, we omit it in the proof. M is the quotient of $\mathcal{C} \otimes_{\mathbb{C}} \mathcal{U}_q(\hat{\mathfrak{g}})$ by the relations generated by $x_{i,r}^{\pm} = \phi_{i,\pm m}^{\pm} - \Psi_{i,\pm m}^{\pm}(u) = k_h - q^{\lambda(h)} = 0$. So the relations between monomials are in $\mathbb{C}[u^{\pm}]$ and $\mathcal{U}_q^u(\hat{\mathfrak{g}}).1 \subset M$ is a $\mathbb{C}[u^{\pm}]$-form of M. Moreover these relations are the same as in the construction of the Verma $\mathcal{U}_q^u(\hat{\mathfrak{g}})$-module M^u as a quotient of $\mathbb{C}[u^{\pm}] \otimes_{\mathbb{C}} \mathcal{U}_q(\hat{\mathfrak{g}})$; and so $\mathcal{U}_q^u(\hat{\mathfrak{g}}).1 \simeq M^u$.

Let us look at L. Denote by L^u the simple $\mathcal{U}_q^u(\hat{\mathfrak{g}})$-module of highest weight $(\lambda, \Psi(u))$. We have $L = M/M'$ (resp. $L^u = M^u/M'^u$) where M' (resp. M'^u) in the maximal proper submodule of M (resp. M^u).

The $\mathcal{C}$-subspace M'' of M generated by M'^u is isomorphic to $\mathcal{C} \otimes_{\mathbb{C}[u^{\pm}]} M'^u$ (because M^u is a $\mathbb{C}[u^{\pm}]$-form of M). As M'' has no vector of weight λ, it is a proper submodule of M and $M'' \subset M'$. Suppose that $M' \neq M''$ and consider $M'/M'' \subset M/M''$. M^u/M'^u is a $\mathbb{C}[u^{\pm}]$-form of M/M''. Let v be a non zero highest weight vector of M'/M'' and let us write : $v = \sum_{\alpha} f_\alpha(u) v_\alpha$ where $v_\alpha \in M^u/M'^u$ and $f_\alpha(u) \in \mathcal{C}$ (as there is a finite number of $f_\alpha(u)$, we can suppose that they are $\mathbb{C}[u^{\pm}]$-linearly independent). For all $i \in I, r \in \mathbb{Z}$, we have $x_{i,r}^{+}.v = 0$ and so for all α, $x_{i,r}^{+}.v_\alpha = 0$. Fix $w_\alpha \in M^u$ whose image in M^u/M'^u is v_α. As for all $i \in I, r \in \mathbb{Z}$, $x_{i,r}^{+}.w_\alpha \in M'^u$, $\mathcal{U}_q^u(\hat{\mathfrak{g}}).w_\alpha$ is a proper submodule of M^u and $w_\alpha \in M'^u$. So $v = 0$, contradiction. So $M' = M''$. In particular $M' \simeq M'^u \otimes_{\mathbb{C}[u^{\pm}]} \mathcal{C}$, $M' \cap M^u = M'^u$.

For v a highest weight vector of L, the $\mathcal{U}_q^u(\hat{\mathfrak{g}}).v \simeq \mathcal{U}_q^u(\hat{\mathfrak{g}}).1 = M^u/(M^u \cap M') = M^u/M'^u = L^u$ is a $\mathbb{C}[u^{\pm}]$-form of L. □

6.5.2 Specializations

Consider $p : \mathcal{E}_{l,u} \to \mathcal{E}_l$ the surjection such that $p((\lambda, \Psi(u))) = (\lambda, \Psi(1))$.

Lemma 6.16. *Let V be in $\mathcal{O}(\mathcal{U}_q'(\hat{\mathfrak{g}}))$. If L is a $\mathbb{C}[u^{\pm}]$-form of V then the specialization $L' = L/(1-u)L$ of L is in $\mathcal{O}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ and $ch_q(L') = p(ch_{q,u}(V))$.*

Proof : Indeed for $(\mu, \gamma(u)) \in QP_{l,u}$ consider $L_{\mu,\gamma(u)} = L \cap V_{\mu,\gamma(u)}$. As $p : L \otimes_{\mathbb{C}[u]} \mathcal{C} \to V$ is an isomorphism, we have $V_{\mu,\gamma(u)} \simeq p^{-1}(V_{\mu,\gamma(u)}) = L_{\mu,\gamma(u)} \otimes_{\mathbb{C}[u]} \mathcal{C}$. In particular $L_{\mu,\gamma(u)}$ is a free $\mathbb{C}[u^{\pm}]$ of rank $\dim_{\mathcal{C}}(V_{\mu,\gamma(u)})$. So $\dim_{\mathbb{C}}(L'_\mu) = \dim_{\mathcal{C}}(V_\mu)$, and $L' \in \mathcal{O}(\mathcal{U}_q(\hat{\mathfrak{g}}))$. We can conclude because

$$L'_{\lambda,\gamma} = \bigoplus_{(\lambda,\gamma(u)) \in p^{-1}((\lambda,\gamma))} (L_{\lambda,\gamma(u)}/(u-1)L_{\lambda,\gamma(u)})$$

□

6.5.3 Proof of theorem 6.2

For $(\lambda, \Psi(u)) \in P^+_{l,u}$, it follows from proposition 6.15 and lemma 6.16 that $p(\mathrm{ch}_{q,u}(L(\lambda, \Psi(u)))) = \mathrm{ch}_q(L)$ where $L \in \mathcal{O}_{\mathrm{int}}(\mathcal{U}_q(\hat{\mathfrak{g}}))$. In particular $p(\mathrm{ch}_{q,u}(L(\lambda, \Psi(u))))$ is in $\mathrm{ch}_q(\mathrm{Rep}^+(\mathcal{U}_q(\hat{\mathfrak{g}})))$ and for $V \in \mathcal{O}_{\mathrm{int}}(\tilde{U}'_q(\hat{\mathfrak{g}}))$ the $p(\mathrm{ch}_{q,u}(V))$ is in $\mathrm{ch}_q(\mathrm{Rep}^+(\mathcal{U}_q(\hat{\mathfrak{g}})))$ (see lemma 6.13).

Consider $V_1, V_2 \in \mathcal{O}_{\mathrm{int}}(\mathcal{U}_q(\hat{\mathfrak{g}}))$. We have seen that :

$$p(\mathrm{ch}_{q,u}(i(V_1) \otimes_0 i(V_2)) \in \mathrm{ch}_q(\mathrm{Rep}^+(\mathcal{U}_q(\hat{\mathfrak{g}})))$$

But

$$p(\mathrm{ch}_{q,u}(i(V_1) \otimes_0 i(V_2)) = \mathrm{ch}_q(V_1)\mathrm{ch}_q(V_2)$$

because the specialization of Δ_u on $\mathcal{U}_q(\hat{\mathfrak{h}})$ at $u = 1$ is $\Delta_{\hat{\mathfrak{h}}}$. This ends the proof of theorem 6.2. □

6.6 Example

We study in detail an example in the case $\mathfrak{g} = sl_2$ where everything is computable thanks to Jimbo's evaluation morphism (see [CP3, CP4]). In this case we have $\mathcal{U}_q(\hat{sl_2}) = \tilde{\mathcal{U}}_q(\hat{sl_2})$.

For $a \in \mathbb{C}^*$ consider $V = L(1 - za) \in \mathcal{O}_{\mathrm{int}}(\mathcal{U}_q(\hat{sl_2}))$. V is two dimensional $V = \mathbb{C}v_0 \oplus \mathbb{C}v_1$ and for $r \in \mathbb{Z}$, $m \geq 1$ the action of $\mathcal{U}_q(\hat{sl_2})$ is given in the following table :

	v_0	v_1
x_r^+	0	$a^r v_0$
x_r^-	$a^r v_1$	0
$\phi^\pm_{\pm m}$	$\pm(q - q^{-1})a^{\pm m} v_0$	$\mp(q - q^{-1})a^{\pm m} v_1$
$k^\pm$	$q^\pm v_0$	$q^\mp v_1$
$\phi^\pm(z)$	$q\frac{1-q^{-2}az}{1-az} v_0$	$q^{-1}\frac{1-q^2 az}{1-az} v_1$

Remark : in the table $\phi^\pm(z) \in \mathcal{U}_q(\hat{\mathfrak{g}})[[z^\pm]]$ acts on $V[[z^\pm]]$.

For $a, b \in \mathbb{C}^*$, let $V = L(1 - za), W = L(1 - zb) \in \mathcal{O}_{\mathrm{int}}(\mathcal{U}_q(\hat{\mathfrak{g}}))$. Consider basis $V = \mathbb{C}v_0 \oplus \mathbb{C}v_1$, $W = \mathbb{C}w_0 \oplus \mathbb{C}w_1$ as in the previous table. The tensor product $\otimes_0$ defines an action of $\mathcal{U}'_q(\hat{sl_2})$ on $X = i(V) \otimes_{\mathcal{C}} i(W)$ (see theorem 6.7). X is a 4 dimensional $\mathcal{C}$-vector space of base $\{v_0 \otimes w_0, v_1 \otimes w_0, v_0 \otimes w_1, v_1 \otimes w_1\}$. The action of $\mathcal{U}'_q(\hat{sl_2})$ is given by ($r \in \mathbb{Z}$) :

	$v_0 \otimes w_0$	$v_1 \otimes w_0$
x_r^+	0	$a^r(v_0 \otimes w_0)$
x_r^-	$u^r b^r(v_0 \otimes w_1) + a^r q\frac{1-q^{-2}a^{-1}ub}{1-a^{-1}ub}(v_1 \otimes w_0)$	$u^r b^r(v_1 \otimes w_1)$
$\phi^\pm(z)$	$q^2\frac{(1-q^{-2}az)(1-q^{-2}buz)}{(1-az)(1-buz)}(v_0 \otimes w_0)$	$\frac{(1-q^2 az)(1-q^{-2}buz)}{(1-az)(1-buz)}(v_1 \otimes w_0)$

	$v_0 \otimes w_1$	$v_1 \otimes w_1$
x_r^+	$b^r u^r q^{-1} \frac{1-q^2a^{-1}bu}{1-uba^{-1}}(v_0 \otimes w_0)$	$a^r(v_0 \otimes w_1) + b^r u^r q \frac{1-q^{-2}a^{-1}bu}{1-uba^{-1}}(v_1 \otimes w_0)$
x_r^-	$a^r q^{-1} \frac{1-q^2ua^{-1}b}{1-a^{-1}ub}(v_1 \otimes w_1)$	0
$\phi^\pm(z)$	$\frac{(1-q^{-2}az)(1-q^2buz)}{(1-az)(1-buz)}(v_0 \otimes w_1)$	$q^{-2}\frac{(1-q^2az)(1-q^2buz)}{(1-az)(1-buz)}(v_1 \otimes w_1)$

Remark : in the table $\phi^\pm(z) \in \mathcal{U}_q(\hat{\mathfrak{g}})[[z^\pm]]$ acts on $X[[z^\pm]]$.

Consider the l-weights $\gamma_a, \gamma'_a, \gamma_b, \gamma'_b \in P_l$ (the $\lambda \in \mathfrak{h}^*$ can be omitted because sl_2 is finite) :

$$\gamma_a^\pm(z) = q\frac{1-q^{-2}az}{1-az} \ , \ \gamma_a'^\pm(z) = q^{-1}\frac{1-q^2az}{1-az} \ , \ \gamma_b^\pm(z) = q\frac{1-q^{-2}bz}{1-bz} \ , \ \gamma_b'^\pm(z) = q^{-1}\frac{1-q^2bz}{1-bz}$$

Consider also $\gamma_a(z)\gamma_b(uz), \gamma'_a(z)\gamma_b(uz), \gamma_a(z)\gamma'_b(uz), \gamma'_a(z)\gamma'_b(uz) \in P_{l,u}$. We see that :

$$\mathrm{ch}_{q,u}(X) = e(\gamma_a(z)\gamma_b(uz)) + e(\gamma'_a(z)\gamma_b(uz)) + e(\gamma_a(z)\gamma'_b(uz)) + e(\gamma_a(z)\gamma'_b(uz))$$

Those l,u-weights are distinct, the l,u-weights spaces are 1 dimensional :

$$X = (X)_{\gamma_a(z)\gamma_b(uz)} \oplus (X)_{\gamma'_a(z)\gamma_b(uz)} \oplus (X)_{\gamma_a(z)\gamma'_b(uz)} \oplus (X)_{\gamma'_a(z)\gamma'_b(uz)}$$

We see that X is of highest weight $\gamma_a(z)\gamma_b(uz) \in P_{l,u}$. Let us prove that it is simple : indeed X has no proper submodule : if for all $r \in \mathbb{Z}$, $x_r^+.(\alpha(v_1 \otimes w_0) + \beta(v_0 \otimes w_1)) = 0$, then for all $r \in \mathbb{Z}$, $\alpha a^r + \beta b^r u^r \frac{1-q^2a^{-1}bu}{1-uba^{-1}} = 0$. In particular $\alpha + \beta\frac{1-q^2a^{-1}bu}{1-uba^{-1}} = 0$ and $a^r - b^r u^r = 0$ for all $r \in \mathbb{Z}$, impossible. So $X \simeq L(\gamma_a(z)\gamma_b(uz))$ as a $\mathcal{U}'_q(\hat{sl_2})$-module. It follows from proposition 6.15 that $\tilde{X} = \mathcal{U}_q^u(\hat{\mathfrak{g}}).(v_0 \otimes w_0) \subset X$ is a $\mathbb{C}[u^\pm]$-form of X.

Let us look explicitly at this $\mathbb{C}[u^\pm]$-form : consider $e_1, e_2, e_3, e_4 \in \tilde{X}$ defined by :

$$e_1 = v_0 \otimes w_0 \ , \ e_2 = x_0^-.e_1 \ , \ e_3 = -a^{-1}x_1^-.e_1 + e_2 \ , \ e_4 = qx_0^-.e_2$$

We have the following formulas : $e_1 = v_0 \otimes w_0$, $e_2 = (v_0 \otimes w_1) + q\frac{1-q^{-2}a^{-1}bu}{1-a^{-1}ub}(v_1 \otimes w_0)$, $e_3 = (1-uba^{-1})(v_0 \otimes w_1)$, $e_4 = (v_1 \otimes w_1)$

Moreover the action of $\mathcal{U}_q^u(\hat{\mathfrak{g}})$ is given by ($r \in \mathbb{Z}$) :

	e_1	e_2
x_r^+	0	$(qa^r\frac{1-(a^{-1}ub)^{r+1}}{1-a^{-1}ub} - q^{-1}bua^{r-1}\frac{1-(a^{-1}ub)^{r-1}}{1-a^{-1}ub})e_1$
x_r^-	$a^r(e_2 - \frac{1-(a^{-1}ub)^r}{1-a^{-1}ub}e_3)$	$(q^{-1}a^r\frac{1-(a^{-1}ub)^{r+1}}{1-a^{-1}ub} - qbua^{r-1}\frac{1-(a^{-1}ub)^{r-1}}{1-a^{-1}ub})e_4$
$\phi^\pm(z)$	$q^2\frac{(1-q^{-2}az)(1-q^{-2}buz)}{(1-az)(1-buz)}e_1$	$\frac{(1-q^2az)(1-q^{-2}buz)}{(1-az)(1-buz)}e_2 + \frac{az(q^2-q^{-2})}{(1-az)(1-buz)}e_3$

	e_3	e_4
x_r^+	$b^r u^r q^{-1}(1-q^2a^{-1}bu)e_1$	$b^r u^r e_2 + a^r\frac{1-(uba^{-1})^r}{1-uba^{-1}}e_3$
x_r^-	$a^r q^{-1}(1-q^2ua^{-1}b)e_4$	0
$\phi^\pm(z)$	$\frac{(1-q^{-2}az)(1-q^2buz)}{(1-az)(1-buz)}e_3$	$q^{-2}\frac{(1-q^2az)(1-q^2buz)}{(1-az)(1-buz)}e_4$

In particular we see that $\mathbb{C}[u^{\pm}]e_1 \oplus \mathbb{C}[u^{\pm}]e_2 \oplus \mathbb{C}[u^{\pm}]e_3 \oplus \mathbb{C}[u^{\pm}]e_4$ is stable by the action of $\mathcal{U}_q^u(\hat{\mathfrak{g}})$, so is equal to $\tilde{X}$. So we have verified that $X \simeq \tilde{X} \otimes_{\mathbb{C}[u^{\pm}]} \mathcal{C}$.

