AN INTRODUCTION TO AFFINE KAC-MOODY ALGEBRAS

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Abstract. In these lectures we give an introduction to affine Kac-Moody algebras, their representations, and applications of this theory.

Contents

1. Introduction

Affine Kac-Moody algebras $\hat{\mathfrak{g}}$ are infinite dimensional analogs of semi-simple Lie algebras $\mathfrak{g}$ and have a central role both in Mathematics (Modular forms, Geometric Langlands program...) and Mathematical Physics (Conformal Field Theory...). These lectures are an introduction to the theory of affine Kac-Moody algebras and their representations with basic results and constructions to enter the theory.

We will first explain how $\hat{\mathfrak{g}}$ appears naturally as a central extension of the loop algebra of a semi-simple Lie algebra $\mathfrak{g}$. Then it is possible to define a system of Chevalley generators which gives a unified point of view on $\hat{\mathfrak{g}}$ and $\mathfrak{g}$. The representation theory of $\hat{\mathfrak{g}}$ is very rich. We study two class of representations :

- the category $\mathcal{O}$ of representations : for example it contains simple highest weight representations.

- the category of finite dimensional representations : for example it contains representations obtained by evaluation from finite dimensional representations of $\mathfrak{g}$.

By construction $\hat{\mathfrak{g}}$ has a central element. It allows to define the level of a simple representation. For example the critical level is of particular importance. We will then study more advanced topics as the fusion product inside the category of a fixed level and applications to Knizhnik-Zamolodchikov equations.

Thus affine Kac-Moody algebras provide a perfect example where abstract mathematical motivations (to build infinite dimensional analogs of semi-simple Lie algebras with analog properties) lead to objects which are closely related to other fields as Mathematical Physics.

In each section we give references where complete proofs can be read. As there is a huge amount of very interesting books and articles on affine Kac-Moody algebras, we listed only a few of them.

Acknowledgments : These are notes from lectures given at the University of Aarhus (Center for the Topology and Quantization of Moduli Spaces) in October 2006. The author would like to thank N. Reshetikhin for the invitation and useful advices during the preparation of these lectures, J.E. Andersen and E. Frenkel for useful references, A. Feragen for indicating typos. He also would like to thank the audience of the lectures for enthusiasm and many interesting questions, and CTQM for the hospitality and excellent working conditions.

2. Quick review on semi-simple Lie algebras

In these lectures, all vectors spaces are over $\mathbb{C}$.

We first recall results and constructions from the classical theory of finite dimensional semi-simple Lie algebras (they are the starting point of the theory of affine Kac-Moody algebras).

2.1. **defi.**

Definition 2.1. *A Lie algebra $\mathfrak{g}$ is a vector space with a bilinear map $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ (called the bracket) satisfying for $x, y, z \in \mathfrak{g}$:*
$[x, y] = -[y, x]$,
$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ *(Jacobi identity).*

For example an algebra $\mathcal{A}$ with the bracket defined by $[a, b] = ab - ba$ is a Lie algebra.

For $\mathfrak{I}, \mathfrak{I}'$ subspaces of a Lie algebra $\mathfrak{g}$, we denote by $[\mathfrak{I}, \mathfrak{I}']$ the subspace of $\mathfrak{g}$ generated by $\{[i, i'] | i \in \mathfrak{I}, i \in \mathfrak{I}'\}$.

A Lie subalgebra of a Lie algebra $\mathfrak{g}$ is a subspace $\mathfrak{g}' \subset \mathfrak{g}$ satisfying $[\mathfrak{g}', \mathfrak{g}'] \subset \mathfrak{g}'$.

An ideal of a Lie algebra $\mathfrak{g}$ is a subspace $\mathfrak{I} \subset \mathfrak{g}$ satisfying $[\mathfrak{I}, \mathfrak{g}] \subset \mathfrak{g}$. For $\mathfrak{I}$ an ideal of $\mathfrak{g}$, $\mathfrak{I}$ and $\mathfrak{g}/\mathfrak{I}$ have an induced structure of a Lie algebra.

In the following we suppose that $\mathfrak{g}$ is finite dimensional.

For $\mathfrak{I} \subset \mathfrak{g}$ an ideal, we define a sequence of ideals $(D_i(\mathfrak{I}))_{i\geq 0}$ by induction : we set $D_0(\mathfrak{I}) = \mathfrak{I}$ and for $i \geq 0$:

$$D_{i+1}(\mathfrak{I}) = [D_i(\mathfrak{I}), D_i(\mathfrak{I})].$$

$(D_i(\mathfrak{I}))_{i\geq 0}$ is called the derived series. $\mathfrak{I}$ is said to be solvable if there is $i \geq 0$ such that $D_i(\mathfrak{I}) = \{0\}$.

Definition 2.2. *A Lie algebra $\mathfrak{g}$ is said to be semi-simple if $\{0\}$ is the unique solvable ideal of $\mathfrak{g}$.*

Example : for $n \geq 2$, consider :

$$sl_n = \{M \in \mathcal{M}_n(\mathbb{C}) | tr(M) = 0\}.$$

It is a Lie subalgebra of $\mathcal{M}_n(\mathbb{C})$ (with the Lie algebra structure coming from the algebra structure; note that sl_n is not a subalgebra of $\mathcal{M}_n(\mathbb{C})$). sl_n is semi-simple.

2.2. **Representations.** A Lie algebra morphism is a linear map which preserves the bracket (ie. $\rho([x,y]) = [\rho(x), \rho(y)]$).

Definition 2.3. *A representation of $\mathfrak{g}$ on a vector space V is a Lie algebra morphism $\rho : \mathfrak{g} \to End(V)$.*

The condition of the definition means :

$$\rho([x,y]) = \rho(x) \circ \rho(y) - \rho(y) \circ \rho(x).$$

One says also that V is a module of $\mathfrak{g}$ or a $\mathfrak{g}$-module.

Examples : For a Lie algebra $\mathfrak{g}$ and $x \in \mathfrak{g}$, $Ad_x : \mathfrak{g} \to \mathfrak{g}$ is defined by $Ad_x(y) = [x,y]$ for $y \in \mathfrak{g}$. The linear map $x \mapsto Ad_x$ defines a structure of $\mathfrak{g}$-module on $\mathfrak{g}$ (called the adjoint representation). Indeed by the Jacobi identity, we have for $x, y, z \in \mathfrak{g}$:

$$[Ad_x, Ad_y](z) = [x,[y,z]] - [y,[x,z]] = [[x,y],z] = Ad_{[x,y]}(z).$$

$\mathbb{C}^n$ is naturally a representation of $sl_{n+1} \subset \mathrm{End}(\mathbb{C}^n)$.

In the following for $\rho : \mathfrak{g} \to \mathrm{End}(V)$ a representation of $\mathfrak{g}$, for $g \in \mathfrak{g}$ and $v \in V$, we denote $\rho(g)(x) = g.x$.

Definition 2.4. *Let V be a representation of $\mathfrak{g}$.*

A submodule V' of V is a subspace $V' \subset V$ such that $g.v' \in V'$ for all $g \in \mathfrak{g}, v' \in V'$.

V is said to be simple if the submodules of V are $\{0\}$ and V.

V said to be semi-simple if V is a direct sum of simple modules.

One of the most important result of the representation theory of finite dimensional semi-simple Lie algebras is :

Theorem 2.5. *A finite dimensional representation of a finite dimensional semi-simple Lie algebra is semi-simple.*

2.3. **Presentation.** A semi-simple Lie algebra has a presentation in terms of Chevalley generators.

