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QUANTUM CHARACTERS FOR QUANTUM AFFINE ALGEBRAS

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1. Introduction

In the case of semi-simple Lie algebras, the structure of the Grothendieck ring of finite dimensional representations of the quantum algebra is well understood: it is analogous to the classic case q=1 and we have a ring homomorphism of characters.

For the general case of Kac-Moody algebras the picture is less clear. In the affine case, E. Frenkel and N. Reshetikhin introduced an injective ring homomorphism of q-characters. It gives informations about the decomposition in Jordan subspaces for a class of commutative elements. The homomorphism of q-characters has a nice symmetry property analogous to the classic action of the Weyl group: the image is the intersection of the kernels of screening operators.

In the ADE case, H. Nakajima, motivated by the geometry of quiver varieties, introduced t-analogs of q-characters. The definition is combinatorial but the proof of the existence uses the geometric theory of quiver varieties which holds only in the simply laced case. In the preprint math.QA/0212257 we propose an algebraic general new approach to q,t-characters motivated by deformed screening operators. The t-deformations are naturally deduced from the algebra structure of $\mathcal{U}_q(\hat{\mathfrak{h}})$: the parameter t is analog to the central charge $c \in \mathcal{U}_q(\hat{\mathfrak{h}})$. This variant of Nakajima's theory allows us to treat the non-simply laced case: in particular the morphism of q,t-characters $\chi_{q,t}$ leads to the construction of a quantization of the Grothendieck ring and to general Nakajima's analogs of Kazhdan-Lusztig polynomials.

2. Homomorphism of characters in finite case

Notations:

$\mathfrak{g}$: simple Lie algebra of rank n, $I = \{1, ..., n\}$

$\mathfrak{h} \subset \mathfrak{g}$: Cartan subalgebra

$\Lambda \subset \mathfrak{h}^*$: lattice of weights of $\mathfrak{g}$

$\omega_i \in \Lambda$: fundamental weights

$\mathrm{Rep}(\mathcal{U}(\mathfrak{g}))$: Grothendieck ring of finite dimensional representations (with $\oplus$ and $\otimes$)

We have an injective homomorphism of rings:

$$\chi : \mathrm{Rep}(\mathcal{U}(\mathfrak{g})) \to \mathbb{Z}[y_i^{\pm}]_{i\in I}$$
$$\chi(V) = \sum_{\lambda = \sum_{i\in I} m_i\omega_i \in \Lambda} \dim(V_\lambda) \prod_{i\in I} y_i^{m_i}$$

where V_λ is the weight space of V:

$$V_\lambda = \{x \in V / \forall h \in \mathfrak{h}, h.x = \lambda(h)x\}$$

We have a symmetry property related to the Weyl groups W:

$$\mathrm{Im}(\chi) = \mathbb{Z}[y_i^{\pm}]_{i\in I}^W \simeq \mathbb{Z}[T_1, ..., T_n]$$

The quantum case is analogeous:

$$\mathrm{Rep}(\mathcal{U}_q(\mathfrak{g})) \simeq \mathrm{Rep}(\mathfrak{g}) \simeq \mathbb{Z}[\Lambda]^W \simeq \mathbb{Z}[T_1, ..., T_n]$$

where $\mathrm{Rep}(\mathcal{U}_q(\mathfrak{g}))$: finite-dimensional representations of type 1.

In the quantum affine case $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$, the picture is less clear.

3. Quantum affine algebra

$\mathfrak{g}$: simple Lie algebra of rank n, $I=\{1,...,n\}$

$(C_{ij})_{1\leq i,j\leq n}=(\frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)})$: Cartan matrix of $\mathfrak{g}$

$\mathfrak{g}\subset\hat{\mathfrak{g}}$: affine Lie algebra, $(C_{i,j})_{0\leq i,j\leq n}$ generalized Cartan matrix of $\hat{\mathfrak{g}}$

$r_i=\frac{(\alpha_i,\alpha_i)}{2}\in\mathbb{Z}$, $B_{i,j}=r_iC_{i,j}$ symmetric matrix