Let us describe the specialization of $\tilde{X}$ at $u = 1$: let $\tilde{X}' = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_4$. The action of $\mathcal{U}_q(\hat{\mathfrak{g}})$ on $\tilde{X}'$ is given by (for $z \in \mathbb{C}, r \in \mathbb{Z}$, we denote $[z]'_r = \frac{1-z^r}{1-z} \in \mathbb{Z}[z^{\pm}]$ $(z \neq 1)$ and $[1]'_r = r$) :

	e_1	e_2
x_r^+	0	$(qa^r[a^{-1}b]'_{r+1} - q^{-1}ba^{r-1}[a^{-1}b]'_{r-1})e_1$
x_r^-	$a^r(e_2 - [a^{-1}b]'_r e_3)$	$(q^{-1}a^r[a^{-1}b]'_{r+1} - qba^{r-1}[a^{-1}b]'_{r-1})e_4$
$\phi^{\pm}(z)$	$q^2\frac{(1-q^{-2}az)(1-q^{-2}bz)}{(1-az)(1-bz)}e_1$	$\frac{(1-q^2az)(1-q^{-2}bz)}{(1-az)(1-bz)}e_2 + \frac{az(q^2-q^{-2})}{(1-az)(1-bz)}e_3$

	e_3	e_4
x_r^+	$b^r q^{-1}(1 - q^2a^{-1}b)e_1$	$b^r e_2 + a^r[a^{-1}b]'_r e_3$
x_r^-	$a^m q^{-1}(1 - q^2a^{-1}b)e_4$	0
$\phi^{\pm}(z)$	$\frac{(1-q^{-2}az)(1-q^2bz)}{(1-az)(1-bz)}e_3$	$q^{-2}\frac{(1-q^2az)(1-q^2bz)}{(1-az)(1-bz)}e_4$

We see that $\tilde{X}' = \mathcal{U}_q(\hat{\mathfrak{g}}).e_1$. Moreover if $ab^{-1} \notin \{q^2, q^{-2}\}$: $\tilde{X}'$ has no proper submodule because the formula $x_m^+(\alpha e_2 + \beta e_3) = 0$ means that for all $r \in \mathbb{Z}$:

$$\alpha(qa^r[a^{-1}b]_{r+1} - q^{-1}ba^{r-1}[a^{-1}b]_r) + \beta b^r q^{-1}(1 - q^2a^{-1}b) = 0$$

which is possible only if $ab^{-1} \in \{q^2, q^{-2}\}$ or $\alpha = \beta = 0$. So :

if $ab^{-1} \notin \{q^2, q^{-2}\}$, $\tilde{X}' \simeq L(\gamma_a\gamma_b)$ is simple and :

$$\mathrm{ch}_q(V)\mathrm{ch}_q(W) = \mathrm{ch}_q(\tilde{X}') = \mathrm{ch}_q(L(\gamma_a\gamma_b))$$

if $ab^{-1} = q^2$ (resp. $ab^{-1} = q^{-2}$), $\mathbb{C}e_3 \subset \tilde{X}'$ (resp. $\mathbb{C}((q^2-1)e_2+e_3) \subset \tilde{X}'$) is a submodule of $\tilde{X}'$ isomorphic to $L(1)$ and :

$$\mathrm{ch}_q(V)\mathrm{ch}_q(W) = \mathrm{ch}_q(\tilde{X}') = \mathrm{ch}_q(L(\gamma_a\gamma_b)) + \mathrm{ch}_q(L(1))$$

Cinquième partie

Monomials of q and q,t-characters for non simply-laced quantum affinizations

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Résumé. Nakajima [N2, N3] a introduit le morphisme de q,t-caractères pour les représentations de dimension finie des algèbres affines quantiques simplement lacées : il s'agit d'une t-déformation du morphisme de q-caractères de Frenkel-Reshetikhin (qui sont des sommes de monômes avec une infinité de variables). Dans [He2] nous avons généralisé la construction des q,t-caractères aux cas non simplement lacés. Dans cette article nous prouvons en premier lieu une conjecture de [He2] : les monômes des q et q,t-caractères des représentations standards sont les mêmes dans les cas non simplement lacés (le cas simplement lacé a été traité dans [N3]) et les coefficients sont positifs. Ces q,t-caractères peuvent en particulier être considérés comme des t-déformations des q-caractères. Dans la preuve nous démontrons pour les algèbres affines quantiques de type A, B, C et les algèbres toroïdales quantiques de type $A^{(1)}$ que les espaces de l-poids des représentations fondamentales sont de dimension 1. Enfin nous prouvons, puis utilisons, une généralisation d'un résultat de [FR3, FM1, N1] : pour une affinisée quantique générale les l-poids d'un module simple de plus haut l-poids sont inférieurs au plus haut l-poids au sens des monômes.

Abstract. Nakajima [N2, N3] introduced the morphism of q,t-characters for finite dimensional representation of simply-laced quantum affine algebras : it is a t-deformation of the Frenkel-Reshetikhin's morphism of q-characters (sum of monomials in infinite variables). In [He2] we generalized the construction of q,t-characters for non simply-laced quantum affine algebras. First in this paper we prove a conjecture of [He2] : the monomials of q and q,t-characters of standard representations are the same in non simply-laced cases (the simply-laced cases were treated in [N3]) and the coefficients are non negative. In particular those q,t-characters can be considered as t-deformations of q-characters. In the proof we show that for quantum affine algebras of type A, B, C and quantum toroidal algebras of type $A^{(1)}$ the l-weight spaces of fundamental representations are of dimension 1. Eventually we show and use a generalization of a result of [FR3, FM1, N1] : for general quantum affinizations we prove that the l-weights of a l-highest weight simple module are lower than the highest l-weight in the sense of monomials.

1 Introduction

Drinfel'd [Dr1] and Jimbo [Jim] associated, independently, to any symmetrizable Kac-Moody algebra $\mathfrak{g}$ and any complex number $q \in \mathbb{C}^*$ a Hopf algebra $\mathcal{U}_q(\mathfrak{g})$ called quantum Kac-Moody algebra.

In this paper we suppose that $q \in \mathbb{C}^*$ is not a root of unity. In the case of a semi-simple Lie algebra $\mathfrak{g}$ of rank n (ie. with a finite Cartan matrix), the structure of the Grothendieck ring $\mathrm{Rep}(\mathcal{U}_q(\mathfrak{g}))$ of finite dimensional representations of the quantum finite algebra $\mathcal{U}_q(\mathfrak{g})$ is well understood. It is analogous to the classical case $q = 1$.

For the general case of Kac-Moody algebras the picture is less clear. The representation theory of the quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$ is of particular interest (see [CP3, CP4]). In this case there is a crucial property of $\mathcal{U}_q(\hat{\mathfrak{g}})$: it has two realizations, the usual Drinfel'd-Jimbo realization and a new realization (see [Dr2, Be]) as a quantum affinization of a quantum finite algebra $\mathcal{U}_q(\mathfrak{g})$.

To study the finite dimensional representations of $\mathcal{U}_q(\hat{\mathfrak{g}})$ Frenkel and Reshetikhin [FR3] introduced q-characters which encode the (pseudo)-eigenvalues of some commuting elements $\phi^{\pm}_{i,\pm m}$ ($m \geq 0$) of the Cartan subalgebra $\mathcal{U}_q(\hat{\mathfrak{h}}) \subset \mathcal{U}_q(\hat{\mathfrak{g}})$ (see also [Kn]) : for V a finite dimensional representations we have :

$$V = \bigoplus_{\gamma \in \mathbb{C}^{I\times\mathbb{Z}}} V_\gamma$$

where for $\gamma = (\gamma^{(\pm)}_{i,\pm m})_{i\in I, m\geq 0}$, V_γ is a simultaneous generalized eigenspace (l-weight space) :

$$V_\gamma = \{x \in V / \exists p \in \mathbb{N}, \forall i \in I, \forall m \geq 0, (\phi^{(\pm)}_{i,\pm m} - \gamma^{(\pm)}_{i,\pm m})^p.x = 0\}$$

The morphism of q-characters is an injective ring homomorphism :

$$\chi_q : \mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}})) \to \mathcal{Y} = \mathbb{Z}[Y^{\pm}_{i,a}]_{i\in I, a\in\mathbb{C}^*}$$

where $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ is the Grothendieck ring of finite dimensional (type 1)-representations of $\mathcal{U}_q(\hat{\mathfrak{g}})$ and $I = \{1, ..., n\}$, and :

$$\chi_q(V) = \sum_{\gamma \in \mathbb{C}^{I\times\mathbb{Z}}} \dim(V_\gamma) m_\gamma$$

where $m_\gamma \in \mathcal{Y}$ depends of γ. In particular $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ is commutative and isomorphic to $\mathbb{Z}[X_{i,a}]_{i\in I, a\in\mathbb{C}^*}$.

In the finite simply laced-case (type ADE) Nakajima [N2, N3] introduced t-analogs of q-characters. The motivations are the study of filtrations induced on representations by (pseudo)-Jordan decompositions, the study of the decomposition in irreducible modules of tensorial products and the study of cohomologies of certain quiver varieties. The morphism of q, t-characters is a $\mathbb{Z}$-linear map :

$$\chi_{q,t} : \mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}})) \to \mathcal{Y}_t = \mathbb{Z}[Y^{\pm}_{i,a}, t^{\pm}]_{i\in I, a\in\mathbb{C}^*}$$

which is a deformation of χ_q and multiplicative in a certain sense. A combinatorial axiomatic definition of q,t-characters is given. But the existence is non-trivial and is proved with the geometric theory of quiver varieties which holds only in the simply laced case.

In [He2] we defined and constructed q,t-characters for a finite (non necessarily simply-laced) Cartan matrix C with a new approach motivated by the non-commutative structure of $\mathcal{U}_q(\hat{\mathfrak{h}}) \subset \mathcal{U}_q(\hat{\mathfrak{g}})$, the study of screening currents of [FR2] and of deformed screening operators $S_{i,t}$ of [He1]. It generalizes the construction of Nakajima to non-simply laced cases.

The quantum affinization process (that Drinfel'd [Dr2] described for constructing the second realization of a quantum affine algebra) can be extended to all symmetrizable quantum Kac-Moody algebras $\mathcal{U}_q(\mathfrak{g})$ (see [Jin, N1, He4]). One obtains a new class of algebras called quantum affinizations : the quantum affinization of $\mathcal{U}_q(\mathfrak{g})$ is denoted by $\mathcal{U}_q(\hat{\mathfrak{g}})$. It has a triangular decomposition [He4]. For example the quantum affinization of a quantum affine algebra is called a quantum toroidal algebra. The quantum affine algebras are the simplest examples and are very special because they are also quantum Kac-Moody algebras. In the following, general quantum affinization means with an invertible quantum Cartan matrix (it includes most interesting cases like affine and toroidal quantum affine algebras, see section 2.2). In [He4] we developed the representation theory of general quantum affinizations and constructed a generalization of the q-characters morphism which appears to be a powerful tool for this investigation. In particular we proved that the new Drinfel'd coproduct leads to the construction of a fusion product on the Grothendieck group.

The results of this paper can be divided in three parts :

1) First we prove that for general quantum affinizations, the l-weights $m' \in \mathcal{Y}$ of a simple module of l-highest weight $m \in \mathcal{Y}$ are lower than m in the sense of monomials (theorem 3.2) : it means that $m'm^{-1}$ is a product of certain $A_{i,l}^{-1} \in \mathcal{Y}$. For C finite, this result was conjectured and partly proved in [FR3], and proved in [N1] (ADE-case) and [FM1] (finite case). In the general case no universal $\mathcal{R}$-matrix has been defined : so we propose a new proof based on the study of $\mathcal{U}_q(\hat{sl_2})$-Weyl modules introduced in [CP6]. This first result is used in the proof of the following points :

2) We prove a conjecture of [He2] : let $\mathcal{U}_q(\hat{\mathfrak{g}})$ be a quantum affine algebra (C finite) and M be a standard module of $\mathcal{U}_q(\hat{\mathfrak{g}})$ (tensorial product of fundamental representations). We prove that the coefficients of $\chi_{q,t}(M)$ are in $\mathbb{N}[t^{\pm}]$ and that the monomials of $\chi_{q,t}(M)$ are the monomials of $\chi_q(M)$ (theorem 7.5) (the case ADE follows from the work of Nakajima [N3] ; in this paper the non-simply laced case is treated.) In particular the q,t-characters for quantum affine algebras have a finite number of monomials, and the q,t-characters of [He2] can be considered as t-deformations of q-characters for all quantum affine algebras. So it is an argument for the existence of a geometric model behind the q,t-characters in non simply-laced cases (in the simply laced-case the standard module can be realized in the K-theory of quivers varieties).

3) In the proof of the conjecture we study combinatorial properties of q-characters : we prove that for quantum affinizations of type $A, B, C, A^{(1)}$ the l-weight spaces of fundamental representations are of dimension 1 (theorem 3.5). Note that this property is not

true in general, for example for type D.

Our proof is based on an investigation of the classical algorithm (see [FM1, He3]) which gives q-characters. The proof is direct without explicit computation. Note that for type A, B, C the result could follow from explicit computation of the specialized $\mathcal{R}$-matrix, as explained in [FR3]. The result should produce the formulas of [FR1]; however with this method it would not be easy to decide if the coefficients are 1 (for example it is not the case for type D_4). Moreover our direct proof can be extend to quantum toroidal algebras of type $A^{(1)}$

This paper is organized as follows : in section 2 we give reminders on representations of quantum affinizations and their q-characters. In section 3 we state and prove theorem 3.2 (the l-weights of a l-highest weight simple module are lower than the highest l-weight in the sense of monomials) and state theorem 3.5 (on q-characters of fundamental representations) and give technical complements. The proof of theorem 3.5 is based on a case by case investigation explained in sections 4, 5, 6. In section 7 we give reminders on q,t-characters and we prove theorem 7.5 (on coefficients of q,t-characters of standard monomials). For the theorem 7.5 type F_4, our proof is based on results obtained by a computer program written with Travis Schedler, and those results are partly given in the appendix (section 8).

2 Reminder

2.1 Representations of quantum affinizations

Let $C = (C_{i,j})_{1\leq i,j\leq n}$ be a symmetrizable (non necessarily finite) Cartan matrix and $I = \{1, ..., n\}$. Let $D = \mathrm{diag}(r_1, ..., r_n)$ such that $B = DC$ is symmetric. We consider $(\mathfrak{h}, \Pi, \Pi^\vee)$ a realization of C, the weight lattice $P \subset \mathfrak{h}^*$, the roots $\alpha_1, ..., \alpha_n \in P$, the set of dominant weights $P^+ \subset P$, the relation $\leq$ on P, the map $\nu : \mathfrak{h}^* \to \mathfrak{h}$ (see [He4]).

Let $q \in \mathbb{C}^*$ not a root of unity. Let $\mathcal{U}_q(\mathfrak{g})$ be the quantum Kac-Moody algebra of Cartan matrix C. Let $\mathcal{U}_q(\hat{\mathfrak{g}}) \supset \mathcal{U}_q(\mathfrak{g})$ be the quantum affinization of $\mathcal{U}_q(\mathfrak{g})$, with generators $x_{i,r}^{\pm}, k_h, c^{\pm\frac{1}{2}}, \phi^{(\pm)}_{i,\pm m}$, where $i \in I, r \in \mathbb{Z}, m \geq 0, h \in \mathfrak{h}$ (see for example [He4]). A $\mathcal{U}_q(\hat{\mathfrak{g}})$-module is said to be integrable if it is integrable as a $\mathcal{U}_q(\mathfrak{g})$-module.

Denote by P_l the set of l-weights, that is to say of couple (λ, Ψ) such that $\lambda \in \mathfrak{h}^*$, $\Psi = (\Psi^{\pm}_{i,\pm m})_{i\in I, m\geq 0}$, $\Psi^{\pm}_{i,\pm m} \in \mathbb{C}$, $\Psi^{\pm}_{i,0} = q_i^{\pm\lambda(\alpha_i^\vee)}$. Note that if C is finite, λ is uniquely determined by Ψ.

A $\mathcal{U}_q(\hat{\mathfrak{g}})$-module V is said to be of l-highest weight $(\lambda, \Psi) \in P_l$ if there is $v \in V$ such that $(i \in I, r \in \mathbb{Z}, m \geq 0, h \in \mathfrak{h})$:

$$x_{i,r}^+.v = 0 \ , \ V = \mathcal{U}_q(\hat{\mathfrak{g}}).v \ , \ \phi^{\pm}_{i,\pm m}.v = \Psi^{\pm}_{i,\pm m} v \ , \ k_h.v = q^{\lambda(h)}.v$$

For $(\lambda, \Psi) \in P_l$, let $L(\lambda, \Psi)$ be the simple $\mathcal{U}_q(\hat{\mathfrak{g}})$-module of l-highest weight (λ, Ψ) (see [He4]).

Let P_l^+ be the set of dominant l-weights, that is to say the set of $(\lambda, \Psi) \in P_l$ such that there exist (Drinfel'd)-polynomials $P_i(z) \in \mathbb{C}[z]$ $(i \in I)$ of constant term 1 satifying in $\mathbb{C}[[z]]$ (resp. in $\mathbb{C}[[z^{-1}]]$) :

$$\sum_{m\geq 0}\Psi^{\pm}_{i,\pm m}z^{\pm m} = q_i^{\deg(P_i)}\frac{P_i(zq_i^{-1})}{P_i(zq_i)}$$

Theorem 2.1. *For $(\lambda, \Psi) \in P_l$, $L(\lambda, \Psi)$ is integrable if and only $(\lambda, \Psi) \in P_l^+$.*

If $\mathfrak{g}$ is finite (case of a quantum affine algebra) it is a result of Chari-Pressley in [CP3, CP4] (moreover in this case the integrable $L(\lambda, \Psi)$ are finite dimensional). If C is simply-laced the result is proved by Nakajima in [N1]. If C is of type $A_n^{(1)}$ (toroidal $\hat{sl}_n$-case) the result is proved by Miki in [M1]. In general the result is proved in [He4].