We start with a Cartan matrix $(C_{i,j})_{1\leq i,j\leq n}$ satisfying

$$C_{i,j}\in\mathbb{Z}\ ,\ C_{i,i}=2\ ,\ (i\neq j\Rightarrow C_{i,j}\leq 0)\ ,\ (C_{i,j}=0\Leftrightarrow C_{j,i}=0),$$

and all principal minors of C are strictly positive :

$$\det((C_{i,j})_{1\leq i,j\leq R})>0 \text{ for } 1\leq R\leq n.$$

Then we consider generators $(e_i)_{1\leq i\leq n}$, $(f_i)_{1\leq i\leq n}$, $(h_i)_{1\leq i\leq n}$. The relations are :

$$[h_i,h_j]=0,$$
$$[e_i,f_j]=\delta_{i,j}h_i,$$
$$[h_i,e_j]=C_{i,j}e_j,$$
$$[h_i,f_j]=-C_{i,j}f_j,$$
$$(Ad_{e_i})^{1-C_{i,j}}(e_j)=0 \text{ for } i\neq j,$$
$$(Ad_{f_i})^{1-C_{i,j}}(f_j)=0 \text{ for } i\neq j.$$

The two last relations are called Serre relations.

Example : Let $e=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $f=\begin{pmatrix}0&0\\1&0\end{pmatrix}$, $h=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$. We have $sl_2=\mathbb{C}e\oplus\mathbb{C}f\oplus\mathbb{C}h$, and we have the relations $[e,f]=h$, $[h,e]=2e$, $[h,f]=-2f$. So we get the above presentation of sl_2 with the Cartan matrix $C=(2)$.

2.4. **Finite dimensional simple representations.** Let

$$\mathfrak{h}=\bigoplus_{1\leq i\leq n}\mathbb{C}h_i\subset\mathfrak{g}.$$

It is a Lie subalgebra of $\mathfrak{g}$ which commutative (that is to say $[x,x']=0$ for any $x,x'\in\mathfrak{h}$). $\mathfrak{h}$ is called a Cartan subalgebra of $\mathfrak{g}$. Let us define

$$P=\{\omega\in\mathfrak{h}^*|\omega(h_i)\in\mathbb{Z},\forall i\in\{1,\cdots,n\}\},$$
$$P^+=\{\omega\in P|\omega(h_i)\geq 0,\forall i\in\{1,\cdots,n\}\}.$$

For $\lambda\in\mathfrak{h}^*$, there exists a unique simple representation $L(\lambda)$ of $\mathfrak{g}$ such that there is $v\in L(\lambda)-\{0\}$ satisfying

$$h_i.v=\lambda(h_i)v\ ,\ e_i.v=0 \text{ for } i\in\{1,\cdots,n\}.$$

λ is called the highest weight of $L(\lambda)$. $L(\lambda)$ is finite dimensional if and only if $\lambda\in P^+$. Moreover all simple finite dimensional representations of $\mathfrak{g}$ are of the form $L(\lambda)$ for one $\lambda\in P^+$.

For complements on this section, the reader may refer to [B, S].

3. Affine Kac-Moody algebras

A natural problem is to generalize the theory of finite dimensional semi-simple Lie algebras to infinite dimensional Lie algebras. A class of infinite dimensional Lie algebras called affine Kac-Moody algebra is of particular importance for this question.

First let us explain the natural geometric construction of affine Kac-Moody as central of extension of loop algebras of semi-simple Lie algebras.

We recall that a central element $c \in \mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ satisfies by definition $[c, g] = 0$ for any $g \in \mathfrak{g}$.

3.1. Loop algebras.

Let $\mathcal{L} = \mathbb{C}[t^{\pm 1}]$ be the algebra of Laurent polynomials. Consider a finite dimensional semi-simple Lie algebra $\mathfrak{g}$.

Definition 3.1. *The loop algebra of $\mathfrak{g}$ is $\mathcal{L}(\mathfrak{g}) = \mathcal{L} \otimes \mathfrak{g}$ with the Lie algebra structure defined by putting for $P, Q \in \mathcal{L}$, $x, y \in \mathfrak{g}$:*

$$[P \otimes x, Q \otimes y] = PQ \otimes [x, y].$$

Remark : $\mathcal{L}(\mathfrak{g})$ is the Lie algebra of polynomial maps from the unit circle to $\mathfrak{g}$, that is why it is called the loop algebra of $\mathfrak{g}$.

3.2. Central extension.

In this subsection we define the affine Kac-Moody $\mathfrak{g}$ as a central extension of $\mathfrak{g}$. To do this construction, we first need a 2-cocycle ν.

We remind that the Killing form of $\mathfrak{g}$ is the symmetric bilinear map $K : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ defined by $K(x, y) = Tr(Ad_x Ad_y)$.

Lemma 3.2. *The Killing form K is Ad-invariant, that is to say for $x, y, z \in \mathfrak{g}$:*

$$K([x, y], z) = K(x, [y, z]).$$

Proof: We have :

$Tr(Ad_{[x,y]} Ad_z) = Tr([Ad_x, Ad_y] Ad_z)$
$= Tr(Ad_x Ad_y Ad_z) - Tr(Ad_y Ad_x Ad_z)$
$= Tr(Ad_x [A_y, Ad_z]) = Tr(Ad_x Ad_{[y,z]})$. □

Remark : K is non degenerated (consequence of Cartan criterion).

In the following we use a normalized version of the Killing form $(x, y) = \frac{1}{h^\vee} K(x, y)$ where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$ (for example for $\mathfrak{g} = sl_n$, we have $h^\vee = n$).

Let us define $(,)_t : \mathcal{L}(\mathfrak{g}) \times \mathcal{L}(\mathfrak{g}) \to \mathcal{L}$ by

$$(Px, Qy)_t = PQ \times (x, y),$$

for $P, Q \in \mathcal{L}$ and $x, y \in \mathfrak{g}$.

The linear maps $\frac{d}{dt} : \mathcal{L}(\mathfrak{g}) \to \mathcal{L}\mathfrak{g}$ and $\mathrm{Res} : \mathcal{L} \to \mathbb{C}$ are defined by

$\frac{d}{dt}(Px) = \frac{dP}{dt}x$ (for $P \in \mathcal{L}$ and $x \in \mathfrak{g}$),
$\mathrm{Res}(t^r) = \delta_{r,-1}$ (for $r \in \mathbb{Z}$).

Definition 3.3. *The bilinear map $\nu : \mathcal{L}(\mathfrak{g}) \times \mathcal{L}(\mathfrak{g}) \to \mathbb{C}$ is defined by :*

$$\nu(f,g) = Res((\frac{df}{dt}, g)_t).$$

Lemma 3.4. *ν is a 2-cocycle on $\mathcal{L}(\mathfrak{g})$, that is to say for $f, g, h \in \mathcal{L}(\mathfrak{g})$,*

$$\nu(f,g) = -\nu(g,f),$$
$$\nu([f,g],h) + \nu([g,h],f) + \nu([h,f],g) = 0.$$

Proof: For $P, Q \in \mathcal{L}$ and $x, y \in \mathfrak{g}$,

$$\nu(x\otimes P, y\otimes Q) + \nu(y\otimes Q, x\otimes P)$$
$$= (x,y)\mathrm{Res}(\frac{dP}{dt}Q + P\frac{dQ}{dt}) = (x,y)\mathrm{Res}(\frac{d(PQ)}{dt}) = 0,$$

and for $P, Q, R \in \mathcal{L}$, $x, y, z \in \mathfrak{g}$:

$\nu([P\otimes x, Q\otimes y], R\otimes z) + \nu([Q\otimes y, R\otimes z], P\otimes x) + \nu([R\otimes z, P\otimes x], Q\otimes y)$
$= ([x,y],z)\mathrm{Res}(\frac{d(PQ)R}{dt}) + ([y,z],x)\mathrm{Res}(\frac{d(QR)P}{dt}) + ([z,x],y)\mathrm{Res}(\frac{d(RP)Q}{dt})$
$= ([x,y],z)\mathrm{Res}(\frac{d(PQ)R}{dt} + \frac{d(QR)P}{dt} + \frac{d(RP)Q}{dt})$
$= ([x,y],z)\mathrm{Res}(\frac{d(PQR)}{dt}) = 0.$ □

Definition 3.5. *The affine Kac-Moody algebra is $\hat{\mathfrak{g}} = \mathcal{L}(\mathfrak{g}) \oplus \mathbb{C}c$ where c is an additional formal central element and the Lie algebra structure is defined by ($f, g \in \mathcal{L}(\mathfrak{g})$) :*

$$[f,g] = [f,g]_{\mathcal{L}(\mathfrak{g})} + \nu(f,g)c,$$

where $[f,g]_{\mathcal{L}(\mathfrak{g})}$ is the bracket in $\mathcal{L}(\mathfrak{g})$.

The skew symmetry for $\hat{\mathfrak{g}}$ is a consequence of the first property of Lemma 3.4, and the Jabobi identity for $\hat{\mathfrak{g}}$ is a consequence of the Jacobi identity for $\mathcal{L}(\mathfrak{g})$ and the second property of Lemma 3.4.

3.3. **Chevalley generators.** In this subsection we give a more algebraic presentation of $\hat{\mathfrak{g}}$ which allows to have a unified point of view on finite dimensional semi-simple Lie algebras and affine Kac-Moody algebras. This is an indication that affine Kac-Moody algebras are the natural generalizations of finite dimensional semi-simple Lie algebras and so it is also a motivation for the definition of affine Kac-Moody algebras.

Theorem 3.6. *$\hat{\mathfrak{g}}$ can be presented by generators $(E_i)_{0\leq i\leq n}$, $(F_i)_{0\leq i\leq n}$, $(H_i)_{0\leq i\leq n}$, and relations :*

$$[H_i, H_j] = 0,$$

$$[E_i, F_j] = \delta_{i,j} H_i,$$
$$[H_i, E_j] = C_{i,j} E_j,$$
$$[H_i, F_j] = -C_{i,j} F_j,$$
$$(Ad_{E_i})^{1-C_{i,j}}(E_j) = 0 \textit{ for } i \neq j,$$
$$(Ad_{F_i})^{1-C_{i,j}}(F_j) = 0 \textit{ for } i \neq j,$$

where $C = (C_{i,j})_{0 \leq i,j \leq n}$ *is an affine Cartan matrix, that it to say* $C_{i,j} \in \mathbb{Z}$, $C_{i,i} = 2$, $(i \neq j \Rightarrow C_{i,j} \leq 0)$, $(C_{i,j} = 0 \Leftrightarrow C_{j,i} = 0)$, *all proper principal minors are strictly positive*

$$det((C_{i,j})_{0 \leq i,j \leq R}) > 0 \textit{ for } 0 \leq R \leq n-1,$$

and $det(C) = 0$.

Moreover we can choose the labeling $\{0, \cdots, n\}$ so that the subalgebra generated by the E_i, F_i, H_i $(1 \leq i \leq n)$ is isomorphic to $\mathfrak{g}$, that is to say $(C_{i,j})_{1 \leq i,j \leq n}$ is the Cartan matrix of $\mathfrak{g}$.

Let us give the general idea of the construction of the Chevalley generators of $\hat{\mathfrak{g}}$. Let e_i, f_i, h_i $(1 \leq i \leq n)$ be Chevalley generators of $\mathfrak{g}$. For $i \in \{1, \cdots, n\}$, we set

$$E_i = 1 \otimes e_i.$$

The point is to define E_0, F_0 and H_0. Consider the following decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha,$$

where for $\alpha \in \mathfrak{h}^*$,

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} | [h, x] = \alpha(x)x, \forall h \in \mathfrak{h}\},$$

and $\Delta = \{\alpha \in \mathfrak{h}^* - \{0\} | \mathfrak{g}_\alpha \neq \{0\}\}$ (it is called the set of roots). For example we have $e_i \in \mathfrak{g}_{\alpha_i}$, $f_i \in \mathfrak{g}_{-\alpha_i}$ where α_i is defined by $\alpha_i(H_j) = C_{i,j}$. By classical results, we have $\dim(\mathfrak{g}_\alpha) = 1$ for $\alpha \in \Delta$, and there is a unique $\theta \in \Delta$ such that $\theta + \alpha_i \notin \Delta \cup \{0\}$ for $i \in \{1, \cdots, n\}$. θ is called the longest root of $\mathfrak{g}$.

Consider ω the linear involution of $\mathfrak{g}$ defined by $\omega(e_i) = -f_i$, $\omega(f_i) = -e_i$, $\omega(h_i) = -h_i$. Consider a bilinear form $(,) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$ defined by $(\alpha_i, \alpha_j) = C_{i,j}/\epsilon_i$ where the ϵ_i are positive integers such that $B = \mathrm{diag}(\epsilon_1, \cdots, \epsilon_n)C$ is symmetric (the ϵ_i are uniquely defined if we assume that there are prime to each other).

Let us choose $f_0 \in \mathfrak{g}_\theta$ such that

$$(f_0, \omega(f_0)) = -\frac{2h^\vee}{(\theta, \theta)},$$

Let us also define :

$$e_0 = -\omega(f_0) \in \mathfrak{g}_{-\theta}.$$

Then we define

$$E_0 = t \otimes e_0 \ , \ F_0 = t^{-1} \otimes f_0 \ , \ H_0 = [E_0, F_0].$$

Example : for $\mathfrak{g} = sl_2$ we have the Cartan matrix $C = (2)$. Let us check that the corresponding affine Cartan matrix for $\hat{sl_2}$ is $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$. We have

$$\hat{sl_2} = (\mathcal{L} \otimes e) \oplus (\mathcal{L} \otimes f) \oplus (\mathcal{L} \otimes h) \oplus \mathbb{C}c.$$

We follow the construction described above : we set $E_1 = 1 \otimes e$, $F_1 = 1 \otimes f$ and $H_1 = 1 \otimes h$. We have $h^\vee = 2$, $\Delta = \{\alpha, -\alpha\}$ where α is defined by $\alpha(h) = 2$. The longest root is $\theta = \alpha$ and so $(sl_2)_\theta = \mathbb{C}e$. So f_0 is of the form $f_0 = \lambda e$ where $\lambda \in \mathbb{C}^*$. Let us compute λ. We have

$$(f_0, \omega(f_0)) = \lambda^2 (e, f) = \lambda^2 K(e, f)/2.$$

To compute $K(e, f) = Tr(Ad_e Ad_f)$, we compute the action

$$(Ad_e Ad_f)(f) = 0 \ , \ (Ad_e Ad_f)(e) = 2e \ , \ (Ad_e Ad_f)(h) = 2h.$$

So $K(e, f) = 4$ and $(f_0, \omega(f_0)) = 2\lambda^2$. We have $(\alpha, \alpha) = 2$, and so by definition we have $(f_0, \omega(f_0)) = -2$, and $\lambda^2 = 1$. Let us fix $\lambda = 1$. So we have :

$$E_0 = t \otimes f \ , \ F_0 = t^{-1} \otimes e.$$

We can also compute :

$$H_0 = [E_0, F_0] = 1 \otimes [f, e] + \nu(t \otimes f, t^{-1} \otimes e)c = -H_1 + (f, e)c = 2c - H_1.$$

Then we can check all relations of Chevalley generators. In particular $[H_1, E_0] = -2E_0$ and $[H_0, E_1] = -2E_1$ give the Cartan matrix. We also have the Serre relations, for example $Ad_{E_1}(E_0) = 2t \otimes h$ so $Ad_{E_1}^2(E_0) = -4t \otimes e$ and $Ad_{E_1}^3(E_0) = 0$.