$q\in\mathbb{C}^*$ not a root of unity, $q_i=q^{r_i}$

Definition 1. *The quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$ is defined by generators $x_{i,m}^{\pm}$ ($1\leq i\leq n$, $m\in\mathbb{Z}$), $k_i^{\pm}$ ($1\leq i\leq n$), $h_{i,m}$ ($1\leq i\leq n$, $m\in\mathbb{Z}^*$), central elements $c^{\pm\frac{1}{2}}$ and relations:*

$$k_ik_j=k_jk_i$$

$$k_ih_{j,m}=h_{j,m}k_i$$

$$k_ix_{j,m}^{\pm}k_i^{-1}=q^{\pm B_{ij}}x_{j,m}^{\pm}$$

$$[h_{i,m},x_{j,m'}^{\pm}]=\pm\frac{1}{m}[mB_{ij}]_qc^{\mp\frac{|m|}{2}}x_{j,m+m'}^{\pm}$$

$$x_{i,m+1}^{\pm}x_{j,m'}^{\pm}-q^{\pm B_{ij}}x_{j,m'}^{\pm}x_{i,m+1}^{\pm}=q^{\pm B_{ij}}x_{i,m}^{\pm}x_{j,m'+1}^{\pm}-x_{j,m'+1}^{\pm}x_{i,m}^{\pm}$$

$$[h_{i,m},h_{j,m'}]=\delta_{m,-m'}\frac{1}{m}[mB_{ij}]_q\frac{c^m-c^{-m}}{q-q^{-1}}$$

$$[x_{i,m}^+,x_{j,m'}^-]=\delta_{ij}\frac{c^{\frac{m-m'}{2}}\phi_{i,m+m'}^+-c^{-\frac{m-m'}{2}}\phi_{i,m+m'}^-}{q_i-q_i^{-1}}$$

$$\sum_{\pi\in\Sigma_s}\sum_{k=0..s}(-1)^k\begin{bmatrix}s\\k\end{bmatrix}_{q_i}x_{i,m_{\pi(1)}}^{\pm}...x_{i,m_{\pi(k)}}^{\pm}x_{j,m'}^{\pm}x_{i,m_{\pi(k+1)}}^{\pm}...x_{i,m_{\pi(s)}}^{\pm}=0$$

where the last relation holds for all $i\neq j$, $s=1-C_{ij}$, all sequences of integers $m_1,...,m_s$. Σ_s is the symmetric group on s letters. For $i\in I$ and $m\in\mathbb{Z}$, $\phi_{i,m}^{\pm}\in\mathcal{U}_q(\hat{\mathfrak{g}})$ is determined by the formal power series in $\mathcal{U}_q(\hat{\mathfrak{g}})[[u]]$ (resp. in $\mathcal{U}_q(\hat{\mathfrak{g}})[[u^{-1}]]$):

$$\sum_{m=0..\infty}\phi_{i,\pm m}^{\pm}u^{\pm m}=k_i^{\pm}exp(\pm(q-q^{-1})\sum_{m'=1..\infty}h_{i,\pm m'}u^{\pm m'})$$

and $\phi_{i,m}^+=0$ for $m<0$, $\phi_{i,m}^-=0$ for $m>0$.

One has an embedding $\mathcal{U}_q(\mathfrak{g})\subset\mathcal{U}_q(\hat{\mathfrak{g}})$ and a Hopf algebra structure on $\mathcal{U}_q(\hat{\mathfrak{g}})$.

4. Finite dimensional representations of $\mathcal{U}_q(\hat{\mathfrak{g}})$

A finite dimensional representation V of $\mathcal{U}_q(\hat{\mathfrak{g}})$ is called of type 1 if c acts as Id and V is of type 1 as a representation of $\mathcal{U}_q(\mathfrak{g})$. We note $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ the Grothendieck ring of finite dimensional representations of type 1.

The operators $\{\phi_{i,\pm m}^{\pm}, i\in I, m\in\mathbb{Z}\}$ commute on V. So we have a pseudo weight space decomposition:

$$V=\bigoplus_{\gamma\in\mathbb{C}^{I\times\mathbb{Z}}\times\mathbb{C}^{I\times\mathbb{Z}}}V_\gamma$$

where for $\gamma=(\gamma^+,\gamma^-)$, V_γ is a simultaneous generalized eigenspace:

$$V_\gamma=\{x\in V/\exists p\in\mathbb{N},\forall i\in\{1,...,n\},\forall m\in\mathbb{Z},(\phi_{i,m}^{\pm}-\gamma_{i,m}^{\pm})^p.x=0\}$$

The $\gamma_{i,m}^{\pm}$ are called pseudo eigen values of V.