Denote by $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ the Grothendieck group of decomposable integrable representations of type 1 (see [He4]). The operators $k_h, \phi^{(\pm)}_{i,\pm m} \in \mathcal{U}_q(\hat{\mathfrak{g}})$ $(h \in \mathfrak{h}, i \in I, m \in \mathbb{Z})$ commute on $V \in \mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$. So we have a l-weight space decomposition :

$$V = \bigoplus_{(\lambda,\gamma)\in P_l} V_{\lambda,\gamma}$$

$$V_{\lambda,\gamma} = \{x \in V_\lambda / \exists p \in \mathbb{N}, \forall i \in I, \forall m \geq 0, (\phi^{(\pm)}_{i,\pm m} - \gamma^{(\pm)}_{i,\pm m})^p.x = 0\}$$
$$\subset V_\lambda = \{v \in V / \forall h \in \mathfrak{h}, k_h.v = q^{\lambda(h)}v\}$$

Let $QP_l^+ \subset P_l$ be the set of $(\mu, \gamma) \in P_l$ such that there exist polynomials $Q_i(z), R_i(z) \in \mathbb{C}[z]$ $(i \in I)$ of constant term 1 such that in $\mathbb{C}[[z]]$ (resp. in $\mathbb{C}[[z^{-1}]]$) :

$$\sum_{m\geq 0}\gamma^{\pm}_{i,\pm m}z^{\pm m} = q_i^{\deg(Q_i)-\deg(R_i)}\frac{Q_i(zq_i^{-1})R_i(zq_i)}{Q_i(zq_i)R_i(zq_i^{-1})}$$

In particular $P_l^+ \subset QP_l^+$.

Proposition 2.2. *Let V be a module in $Rep(\mathcal{U}_q(\hat{\mathfrak{g}}))$ and $(\mu, \gamma) \in P_l$. If $dim(V_{\mu,\gamma}) > 0$ then $(\mu, \gamma) \in QP_l^+$.*

The result is proved in [FR3] for C finite. The generalization is straightforward (see [He4]).

2.2 q-characters

Let z be an indeterminate. We denote $z_i = z^{r_i}$ and for $l \in \mathbb{Z}$, $[l]_z = \frac{z^l - z^{-l}}{z - z^{-1}} \in \mathbb{Z}[z^{\pm}]$. Let $C(z)$ be the quantized Cartan matrix defined by $(i \neq j \in I)$:

$$C_{i,i}(z) = z_i + z_i^{-1}\ ,\ C_{i,j}(z) = [C_{i,j}]_z$$

In the rest of this paper we suppose that $C(z)$ is invertible, that is to say $\det(C(z)) \neq 0$. We have seen in lemma 6.4 of [He3] that the condition $(C_{i,j} < -1 \Rightarrow -C_{j,i} \leq r_i)$ implies that $\det(C(z)) \neq 0$. In particular finite and affine Cartan matrices (where we impose $r_1 = r_2 = 2$ for $A_1^{(1)}$) satisfy this condition.

Consider formal variables $Y_{i,a}^{\pm}$ ($i \in I, a \in \mathbb{C}^*$) and k_ω ($\omega \in \mathfrak{h}$) ($k_0 = 1$). Let A be the commutative group of monomials of the form $m = \prod_{i \in I, a \in \mathbb{C}^*} Y_{i,a}^{u_{i,a}(m)} k_{\omega(m)}$ where a finite number of $u_{i,a}(m) \in \mathbb{Z}$ are non zero, $\omega(m) \in \mathfrak{h}$ (the coweight of m), and such that for $i \in I : \alpha_i(\omega(m)) = r_i \sum_{a \in \mathbb{C}^*} u_{i,a}(m)$.

For $(\mu, \Gamma) \in QP_l^+$ we define $Y_{\mu,\Gamma} \in A$ by :

$$Y_{\mu,\Gamma} = k_{\nu(\mu)} \prod_{i \in I, a \in \mathbb{C}^*} Y_{i,a}^{\beta_{i,a} - \gamma_{i,a}}$$

where $\beta_{i,a}, \gamma_{i,a} \in \mathbb{Z}$ are defined by $Q_i(u) = \prod_{a \in \mathbb{C}^*} (1 - ua)^{\beta_{i,a}}$, $R_i(u) = \prod_{a \in \mathbb{C}^*} (1 - ua)^{\gamma_{i,a}}$.

For $(\mu, \Gamma) \in QP_L^+$ and V a $\mathcal{U}_q(\hat{\mathfrak{g}})$-module, we denote $V_m = V_{\mu,\Gamma}$ where $m = Y_{\mu,\Gamma}$.

For $\chi \in A^{\mathbb{Z}}$ we say $\chi \in \mathcal{Y}$ if for $\lambda \in \mathfrak{h}$, there is a finite number of monomials of χ such that $\omega(m) = \lambda$ and there is a finite number of element $\lambda_1, ..., \lambda_s \in \mathfrak{h}^*$ such that the coweights of monomials of χ are in $\bigcup_{j=1...s} \nu(D(\lambda_j))$ (where $D(\lambda_j) = \{\mu \in \mathfrak{h}^* / \mu \leq \lambda_j\}$). In particular $\mathcal{Y}$ has a structure of $\mathfrak{h}$-graded $\mathbb{Z}$-algebra.

Definition 2.3. *For $V \in Rep(\mathcal{U}_q(\hat{\mathfrak{g}}))$ a representation, the q-character of V is :*

$$\chi_q(V) = \sum_{(\mu,\Gamma) \in QP_l^+} dim(V_{\mu,\Gamma}) Y_{\mu,\Gamma} \in \mathcal{Y}$$

If C is finite the construction is given in [FR3] and it is proved that $\chi_q : \mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}})) \to \mathcal{Y}$ is an injective ring homomorphism (with the ring structure on $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ deduced from the Hopf algebra structure of $\mathcal{U}_q(\hat{\mathfrak{g}})$).

For general C, χ_q is defined in [He4]. A priori there is no ring structure on $\mathcal{Y}$ that comes from a tensor product, but we proved [He4] :

Theorem 2.4. *The image $Im(\chi_q) \subset \mathcal{Y}$ is a subalgebra of $\mathcal{Y}$.*

Let $*$ be the induced commutative product on $\mathrm{Im}(\chi_q) \subset \mathcal{Y}$. Using a deformation of the new Drinfel'd coproduct we proved in [He4] :

Theorem 2.5. *For $(\lambda, \Psi), (\lambda', \Psi') \in P_l^+$ we have :*

$$L(\lambda, \Psi) * L(\lambda', \Psi') = L(\lambda + \lambda', \Psi\Psi') + \sum_{(\mu,\Phi) \in P_l^+ / \mu < \lambda + \lambda'} Q_{\lambda,\Psi,\lambda',\Psi'}(\mu, \Phi) L(\mu, \Phi)$$

where the integers $Q_{\lambda,\Psi,\lambda',\Psi'}(\mu, \Phi) \geq 0$.

2.3 Notations and technical tools

For $i \in I$ and $a \in \mathbb{C}^*$ we set :

$$A_{i,a} = k_{r_i\alpha_i^\vee} Y_{i,aq_i^{-1}} Y_{i,aq_i} \prod_{j/C_{j,i}<0} \prod_{r=C_{j,i}+1,C_{j,i}+3,...,-C_{j,i}-1} Y_{j,aq^r}^{-1} \in A$$

The $A_{i,l}$ are algebraically independent (see [He2]). Let $\mathcal{A} = \mathbb{Z}[A_{i,a}^{-1}]_{i\in I,a\in\mathbb{C}^*} \subset \mathcal{Y}$.

For a product $M \in A$ such that $\omega(M) \in \mathbb{Z}\alpha_1 \oplus ... \oplus \mathbb{Z}\alpha_n$, denote $\omega(M) = -v_1(M)\alpha_1 - ... - v_n(M)\alpha_n$ and $v(M) = v_1(M) + ... + v_n(M)$. v defines a $\mathbb{N}$-gradation on $\mathcal{A}$.

Definition 2.6. *For $m, m' \in A$, we say that $m \geq m'$ if $m'm^{-1} \in \mathcal{A}$.*

For $m \in A$ and $J \subset I$, denote $u_J(m) = \sum_{j\in J,a\in\mathbb{C}^*} u_{j,a}(m)$, $m^{(J)} = k_{\omega(m)} \prod_{j\in J,a\in\mathbb{C}^*} Y_{j,a}^{u_{j,a}(m)}$ and $(j \in I)$:

$$u_j^\pm(m) = \pm \sum_{l\in\mathbb{Z}/\pm u_{j,l}(m)>0} u_{j,l}(m)\ ,\ u_J^\pm(m) = \sum_{j\in J} u_j^\pm(m)$$

For $J \subset I$, denote $B_J \subset A$ the set of J-dominant monomials (ie $\forall j \in J, l \in \mathbb{Z}$, $u_{j,l}(m) \geq 0$) and $B = B_I$. Note that for $(\lambda, \Psi) \in QP_l^+$: $((\lambda, \Psi) \in P_l^+ \Leftrightarrow Y_{\lambda,\Psi} \in B)$.

For $m \in B$ denote $V_m = L(\lambda, \Psi) \in \mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ where $(\lambda, \Psi) \in P_l^+$ is given by $Y_{\lambda,\Psi} = m$. In particular for $i \in I, a \in \mathbb{C}^*$ denote $V_i(a) = V_{k_{\nu(\Lambda_i)}Y_{i,a}}$ and $X_{i,a} = \chi_q(V_{i,a})$. The simple modules $V_i(a)$ are called fundamental representations.

Denote $M_m = \prod_{i\in I,a\in\mathbb{C}^*} V_{i,a}^{*u_{i,a}(m)} \in \mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$. We have $\chi_q(M_m) = \prod_{i\in I,a\in\mathbb{C}^*} X_{i,a}^{u_{i,a}(m)}$.

For $J \subset I$ we denote by $\mathfrak{g}_J$ the Kac-Moody algebra of Cartan matrix $(C_{i,j})_{i,j\in J}$ and by χ_q^J the morphism of q-characters for $\mathcal{U}_q(\hat{\mathfrak{g}}_J) \subset \mathcal{U}_q(\hat{\mathfrak{g}})$. Let us recall the definition of the morphism τ_J (section 3.3 in [FM1] for finite case and [He4] for general case) :

We suppose that $\mathfrak{g}_J$ is finite. Let $\mathfrak{h}_J^\perp = \{\omega \in \mathfrak{h}/\forall i \in J, \alpha_i(\omega) = 0\}$ and $\mathfrak{h}_J = \bigoplus_{i\in J} \mathbb{Q}\Lambda_i^\vee$. Consider formal variables k'_ω ($\omega \in \mathfrak{h}_J$), k_ω ($\omega \in \mathfrak{h}_J^\perp$), $Y_{i,a}^\pm$ ($i \in J, a \in \mathbb{C}^*$), $Z_{j,c}$ ($j \in I - J$, $c \in \mathbb{C}^*$). Let $A^{(J)}$ be the commutative group of monomials :

$$m = k'_{\omega'(m)} k_{\omega(m)} \prod_{i\in J,a\in\mathbb{C}^*} Y_{i,a}^{u_{i,a}(m)} \prod_{j\in I-J,c\in\mathbb{C}^*} Z_{j,c}^{z_{j,c}(m)}$$

where a finite number of $u_{i,a}(m), z_{j,c}(m), r(m) \in \mathbb{Z}$ are non zero, $\omega(m) \in \mathfrak{h}_J^\perp$ and such that for $i \in J$, $\alpha_i(\omega'(m)) = r_i u_i(m) = r_i \sum_{a\in\mathbb{C}^*} u_{i,a}(m)$.

Let $\tau_J : A \to A^{(J)}$ be the group morphism defined formally by ($j \in I$, $a \in \mathbb{C}^*$, $\lambda \in \mathfrak{h}$) :

$$\tau_J(Y_{j,a}) = Y_{j,a}^{\epsilon_{j,J}} \prod_{k\in I-J} \prod_{r\in\mathbb{Z}} Z_{k,aq^r}^{p_{j,k}(r)}\ ,\ \tau_J(k_\lambda) = k'_{\sum_{i\in J}\alpha_i(\lambda)\Lambda_i^\vee} k_{\lambda-\sum_{i\in J}\alpha_i(\lambda)\Lambda_i^\vee}$$

where $j \in J \Leftrightarrow \epsilon_{j,J} = 1$ and $j \notin J \Leftrightarrow \epsilon_{j,J} = 0$. The $p_{i,j}(r) \in \mathbb{Z}$ are defined in the following way : we write $\tilde{C}(z) = \frac{\tilde{C}'(z)}{d(z)}$ where $d(z), \tilde{C}'_{i,j}(z) \in \mathbb{Z}[z^\pm]$ and $(D(z)\tilde{C}'(z))_{i,j} = \sum_{r\in\mathbb{Z}} p_{i,j}(r) z^r$.

It is proved in [FM1] (finite case) and in [He4] (the proof is given for the $\tau_{\{i\}}$ $(i \in I)$, but the proof for τ_J $(J \subset I$, $\mathfrak{g}_J$ finite) is the same) :

Lemma 2.7. *Consider V a module in $Rep(\mathcal{U}_q(\hat{\mathfrak{g}}))$ and a decomposition $\tau_J(\chi_q(V)) = \sum\limits_k P_k Q_k$ where $P_k \in \mathbb{Z}[Y_{i,a}^{\pm}, k'_h]_{i\in J, a\in\mathbb{C}^*, h\in\mathfrak{h}_J}$, Q_k is a monomial in $\mathbb{Z}[Z_{j,c}^{\pm}, k_h]_{j\in I-J, c\in\mathbb{C}^*, h\in\mathfrak{h}_J^{\perp}}$ and all monomials Q_k are distinct. Then there exist a $\mathcal{U}_q(\hat{\mathfrak{g}}_J)$-module $\bigoplus\limits_k V_k$ isomorphic to the restriction of V to $\mathcal{U}_q(\hat{\mathfrak{g}}_J)$ and such that for all k, $\chi_q^J(V_k) = P_k$.*

2.4 Classical algorithm

Consider $\mathfrak{K} = \bigcap\limits_{i\in I} \mathfrak{K}_i \subset \mathcal{Y}$ where $\mathfrak{K}_i = \mathrm{Ker}(S_i) \subset \mathcal{Y}$ is the kernel of the screening operator S_i (see [He4]).

Theorem 2.8. *We have $\mathfrak{K} = Im(\chi_q)$ and it is a subalgebra of $\mathcal{Y}$.*

The result in proved in [FM1] for C finite and in [He4] in general. Note that for $m \in B_i$, there is a unique $F_i(m) \in \mathfrak{K}_i$ such that m is the unique i-dominant monomial of $F_i(m)$ (see [He2]).

In [He2] a classical algorithm (and also a t-deformation of it) is proposed : if it is well-defined, it gives for $m \in B$ a $F(m) \in \mathfrak{K}$ such that m is the unique dominant monomial of $F(m)$. Such an algorithm was first used in [FM1] for finite case (see also [He3]). Note that if $F(m)$ exists, it is unique (see [He2]). Let us describe this algorithm : first for $m \in B$ we have to define the set D_m :

Definition 2.9. *For $m \in B$, we say that $m' \in D_m$ if and only if there is a finite sequence $(m_0 = m, m_1, ..., m_R = m')$, such that for all $1 \leq r \leq R$, there is $j \in I$ such that $m_{r-1} \in B_j$ and m_r is a monomial of $F_j(m_{r-1})$.*

In particular the set D_m is countable (see [He2]) and $m' \in D_m \Rightarrow m' \leq m$. Denote $D_m = \{m_0 = m, m_1, m_2, ...\}$ such that $m_r \leq m_{r'} \Rightarrow r \geq r'$.

For $r, r' \geq 0$ and $j \in I$ denote $[F_j(m_{r'})]_{m_r} \in \mathbb{Z}$ the coefficient of m_r in $F_j(m_{r'})$.

We call classical algorithm the following inductive definition of the sequences $(s(m_r))_{r\geq 0} \in \mathbb{Z}^{\mathbb{N}}$, $(s_j(m_r))_{r\geq 0} \in \mathbb{Z}^{\mathbb{N}}$ $(j \in I)$: $s(m_0) = 1$, $s_j(m_0) = 0$ and for $r \geq 1, j \in I$:

$$s_j(m_r) = \sum_{r'<r} (s(m_{r'}) - s_j(m_{r'}))[F_j(m_{r'})]_{m_r}$$

$$m_r \notin B_j \Rightarrow s(m_r) = s_j(m_r) \ , \ m_r \in B \Rightarrow s(m_r) = 0$$

It follows from theorem 2.8 that the classical algorithm is weel-defined and for all $m \in B$, $F(m) \in \mathfrak{K}$ exists (see section 5.5.4 in [He4]).

3 Monomials of q-characters

In this section we state the two main results on q-characters of this paper : theorems 3.2 and 3.5.

3.1 First result

In this section we prove that for m' a l-weight of V_m we have $m' \leq m$ (theorem 3.2). This result is proved in [FR3, FM1] for C finite. In the general case a universal $\mathcal{R}$-matrix has not been defined so we propose a new proof based on the Weyl modules introduced in [CP6].

Definition 3.1. *For $m \in B$, denote $L(m) = \chi_q(V_m)$ and by $D(m)$ the set of monomials of $L(m)$.*

The partial order on monomials is set in definition 2.6.