For complements on this section, the reader may refer to [K1, M, K2].

4. Representations of Lie algebras

Affine Kac-Moody algebras have a very rich representation theory which have applications in several fields of Mathematics and Mathematics Physics. First let us explain how to construct analogs of highest weight representations of semi-simple Lie algebras.

4.1. **Triangular decomposition of $\hat{\mathfrak{g}}$.** $\mathfrak{g}$ has the classical triangular decomposition $\mathfrak{g} = n^+ \oplus \mathfrak{h} \oplus n^-$ where n^+ (resp. n^-, $\mathfrak{h}$) is the Lie subalgebra of $\mathfrak{g}$ generated by the $(e_i)_{1\leq i\leq n}$ (resp. $(f_i)_{1\leq i\leq n}$, $(h_i)_{1\leq i\leq n}$).

Consider the following subspaces of $\hat{\mathfrak{g}}$:

$$\hat{n}^+ = t\mathbb{C}[t] \otimes (n^- \oplus \mathfrak{h}) \oplus \mathbb{C}[t] \otimes n^+,$$
$$\hat{n}^- = t^{-1}\mathbb{C}[t^{-1}] \otimes (n^+ \oplus \mathfrak{h}) \oplus \mathbb{C}[t^{-1}] \otimes n^-,$$
$$\hat{\mathfrak{h}} = (1 \otimes \mathfrak{h}) \oplus \mathbb{C}c.$$

Lemma 4.1. *$\hat{n}^+$, $\hat{n}^-$, $\hat{\mathfrak{h}}$ are Lie subalgebras of $\hat{\mathfrak{g}}$, and we have the triangular decomposition*

$$\hat{\mathfrak{g}} = \hat{n}^+ \oplus \hat{\mathfrak{h}} \oplus \hat{n}^-.$$

Note that we get the same decomposition by using Chevalley generators, that it to say $\hat{n}^+$ (resp. $\hat{n}^-$, $\hat{\mathfrak{h}}$) is the Lie subalgebra of $\hat{\mathfrak{g}}$ generated by the E_i (resp. the F_i, the H_i) for $0 \leq i \leq n$. Indeed for $i \in \{1, \cdots, n\}$, we have $E_i = 1 \otimes e_i \in \hat{n}^+$, $F_i = 1 \otimes f_i \in \hat{n}^-$ and $H_i = 1 \otimes h_i \in \hat{\mathfrak{h}}$. Moreover

$$E_0 = t \otimes e_0 \in t \otimes n^- \subset \hat{n}^+,$$
$$F_0 = t^{-1} \otimes f_0 \in t^{-1} \otimes n^+ \subset \hat{n}^-,$$
$$H_0 = [E_0, F_0] \in 1 \otimes [e_0, f_0] + \mathbb{C}c \subset \hat{\mathfrak{h}}.$$

4.2. **The extended algebra $\tilde{\mathfrak{g}}$.** The simple roots $\alpha_i \in \hat{\mathfrak{h}}^*$ are defined by $\alpha_i(H_j) = C_{i,j}$ for $0 \leq i, j \leq n$. As $\det(C) = 0$, the simple roots are not linearly independent. For example for $\hat{sl}_2$, we have $\alpha_0 + \alpha_1 = 0$. For the following constructions, we need linearly independent simple roots. That is why we consider the extended affine Lie algebra

$$\tilde{\mathfrak{g}} = \hat{\mathfrak{g}} \oplus \mathbb{C}d,$$

with the additional derivation element d. We extend the Lie algebra structure of $\hat{\mathfrak{g}}$ to $\tilde{\mathfrak{g}}$ by the relations

$$[d, P(t) \otimes x] = t\frac{dP(t)}{dt} \otimes x \ , \ [d, c] = 0,$$

for $P(t) \in \mathcal{L}$ and $x \in \mathfrak{g}$. We have the new Cartan subalgebra $\tilde{\mathfrak{h}} = \hat{\mathfrak{h}} \otimes \mathbb{C}d$. It is a commutative Lie subalgebra of $\tilde{\mathfrak{g}}$ of dimension $n + 2$. We have the corresponding triangular decomposition :

$$\tilde{\mathfrak{g}} = \hat{n}^+ \oplus \tilde{\mathfrak{h}} \oplus \hat{n}^-.$$

Let us define the new simple roots $\alpha_i \in \tilde{\mathfrak{h}}^*$ for $i \in \{0, \cdots, n\}$. The action of α_i on $\hat{\mathfrak{h}}$ has already been defined, and so we have to specify the $\alpha_i(d)$. The condition that leads to the computation is that the E_i

should be of weight α_i for the adjoint representation, that is to say $[H, E_i] = \alpha_i(H)E_i$ for any $H \in \tilde{\mathfrak{h}}^*$. In particular we get for $i \neq 0$, $\alpha_i(d)E_i = [d, 1 \otimes e_i] = 0$ and so $\alpha_i(d) = 0$. We also get

$$\alpha_0(d)E_0 = [d, t \otimes e_0] = t \otimes e_0 = E_0,$$

and so $\alpha_0(d) = 1$.

4.3. **Category $\mathcal{O}$ of representations.**

Definition 4.2. *A module V of $\tilde{\mathfrak{g}}$ is said to be $\tilde{\mathfrak{h}}$-diagonalizable if we have a decomposition $V = \bigoplus_{\lambda \in \tilde{\mathfrak{h}}^*} V_\lambda$ where for $\lambda \in \tilde{\mathfrak{h}}^*$:*

$$V_\lambda = \{v \in V | h.v = \lambda(h)v, \forall h \in \tilde{\mathfrak{h}}\}.$$

V_λ is called the weight space of weight λ of V. The set of weight of V is

$$wt(V) = \{\lambda \in \tilde{\mathfrak{h}}^* | V_\lambda \neq \{0\}\}.$$

We define a partial ordering $\preceq$ on $\tilde{\mathfrak{h}}^*$ by

$$\lambda \preceq \mu \Leftrightarrow \mu = \lambda + \sum_{0 \leq i \leq n} m_i \alpha_i,$$

where $m_i \in \mathbb{Z}$, $m_i \geq 0$.

For $\lambda \in \tilde{\mathfrak{h}}^*$, we set $D(\lambda) = \{\mu \in \tilde{\mathfrak{h}}^* | \mu \preceq \lambda\}$.

Definition 4.3. *The category $\mathcal{O}$ is the category of $\tilde{\mathfrak{g}}$-modules V satisfying :*

1) V is $\tilde{\mathfrak{h}}$-diagonalizable,

2) the weight spaces of V are finite-dimensional,

3) there is a finite number of $\lambda_1, \cdots, \lambda_s \in \tilde{\mathfrak{h}}^*$ such that

$$wt(V) \subset \bigcup_{1 \leq i \leq s} D(\lambda_i).$$

The category $\mathcal{O}$ is stable by submodules and quotients.