Theorem 1. *(**Chari, Pressley** 94) Every simple representation $V \in Rep(\mathcal{U}_q(\hat{\mathfrak{g}}))$ is a highest weight representation V, that is to say there is $v_0 \in V$ (highest weight vector) $\gamma_{i,m}^{\pm} \in \mathbb{C}$ (highest weight) such that:*

$$V = \mathcal{U}_q(\hat{\mathfrak{g}}).v_0 \ , \ c^{\frac{1}{2}}.v_0 = v_0$$

$$\forall i \in I, m \in \mathbb{Z}, x_{i,m}^{+}.v_0 = 0 \ , \ \ , \phi_{i,m}^{\pm}.v_0 = \gamma_{i,m}^{\pm}v_0$$

Moreover we have an I-uplet $(P_i(u))_{i\in I}$ of (Drinfeld-)polynomials such that $P_i(0) = 1$ and:

$$\gamma_i^{\pm}(u) = \sum_{m\in\mathbb{N}} \gamma_{i,\pm m}^{\pm} u^{\pm} = q_i^{\deg(P_i)} \frac{P_i(uq_i^{-1})}{P_i(uq_i)} \in \mathbb{C}[[u^{\pm}]]$$

and $(P_i)_{i\in I}$ parametrizes simple modules in $Rep(\mathcal{U}_q(\hat{\mathfrak{g}}))$.

Theorem 2. *(**Frenkel, Reshetikhin** 98) The eigenvalues $\gamma_i(u)^{\pm} \in \mathbb{C}[[u]]$ of a representation $V \in Rep(\mathcal{U}_q(\hat{\mathfrak{g}}))$ have the form:*

$$\gamma_i^{\pm}(u) = q_i^{deg(Q_i)-deg(R_i)} \frac{Q_i(uq_i^{-1})R_i(uq_i)}{Q_i(uq_i)R_i(uq_i^{-1})}$$

where $Q_i(u), R_i(u) \in \mathbb{C}[u]$ and $Q_i(0) = R_i(0) = 1$.

Note that the polynomials Q_i, R_i are uniquely defined by γ. We note $Q_{\gamma,i}, R_{\gamma,i}$ the polynomials associated to γ.

Example: We suppose $\mathfrak{g} = sl_2$ and so $I = \{1\}$. We do explicit computations with the help of Jimbo's evaluation homomorphism $ev_a : \mathcal{U}_q(\hat{sl_2}) \to \mathcal{U}_q(sl_2)$.

For Drinfeld-polynomial $P(u) = 1 - ua$ we have the $\mathcal{U}_q(\hat{sl_2})$-module $M_a = \mathbb{C}v_0 \oplus \mathbb{C}v_1$:

v_0 is the highest weight vector: $\phi^{\pm}(u).v_0 = q\frac{1-uaq^{-1}}{1-uaq}v_0 = q\frac{P(uq^{-1})}{P(uq)}v_0$.

v_1 is a simultaneous eigenvector:

$$\phi^{\pm}(u).v_1 = q^{-1}\frac{1-uaq^3}{1-uaq}v_1 = q^{-1}\frac{Q(uq^{-1})R(uq)}{Q(uq)R(uq^{-1})}v_1$$

where

$$Q(u) = (1 - ua) \ , \ R(u) = (1 - uaq^2)(1 - ua)$$

5. Frenkel-Reshetikhin's q-characters

Let $\mathcal{Y}$ be the commutative ring $\mathcal{Y} = \mathbb{Z}[Y_{i,a}^{\pm}]_{i\in I, a\in\mathbb{C}^*}$.

Definition 2. *For $V \in Rep(\mathcal{U}_q(\hat{\mathfrak{g}}))$ a representation, the q-character $\chi_q(V)$ of V is:*

$$\chi_q(V) = \sum_{\gamma\in\mathbb{C}^{I\times\mathbb{Z}}\times\mathbb{C}^{I\times\mathbb{Z}}} dim(V_\gamma) \prod_{i\in I, a\in\mathbb{C}^*} Y_{i,a}^{\lambda_{\gamma,i,a}-\mu_{\gamma,i,a}} \in \mathcal{Y}$$

where

$$Q_{\gamma i}(z) = \prod_{a\in\mathbb{C}^*}(1-za)^{\lambda_{\gamma,i,a}} \ , \ R_{\gamma i}(z) = \prod_{a\in\mathbb{C}^*}(1-za)^{\mu_{\gamma,i,a}}$$

Example: $\chi_q(M_a) = Y_a + Y_{aq^2}^{-1}$

Theorem 3. *(**Frenkel, Reshetikhin** 98) The map*

$$\chi_q : Rep(\mathcal{U}_q(\hat{\mathfrak{g}})) \to \mathcal{Y}$$

is an injective ring homomorphism and the following diagram is commutative:

$$\begin{array}{ccc} Rep(\mathcal{U}_q(\hat{\mathfrak{g}})) & \xrightarrow{\chi_q} & \mathbb{Z}[Y_{i,a}^{\pm}]_{i\in I,a\in\mathbb{C}^*} \\ \downarrow res & & \downarrow \beta \\ Rep(\mathcal{U}_q(\mathfrak{g})) & \xrightarrow{\chi} & \mathbb{Z}[y_i^{\pm}]_{i\in I} \end{array}$$

where β is the ring homomorphism such that $\beta(Y_{i,a}) = y_i$ ($i \in I, a \in \mathbb{C}^*$).