Theorem 3.2. *For $m \in B$ and $m' \in D(m)$, we have $m' \leq m$.*

In this section 3 we prove this theorem. First let us show some lemmas which will be useful :

Lemma 3.3. *Let V be a $\mathcal{U}_q(\hat{\mathfrak{g}})$-module and $W \subset V$ a $\mathcal{U}_q(\hat{\mathfrak{h}})$-submodule of V. Then for $i \in I$, $W_i' = \sum_{r \in \mathbb{Z}} x_{i,r}^-.W$ is a $\mathcal{U}_q(\hat{\mathfrak{h}})$-submodule of V.*

Proof : For $w \in W$, $j \in J$, $m, r \in \mathbb{Z}$ ($m \neq 0$), $h \in \mathfrak{h}$ we have :

$$h_{j,m}.(x_{i,r}^-.w) = x_{i,r}^-.(h_{j,m}.w) - \frac{1}{m}[mB_{i,j}]_q x_{i,m+r}^-.w \in W_i'$$

$$k_h.(x_{i,r}^-.w) = x_{i,r}^-.(q^{\alpha_i(h)}k_h.w) \in W_i'$$

□

Note that q-character of an (integrable) $\mathcal{U}_q(\hat{\mathfrak{h}})$-module is well-defined (see section 5.4 of [He4]).

Lemma 3.4. *Suppose that $\mathfrak{g} = sl_2$ and let L be a finite dimensional $\mathcal{U}_q(\hat{\mathfrak{g}})$-module ($\Lambda^\vee$ is the fundamental coweight).*

(i) If L is of l-highest weight M then $L_{m'} \neq \{0\}$ implies $m' \leq M$.

(ii) For $p \in \mathbb{Z}$, let $L_p = \sum_{\lambda \in P^*/\lambda(\Lambda^\vee) \geq p} L_\lambda$ and $L_p' = \sum_{r \in \mathbb{Z}} x_r^-.L_p$. Then L_p, L_p' are $\mathcal{U}_q(\hat{\mathfrak{h}})$-submodule of L and $(L_p')_{m'} \neq 0 \Rightarrow \exists m, m' \leq m$ and $(L_p)_m \neq \{0\}$.

Proof : (i) Consider the Weyl module $W_q(M)$ of l-highest weight M defined in [CP6] : $W_q(M)$ is the universal finite dimensional module of l-highest weight M such that all finite dimensional module of highest l-weight M is a quotient of $W_q(M)$. In particular L

is a quotient of $W_q(M)$. So it suffices to study $W_q(M)$. For $\mathcal{U}_q(\hat{sl}_2)$, the Weyl modules are explicitly described in [CP7] : in particular the dimension of $W_q(M)$ is 2^m where $m = u(M) = \sum_{i\in I, a\in\mathbb{C}^*} u_{i,a}(M)$. But (see [VV2, AK, FM1]) there is a standard module (tensorial product of fundamental representations) of highest l-weight M. The dimension of such a standard module is 2^m and it is a quotient of $W_q(M)$. So $W_q(M)$ is isomorphic to a standard module. The q-character of a standard module is known (see section 2.3), in particular for a l-weight m' of $W_q(M)$ we have $m' \leq M$.

(ii) L_p is a $\mathcal{U}_q(\hat{\mathfrak{h}})$-submodule of L because the action of $\mathcal{U}_q(\hat{\mathfrak{h}})$ does not change the weight, so it follows from lemma 3.3 that L'_p is a $\mathcal{U}_q(\hat{\mathfrak{h}})$-submodule of L. Let us prove the second point by induction on $\dim(L_p)$: if $L_p = \{0\}$ we have $L'_p = \{0\}$. In general let v be a l-highest weight vector of L_p (there is at least one, see the proof of proposition 5.2 in [He4]) and denote by M his l-weight. Consider $V = \mathcal{U}_q(\hat{\mathfrak{g}}).v$. It is a l-highest weight module and so it follows from (i) that $V_m \neq \{0\} \Rightarrow m \leq M$. We can use the induction hypothesis with $L^{(1)} = L/V$ and we get the result because $\chi_q(L) = \chi_q(V) + \chi_q(L^{(1)})$. $\square$

End of the proof of theorem 3.2 :

We prove the result by induction on $v(m'm^{-1}) \geq 0$. For $v(m'm^{-1}) = 0$ we have $m' = m$. In general suppose that the result is known for $v(m'm^{-1}) \leq p$ and let $W = \sum_{m'/v(m'm^{-1})\leq p} (V_m)_{m'}$.

Note that W is a $\mathcal{U}_q(\hat{\mathfrak{h}})$-submodule of V_m. It follows from the triangular decomposition of $\mathcal{U}_q(\hat{\mathfrak{g}})$ (see [He4]) that :

$$\bigoplus_{m'/v(m'm^{-1})=p+1} (V_m)_{m'} \subset \sum_{i\in I} W'_i \text{ where } W'_i = \sum_{r\in\mathbb{Z}} x^-_{i,r}.W$$

For $i \in I$, W'_i is a $\mathcal{U}_q(\hat{\mathfrak{h}})$-submodule of V_m (lemma 3.3). In particular $W'_i = \bigoplus_{m'} (W'_i \cap (V_m)'_m) = \bigoplus_{m'} (W_i)_{m'}$ and it suffices to show that for $i \in I$, $(W'_i)_{m'} \neq \{0\} \Rightarrow m' \leq m$.

Consider the decomposition of lemma 2.7 with $J = \{i\}$: $V = \bigoplus_k V_k$. We have $W = \bigoplus_k (V_k \cap W)$ and so $W'_i = \bigoplus_k (V_k \cap W'_i)$ (because V_k is a sub $\mathcal{U}_q(\hat{\mathfrak{g}}_i)$-module of V_m). So we can use the (ii) of lemma 3.4 to the $\mathcal{U}_{q_i}(\hat{sl}_2)$-module V_k with p_k such that $(V_k)_{p_k} = V_k \cap W$. We get that for m a monomial of $\chi^i_q(W'_i)$ there are m' a monomial $\chi^i_q(W)$ and $m'' \in \mathbb{Z}[Y^{-1}_{i,a} Y^{-1}_{i,aq_i^2} (k^{(i)}_{2r_i})^{-1}]_{a\in\mathbb{C}^*}$ such that $m = m'm''$ (the $k^{(i)}_r$ are the k'_h for $\mathcal{U}_{q_i}(\hat{sl}_2)$, see [He4]). It follows from the lemma 5.9 of [He4] (see also [FM1]) that $\tau_i(A_{i,aq_i}) = Y_{i,a} Y_{i,aq_i^2} k^{(i)}_{2r_i}$. So for M a monomial of $\chi_q(W'_i)$ there is M' a monomial of $\chi_q(W)$ such that $M \leq M'$. $\square$

3.2 Second result

Theorem 3.5. *Let $\mathfrak{g}$ be of type A_n ($n \geq 1$), $A^{(1)}_l$ ($l \geq 2$), B_n ($n \geq 2$) or C_n ($n \geq 2$). Let $i \in I, a \in \mathbb{C}^*$. Then for $m \in D(Y_{i,a})$, for all $j \in I, l \in \mathbb{Z}$, $u_{j,l}(m) \leq 1$. In particular all coefficients of $L(Y_{i,a})$ are equal to 1 and all l-weight space of $V_i(a)$ are of dimension 1.*

The last part of the result for type A_n is established in [N4].

Note that for type D_n the statement is false : for example for the type D_4, the monomial $Y_{2,2}Y_{2,4}^{-1}$ has a coefficient 2 in $\chi_q(V_2(q^0))$ (see the figure 1 in [N2]). For type F_4 it is also false (see section 8).

Let us explain the main points of the proof : it is based on the study of the classical algorithm in a case by case investigation : for type A_n a proof is given in [He2] and recalled in section 4. The result for type $A_l^{(1)}$ is proved in section 4, the result for type B_n is proved in section 5, the result for type C_n in section 6. In each case we suppose the existence of a $m \in D(Y_{i,a})$, such that there is $j \in I, l \in \mathbb{Z}$, $u_{j,l}(m) \geq 2$. The classical algorithm starts from the highest weight monomial. In our proof we look at a monomial m with $u_{j,l}(m) \geq 2$ and show inductively that it implies a condition on some monomials of higher weight. In particular it leads to a contradiction on the highest weight monomial.

Note that for type A_n, B_n, C_n the result could follow from explicit computation of $\chi_q(V_i(a))$. We would have to compute the specialized $\mathcal{R}$-matrix, as explained in [FR3]. The result should produce the formulas of [FR1]. However with this method it would not easy to decide if the coefficients are 1 (for example it is not the case for type D_4). In this paper the proof is direct without explicit computation. In particular it allows us to extend the proof to $A_l^{(1)}$.

3.3 Notations

In the following (sections 3, 4, 5, 6) we can omit the terms k_λ because we work in a set $D(m)$ or D_m : indeed m' such that $m' \leq m$ is uniquely determined by m and the $v_{i,l}(m'm^{-1})$.

For $J \subset I$, $j \in J, a \in \mathbb{C}^*$ consider $A_{j,a}^{J,\pm} = (A_{j,a}^{\pm})^{(J)}$. Define

$$\mu_J^I : \mathbb{Z}[A_{j,a}^{J,\pm}]_{j\in J, a\in\mathbb{C}^*} \to \mathbb{Z}[A_{j,a}^{\pm}]_{j\in J,a\in\mathbb{C}^*}$$

the ring morphism such that $\mu_J^I(A_{j,a}^{J,\pm}) = A_{j,a}^{\pm}$. For $m \in B_J$, denote $L^J(m^{(J)})$ defined for $\mathfrak{g}_J$ ($\mathfrak{g}_J$ is the Kac-Moody algebra of Cartan matrix $(C_{i_1,i_2})_{i_1,i_2\in J}$). Define :

$$L_J(m) = m^{(I-J)}\mu_J^I((m^{(J)})^{-1}L^J(m^{(J)}))$$

Definition 3.6. *For $J \subset I$ and $m \in B_J$, denote by $D_J(m)$ the set of monomials of $L_J(m)$.*

For $J = \{i\}$ and $m \in B_i$, an explicit description of $D_i(m)$ is given in [FR3] : a $\sigma \subset \mathbb{Z}$ is called a 2-segment if σ is of the form $\sigma = \{l, l+2, ..., l+2k\}$ where $l \in \mathbb{Z}, k \geq 0$. Two 2-segment are said to be in special position if their union is a 2-segment that properly contains each of them. All finite subset of $\mathbb{Z}$ with multiplicity $(l, u_l)_{l\in\mathbb{Z}}$ ($u_l \geq 0$) can be broken in a unique way into a union of 2-segments which are not in pairwise special position. For $m \in B_i$ and $r \in \{1, ..., 2r_i\}$, consider $(\sigma_j^{(r)})_j$ the decomposition of the $(l, u_{r+2r_il}(m))_{l\in\mathbb{Z}}$ as above. Let $m^{(i)} = \prod_{r=1,...,2r_i}\prod_j m_{r,j}$ where $m_{r,j} = \prod_{l\in\sigma_j^{(r)}} Y_{i,r+2r_il}$, and we have :

$$D_i(m) = m^{(I-\{i\})}\prod_{r=1,...,2r_i}\prod_j D_i(m_{r,j})$$

where for $m = \prod_{k=1...r} Y_{i,l+2r_ik}$. So $D_i(m)$ is :

$$\{mA^{-1}_{i,l+2r_ik+r_i}, mA^{-1}_{i,l+2r_ik+r_i}A^{-1}_{i,l+2r_i(k-1)+r_i}, ..., mA^{-1}_{i,l+2r_ik+r_i}A^{-1}_{i,l+2r_i(k-1)+r_i}...A^{-1}_{i,l+r_i}\}$$

In particular :

Lemma 3.7. *For $m \in B_i$ such that $\forall l \in \mathbb{Z}, u_{i,l}(m) \leq 1$, we have $F_i(m) = L_i(m)$.*

Definition 3.8. *Let $J \subset I$ and $i \in I, a \in \mathbb{C}^*$. For $m, m' \in D(Y_{i,a})$, we denote :*

$m \to_J m'$ (or $m' \leftarrow_J m$) if $m \in B_J$ and $m' \in D_J(m)$.

$m \rightharpoonup_J m'$ (or $m' \leftharpoonup_J m$) if $v(m'Y_{i,a}^{-1}) \geq v(mY_{i,a}^{-1})$ and $\exists m'' \in D(Y_{i,a})$ such that $m'' \to_J m$ and $m'' \to_J m'$.

In particular $m \to_J m'$ implies $m \rightharpoonup_J m'$. For $J = \{j\}$ (resp. $J = I$) we denote $\to_j, \rightharpoonup_J$ (resp. $\to, \rightharpoonup$).

3.4 Technical complements

Proposition 3.9. *For $m \in B$ and $J \subset I$ such that $\mathfrak{g}_J$ is finite, there is a unique decomposition :*

$$L(m) = \sum_{m' \in B_J \cap D(m)} \lambda_J(m') L_J(m')$$

where $\lambda_J(m') \geq 0$.

Proof : Consider the decomposition of lemma 2.7 with $J : V = \bigoplus_k V_k$. We can decompose each V_k in a sum of simple $\mathcal{U}_q(\hat{\mathfrak{g}}_J)$-modules in the Grothendieck group : $\chi_q^J(V_k) = \sum_{k'} \lambda_{k,k'} L^J(m_{k,k'})$ where $m_{k.k'} \in B_J$ and $\lambda_{k,k'} \geq 0$. In particular :

$$\tau_J^{-1}(P_kQ_k) = \sum_{k'} \lambda_{k,k'} L_J(\tau_J^{-1}(m_{k,k'}Q_k))$$

(consequence of lemma 5.9 of [He4]). For the uniqueness the $L_J(m')$ ($m' \in B_J$) are linearly independent. □

We say that a monomial $m \in B$ is right (resp. left) negative if : for $b \in \mathbb{C}^*$ such that ($\exists j \in I$, $u_{j,b}(m') \neq 0$ and $\forall k \in I, l > 0$ (resp. $l < 0$), $u_{k,bq^l}(m') = 0$), we have $\forall k \in I$, $u_{k,b}(m') \leq 0$ (see [FM1]). A product of right (resp. left) negative monomials if right (resp. left) negative.

Corollary 3.10. *For $i \in I, a \in \mathbb{C}^*$ and $m' \in D(Y_{i,a})$, we have :*

1) for $J \subset I$ such that $\mathfrak{g}_J$ is finite, there is $m'' \to_J m'$.

2) there is a finite sequence $Y_{i,a} = m_0 > m_1 > m_2 > ... > M_R = m'$ such that for all $1 \leq r \leq R$, $\exists j_r \in I$, $m_{r-1} \to_{j_r} m_r$.

3) if $m' \neq Y_{i,a}$, then m' is right negative.

4) for $b \in \mathbb{Z}$ and $j \in I$, we have $u_{j,b}(m') \neq 0 \Rightarrow b \in aq^{\mathbb{Z}}$

Note that the (1) will be used a lot in the following. For C finite those results are proved in [FM1].

Proof : 1) Consequence of proposition 3.9.

2) We use (1,3) inductively.

3) For $m' \in D_{Y_{i,a}} - \{Y_{i,a}\}$, we have $m' < Y_{i,a}$ (theorem 3.2) and m' is right or left negative, so not dominant. So as in [FM1] m' is right negative.

4) As for $m \in B_j \cap \mathbb{Z}[Y_{i,aq^m}]_{m\in\mathbb{Z}}$ implies $L_j(m) \in \mathbb{Z}[Y_{i,aq^m}]_{m\in\mathbb{Z}}$ (see section 3.3), we have $M \in B \cap \mathbb{Z}[Y_{i,aq^m}]_{m\in\mathbb{Z}}$ implies $D_m \subset \mathbb{Z}[Y_{i,aq^m}]_{m\in\mathbb{Z}}$ (see also [FM1]). $\square$

As a right negative monomial is not dominant, we have :

Corollary 3.11. *For $i \in I, a \in \mathbb{C}^*$, $L(Y_{i,a}) = F(Y_{i,a})$ has a unique dominant monomial $Y_{i,a}$.*

For $c \in \mathbb{C}^*$, let $\beta_c : \mathcal{Y} \to \mathcal{Y}$ be the ring morphism such that $\beta_c(Y_{i,a}) = Y_{i,ac}$.

Proposition 3.12. *For $a, b \in \mathbb{C}^*$, $L(Y_{i,a}) = \beta_{ab^{-1}}(L(Y_{i,b}))$.*

Proof : For $c \in \mathbb{C}^*$, we have $\beta_c(\mathfrak{K}) = \mathfrak{K}$ (see [FM1, He3]). $\square$

It suffices to study (see (4) of corollary 3.10 and [He3]) :

$$\chi_q : \mathbb{Z}[X_{i,q^l}]_{i\in I, l\in\mathbb{Z}} \to \mathbb{Z}[Y_{i,q^l}]_{i\in I, l\in\mathbb{Z}}$$

In the following we denote $\mathrm{Rep} = \mathbb{Z}[X_{i,q^l}]_{i\in I,l\in\mathbb{Z}}$, $X_{i,l} = X_{i,q^l}$, $\mathcal{Y} = \mathbb{Z}[Y^{\pm}_{i,q^l}]_{i\in I, l\geq 0}$, $Y^{\pm}_{i,l} = Y^{\pm}_{i,q^l}$. A Rep-monomials is a product of the $X_{i,l}$.