For V_1, V_2 representations of $\tilde{\mathfrak{g}}$, we can define a structure of $\tilde{\mathfrak{g}}$-module on $V_1 \otimes V_2$, by using the coproduct $\Delta : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}}$:

$$\Delta(g) = g \otimes 1 + 1 \otimes g \text{ for } g \in \tilde{\mathfrak{g}}.$$

Proposition 4.4. *If V_1, V_2 are modules in the category $\mathcal{O}$, $V_1 \oplus V_2$ and $V_1 \otimes V_2$ are in the category $\mathcal{O}$.*

The result follows from the following observations : for $\lambda \in \tilde{\mathfrak{h}}^*$, we have

$$(V_1 \oplus V_2)_\lambda = (V_1)_\lambda \oplus (V_2)_\lambda,$$
$$(V_1 \otimes V_2)_\lambda = \bigoplus_{\mu \in \tilde{\mathfrak{h}}^*} (V_1)_\mu \otimes (V_2)_{\lambda - \mu},$$

$$D(\lambda) + D(\lambda') = D(\lambda + \lambda') \text{ for } \lambda, \lambda' \in \tilde{\mathfrak{h}}^*.$$

4.4. **Verma modules and simple highest weight modules.** In this subsection we study important examples of modules in the category $\mathcal{O}$.

Let $\mathcal{U}(\hat{\mathfrak{g}})$ be the enveloping algebra of $\hat{\mathfrak{g}}$: it is defined by generators E_i, F_i, H_i (where $0 \leq i \leq n$), and the same relations than $\hat{\mathfrak{g}}$ described in Theorem 3.6 where $[a, b]$ means $ab - ba$. One can define in the same way $\mathcal{U}(\tilde{\mathfrak{g}})$.

For $\lambda \in \tilde{\mathfrak{h}}^*$, let

$$J(\lambda) = \mathcal{U}(\tilde{\mathfrak{g}})\hat{n}^+ + \sum_{h \in \tilde{\mathfrak{h}}^*} \mathcal{U}(\tilde{\mathfrak{g}})(h - \lambda(h)) \subset \mathcal{U}(\tilde{\mathfrak{g}}).$$

As it is a left ideal of $\mathcal{U}(\tilde{\mathfrak{g}})$, $M(\lambda) = \mathcal{U}(\tilde{\mathfrak{g}})/J(\lambda)$ has a natural structure of a $\mathcal{U}(\tilde{\mathfrak{g}})$-module by left multiplication. $M(\lambda)$ is called a Verma module.

Proposition 4.5. *$M(\lambda)$ is in the category $\mathcal{O}$ and has a unique maximal proper submodule $N(\lambda)$.*

Proof: We have $wt(M(\lambda)) \subset D(\lambda)$ and for $\mu = \lambda - \sum_{i=0 \cdots n} m_i \alpha_i \in D(\lambda)$ (where $m_i \geq 0$) and $m = \sum_{0 \leq i \leq n} m_i$ we have

$$(M(\lambda))_\mu \subset \sum_{(i_1, \cdots, i_m) \in \{0, \cdots, n\}^m} \mathbb{C} F_{i_1} \cdots F_{i_m}.$$

Note that $(M(\lambda))_\lambda = \mathbb{C}.1$ is of dimension 1 and that $M(\lambda)$ is generated by $(M(\lambda))_\lambda$. In particular a proper submodule N of $M(\lambda)$ satisfies $N_\lambda = \{0\}$. In particular the sum N_{max} of all proper submodules of $M(\lambda)$ satisfies $(N_{max})_\lambda = 0$ and so is proper. This gives the existence and unicity of the maximal proper submodule of $M(\lambda)$: $N(\lambda) = N_{max}$. □

As a consequence of the proposition, $M(\lambda)$ has a unique simple quotient

$$L(\lambda) = M(\lambda)/N(\lambda).$$

Proposition 4.6. *$L(\lambda)$ is in the category $\mathcal{O}$ and all simple modules of the category $\mathcal{O}$ are of the form $L(\lambda)$ for a certain λ.*

Proof: $L(\lambda)$ is in $\mathcal{O}$ as a quotient of a module in $\mathcal{O}$. Consider L a simple module in $\mathcal{O}$ and let λ maximal for $\preceq$ in $wt(\lambda)$ (it exists by property 3) of definition 4.3). Let $v \in L_\lambda - \{0\}$. Then we have $\hat{n}^+.v = \{0\}$ and so $L = \mathcal{U}(\tilde{\mathfrak{g}}).v$ is a quotient of $M(\lambda)$. As L is simple, this quotient is isomorphic to $L(\lambda)$. □

4.5. **Characters and integrable representations.** The character of a module V in $\mathcal{O}$ is by definition

$$\chi(V) = \sum_{\lambda\in\tilde{\mathfrak{h}}^*} \dim(V_\lambda)e(\lambda),$$

where the $e(\lambda)$ are formal elements.

In general a representation V in $\mathcal{O}$ has no composition series

$$V_0 = V \supset V_1 \supset V_2 \supset \cdots$$

where V_i/V_{i+1} is simple for $i \geq 0$. But we have the following "replacement" :

Lemma 4.7. *For V a representation in $\mathcal{O}$ and $\lambda \in \tilde{\mathfrak{h}}^*$, there are submodules $\{0\} = V_0 \subset V_1 \subset \cdots V_{t-1} \subset V_t = V$ such that for $1 \leq i \leq t$, $V_i/V_{i-1} \simeq L(\lambda_i)$ for an $\lambda_i \succeq \lambda$, or $(V_i/V_{i-1})_\mu = \{0\}$ for all $\mu \succeq \lambda$.*

This is proved by induction on $\sum_{\mu\succeq\lambda} \dim(V_\mu)$. As a consequence of this result, for V in $\mathcal{O}$ the character $\chi(V)$ of V is a sum of characters of simple representations in $\mathcal{O}$.

In general $L(\lambda)$ is not finite dimensional : the notion of finite dimensional representations has to be replaced by the notion of integrable representations in the category $\mathcal{O}$.

Definition 4.8. *A representation V of $\tilde{\mathfrak{g}}$ is said to be integrable if*

i) V is $\tilde{\mathfrak{h}}$-diagonalizable,

ii) for $\lambda \in \tilde{\mathfrak{h}}^*$, $dim(V_\lambda) < \infty$,

iii) for all $\lambda \in wt(V)$, for all $i \in \{0, \cdots, n\}$, there is $M \geq 0$ such that for $m \geq M$, $\lambda + m\alpha_i \notin wt(V)$ and $\lambda - m\alpha_i \notin wt(V)$.

The character of the simple integrable representations in the category $\mathcal{O}$ satisfy remarkable combinatorial identities (related to MacDonald identities).

4.6. **Evaluation representations.** Let $a \in \mathbb{C}^*$. For V a finite dimensional representation of $\mathfrak{g}$, one defines a structure of $\hat{\mathfrak{g}}$-modules on V ($v \in V$, $x \in \mathfrak{g}$, $P(t) \in \mathcal{L}$) :

$$(P(t)x).v = P(a)x.v \ , \ c.v = 0.$$

This module is denoted by $V(a)$ and is called an evaluation representation (as we evaluate $P(t)$ at a). Note that in general the structure can not be extended to a representation of $\tilde{\mathfrak{g}}$.

Example : for $\mathfrak{g} = sl_2$ and $m \geq 0$, consider V_m the $m+1$ simple finite dimensional representation of sl_2. Let $a \in \mathbb{C}^*$. Then the $\hat{sl_2}$-module $V_m(a)$ is defined by :

$$V_m(a) = \mathbb{C}v_0 \oplus \mathbb{C}v_1 \oplus \cdots \oplus \mathbb{C}v_m,$$

$$(Kt^r).v_j = a^r(m-2j)v_j,$$
$$(Et^r).v_j = a^r(m-j+1)v_{j-1},$$
$$(Ft^r).v_j = a^r(j+1)v_{j+1},$$
$$c.v_j = 0,$$

where we denote $v_{-1} = v_{m+1} = 0$.

We can see that $V_m(a)$ is $\hat{\mathfrak{h}}$-diagonalizable (the v_j are weight vectors). However it is not an highest weight representation (that is to say it is not generated by a weight vector v satisfying $\hat{n}^+.v = \{0\}$) as we have

$$E_0.v_0 = (t \otimes f)v_0 = av_1 \neq 0.$$

To understand there representations from a different "highest weight" point of view, we have to use the second triangular decomposition of $\hat{\mathfrak{g}}$:

$$\hat{\mathfrak{g}} = \mathcal{L}n^+ \oplus (\mathcal{L}\mathfrak{h} \oplus \mathbb{C}c) \oplus \mathcal{L}n^-.$$

Note that the new Cartan subalgebra $\mathcal{L}\mathfrak{h} \oplus \mathbb{C}c$ is infinite dimensional, and so the corresponding theory has major differences with the usual one. For example the notion of "highest weight" has to be replaced by series of complex numbers corresponding to the eigenvalues of the $(t^r \otimes H_i)_{1\leq i\leq n, r\in\mathbb{Z}}$.

Let us look at a similar construction of infinite dimensional "vertex" representations. Let us consider the algebra $\tilde{\mathfrak{g}}'$ which is defined exactly as $\tilde{\mathfrak{g}}$, except that we use $\mathbb{C}((t))$ instead of $\mathcal{L}$. We can define in the same way the category $\mathcal{O}$, the simple highest weight representations and the level for $\tilde{\mathfrak{g}}'$. For V a finite dimensional representation of $\mathfrak{g}$ we can define a structure of $\tilde{\mathfrak{g}}'$-module on $V((z)) = \mathbb{C}((z)) \otimes V$ by ($v \in V$, $x \in \mathfrak{g}$, $f(z), g(z) \in \mathbb{C}((z))$) :

$$(f(z) \otimes x).(g(z) \otimes v) = f(z)g(z) \otimes (x.v),$$
$$c.(g(z) \otimes v) = 0 \ , \ d.(g(z) \otimes v) = z\frac{dg(z)}{dz} \otimes v.$$

Analog representations can be considered for $\hat{\mathfrak{g}}'$ (analog of $\tilde{\mathfrak{g}}'$ for $\hat{\mathfrak{g}}$).

4.7. **Level of representations.** Let us recall the statement of the well-known Schur lemma :

Lemma 4.9. *A central element c of a Lie algebra acts as a scalar on a simple finite dimensional representation L.*

Proof: c has an eigenvalue λ, and $\mathrm{Ker}(c - \lambda \mathrm{Id})$ is a submodule of L, so $L = \mathrm{Ker}(c - \lambda\mathrm{Id})$. □

The result holds for $\tilde{\mathfrak{g}}$-modules which are $\tilde{\mathfrak{h}}$-diagonalizable with finite dimensional weight spaces. In particular, $c \in \tilde{\mathfrak{g}}$ acts as a scalar $k \in \mathbb{C}$ on a simple representation V of the category $\mathcal{O}$.

Definition 4.10. *A representation V is said to be of level k if c acts as kId on V.*

All simple representations of the category $\mathcal{O}$ have a level. The category of modules of the category $\mathcal{O}$ of level k is denoted by $\mathcal{O}_k$.

Note that one can define the same notion for $\hat{\mathfrak{g}}$ and then for example the evaluations representations have level 0.

Note that $\lambda \in \tilde{\mathfrak{h}}^*$ is characterized by the image of λ in $\mathfrak{h}^*$, the level $\lambda(c) \in \mathbb{C}$ and $\lambda(d) \in \mathbb{C}$. So the data of an element of $\tilde{\mathfrak{h}}^*$ is equivalent to the data of (λ, k, k') where $\lambda \in \mathfrak{h}^*$ and $k, k' \in \mathbb{C}$. In particular the notation $L_{\lambda,k,k'}$ is used for the simple modules of the category $\mathcal{O}$.

The level $k = -h^\vee$ (where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$, see above) is particular as the center of $\hat{\mathfrak{g}}/\hat{\mathfrak{g}}(c-k)$ is large and the representation theory changes drastically at this level (see below). This level is called critical level. For example it is of particular importance for application to Conformal Field Theory and Geometric Langlands Program.

Unless the category $\mathcal{O}$ is stable by tensor product, the category $\mathcal{O}_k$ is not stable by tensor product (except for $k = 0$). Indeed from $\Delta(c) = c \otimes 1 + 1 \otimes c$ we get that for V_1, V_2 representations respectively in $\mathcal{O}_{k_1}$, $\mathcal{O}_{k_2}$, the module $V_1 \otimes V_2$ is in $\mathcal{O}_{k_1+k_2}$. This is one motivation to study the fusion product in the next section.

For complements on this section, the reader may refer to [C, EFK, Fr, K1, K2].

5. Fusion product, conformal blocks and Knizhnik-Zamolodchikov equations

In this section we give a glimpse on examples of mode advanced subjects related to representations of affine Kac-Moody algebras.

In the following "KZ" means Knizhnik-Zamolodchikov.

5.1. **Construction of the fusion product.**

Definition 5.1. *A representation V of $\hat{\mathfrak{g}}'$ is said to be smooth if for all $v \in V$, there is $N \geq 0$ such that for all $g_1, \cdots, g_N \in \mathfrak{g}$:*

$$(t \otimes g_1)(t \otimes g_2) \cdots (t \otimes g_N).v = 0.$$

(The vectors v with such a property are called smooth vectors.)

In this section we explain how to construct a smooth module in $\mathcal{O}_k$ as a fusion of two smooth modules in $\mathcal{O}_k$. In the following the level k of considered representations of $\hat{\mathfrak{g}}$ satisfies $k \notin -h^\vee + \mathbb{Q}_{\geq 0}$.

Let us give a "reason" why this restriction is required : let V be a smooth module in $\mathcal{O}_k$ such that $\dim(\bigcap_{g \in \mathfrak{g}} \mathrm{Ker}(t \otimes g)) < \infty$. If $k \notin$

$-h^\vee + \mathbb{Q}$ then V is semi-simple and if $k \notin -h^\vee + \mathbb{Q}_{\geq 0}$ then V has a finite composition series.

However if we restrict to certains subcategories of smooth representations in $\mathcal{O}_k$, an analog construction holds for any k (see [Fi]).

Consider $V_1, \cdots, V_n$ smooth representations of $\hat{\mathfrak{g}}'$ of level k. Let $s_1, \cdots, s_{n+1} \in \mathbb{P}_1(\mathbb{C})$ distinct and $C' = \mathbb{P}_1(\mathbb{C}) - \{s_1, \cdots, s_{n+1}\}$. Consider the algebra R of functions $f : C' \to \mathbb{C}$ which are regular outside the points $s_1, \cdots, s_{n+1}$ and which are meromorphic at these points.