Lemma 3.13. *For $m \in B$, we have :*

$$D(m) \subset \prod_{j\in I, l\in\mathbb{Z}} D(Y_{j,l})^{u_{j,l}(m)}$$

Proof : $\prod\limits_{j\in I,l\in\mathbb{Z}} D(Y_{j,l})^{u_{j,l}(m)}$ is the set of monomials of $\chi_q(M_m)$. Then see theorem 2.5. $\square$

Lemma 3.14. *Let $m_1, m_2 \in B_i$ such that $\forall l \in \mathbb{Z}$, $u_{i,l}(m_1) \leq 1$ and $u_{i,l}(m_2) \leq 1$. Then $D_i(m_1) \cap D_i(m_2) = \emptyset \Leftrightarrow m_1 \neq m_2$.*

Proof : Let us write $m_1^{(i)} = \prod\limits_{r=1,\dots,2r_i} \prod\limits_j m^{(r)}_{\sigma_j^{(r)}}$ as in section 3.3 . Denote $\overline{\sigma_j^{(r)}} = \sigma_j^{(r)} \cup \{\max(\sigma_j^{(r)}) + 2r_i\}$. It follows from the hypothesis of the lemma that $(j,r) \neq (j',r') \Rightarrow \overline{\sigma_j^{(r)}} \cap \overline{\sigma_{j'}^{(r')}} = \emptyset$. Moreover for $m' \in D_i(m_{\sigma_j^{(r)}})$, we have $u_{i,r+2lr_i}(m') \neq 0 \Rightarrow \exists j, l \in \overline{\sigma_j^{(r)}}$. In particular the given of m' suffices to determine the $\overline{\sigma_j^{(r)}}$: for example we can find the set $\mathcal{M} = \{\max(\overline{\sigma_j^{(r)}})/j, r\}$ and $\mathcal{M}' = \{\min(\overline{\sigma_j^{(r)}})/j, r\}$ in the following way :

if $u_{i,l}(m')=1$ and $u_{i,l+2r_i}(m')=0$ and $u_{i,l+4r_i}(m')\geq 0$, then $l+2r_i\in\mathcal{M}$
if $u_{i,l}(m')=-1$ and $u_{i,l+2r_i}(m')\geq 0$, then $l\in\mathcal{M}$
if $u_{i,l}(m')=-1$ and $u_{i,l-2r_i}(m')=0$ and $u_{i,l-4r_i}(m')\leq 0$, then $l-2r_i\in\mathcal{M}'$
if $u_{i,l}(m')=1$ and $u_{i,l-2r_i}(m')\leq 0$, then $l\in\mathcal{M}'$

So if $m'\in D_i(m_1)\cap D_i(m_2)$, we have the same decomposition for m_1 and m_2, that is to say $m_1=m_2$. □

Lemma 3.15. *Let $m\in B$, $m'\in D(m)\cap B_j$. We suppose that for all $m''\in D(m)$ such that $v(m''m^{-1})<v(m'm^{-1})$, all $i\in I, l\in\mathbb{Z}$ we have $u_{i,l}(m'')\leq 1$. Then $D_j(m')\subset D(m)$.*

Proof : Let $p=v(m'm^{-1})$. Consider the decomposition of proposition 3.9 with $J=\{j\}$: $L(m)=\sum_{M\in B_j\cap D(m)}\lambda_j(M)L_j(M)$. It follows from the hypothesis and from the lemma 3.7 that for $v(Mm^{-1})<p$, $m'\notin D_j(M)$. So $\lambda_j(m')>0$, and $D_j(m')\subset D(m)$. □

Proposition 3.16. *Let $i\in I$ such that all $m\in D(Y_{i,L})$ satisfies : for $j\in I$, if $m\in B_j$ then $\forall l\in\mathbb{Z}, u_{j,l}(m)\leq 1$. Then all coefficients of $L(Y_{i,L})$ are equal to 1.*

Proof : We can compute the coefficients of $L(Y_{i,L})=F(Y_{i,L})$ thanks to the classical algorithm (see section 3.3) : let us show by induction on $v(mY_{i,L}^{-1})$ that the coefficients of m is equal to 1. For a monomial $m<Y_{i,L}$, there is $j\in I$ such that $m\notin B_j$. There is $M\rightarrow_j m$. It follows from the lemma 3.14 that M is entirely determined by m. So the coefficient of m is the coefficient of M in $L(Y_{i,l})$ multiplied with the coefficient of m in $L_j(M)=F_j(M)$, that is to say 1 (section 3.3). □

4 Type A_n, $A_l^{(1)}$

Proposition 4.1. *The property of theorem 3.5 is true for $\mathfrak{g}$ of type A_n $(n\geq 1)$ or $A_l^{(1)}$ $(l\geq 2)$.*

Technical consequences of this result which will be used in the following are discussed in section 4.3.

4.1 Type A_n

Let $n\geq 1$ and $\mathfrak{g}$ of type A_n. For $i\in\{2,...,n-1\}$, $l\in\mathbb{Z}$:

$$A_{i,l}=Y_{i,l+1}Y_{i,l-1}Y_{i+1,l}^{-1}Y_{i-1,l}^{-1}$$

$$A_{1,l}=Y_{1,l+1}Y_{1,l-1}Y_{2,l}^{-1}\ ,\ A_{n,l}=Y_{n,l+1}Y_{n,l-1}Y_{n-1,l}^{-1}$$

In particular for all $i\in I, l\in\mathbb{Z}$, $u(A_{i,l}^{-1})\leq 0$. So $m\leq m'\Rightarrow u(m)\leq u(m')$.

We can suppose $Y_{i,L}=Y_{i,0}$ (proposition 3.12).

Lemma 4.2. *For $m \in B$ and $m' \in D(m)$ we have $u(m) \geq v_n(m'm^{-1})$.*

Proof : For all $i \in I$, we have $\omega_i + \omega_{n+1-i} \in \alpha_n + \sum\limits_{j \leq n-1} \mathbb{Z}\alpha_j$ (see [Bo]).

Consider $m' \in D(m)$. It follows from the lemma 6.8 of [FM1] that $\omega(m) \geq \omega(m') \geq -\sum\limits_{i\in I} u_i(m)\omega_{n+1-i}$, and so $-\omega(m'm^{-1}) \leq \sum\limits_{i\in I} u_i(m)(\omega_i+\omega_{n+1-i})$. So $v_n(m'm^{-1}) \leq \sum\limits_{i\in I} u_i(m) = u(m)$. $\square$

Lemma 4.3. *For $j \in I$, if $m \in B_j \cap D(Y_{i,0})$ then $u_j(m) \leq 1$.*

Proof : Suppose there is $j \in I$ and $m_1 \in B_j \cap D(Y_{i,0})$ such that $u_j(m_1) \geq 2$. Let $J_1 = \{k \in I/k < j\}$, $J_2 = \{k \in I/k > j\}$ and $J = J_1 \cup J_2$. Let $m_2 \to_J m_1$ and $v = v_{j-1}(m_1 m_2^{-1}) + v_{j+1}(m_1 m_2^{-1})$. It follows from lemma 4.2 (for $\mathfrak{g}_{J_1}$ and $\mathfrak{g}_{J_2}$) that $u_{J_1}(m_2) + u_{J_2}(m_2) \geq v$. Moreover we have $u_j(m_2) = u_j(m_1) - v \geq 2 - v$. So $u(m_2) = u_{J_1}(m_2) + u_j(m_2) + u_{J_2}(m_2) \geq 2$, contradiction because $m_2 \leq Y_{i,0}$. $\square$

The proposition 4.1 for type A_n follows from proposition 3.16 and lemma 4.3.

4.2 Type $A_l^{(1)}$

Let $l \geq 2$ and $\mathfrak{g}$ of type $A_l^{(1)}$. For $i \in I$, $l \in \mathbb{Z}$ (where $Y_{-1,L} = Y_{n,L}$, $Y_{n+1,L} = Y_{0,L}$) :

$$A_{i,l} = Y_{i,l+1} Y_{i,l-1} Y_{i+1,l}^{-1} Y_{i-1,l}^{-1}$$

In particular for all $i \in I, l \in \mathbb{Z}$, $u(A_{i,l}^{-1}) \leq 0$. So $m \leq m' \Rightarrow u(m) \leq u(m')$.

We have an analog of lemma 4.3 by putting in the proof $J = I - \{j\}$ instead of $J_1 \cup J_2$. In particular we get proposition 4.1 for type $A_l^{(1)}$.

4.3 Consequences

In this section $\mathfrak{g}$ is general and consider $J \subset I$ such that $\mathfrak{g}_J$ is of type A_m $(m \leq n)$. We prove technical results which will be useful in the following. Let $i \in I, a \in \mathbb{C}^*$.

Lemma 4.4. *Let $m \in B_J$, $j \in J$ and $m' \in B_j$ such that $m' \in D_J(m)$. We have $u_J(m) \geq u_j(m')$.*

Proof : It follows from lemma 3.13 that we can write :

$$m' = m^{(I-J)} \prod_{k \in J, l \in \mathbb{Z}} m'_{k,l,1} ... m'_{k,l,u_{k,l}(m)}$$

where $m'_{k,l,1} \in D_J(Y_{k,l})$. For $\alpha \in J \times \mathbb{Z} \times \mathbb{N}$, it follows from lemma 4.3 that $u_{j,L}^+(m'_\alpha) \geq 1 \Rightarrow (m'_\alpha)^{(j)} = Y_{j,L}$. So there are $\alpha_1, ..., \alpha_{u_j(m')}$ such that the $(m'_{\alpha_p})^{(j)} = Y_{j,l_p}$ and $m'_{\alpha_1} ... m'_{\alpha_{u_j(m')}} = (m')^{(j)}$. So $u_j(m') \leq \sum\limits_{k \in J, l \in \mathbb{Z}} u_{k,l}(m) = u_J(m)$. $\square$

Lemma 4.5. *Let $M \in B_J$ such that $u_J(M) \geq 2$. The following properties are equivalent :*

(i) there are $j \in J$, $l \in \mathbb{Z}$, $M_1 \in D_J(M) \cap B_j$ such that $u_{j,l}(M_1) \geq 2$

(ii) there are $M' \in D_J(M) \cap B_J$, $i_1, i_2 \in J$, $l_1, l_2 \in \mathbb{Z}$ such that $i_2 - i_1 \geq |l_1 - l_2|$ and $(i_2 - i_1) - (l_2 - l_1)$ is even and $u_{i_1,l_1}(M') \geq 1$, $u_{i_2,l_2}(M') \geq 1$.

Moreover one can choose M' such that $M_1 \in D_J(M')$.

Proof : We can suppose that $\mathfrak{g} = \mathfrak{g}_J$ is of type A_n. For $K \subset I$, in this proof the notation $\rightarrow_K, \rightharpoonup_K$ is defined as in definition 3.8 by putting $D(M)$ instead of $D(Y_{i,a})$.

Let us show that $(ii) \Rightarrow (i)$: if $i_2 = i_1$ we have $u_{i_1,l_1}(M') \geq 2$. If $i_2 - i_1 > 0$, suppose that (i) is not true. In this situation we can use the lemma 3.15. Consider the integers :

$$K = \frac{(i_2 - i_1) + (l_2 - l_1)}{2} \text{ , } K' = \frac{(i_2 - i_1) + (l_1 - l_2)}{2}$$

We have $K, K' \geq 0$. Denote $i = i_1 + K = i_2 - K'$, $l = l_1 + K = l_2 + K'$ and consider :

$$V = A^{-1}_{i_1,l_1+1} A^{-1}_{i_1+1,l_1+2} \ldots A^{-1}_{i_1+(K-1),l_1+K} A^{-1}_{i_2,l_2+1} A^{-1}_{i_2-1,l_2+2} \ldots A^{-1}_{i_2-(K'-1),l_2+K'}$$

There is $M_1 \in D(M')$ such that $M_1 \leq M'V$ and $v_i(M_1(M')^{-1}) = 0$ (lemma 3.15). In particular $M_1 \in B_i$ and $u_{i,l}(M_1) \geq 2$, contradiction.

Let us show that $(i) \Rightarrow (ii)$: it follows from lemma 3.13 and proposition 7.1 that we can suppose that $u(M) = 2$. Denote $M = Y_{i_1,l_1} Y_{i_2,l_2}$, and let us show the result by induction on n. For $n = 1$ we have $M_1 = M$ and (ii) is clear. In general let $M_1 \in D(M) \cap B_j$ such that $M_1^{(j)} = Y_{j,l}^2$. We can suppose that $v(M_1 M^{-1})$ is minimal. If M_1 is dominant, we put $M_1 = M'$. Otherwise consider $J' = \{1, ..., n-1\}$ if $j \leq n-1$, and $J' = \{2, ..., n\}$ if $j = n$ (we suppose that $j \leq n-1$, the case $j = n$ can be treated in the same way). Let $M_2 \rightarrow_{J'} M_1$. The induction with $\mathfrak{g}_{J'}$ of type A_{n-1} gives that $M_2^{(J')} = Y_{i_1,l_1} Y_{i_2,l_2}$ where $i_2 - i_1 \geq |l_1 - l_2|$ and $(i_2 - i_1) - (l_2 - l_1)$ is even. We have $u(M_2) \leq 2$ and so $u_n(M_2) = u(M_2) - u_{J'}(M_2) \leq 0$. If $M_2^{(n)} = 1$, we put $M_2 = M'$. Otherwise it follows from the lemma 4.4 that we are in one the following cases α, β, γ :

α) if $M_2^{(n)} = Y^{-1}_{n,K_1} Y^{-1}_{n,K_2}$, we have :

$$M_2 \leftarrow_n M_3 = Y_{i_1,l_1} Y_{i_2,l_2} Y^{-1}_{n-1,K_1-1} Y^{-1}_{n-1,K_2-1} Y_{n,K_1-2} Y_{n,K_2-2}$$

If $Y_{i_2,l_2} \neq Y_{n-1,K_1-1}$ and $Y_{i_2,l_2} \neq Y_{n-1,K_2-1}$, there is $M_4 = M_3(M_1 M_2^{-1}) \in D(M_3)$ (lemma 3.15) such that $M_4^{(j)} = Y_{j,l}^2$ and $v(M_4 M^{-1}) < v(M_1 M^{-1})$, contradiction. So for example we have $M_3 = Y_{i_1,l_1} Y^{-1}_{n-1,K_2-1} Y_{n,K_1-2} Y_{n,K_2-2}$ and $i_2 = n-1$, $l_2 = K_1 - 1$. We have :

$$M_3 \leftarrow_{\{i_1+1,...,n-1\}} M_4 = Y_{i_1,l_1} Y^{-1}_{i_1,K_2-1-(n-i_1-1)} Y_{i_1+1,K_2-2-(n-i_1-1)} Y_{n,K_1-2}$$

If $Y_{i_1,l_1} Y^{-1}_{i_1,K_2-1-(n-i_1-1)} \neq 1$, there is :

$$M_4 \leftarrow_{\{1,...,i_1\}} M_5 = Y_{1,K_3} Y_{i_1,l_1} Y^{-1}_{i_1+1,K_2-2-(n-i_1-1)} Y_{i_1+1,K_2-2-(n-i_1-1)} Y_{n,K_1-2}$$

and $u(M_5) = 3$, impossible. So $l_1 = K_2 - 1 - (n - i_1 - 1)$ and $M_4 = Y_{i'_1,l'_1} Y_{i'_2,l'_2}$ where $i'_1 = i_1 + 1$, $i'_2 = n$, $l'_1 = K_2 - 2 - (n - i_1 - 1)$, $l'_2 = K_1 - 2$. Let $M' = M_4$. We have $i'_2 - i'_1 = i_2 - i_1 \geq |l_2 - l_1| = |l'_2 - l'_1|$ and $(i'_2 - i'_1) - (l'_2 - l'_1) = (i_2 - i_1) - (l_2 - l_1)$ is even.

β) if $M_2^{(n)} = Y_{n,K_1} Y_{n,K_2}^{-1}$, we have :

$$M_2 \leftarrow_n M_3 = Y_{i_1,l_1} Y_{i_2,l_2} Y_{n-1,K_2-1}^{-1} Y_{n,K_1-2} Y_{n,K_2-2}$$

and $u(M_3) = 3$, impossible.

γ) if $M_2^{(n)} = Y_{n,K_1}^{-1}$, we have :

$$M_2 \leftarrow_n M_3 = Y_{i_1,l_1} Y_{i_2,l_2} Y_{n-1,K_1-1}^{-1} Y_{n,K_1-2}$$

If $Y_{n-1,K_1-1} \neq Y_{i_1,l_1}$ and $Y_{n-1,K_1-1} \neq Y_{i_2,l_2}$, there is $M_4 = M_3(M_1 M_2^{-1}) \in D(M_3)$ (lemma 3.15) such that $M_4^{(j)} = Y_{j,l}^2$ and $v(M_4 M^{-1}) < v(M_1 M^{-1})$, contradiction. So for example we have $M_3 = Y_{i_1,l_1} Y_{n,K_1-2}$ and $i_2 = n - 1$, $l_2 = K_1 - 1$. Let $M' = M_3$ and we have $n - i_1 = i_2 - i_1 + 1 \geq |l_2 - l_1| + 1 \geq |(K_1 - 2) - l_1 + 1| + 1 \geq |(K_1 - 2) - l_1|$ and $n - i_1 - ((K_1 - 2) - l_1) = (i_2 - i_1) - (l_2 - l_1)$ is even.