Let us define the bilinear map $\{,\} : R \times R \to \mathbb{C}$ by $(f_1, f_2 \in R)$:

$$\{f_1, f_2\} = \mathrm{Res}_{n+1}(f_2 d(f_1))$$

where Res_i is the residue of the expansion at the point s_i. By the residue theorem we have also

$$\{f_1, f_2\} = -\sum_{i=1\cdots n} \mathrm{Res}_i(f_2 d(f_1)). \tag{1}$$

By using the standard properties of differentials, we have :

Lemma 5.2. $\{,\}$ *is skew symmetric*

$$\{f_1, f_2\} = -\{f_2, f_1\}, \text{ for all } f_1, f_2 \in R,$$

and is a cocycle, that is to say :

$$\{f_1 f_2, f_3\} + \{f_2 f_3, f_1\} + \{f_3 f_1, f_2\} = 0 \text{ for all } f_1, f_2, f_3 \in R.$$

Let $\Gamma = (R \otimes \mathfrak{g}) \oplus \mathbb{C}c$ the Lie algebra such that c is central and for $f, g \in R$, $x, y \in \mathfrak{g}$:

$$[fx, gy] = fg[x, y] + \{f, g\}(x, y)c.$$

Consider n copies $\hat{\mathfrak{g}}'_1, \cdots, \hat{\mathfrak{g}}'_n$ of $\hat{\mathfrak{g}}'$ with respective central elements $c_1, \cdots, c_n$ and affine parameter $t - s_i$ (if $s_i = \infty$, $t - s_i$ means t^{-1}). We define $(\hat{\mathfrak{g}}')^n$ as the Lie algebra

$$(\hat{\mathfrak{g}}')^n = (\hat{\mathfrak{g}}'_1 \oplus \cdots \oplus \hat{\mathfrak{g}}'_n)/(c_1 = c_2 = \cdots = c_n).$$

The image of the central elements c_i in $(\hat{\mathfrak{g}}')^n$ is denoted by $c \in (\hat{\mathfrak{g}}')^n$.

Lemma 5.3. *We have two Lie algebra morphisms* $p_1 : \Gamma \to \hat{\mathfrak{g}}'$, $p_2 : \Gamma \to (\hat{\mathfrak{g}}')^n$ *defined by* $(f \in R,\ x \in \mathfrak{g})$ *:*

$$p_1(fx) = (f)_{n+1} x \ ,\ p_1(c) = c,$$

$$p_2(fx) = \sum_{1 \leq i \leq n} (f)_i x \ ,\ p_2(c) = -c,$$

where $(f)_i \in \mathbb{C}((t - s_i))$ *is the expansion of* f *at* s_i.

Now let $W = V_1 \otimes V_2 \otimes \cdots \otimes V_n$.

Lemma 5.4. *W has a structure of $(\hat{\mathfrak{g}}')^n$-module defined by :*

$$g_i.(v_1, \cdots, v_N) = (v_1, \cdots, v_{i-1}, g_i v_i, v_{i+1}, \cdots, v_N) \text{ for } g_i \in \hat{\mathfrak{g}}'_i,$$

$$c.(v_1, \cdots, v_N) = k(v_1, \cdots, v_N).$$

So W is also a Γ-module of level $-k$ by Lemma 5.3.

For $N \geq 0$, let W_N be the subspace of W generated by the elements of the form

$$(f_1 \otimes g_1)((f_2 \otimes g_2)(\cdots(f_N \otimes g_N)(w))\cdots),$$

where $w \in W$, $g_1, \cdots, g_N \in \mathfrak{g}$ and $f_1, \cdots, f_N \in R$ have an expansion in $(t - s_{n+1})\mathbb{C}[[(t - s_{n+1})]]$ at s_{n+1}.

We get inclusions $\cdots \subset W_2 \subset W_1 \subset W$ and so a sequence of linear maps

$$(W/W_1) \leftarrow (W/W_2) \leftarrow \cdots .$$

The projective limit is denoted by $\hat{W}$.

Let us define a structure of $\hat{\mathfrak{g}}'$-module of level $-k$ on $\hat{W}$. An element $w \in \hat{W}$ can be written in the form $w = (w_1, w_2, \cdots)$ where $w_i \in W$ and $w_{i+1} - w_i \in W_i$. Let $f \in \mathbb{C}((t - s_{n+1}))$ and $g \in \mathfrak{g}$. Consider $f_i \in R$ such that $f - (f_i)_{n+1} \in (t - s_{n+1})^i\mathbb{C}[[(t - s_{n+1})]]$ and $q \geq 0$ such that $f \in (t - s_{n+1})^{-q}\mathbb{C}[[(t - s_{n+1})]]$. Then we define :

$$(f \otimes g)w = ((f_1 \otimes g)w_{q+1}, (f_2 \otimes g)w_{q+2}, \cdots),$$

where $(f_i \otimes g)w_{q+i}$ is given by the Γ-module structure on W.

Proposition 5.5. *The above formula is well-defined and gives a structure of $\hat{\mathfrak{g}}'$-module on $\hat{W}$.*

By Lemma 5.3, $\hat{W}$ is also a Γ-module.

Remark : to construct a tensor category structure, it is necessary to consider the $\hat{\mathfrak{g}}$-module $\hat{W}^\omega$ obtained from the $\hat{\mathfrak{g}}$-module $\hat{W}$ by the twisting $\omega : \hat{\mathfrak{g}} \rightarrow \hat{\mathfrak{g}}$ satisfying $\omega(t^r \otimes g) = (-t)^{-r} \otimes g$, $\omega(c) = -c$, and then to consider the submodule of smooth vectors of $\hat{W}^\omega$. We get a smooth module of level k.

It is remarkable that the corresponding tensor category is related to certain categories of representations of quantum groups.

For complements see [D2, KL1, KL2, CP].

5.2. **Solutions of KZ equations - examples.** In this section and next section we suppose that $k \neq h^\vee$.

Suppose that we have $W_0, \cdots, W_m$ highest weight representations of $\tilde{\mathfrak{g}}$ of level k and whose restricted highest weights to $\mathfrak{h}$ are respectively

$\lambda_0, \cdots, \lambda_m \in \mathfrak{h}^*$. Let $V_1, \cdots, V_m$ be simple finite dimensional representations of $\mathfrak{g}$ of corresponding highest weights $\mu_1, \cdots, \mu_m$. We consider intertwining operators :

$$I_j(z) : W_j \to W_{j-1} \otimes V_j((z)),$$

that is to say we have for $r \in \mathbb{Z}$, $g \in \mathfrak{g}$

$$I_j(z) \circ (t^r \otimes g) = ((t^r \otimes g) \otimes 1 + 1 \otimes (z^r \otimes g)) \circ I_j(z).$$

Whe have the existence of such non trivial operators when $\lambda_j = \lambda_{j-1} = \mu_j$ which are constructed by induction (for the partial ordering on weights) on the weight spaces of W_j.

Let us define the Casimir element. Let

$$\Psi : \mathfrak{g} \otimes \mathfrak{g} \to \mathrm{End}(\mathfrak{g}),$$

$$(g_1 \otimes g_2) \mapsto (g \mapsto (g_1, g) g_2).$$

We denote $T = \Psi^{-1}(\mathrm{Id})$ and T' the image of T by multiplication in $\mathcal{U}(\mathfrak{g})$. For $(b_k)_k$ a basis of $\mathfrak{g}$ and $(b_k^*)_k$ the dual basis of $\mathfrak{g}$ relatively to $(,)$ defined by $(b_k^*, b_{k'}) = \delta_{k,k'}$, we have $T = \sum_k b_k^* \otimes b_k$ and so $T' = \sum_k b_k^* b_k$. Note that in general $T' \in \mathcal{U}(\mathfrak{g})$ is not in $\mathfrak{g}$.

For λ a weight, let C_λ the scalar corresponding to the action of T' on the simple $\mathfrak{g}$-module of highest weight λ. The conformal weight of λ is

$$h(\lambda) = \frac{C_\lambda}{2(k + h^\vee)}.$$

We replace $I_j(z)$ by $I_j(z) z^{h(\mu_j) + h(\lambda_j) - h(\lambda_{j-1})}$.