For the last point, the arguments of this proof can be used for any $M' \in B$ such that $M_1 \in D_J(M')$. □

5 Type B

5.1 Statement

In this section $\mathfrak{g}$ is of type B_n ($n \geq 2$,) (see the definition in section 1). For $i \in \{2, ..., n-2\}$, $l \in \mathbb{Z}$:

$$A_{i,l} = Y_{i,l+2} Y_{i,l-2} Y_{i+1,l}^{-1} Y_{i-1,l}^{-1} \ , \ A_{1,l} = Y_{1,l+2} Y_{1,l-2} Y_{2,l}^{-1} \ , \ A_{n,l} = Y_{n,l+1} Y_{n,l-1} Y_{n-1,l}^{-1}$$

$$A_{n-1,l} = Y_{n-1,l+2} Y_{n-1,l-2} Y_{n-2,l}^{-1} Y_{n,l-1}^{-1} Y_{n,l+1}^{-1}$$

In this section we prove :

Proposition 5.1. *The property of theorem 3.5 is true for $\mathfrak{g}$ of type B_n ($n \geq 2$).*

Denote $J = \{1, ..., n-1\}$. We can suppose $Y_{i,L} = Y_{i,0}$ (proposition 3.12).
As $u(A_{n-1,l}^{-1}) > 0$, the $m' \leq m$ does not imply $u(m') \leq u(m)$.

5.2 Proof of the proposition 5.1

Suppose that there is $m \in D(Y_{i,0})$ such that there is $j \in J, l \in \mathbb{Z}$, $u_{j,l}(m) \geq 2$, and let m such that $v(mY_{i,0}^{-1})$ is minimal with this property.

Lemma 5.2. *There is $M \in D(Y_{i,0})$ such that $v(MY_{i,0}^{-1}) < v(mY_{i,0}^{-1})$ and $\exists l' \in \mathbb{Z}$, $u_{n,l'}(M) \geq 2$.*

Proof : Suppose that M does not exists. Let $m_1 \rightarrow_J m$. It follows from lemma 4.5 that $m_1 = m_1' Y_{i_1,l_1} Y_{i_2,l_2}$ where $m_1' \in B_J$, $2(i_2 - i_1) \geq |l_1 - l_2|$, $(i_2 - i_1) - (l_2 - l_1)/2$ is even. Let $m_2 \rightharpoonup_n m_1$ such that $v(m_1 m_2^{-1}) = 1$ (m_2 exists because it follows from the hypothesis and lemma 3.15 that $M_2 \rightarrow_n m_2$ implies $D_n(M_2) \subset D(Y_{i,0})$). We have $m_2 = m_2' Y_{i_1,l_1} Y_{i_2,l_2} Y_{n-1,L}^{-1} Y_{n,L-1}$ where $m_2' \in B_J$ and $u_{n,L-1}(m_2') \geq 0$. If $Y_{i_1,l_1} Y_{i_2,l_2} Y_{n-1,L}^{-1} \notin B$, there is $m_2 \rightharpoonup_J M_2 = m_2(mm_1^{-1})$ (lemma 3.15) such that $u_{j,l}(M_2) \geq 2$ and $v(M_2 Y_{i,0}^{-1}) < v(mY_{i,0}^{-1})$, contradiction. So for example $Y_{i_2,l_2} = Y_{n-1,L}$, and $m_2 \in B_J$. m_2 is not dominant (because we would have $u(m_2) \geq 2$), so there is $m_3 \rightharpoonup_n m_2$ such that $v(m_2 m_3^{-1}) = 1$ (same argument as above for the existence of m_3). We have $m_3 = m_3' Y_{i_1,l_1} Y_{n-1,L'}^{-1} Y_{n,L-1} Y_{n,L'-1}$ where $m_3' \in B_J$ and $u_{n,L-1}(m_3') \geq 0$, $u_{n,L'-1}(m_3') \geq 0$.

if we can choose $L' \neq L + 2$, the same argument gives $Y_{i_1,l_1} = Y_{n-1,L'}$ because :

$$m_4 = Y_{i_1,l_1} Y_{i_2,l_2} Y_{n-1,L'}^{-1} Y_{n,L-1'} m_4' \rightharpoonup_n m_1$$

where $m_4' \in B_J$. So we have $i_1 = i_2 = n - 1$, so $l_1 = l_2$ and $L' = L$, ie $u_{n,L}(m_3) \geq 2$, contradiction.

if we can not choose $L' \neq L + 2$, we can not use the same argument (because : $1 \notin L_n(Y_{n,L-1} Y_{n,L+1})$). We have ($k \geq 1$) :

$$m_3 \leftarrow_n m_5 = m_5' Y_{i_1,l_1} Y_{n-1,L+2}^{-1} Y_{n-1,L+4}^{-1} ... Y_{n-1,L-1+2k+1}^{-1} Y_{n,L-1} Y_{n,L+1} ... Y_{n,L-1+2k}$$

where $m_5' \in B$. Suppose that $m_5' Y_{i_1,l_1} Y_{n-1,L+4}^{-1} \notin B$. Then we have :

$$m_5 \leftharpoondown_{n-1} m_6 = m_5' Y_{i_1,l_1} Y_{n-1,L} Y_{n-1,L+2}^{-1} Y_{n-1,L+6}^{-1} ... Y_{n-1,L-1+2k+1}^{-1} Y_{n,L-1} Y_{n,L+5} ... Y_{n,L-1+2k}$$

But $u_{n-1,L}(m_6 A_{n,L}^{-1}) = 2$, contradiction. In the same way we prove by induction that $m_5' Y_{i_1,l_1} Y_{n-1,L+4}^{-1} Y_{n-1,L+8}^{-1} ... \in B$ and so :

$$m_5 = m_5'' Y_{i_1,l_1} Y_{n-1,L+2}^{-1} Y_{n-1,L+6}^{-1} ... Y_{n-1,L+2+4K'}^{-1} Y_{n,L-1} Y_{n,L+1} ... Y_{n,L-1+2k}$$

where $m_5'' \in B$. Suppose that $m_5 \notin B$. As m_5 is right negative, we have $L + 2 + 4K' = L + 2k$ and so $k = 1 + 2K'$. Consider $m_7 \rightarrow_J m_5$. We get that m_7 is dominant, and so $m_7 = Y_{i,0}$. Let $K'' = u_-(m_5) \leq K'$. We have $1 = u(m_7) \geq u_J(m_7) + u_n(m_7) \geq K'' + (k + 1 - 2K'') = 1 + k - K'' \geq k - K' \geq 1 + K'$. So $K' = 0$ and $k = 1$. In particular $m_5 = m_5'' Y_{i_1,l_1} Y_{n-1,L+2}^{-1} Y_{n,L-1} Y_{n,L+1}$ and so we have $i_1 = n - 1 - j$, $l_1 = L + 2 - 2j$ where $j \geq 0$ (otherwise we would have $u(m_7) \geq 2$). It implies $i_2 - i_1 - (l_2 - l_1)/2 = j - (j - 1) = 1$ not even, contradiction. So $m_5 \in B$ and $m_5 = Y_{i,0}$. But $u(m_5) \geq u_n(m_5) \geq 2$, contradiction. $\square$

Lemma 5.3. *Let $j \in I$ and $m \in D(Y_{i,0}) \cap B_j$ such that $u_j(m) = 2$. For $L, L' \in \mathbb{Z}$ such that $m^{(j)} = Y_{j,L} Y_{j,L'}$, we have $L \neq L'$.*

Proof : It follows from lemma 5.2 that we can suppose that $j = n$ and that for $v(m'Y_{i,0}^{-1}) \leq v(mY_{i,0}^{-1})$, for all $j \in J, l \in \mathbb{Z}$, $u_{j,l}(m') \leq 1$. Let M such that $\exists l \in \mathbb{Z}$, $u_{n,l}(M) \geq 2$ and suppose that $v(MY_{i,0}^{-1})$ is minimal with this property. Let L be maximal such that $u_{n,L-1}(M) \geq 2$. First it follows from lemma 4.5 that $M \in B_n$. We have $M \to_n M' = MA_{n,L}^{-1}$ and the coefficient of M' in $L_n(M)$ is at least 2. Suppose that there is $j \in J$ such that $M' \notin B_j$. Let $M'' \to_j M'$. It follows from lemma 3.14 that M'' is uniquely determined by M', and that the coefficient of M' in $F_j(M')$ is 1. But the coefficient of M'' in $L(Y_{i,0})$ is 1, so it follows from the proposition 3.9 that the coefficient of M' is 1, contradiction. So $M' \in B_J$. So $M = Y_{n-1,L}^{-1}\tilde{M}$ where $\tilde{M} \in B$ and $u_{n,L-1}(\tilde{M}) \geq 2$. As $u_n(M) \geq 2$, $M \notin B$. So there is $M_0 \to_J M$. We have :

$$M_0 = \tilde{M}_0 Y_{n,L-1} Y_{n,L-3}^{-1}$$

where $\tilde{M}_0 \in B$ and $u_J(\tilde{M}_0) \geq 1$. But M_0 is not right negative, so M_0 is dominant (corollary 3.10). But $u(M_0) \geq u_J(\tilde{M}_0) + u_{n,L-1}(M_0) \geq 2$, contradiction. □

So the proposition 5.1 follows from proposition 3.16 and lemma 5.3.

5.3 Complement : degree of monomials

The aim of this section is to prove that the degrees are bounded (it is a complement independent of the proof of theorem 3.5) :

Proposition 5.4. *For $j \in I$ and $m \in B_j \cap D(Y_{i,0})$, then $u_j(m) \leq 2$.*

Note that it follows from proposition 5.1 that we can use the lemma 3.15.

For $m \in A$ denote $w(m) = (u_J^+(m), u_J^-(m), u_n^+(m), u_n^-(m))$.

Suppose that there is $j \in J$ and $m_0 \in D(Y_{i,0}) \cap B_j$ such that $u_j(m_0) \geq 3$. It follows from lemma 4.4 that there is $m \to_J m_0$ such that $u_J(m) \geq 3$. Suppose that $v(mY_{i,0}^{-1})$ is minimal for this property.

Lemma 5.5. *There is $M \in D(Y_{i,0}) \cap B_n$ such that $M > m$ and $u_n(M) \geq 3$.*

Proof : We have $m \in B_J$ and $u_J(m) = 3$: $m = Y_{i_1,l_1}Y_{i_2,l_2}Y_{i_3,l_3}m'$ where $(m')^{(J)} = 1$. There is

$$m \leftharpoonup_n m_1 = Y_{i_1,l_1}Y_{i_2,l_2}Y_{i_3,l_3}Y_{n-1,L}^{-1}m_1'Y_{n,L-1}$$

where $(m_1')^{(J)} = 1$ and $u_{n,L-1}(m_1') \geq 0$. If $Y_{i_1,l_1}Y_{i_2,l_2}Y_{i_3,l_3}Y_{n-1,L}^{-1} \notin B$, there is $M_1 \to_J m_1$ such that $u_J(M_1) \geq 3$, contradiction. So for example $m_1 = Y_{i_1,l_1}Y_{i_2,l_2}m_1'Y_{n,l_3-1}$. There is

$$m_1 \leftharpoonup_n m_2 = Y_{i_1,l_1}Y_{i_2,l_2}Y_{n-1,L'}^{-1}m_2'Y_{n,l_3-1}Y_{n,L'-1}$$

where $(m_2')^{(J)} = 1$ and $u_{n,l_3-1}(m_2') \geq 0$, $u_{n,L'-1}(m_2') \geq 0$. It follows from lemma 5.3 that $l_3 \neq L'$. If $L' = l_3+1$, we have $m_2'' \to_J m_2$ where m_2'' is dominant and $u(m_2'') \geq u_J(m_2'') \geq 2$, contradiction. So we see as above that for example $m_2 = Y_{i_1,l_1}m_2'Y_{n,l_3-1}Y_{n,l_2-1}$. In the same way

$$m_2 \leftharpoonup_n m_3 = m_3'Y_{i_1,l_1}Y_{n-1,L''}^{-1}Y_{n,l_3-1}Y_{n,l_2-1}Y_{n,L''-1}$$

where $(m_3')^{(J)} = 1$ and $u_{n,l_3-1}(m_3') \geq 0$, $u_{n,l_2-1}(m_3') \geq 0$, $u_{n,l_1-1}(m_3') \geq 0$. We can conclude with lemma 4.4. □

For $m \in A$, denote $w(m) = (u_J^+(m), u_J^-(m), u_n^+(m), u_n^-(m))$.

End of the proof of proposition 5.4 :

Suppose that there is $M \in D(Y_{i,0}) \cap B_n$ such that $u_n(M) \geq 3$. Suppose that $v(MY_{i,0}^{-1})$ is minimal for this property. It follows from lemma 5.5 that for $M' \in D(Y_{i,0})$, $M' \geq M$ for $j \in J, l \in \mathbb{Z}$, $u_{j,l}(M') \leq 2$.

We have $M \in B_{\{1,...,n-2\}}$ (if not we would have $M' \rightarrow_{\{1,...,n-2\}} M$ with $M' > M$ and $u_n(M') \geq 3$).

If $u_n(M) = 3$: there is $M \leftarrow_J M_1$. If $v_{n-1}(M_1M^{-1}) = 1$, we have $w(M_1) = (a, 0, 3, 2)$ or $(a, 0, 2, 1)$ or $(a, 0, 1, 0)$ where $a = 1$ or 2. For the first two cases we have $u_n^+(M_1) + u_n^-(M_1) \geq 3$, so there is $M_1' \rightarrow_n M_1$ such that $u_n(M_1') \geq 3$, contradiction. For the last case M_1 is dominant with $u(M_1) \geq 2$, contradiction. So $v_{n-1}(M_1M^{-1}) = 2$ and $w(m_1) = (2, 0, 3, 4)$ or $(2, 0, 2, 3)$ or $(2, 0, 1, 2)$ or $(2, 0, 0, 1)$. As above we have $w(M_1) = (2, 0, 0, 1)$. Let $M_2 \rightarrow_n M_1$. If $w(M_2) = (1, 0, 1, 0)$, M_2 is dominant with $u(M_2) \geq 2$, contradiction. So $w(M_2) = (2, 1, 1, 0)$. Let $M_3 \rightarrow_J M_2$. If $w(M_3) = (2, 0, 1, 2)$, there is $M_3' > M_3$ such that $u_n(M_3') \geq 3$, contradiction. So $w(M_3) = (2, 0, 0, 1)$. We continue and we get an infinite sequence such that $w(M_{2k}) = (2, 1, 1, 0)$ and $w(M_{2k+1}) = (2, 0, 0, 1)$. Contradiction because the sequence $v(M_kY_{i,0}^{-1}) \geq 0$ decreases strictly.

If $u_n(M) = 4$: there is $M \leftarrow_J M_1$. If $v_{n-1}(M_1M^{-1}) = 1$, we have $w(M_1) = (a, 0, 4, 2)$ or $(a, 0, 3, 1)$ or $(a, 0, 2, 0)$ where $a = 1$ or 2. We see as above that it is impossible. So $v_{n-1}(M_1M^{-1}) = 2$ and $w(m_1) = (2, 0, 4, 4)$ or $(2, 0, 3, 3)$ or $(2, 0, 2, 2)$ or $(2, 0, 1, 1)$. As above we have $w(M_1) = (2, 0, 1, 1)$. Let $M_2 \rightarrow_n M_1$. If $w(M_2) = (1, 0, 2, 0)$, M_2 is dominant with $u(M_2) \geq 2$, contradiction. So $w(M_2) = (2, 1, 2, 0)$. Let $M_3 \rightarrow_J M_2$. If $w(M_3) = (2, 0, 2, 2)$, there is $M_3' > M_3$ such that $u_n(M_3') \geq 3$, contradiction. So $w(M_3) = (2, 0, 1, 1)$. We continue and we get an infinite sequence such that $w(M_{2k}) = (2, 1, 2, 0)$ and $w(M_{2k+1}) = (2, 0, 1, 1)$. Contradiction because the sequence $v(M_kY_{i,0}^{-1}) \geq 0$ decreases strictly. □

6 Type C

6.1 Statement

Let $\mathfrak{g}$ be of type C_n $(n \geq 2)$. For $i \in \{2, ..., n-1\}$, $l \in \mathbb{Z}$:

$$A_{i,l} = Y_{i,l+1}Y_{i,l-1}Y_{i-1,l}^{-1}Y_{i+1,l}^{-1}$$

$$A_{1,l} = Y_{1,l+1}Y_{1,l-1}Y_{2,l}^{-1} \text{ , } A_{n,l} = Y_{n,l+2}Y_{n,l-2}Y_{n-1,l+1}^{-1}Y_{n-1,l-1}^{-1}$$

In particular for all $i \in I, l \in \mathbb{Z}$, $u(A_{i,l}^{-1}) \leq 0$. So $m \leq m' \Rightarrow u(m) \leq u(m')$.

In this section we prove :

Proposition 6.1. *The property of theorem 3.5 is true for $\mathfrak{g}$ of type C_n $(n \geq 2)$.*

Denote $J = \{1, ..., n-1\} \subset I$.

6.2 Proof of proposition 6.1

We can suppose $Y_{i,L} = Y_{i,0}$ (proposition 3.12).

Lemma 6.2. *(i) For $m \in B_n \cap D(Y_{i,0})$, we have $u_n(m) \leq 1$.*

(ii) For $j \leq n-1$ and $m \in B_j \cap D(Y_{i,0})$, we have $u_j(m) \leq 2$.

(iii) Let $j \leq n-1$ and $m \in D(Y_{i,0}) \cap B_j$ such that $u_j(m) = 2$. For $L, L' \in \mathbb{Z}$ such that $m^{(j)} = Y_{j,L}Y_{j,L'}$, we have $L \neq L'$.

Proof : (i) suppose that there is $m_1 \in B_n \cap D(Y_{i,0})$ such that $u_n(m_1) \geq 2$. Let $m_2 \rightarrow_J m_1$. We have $u_n(m_2) \geq 2 - v_{n-1}(m_2 m_1^{-1})$. But $u_J(m_2) \geq v_{n-1}(m_2 m_1^{-1})$ (lemma 4.2) and so $u(m_2) = u_J(m_2) + u_n(m_2) \geq 2$. As $Y_{i,0} \geq m_2$ it is impossible.

(ii) suppose that there is $j \leq n-1$ and $m_1 \in B_j \cap D(Y_{i,0})$ such that $u_j(m_1) \geq 3$. Let $m_2 \rightarrow_J m_1$. Then we have $u_J(m_2) \geq 3$ (lemma 4.4) and so $u(m_2) \geq 3 + u_n(m_1) \geq 2$ (it follows from (i) that $u_n(m_1) \geq -1$). Contradiction.

(iii) let $j \neq n$ and $m_1 \in D(Y_{i,0}) \cap B_j$ such that $m_1^{(j)} = Y_{j,L}^2$. We can suppose that $v(m_1 m^{-1})$ is minimal. Let $m_2 \rightarrow_J m_1$. It follows from lemma 4.5 for $\mathfrak{g}_J$ of type A_{n-1} that $m_2^{(J)} = Y_{i_1,L_1}Y_{i_2,L_2}$ with $i_2 - i_1 \geq |L_1 - L_2|$ and $(i_2 - i_1) - (L_2 - L_1)$ is even. As $u(m_2) \leq 1$ and $u_n(m_2) \geq -1$, we have $u_n(m_2) = -1$ and $u_J(m_2) = 2$. In particular $m_2 = Y_{i_1,L_1}Y_{i_2,L_2}Y_{n,K}^{-1}$. There is $m_2 \leftarrow_n m_3 = Y_{i_1,L_1}Y_{i_2,L_2}Y_{n-1,K-1}^{-1}Y_{n-1,K-3}^{-1}Y_{n,K-4}$. We are in one the following cases $\alpha, \beta, \gamma, \delta$:

α) $Y_{i_1,L_1}Y_{i_2,L_2}Y_{n-1,K-1}^{-1}Y_{n-1,K-3}^{-1} = 1$: impossible because $i_1 = i_2 \Rightarrow L_1 = L_2$.

β) $Y_{i_1,L_1}Y_{i_2,L_2}Y_{n-1,K-1}^{-1}Y_{n-1,K-3}^{-1} = Y_{i_1,L_1}Y_{n-1,K-1}^{-1} \neq 1$ (or in the same way $Y_{i_2,L_2}Y_{n-1,K-1}^{-1} \neq 1$). There is $m_3 \leftarrow_{n-1} m_4 = m_3 A_{n-1,K-2}$. In particular $m_4^{(n)} = Y_{n,K-4}Y_{n,K-2}^{-1}$, contradiction with (i).

γ) $Y_{i_1,L_1}Y_{i_2,L_2}Y_{n-1,K-1}^{-1}Y_{n-1,K-3}^{-1} = Y_{i_1,L_1}Y_{n-1,K-3}^{-1} \neq 1$ (or in the same way $Y_{i_2,L_2}Y_{n-1,K-3}^{-1} \neq 1$). In particular $i_2 = n-1$, $L_2 = K-1$ and $m_3 = Y_{i_1,L_1}Y_{n-1,K-3}^{-1}Y_{n,K-4}$. Let $J' = \{i_1+1, ..., n-1\}$ (J' can be empty) and :

$$m_3 \leftarrow_{J'} m_4 = Y_{i_1,L_1}Y_{i_1,K-2-n+i_1}^{-1}Y_{i_1+1,K-3-n+i_1}$$

If $Y_{i_1,L_1}Y_{i_1,K-2-n+i_1}^{-1} \neq 1$, let $m_5 \rightarrow_{i_1} m_4$. If $i_1 = 1$, we have $u(m_5) = 2$, impossible. If $i_1 \geq 2$, we have $m_5 = Y_{i_1-1,K-3-n+i_1}^{-1}Y_{i_1,L_1}Y_{i_1,K-4-n+i_1}$. Let $J'' = \{1, ..., i_1 - 1\}$ and $m_5 \leftarrow_{J''} m_6 = Y_{1,K'}Y_{i_1,L_1}$. We have $u(m_6) = 2$, impossible. So $Y_{i_1,L_1} = Y_{i_1,K-2-n+i_1}$, that is to say $L_1 = K-2-n+i_1$. So $L_2 - L_1 = n - i_1 + 1 = i_2 - i_1 + 2 > i_2 - i_1$, contradiction.

δ) $\{(i_1, L_1), (i_2, L_2)\} \cap \{(n-1, K-1), (n-1, K-3)\}$ is empty : there is $m_3 \rightarrow_J m_4 = m_3(m_1 m_2^{-1})$ such that $m_4^{(j)} = Y_{j,l}^2$ (lemma 3.15) and $v(m_4 m^{-1}) < v(m_1 m^{-1})$, contradiction. □

The proposition 6.1 follows from proposition 3.16 and lemma 6.2.

7 Application to q,t-characters

In this section we state and prove the main result of this paper on q,t-characters (theorem 7.5).

7.1 Reminder on q,t-characters [N2, N3, He1, He2, He3]

We define the product $*_t$ on $A \times (\mathcal{A} \otimes \mathbb{Z}[t\pm])$ such that : for $(m,v),(m',v') \in A \times \mathcal{A}$ (m,m',v,v' monomials) :

$$(m,v) *_t (m',v') = t^{D((m,v),(m',v'))}(mm',vv')$$

where $D((m,v),(m',v'))$ is :

$$\sum_{i\in I, l\in\mathbb{Z}} 2u_{i,l+r_i}(m)v_{i,l}(v') + 2v_{i,l+r_i}(v)u_{i,l}(m') + v_{i,l+r_i}(v)u_{i,l}(v') + u_{i,l+r_i}(v)v_{i,l}(v')$$

(see [N3] for the ADE-case and [He2, He3] for other cases ; if $B(z)$ is not symmetric, the definition is slightly different as explained in [He3], section 7.3.3).

Let $\mathcal{Y}_t = \mathcal{Y} \otimes_{\mathbb{Z}} \mathbb{Z}[t^\pm]$. One can define $\mathfrak{K}_{i,t}, \mathfrak{K}_t \subset \mathcal{Y}_t$ with deformed screening operators (see [He1, He3]).

Definition 7.1. *We say that a $\mathbb{Z}$-linear map $\chi_{q,t} : Rep \rightarrow \mathcal{Y}_t$ is a morphism of q,t-characters if :*

1) For M a Rep-monomial define $m = \prod_{i\in I, l\in\mathbb{Z}} (Y_{i,l})^{x_{i,l}(M)} \in B$. We have :

$$\chi_{q,t}(M) = m + \sum_{m'<m} a_{m'}(t)m' \text{ (where } a_{m'}(t) \in \mathbb{Z}[t^\pm])$$

2) The image of $\chi_{q,t}$ is contained in $\mathfrak{K}_t$.

3) Let M_1, M_2 be Rep-monomials. If $max\{l/\sum_{i\in I} x_{i,l}(M_1) > 0\} \leq min\{l/\sum_{i\in I} x_{i,l}(M_2) > 0\}$ then :

$$(M_1M_2, (M_1M_2)^{-1}\chi_{q,t}(M_1M_2)) = (M_1, M_1^{-1}\chi_{q,t}(M_1)) *_t (M_2, M_2^{-1}\chi_{q,t}(M_2))$$

Those properties are generalizations of the axioms that Nakajima [N3] defined for the ADE-case.

Theorem 7.2. *([N3, He2, He3]) For C such that $i \neq j \Rightarrow C_{i,j}C_{j,i} \leq 3$, there is a unique morphism of q,t-characters.*

This result (among others) was proved by Nakajima [N3] for C of type ADE. For C finite it is proved in [He2], and for C such that $i \neq j \Rightarrow C_{i,j}C_{j,i} \leq 3$ in [He3]. The existence of

$\chi_{q,t}$ for symmetric toroidal type is also mentioned in [N5] (it includes quantum affine and toroidal algebras except $A_1^{(1)}, A_2^{(2)}$). A generalization of this result is given in section 7.3.

In [He2] we defined a t-deformed algorithm : for $m \in B$, if it is well-defined it gives an element $F_t(m) \in \mathfrak{K}_t$ such that m is the unique dominant monomial of $F_t(m)$ (an algorithm was also used by Nakajima in the ADE-case in [N2]). If we set $t = 1$ we get the classical algorithm. It follows from theorem 7.2 that the t-deformed algorithm is well defined if $i \neq j \Rightarrow C_{i,j}C_{j,i} \leq 3$. We proved in [He2] that if the t-deformed algorithm is well-defined, for $i \in I, j \in I, l \in \mathbb{Z} : F_t(Y_{i,l})F_t(Y_{j,l}) = F_t(Y_{j,l})F_t(Y_{i,l})$.

Note that $\chi_{q,t}$ is injective and we have (see [He2]) :

$$\chi_{q,t}(\prod_{i\in I, l\in\mathbb{Z}} X_{i,l}^{x_{i,l}}) = \overrightarrow{\prod_{l\in\mathbb{Z}}}\prod_{i\in I} F_t(Y_{i,l})^{x_{i,l}}$$

$$= ...(\prod_{i\in I} F_t(Y_{i,l-1})^{x_{i,l-1}})(\prod_{i\in I} F_t(Y_{i,l})^{x_{i,l}})(\prod_{i\in I} F_t(Y_{i,l+1})^{x_{i,l+1}})... \tag{33}$$

7.2 Technical complement

Proposition 7.3. *(i) Let $m \in B_j$ such that for all $l \in \mathbb{Z}$, $u_{j,l}(m) \leq 1$. Then $F_{i,t}(m) = F_i(m) = L_i(m)$ and all coefficients are equal to 1.*

(ii) Let $i \in I$ such that all $m \in D(Y_{i,L})$ satisfies : for $j \in I$, if $m \in B_j$ then $\forall l \in \mathbb{Z}, u_{j,l}(m) \leq 1$. Then $F_t(Y_{i,L}) = F(Y_{i,L}) = L(Y_{i,L}) \in \mathcal{Y}_t$ is in $\mathfrak{K}_t$ and all coefficients are equal to 1.

Proof : (i) Direct consequence of the lemma 4.13 of [He2].

(ii) Let j be in I and consider the decomposition of proposition 3.9 :

$$L(Y_{i,L}) = \sum_{m'\in B_j\cap D(Y_{i,L})} \lambda_j(m')L_j(m')$$

But it follows from (i) that $m' \in B_j \cap D(Y_{i,L})$ implies that $L_j(m') = F_{j,t}(m')$. And so :

$$L(Y_{i,l}) = \sum_{m'\in B_j\cap D(Y_{i,L})} \lambda_j(m')F_{j,t}(m') \in \mathfrak{K}_{j,t}$$

So $L(Y_{i,L}) \in \mathfrak{K}_t$ and $F_t(Y_{i,L}) = L(Y_{i,L}) = F(Y_{i,l})$. □

7.3 New results for q,t-characters

It follows also from theorem 3.5 and proposition 7.3 :

Proposition 7.4. *Let $\mathfrak{g}$ be of type A_n $(n \geq 1)$, $A_l^{(1)}$ $(l \geq 2)$, B_n $(n \geq 2)$ or C_n $(n \geq 2)$. For $i \in I, a \in \mathbb{C}^*$, we have $\chi_{q,t}(V_i(a)) = \chi_q(V_i(a))$ and all coefficients are equal to* 1.

We prove a conjecture of [He2] :

Theorem 7.5. *Let $\mathcal{U}_q(\hat{\mathfrak{g}})$ be a quantum affine algebra (C finite) and M be a standard module of $\mathcal{U}_q(\hat{\mathfrak{g}})$. The coefficients of $\chi_{q,t}(M)$ are in $\mathbb{N}[t^{\pm}]$ and the monomials of $\chi_{q,t}(M)$ are the monomials of $\chi_q(M)$.*

In particular the q,t-characters for quantum affine algebras have a finite number of monomials and this result shows that the q,t-characters of [He2] can be considered as a t-deformation of q-characters for all quantum affine algebras. In particular it is an argument for the existence of a geometric model behind the q,t-characters in non simply-laced cases.

Proof : If follows from formula (33) in section 7.1 that it suffices to look at the $F_t(Y_{i,l})$. We do it with a case by case investigation :

the case ADE follows from the work of Nakajima [N3]

the case BC follows from theorem 3.5 and proposition 7.4 (ii)

the case G_2 follows from an explicit computation in [He2]

the case F_4 follows from an explicit computation on computer (see section 8). □

8 Appendix : explicit computations on computer for type F_4

The proof of theorem theorem 7.5 for type F_4 is based on an explicit computations on computer (program written in C with Travis Schedler).

For type F_4 there are 4 fundamental representations (see [Bo] for the numbers on the Dynkin diagram) : $\dim(V_1(a)) = 26$ (26 monomials), $\dim(V_1(a)) = 299$ (283 monomials), $\dim(V_1(a)) = 1703$ (1532 monomials), $\dim(V_1(a)) = 53$ (53 monomials). We checked that the coefficients are in $\mathbb{N}[t^{\pm}]$. We give an explicit list of terms of fundamental representations of type F_4 whose coefficient is not 1 (the complete list of monomials can be found on http ://www.dma.ens.fr/∼dhernand/f4monomials.pdf). We can see that the coefficient are all $(t + t^{-1}) \in \mathbb{N}[t^{\pm}]$. They appear only in fundamental representations 2 and 3 :

Fundamental Representation 2 :

Monomial 70 : $(t^{-1} + t)\ Y_{1,10}Y_{2,7}Y_{2,9}^{-1}Y_{2,11}^{-1}Y_{4,6}$
Monomial 87 : $(t^{-1} + t)\ Y_{1,12}^{-1}Y_{2,7}Y_{2,9}^{-1}Y_{4,6}$
Monomial 89 : $(t^{-1} + t)\ Y_{1,10}Y_{2,7}Y_{2,9}^{-1}Y_{2,11}^{-1}Y_{3,8}Y_{4,10}^{-1}$
Monomial 105 : $(t^{-1} + t)\ Y_{1,12}^{-1}Y_{2,7}Y_{2,9}^{-1}Y_{3,8}Y_{4,10}^{-1}$
Monomial 109 : $(t^{-1} + t)\ Y_{1,10}Y_{2,7}Y_{3,12}^{-1}$
Monomial 120 : $(t^{-1} + t)\ Y_{1,8}Y_{1,10}Y_{2,9}^{-1}Y_{3,8}Y_{3,12}^{-1}$
Monomial 124 : $(t^{-1} + t)\ Y_{1,12}^{-1}Y_{2,7}Y_{2,11}Y_{3,12}^{-1}$
Monomial 142 : $(t^{-1} + t)\ Y_{2,7}Y_{2,13}^{-1}$
Monomial 143 : $(t^{-1} + t)\ Y_{1,8}Y_{1,12}^{-1}Y_{2,9}^{-1}Y_{2,11}Y_{3,8}Y_{3,12}^{-1}$

Monomial 151 : $(t^{-1}+t)\ Y_{1,10}^{-1}Y_{1,12}^{-1}Y_{2,11}Y_{3,8}Y_{3,12}^{-1}$
Monomial 155 : $(t^{-1}+t)\ Y_{1,8}Y_{2,9}^{-1}Y_{2,13}^{-1}Y_{3,8}$
Monomial 168 : $(t^{-1}+t)\ Y_{1,10}^{-1}Y_{2,13}^{-1}Y_{3,8}$
Monomial 173 : $(t^{-1}+t)\ Y_{1,8}Y_{2,11}Y_{2,13}^{-1}Y_{3,12}^{-1}Y_{4,10}$
Monomial 188 : $(t^{-1}+t)\ Y_{1,10}^{-1}Y_{2,9}Y_{2,11}Y_{2,13}^{-1}Y_{3,12}^{-1}Y_{4,10}$
Monomial 193 : $(t^{-1}+t)\ Y_{1,8}Y_{2,11}Y_{2,13}^{-1}Y_{4,14}^{-1}$
Monomial 206 : $(t^{-1}+t)\ Y_{1,10}^{-1}Y_{2,9}Y_{2,11}Y_{2,13}^{-1}Y_{4,14}^{-1}$

Fundamental Representation 3 :