Let us define $T_{j,i}$ the image of T in

$$\mathrm{Id} \otimes \cdots \otimes \mathrm{Id} \otimes \mathrm{End}(V_j) \otimes \mathrm{Id} \otimes \cdots \otimes \mathrm{Id} \otimes \mathrm{End}(V_i) \otimes \mathrm{Id} \otimes \cdots \otimes \mathrm{Id},$$

that is to say for $T = \sum_k b_k^* \otimes b_k$, b_k^* acts on $\mathrm{End}(V_j)$ and b_k on $\mathrm{End}(V_i)$.

Consider formal variables $z_1, \cdots, z_m$ and the map

$$f : W_m \to W_0 \otimes V_1((z_1)) \otimes V_2((z_2)) \otimes \cdots \otimes V_m((z_m)),$$

$$\begin{aligned} f =& (I_{V_1, W_0, W_1}(z_1) \otimes \mathrm{Id}^{\otimes (m-1)}) \circ \cdots \\ & \circ (I_{V_{m-1}, W_{m-2}, W_{m-1}}(z_{m-1}) \otimes \mathrm{Id}) \circ I_{V_m, W_{m-1}, W_m}(z_m). \end{aligned}$$

We can look at f as a matrix with coefficients in

$$V_1((z_1)) \otimes V_2((z_2)) \otimes \cdots \otimes V_m((z_m)).$$

Proposition 5.6. *f satisfies the KZ equation :*

$$\frac{\partial f}{\partial z_j} = \frac{1}{k + h^\vee} \sum_{i \neq j} \frac{T_{j,i}}{z_j - z_i}(f).$$

Let us look at a simple explicit example of a solution. Let $\mathfrak{g} = sl_2$. We have $h^\vee = 2$. As $(e,f) = 2$, $(h,h) = 4$ and $(e,h) = (f,h) = (e,e) = (f,f) = 0$, we have

$$T = \frac{1}{2}(e \otimes f + f \otimes e + \frac{1}{2} h \otimes h).$$

Let $V_1, \cdots, V_N$ be highest weight vectors representations, $v_i \in V_i$ an highest weight vector and $\lambda_i \geq 1$ the highest weight of V_i (here P^+ is identified with $\mathbb{N}$). Let

$$v = v_1 \otimes \cdots \otimes v_n.$$

We have $T_{i,j} v = \frac{\lambda_i \lambda_j}{4} v$. Then

$$\Psi(z_1, \cdots, z_N) = \prod_{i<j} (z_i - z_j)^{\lambda_i \lambda_j / (4(2+k))} v,$$

is a solution of the KZ equation. Indeed $\frac{\partial \Psi(z_1, \cdots, z_N)}{\partial z_j}$ is equal to

$$\Psi(z_1, \cdots, z_N)(\sum_{i<j} \frac{\lambda_i \lambda_j}{4(2+k)} \frac{1}{-z_i + z_j} + \sum_{i>j} \frac{\lambda_i \lambda_j}{4(2+k)} \frac{1}{z_j - z_i})$$

$$= \frac{1}{2+k} \sum_{i \neq j} \frac{T_{j,i}}{z_j - z_i}(f).$$

For complements see [KZ, FR, EFK, CP].

5.3. **Conformal blocks, space of coinvariants and KZ connection.** The constructions of this section are closely related to the construction of section 5.1. But the context is different as we want to focus on applications on KZ equations (some objects are defined both in this section and in section 5.1).

The Wess-Zumino-Witten (WZW) model is a model of Conformal Field Theory whose solutions are realized by affine Kac-Moody algebras (Conformal field theory has important applications in string theory, statistical mechanics, and condensed matter physics). The Hilbert space of the WZW model is $\bigotimes_{\lambda \in P^+} L_{\lambda,k} \otimes L_{\lambda,k}$. To study the WZW model, important algebraic objects are the spaces of conformal blocks defined bellow.

Consider $s_1, \cdots, s_n \in \mathbb{P}_1(\mathbb{C}) - \{\infty\}$ distinct. $\hat{\mathfrak{g}}'_i$, Γ, $(\hat{\mathfrak{g}}')^{(n)}$ are defined as in section 5.1. We set $s_{n+1} = \infty$

We consider $(t - s_i)$ as the affine parameter. For M a representation of $\mathfrak{g}$, M has a structure of $(\mathfrak{g}[[(t-s_i)]] \oplus \mathbb{C}.c)$-module such that $t\mathfrak{g}[[(t-s_i)]](M) = 0$ and c acts as k on M. Then we consider the $\hat{\mathfrak{g}}'$-module $\hat{M} = \mathrm{Ind}_{\mathfrak{g}[[(t-s_i)]] \oplus \mathbb{C}c}^{\hat{\mathfrak{g}}'} M$. As a vector space $\hat{M} = \mathcal{U}((t-s_i)^{-1}\mathbb{C}[(t -

$s_i)^{-1}] \otimes \mathfrak{g}) \otimes_{\mathbb{C}} M$. For example if M is a Verma module of $\mathfrak{g}$ then $\hat{M}$ is a Verma module of $\hat{\mathfrak{g}}'$ of level k. We denote this structure by $(\hat{M})_{s_i}$.

Let $M_1, \cdots, M_n$ be representations of $\mathfrak{g}$. $\bigotimes_{i=1,\cdots,n}(\hat{M})_{s_i}$ is a Γ-module. The space of conformal blocks $C_k^0(s_1, \cdots, s_n, M_1, \cdots, M_n)$ is the subspace of the space of linear functions

$$\phi : \bigotimes_{i=1,\cdots,n} (\hat{M})_{s_i} \to \mathbb{C}$$

which are invariant by the Lie subalgebra of Γ of maps which vanish at ∞, that is to say :

$$\phi(g.y) = 0 \text{ for all } g \in \Gamma, g(\infty) = 0.$$

This is called the Ward identity.

Lemma 5.7. *The restriction to $\bigotimes_{i=1,\cdots,n} M_i$ defines an isomorphism*

$$C_k^0(s_1, \cdots, s_n, M_1, \cdots, M_n) \simeq (\bigotimes_{i=1,\cdots,n} M_i)^*.$$

Note that the notion of conformal blocks can also be defined as subspaces of $(\bigotimes_{i=1,\cdots,n} M_i)^*$: simple quotients of $\hat{M}_i$ are considered instead of $\hat{M}_i$ and the invariance for the whole Γ is required, and so with this definition we get a subspace of $\mathfrak{g}$-invariant elements of $(\bigotimes_{i=1,\cdots,n} M_i)^*$ (for instance see [U]).

The space of coinvariants is by definition the dual space of the space of conformal blocks.

Now we suppose that $(s_1, \cdots, s_n)$ can vary in

$$\mathcal{C}^n = \mathbb{C}^n - \bigcup_{i \neq j} (s_i = s_j).$$

We consider $C_k^0(M_1, \cdots, M_n)$ the trivial bundle on $\mathcal{C}^n$ with fibers

$$(\bigotimes_{i=1,\cdots,n} M_i)^*.$$

The following connection on $C_k^0(M_1, \cdots, M_n)$ arises naturally from general results on conformal blocks :

$$\bigtriangledown_j = \frac{\partial}{\partial s_j} - \frac{1}{k + h^\vee} \sum_{i \neq j} \frac{T_{j,i}}{s_j - s_i},$$

where the operators $T_{i,j}$ are defined as in the previous section. This connection is called the KZ connection.

Lemma 5.8. *The KZ connection is flat, the operators commute :*

$$[\nabla_i, \nabla_j] = 0.$$

For complements see [FFR, Fr, FB, U].

References

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