Monomial 64 : $(t^{-1}+t)\ Y_{1,3}Y_{1,9}Y_{2,6}Y_{2,8}^{-1}Y_{2,10}^{-1}Y_{4,5}$
Monomial 90 : $(t^{-1}+t)\ Y_{1,5}^{-1}Y_{1,9}Y_{2,4}Y_{2,6}Y_{2,8}^{-1}Y_{2,10}^{-1}Y_{4,5}$
Monomial 91 : $(t^{-1}+t)\ Y_{3,5}Y_{3,9}^{-1}Y_{4,5}$
Monomial 93 : $(t^{-1}+t)\ Y_{1,3}Y_{1,11}^{-1}Y_{2,6}Y_{2,8}^{-1}Y_{4,5}$
Monomial 96 : $(t^{-1}+t)\ Y_{1,3}Y_{1,9}Y_{2,6}Y_{2,8}^{-1}Y_{2,10}^{-1}Y_{3,7}Y_{4,9}^{-1}$
Monomial 117 : $(t^{-1}+t)\ Y_{1,5}^{-1}Y_{1,11}^{-1}Y_{2,4}Y_{2,6}Y_{2,8}^{-1}Y_{4,5}$
Monomial 125 : $(t^{-1}+t)\ Y_{1,5}^{-1}Y_{1,9}Y_{2,4}Y_{2,6}Y_{2,8}^{-1}Y_{2,10}^{-1}Y_{3,7}Y_{4,9}^{-1}$
Monomial 126 : $(t^{-1}+t)\ Y_{3,5}Y_{3,7}Y_{3,9}^{-1}Y_{4,9}^{-1}$
Monomial 128 : $(t^{-1}+t)\ Y_{1,3}Y_{1,11}^{-1}Y_{2,6}Y_{2,8}^{-1}Y_{3,7}Y_{4,9}^{-1}$
Monomial 138 : $(t^{-1}+t)\ Y_{1,3}Y_{1,9}Y_{2,6}Y_{3,11}^{-1}$
Monomial 152 : $(t^{-1}+t)\ Y_{1,5}^{-1}Y_{1,11}^{-1}Y_{2,4}Y_{2,6}Y_{2,8}^{-1}Y_{3,7}Y_{4,9}^{-1}$
Monomial 159 : $(t^{-1}+t)\ Y_{2,8}Y_{2,10}Y_{3,5}Y_{3,9}^{-1}Y_{3,11}^{-1}$
Monomial 162 : $(t^{-1}+t)\ Y_{1,3}Y_{1,7}Y_{1,9}Y_{2,8}^{-1}Y_{3,7}Y_{3,11}^{-1}$
Monomial 165 : $(t^{-1}+t)\ Y_{1,3}Y_{1,11}^{-1}Y_{2,6}Y_{2,10}Y_{3,11}^{-1}$
Monomial 166 : $(t^{-1}+t)\ Y_{1,5}^{-1}Y_{1,9}Y_{2,4}Y_{2,6}Y_{3,11}^{-1}$
Monomial 194 : $(t^{-1}+t)\ Y_{1,5}^{-1}Y_{1,11}^{-1}Y_{2,4}Y_{2,6}Y_{2,10}Y_{3,11}^{-1}$
Monomial 208 : $(t^{-1}+t)\ Y_{1,5}^{-1}Y_{1,7}Y_{1,9}Y_{2,4}Y_{2,8}^{-1}Y_{3,7}Y_{3,11}^{-1}$
Monomial 209 : $(t^{-1}+t)\ Y_{1,11}Y_{2,8}Y_{2,12}^{-1}Y_{3,5}Y_{3,9}^{-1}$
Monomial 220 : $(t^{-1}+t)\ Y_{1,3}Y_{2,6}Y_{2,12}^{-1}$
Monomial 221 : $(t^{-1}+t)\ Y_{1,3}Y_{1,7}Y_{1,11}^{-1}Y_{2,8}^{-1}Y_{2,10}Y_{3,7}Y_{3,11}^{-1}$
Monomial 237 : $(t^{-1}+t)\ Y_{1,9}Y_{1,11}Y_{2,10}^{-1}Y_{2,12}^{-1}Y_{3,5}$
Monomial 238 : $(t^{-1}+t)\ Y_{1,7}Y_{1,9}Y_{2,6}^{-1}Y_{2,8}^{-1}Y_{3,5}Y_{3,7}Y_{3,11}^{-1}$
Monomial 251 : $(t^{-1}+t)\ Y_{1,5}^{-1}Y_{2,4}Y_{2,6}Y_{2,12}^{-1}$
Monomial 252 : $(t^{-1}+t)\ Y_{1,3}Y_{1,9}^{-1}Y_{1,11}^{-1}Y_{2,10}Y_{3,7}Y_{3,11}^{-1}$
Monomial 253 : $(t^{-1}+t)\ Y_{1,5}^{-1}Y_{1,7}Y_{1,11}^{-1}Y_{2,4}Y_{2,8}^{-1}Y_{2,10}Y_{3,7}Y_{3,11}^{-1}$
Monomial 257 : $(t^{-1}+t)\ Y_{1,3}Y_{1,7}Y_{2,8}^{-1}Y_{2,12}^{-1}Y_{3,7}$
Monomial 281 : $(t^{-1}+t)\ Y_{1,13}^{-1}Y_{2,8}Y_{3,5}Y_{3,9}^{-1}$
Monomial 289 : $(t^{-1}+t)\ Y_{1,5}^{-1}Y_{1,9}^{-1}Y_{1,11}^{-1}Y_{2,4}Y_{2,10}Y_{3,7}Y_{3,11}^{-1}$
Monomial 294 : $(t^{-1}+t)\ Y_{1,7}Y_{1,9}Y_{3,7}Y_{3,9}^{-1}Y_{3,11}^{-1}Y_{4,7}$
Monomial 296 : $(t^{-1}+t)\ Y_{1,9}Y_{1,11}Y_{2,6}Y_{2,8}Y_{2,10}^{-1}Y_{2,12}^{-1}Y_{3,9}^{-1}Y_{4,7}$
Monomial 298 : $(t^{-1}+t)\ Y_{1,3}Y_{1,9}^{-1}Y_{2,12}^{-1}Y_{3,7}$
Monomial 300 : $(t^{-1}+t)\ Y_{1,5}^{-1}Y_{1,7}Y_{2,4}Y_{2,8}^{-1}Y_{2,12}^{-1}Y_{3,7}$
Monomial 303 : $(t^{-1}+t)\ Y_{1,7}Y_{1,11}^{-1}Y_{2,6}^{-1}Y_{2,8}^{-1}Y_{2,10}Y_{3,5}Y_{3,7}Y_{3,11}^{-1}$

Monomial 320 : $(t^{-1}+t)\ Y_{1,9}Y_{1,13}^{-1}Y_{2,10}^{-1}Y_{3,5}$
Monomial 332 : $(t^{-1}+t)\ Y_{1,3}Y_{1,7}Y_{2,10}Y_{2,12}^{-1}Y_{3,11}^{-1}Y_{4,9}$
Monomial 351 : $(t^{-1}+t)\ Y_{1,5}^{-1}Y_{1,9}^{-1}Y_{2,4}Y_{2,12}^{-1}Y_{3,7}$
Monomial 353 : $(t^{-1}+t)\ Y_{1,7}Y_{2,6}^{-1}Y_{2,8}^{-1}Y_{2,12}^{-1}Y_{3,5}Y_{3,7}$
Monomial 359 : $(t^{-1}+t)\ Y_{1,9}^{-1}Y_{1,11}^{-1}Y_{2,6}^{-1}Y_{2,10}Y_{3,5}Y_{3,7}Y_{3,11}^{-1}$
Monomial 361 : $(t^{-1}+t)\ Y_{1,7}Y_{1,11}^{-1}Y_{2,10}Y_{3,7}Y_{3,9}^{-1}Y_{3,11}^{-1}Y_{4,7}$
Monomial 362 : $(t^{-1}+t)\ Y_{1,11}^{-1}Y_{1,13}^{-1}Y_{3,5}$
Monomial 368 : $(t^{-1}+t)\ Y_{1,7}Y_{1,9}Y_{3,7}Y_{3,11}^{-1}Y_{4,11}^{-1}$
Monomial 370 : $(t^{-1}+t)\ Y_{1,9}Y_{1,11}Y_{2,6}Y_{2,8}Y_{2,10}^{-1}Y_{2,12}^{-1}Y_{4,11}^{-1}$
Monomial 382 : $(t^{-1}+t)\ Y_{1,3}Y_{1,9}^{-1}Y_{2,8}Y_{2,10}Y_{2,12}^{-1}Y_{3,11}^{-1}Y_{4,9}$
Monomial 384 : $(t^{-1}+t)\ Y_{1,5}^{-1}Y_{1,7}Y_{2,4}Y_{2,10}Y_{2,12}^{-1}Y_{3,11}^{-1}Y_{4,9}$
Monomial 394 : $(t^{-1}+t)\ Y_{1,9}Y_{1,13}^{-1}Y_{2,6}Y_{2,8}Y_{2,10}^{-1}Y_{3,9}^{-1}Y_{4,7}$
Monomial 399 : $(t^{-1}+t)\ Y_{1,3}Y_{1,7}Y_{2,10}Y_{2,12}^{-1}Y_{4,13}^{-1}$
Monomial 414 : $(t^{-1}+t)\ Y_{1,9}^{-1}Y_{2,6}^{-1}Y_{2,12}^{-1}Y_{3,5}Y_{3,7}$
Monomial 422 : $(t^{-1}+t)\ Y_{1,5}^{-1}Y_{1,9}^{-1}Y_{2,4}Y_{2,8}Y_{2,10}Y_{2,12}^{-1}Y_{3,11}^{-1}Y_{4,9}$
Monomial 428 : $(t^{-1}+t)\ Y_{1,9}^{-1}Y_{1,11}^{-1}Y_{2,8}Y_{2,10}Y_{3,7}Y_{3,9}^{-1}Y_{3,11}^{-1}Y_{4,7}$
Monomial 431 : $(t^{-1}+t)\ Y_{1,7}Y_{1,11}^{-1}Y_{2,10}Y_{3,7}Y_{3,11}^{-1}Y_{4,11}^{-1}$
Monomial 432 : $(t^{-1}+t)\ Y_{1,11}^{-1}Y_{1,13}^{-1}Y_{2,6}Y_{2,8}Y_{3,9}^{-1}Y_{4,7}$
Monomial 436 : $(t^{-1}+t)\ Y_{1,7}Y_{2,6}^{-1}Y_{2,10}Y_{2,12}^{-1}Y_{3,5}Y_{3,11}^{-1}Y_{4,9}$
Monomial 438 : $(t^{-1}+t)\ Y_{1,7}Y_{2,12}^{-1}Y_{3,7}Y_{3,9}^{-1}Y_{4,7}$
Monomial 461 : $(t^{-1}+t)\ Y_{1,3}Y_{1,9}^{-1}Y_{2,8}Y_{2,10}Y_{2,12}^{-1}Y_{4,13}^{-1}$
Monomial 463 : $(t^{-1}+t)\ Y_{1,5}^{-1}Y_{1,7}Y_{2,4}Y_{2,10}Y_{2,12}^{-1}Y_{4,13}^{-1}$
Monomial 469 : $(t^{-1}+t)\ Y_{1,9}Y_{1,13}^{-1}Y_{2,6}Y_{2,8}Y_{2,10}^{-1}Y_{4,11}^{-1}$
Monomial 495 : $(t^{-1}+t)\ Y_{1,5}^{-1}Y_{1,9}^{-1}Y_{2,4}Y_{2,8}Y_{2,10}Y_{2,12}^{-1}Y_{4,13}^{-1}$
Monomial 498 : $(t^{-1}+t)\ Y_{1,9}Y_{1,11}^{-1}Y_{1,13}^{-1}Y_{2,6}Y_{2,10}^{-1}Y_{4,7}$
Monomial 500 : $(t^{-1}+t)\ Y_{1,9}^{-1}Y_{2,8}Y_{2,12}^{-1}Y_{3,7}Y_{3,9}^{-1}Y_{4,7}$
Monomial 502 : $(t^{-1}+t)\ Y_{1,7}Y_{2,8}Y_{2,10}Y_{2,12}^{-1}Y_{3,9}^{-1}Y_{3,11}^{-1}Y_{4,7}Y_{4,9}$
Monomial 505 : $(t^{-1}+t)\ Y_{1,9}^{-1}Y_{2,6}^{-1}Y_{2,8}Y_{2,10}Y_{2,12}^{-1}Y_{3,5}Y_{3,11}^{-1}Y_{4,9}$
Monomial 512 : $(t^{-1}+t)\ Y_{1,9}^{-1}Y_{1,11}^{-1}Y_{2,8}Y_{2,10}Y_{3,7}Y_{3,11}^{-1}Y_{4,11}^{-1}$
Monomial 514 : $(t^{-1}+t)\ Y_{1,11}^{-1}Y_{1,13}^{-1}Y_{2,6}Y_{2,8}Y_{4,11}^{-1}$
Monomial 526 : $(t^{-1}+t)\ Y_{1,7}Y_{2,6}^{-1}Y_{2,10}Y_{2,12}^{-1}Y_{3,5}Y_{4,13}^{-1}$
Monomial 528 : $(t^{-1}+t)\ Y_{1,7}Y_{2,12}^{-1}Y_{3,7}Y_{4,11}^{-1}$
Monomial 564 : $(t^{-1}+t)\ Y_{2,10}^{-1}Y_{2,12}^{-1}Y_{3,7}Y_{4,7}$
Monomial 575 : $(t^{-1}+t)\ Y_{1,7}Y_{1,9}Y_{1,11}^{-1}Y_{1,13}^{-1}Y_{2,8}^{-1}Y_{2,10}^{-1}Y_{3,7}Y_{4,7}$
Monomial 577 : $(t^{-1}+t)\ Y_{1,9}^{-1}Y_{2,8}^{2}Y_{2,10}Y_{2,12}^{-1}Y_{3,9}^{-1}Y_{3,11}^{-1}Y_{4,7}Y_{4,9}$
Monomial 581 : $(t^{-1}+t)\ Y_{1,9}Y_{1,11}^{-1}Y_{1,13}^{-1}Y_{2,6}Y_{2,10}^{-1}Y_{3,9}Y_{4,11}^{-1}$
Monomial 583 : $(t^{-1}+t)\ Y_{1,9}^{-1}Y_{2,8}Y_{2,12}^{-1}Y_{3,7}Y_{4,11}^{-1}$
Monomial 586 : $(t^{-1}+t)\ Y_{1,7}Y_{2,8}Y_{2,10}Y_{2,12}^{-1}Y_{3,9}^{-1}Y_{4,7}Y_{4,13}^{-1}$
Monomial 591 : $(t^{-1}+t)\ Y_{1,9}^{-1}Y_{2,6}^{-1}Y_{2,8}Y_{2,10}Y_{2,12}^{-1}Y_{3,5}Y_{4,13}^{-1}$
Monomial 622 : $(t^{-1}+t)\ Y_{1,7}Y_{2,8}Y_{2,10}Y_{2,12}^{-1}Y_{3,11}^{-1}Y_{4,9}Y_{4,11}^{-1}$
Monomial 648 : $(t^{-1}+t)\ Y_{2,10}^{-1}Y_{2,12}^{-1}Y_{3,7}Y_{3,9}Y_{4,11}^{-1}$
Monomial 656 : $(t^{-1}+t)\ Y_{2,8}Y_{2,12}^{-1}Y_{3,11}^{-1}Y_{4,7}Y_{4,9}$

Monomial 657 : $(t^{-1}+t)\ Y_{1,7}Y_{1,9}Y_{1,11}^{-1}Y_{1,13}^{-1}Y_{2,8}^{-1}Y_{2,10}^{-1}Y_{3,7}Y_{3,9}Y_{4,11}^{-1}$
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Monomial 1044 : $(t^{-1}+t)\ Y_{1,7}Y_{1,9}Y_{1,11}^{-1}Y_{2,10}Y_{2,14}^{-1}Y_{4,13}^{-1}$
Monomial 1058 : $(t^{-1}+t)\ Y_{1,11}Y_{1,15}Y_{2,8}Y_{2,10}^{-1}Y_{2,12}^{-1}Y_{2,16}^{-1}Y_{3,9}Y_{4,11}^{-1}$
Monomial 1060 : $(t^{-1}+t)\ Y_{1,9}Y_{1,11}Y_{2,10}^{-1}Y_{2,12}^{-1}Y_{3,9}Y_{3,11}Y_{3,13}^{-1}Y_{4,13}^{-1}$

Monomial 1074 : $(t^{-1}+t)$ $Y_{1,11}Y_{1,17}^{-1}Y_{2,8}Y_{2,10}^{-1}Y_{2,12}^{-1}Y_{4,7}$
Monomial 1075 : $(t^{-1}+t)$ $Y_{1,13}^{-1}Y_{1,15}Y_{2,8}Y_{2,10}^{-1}Y_{2,16}^{-1}Y_{4,7}$
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Monomial 1137 : $(t^{-1}+t)$ $Y_{1,9}^{-1}Y_{1,11}^{-1}Y_{2,8}Y_{2,10}Y_{2,12}^{-1}Y_{2,14}^{-1}Y_{4,9}$
Monomial 1138 : $(t^{-1}+t)$ $Y_{1,11}^{-1}Y_{1,13}^{-1}Y_{3,9}Y_{3,13}^{-1}Y_{4,9}$
Monomial 1140 : $(t^{-1}+t)$ $Y_{1,9}Y_{1,11}Y_{2,10}^{-1}Y_{2,14}Y_{3,9}Y_{3,13}^{-1}Y_{3,15}^{-1}$
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