Example:

$$\chi_q(M_{aq^2}\otimes M_a)=(Y_{aq^3}+Y_{aq}^{-1})(Y_{aq}+Y_{aq^{-1}}^{-1})$$

$$=Y_{aq^3}Y_{aq}+Y_{aq^3}Y_{aq^{-1}}^{-1}+Y_{aq}^{-1}Y_{aq^{-1}}^{-1}+1=\chi_q(N)+\chi_q(\text{trivial module})$$

where N is the simple module with Drinfeld polynomial $P(u)=(1-aq^3u)(1-aqu)$. In particular with $dim(N)=3$.

Note that $M_{aq^2}\otimes M_a\neq N\oplus$ triv. is not semi-simple. In fact we have an exact sequence:

$$0\to N\to M_{aq^2}\otimes M_a\to \text{trivial module}\to 0$$

Example:

$$\chi_q(M_a\otimes M_a)=Y_a^2+2Y_aY_{aq^2}^{-1}+Y_{aq^2}^{-1}$$

In particular $M_a\otimes M_a$ is the simple module with Drinfeld polynomial $P(u)=(1-au)^2$.

Definition 3. *For $i\in I, a\in\mathbb{C}^*$ we note $V_{i,a}$ the simple module with Drinfeld polynomials $P_j(u)=\delta_{i,j}(1-ua)$. Those simple modules are called fundamental representations.*

We note $X_{i,a}=\chi_q(V_{i,a})\in\mathcal{Y}$.

Corollary 1. *(***Frenkel, Reshetikhin** *98) The ring $Rep(\mathcal{U}_q(\hat{\mathfrak{g}}))$ is commutative and isomorphic to $\mathbb{Z}[X_{i,a}]_{i\in I,a\in\mathbb{C}^*}$.*

We say that $m\in\mathcal{Y}$ is a dominant monomial if it is of the form $m=\prod_{i\in I,a\in\mathbb{C}^*}Y_{i,a}^{u_{i,a}(m)}$ with $u_{i,a}(m)\geq 0$. For m a dominant monomial we note $M_m\in\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ the module $\bigotimes_{i\in I,a\in\mathbb{C}^*}V_{i,a}^{\otimes u_{i,a}(m)}$. It is called a standard module and his q-character is $\chi_q(M_m)=\prod_{i\in I,a\in\mathbb{C}^*}X_{i,a}^{u_{i,a}(m)}$.

6. Nakajima's q,t-characters

In the ADE-case Nakajima defined a $\mathbb{Z}[t^{\pm}]$-linear map :

$$\chi_{q,t}:\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))\otimes_{\mathbb{Z}}\mathbb{Z}[t^{\pm}]\to\mathcal{Y}_t=\mathbb{Z}[Y_{i,a}^{\pm},t^{\pm}]_{i\in I,a\in\mathbb{C}^*}$$

such that $(\chi_{q,t})_{t=1}=\chi_q$ but $\chi_{q,t}\neq\chi_q$. In particular it leads to the construction of :

-a quantization of $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$

-an involution of $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))\otimes\mathbb{Z}[t^{\pm}]$

-canonical invariant basis of $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))\otimes\mathbb{Z}[t^{\pm}]$

-analogs of Kazdhan-Lusztig polynomials

Nakajima gave a combinatorial axiomatic definition of q,t-characters, but the existence is non-trivial and is proved with the geometric theory of quiver varieties which holds only in the ADE-case.

In the preprint math.QA/0212257 we propose a construction of $\chi_{q,t}$ without quiver varieties; in particular we extend the applications to the non-simply-laced case. In the following we give a sketch of the construction.

7. Algebraic construction of q,t-characters in the general case

We set $\mathrm{Rep}=\mathbb{Z}[X_{i,l}]_{i\in I,l\in\mathbb{Z}}$ and $\mathrm{Rep}_t=\mathrm{Rep}\otimes\mathbb{Z}[t^{\pm}]$

7.1. **Quantization of $\mathcal{Y}$.** Let Z_q be the $\mathbb{C}$-algebra defined by generators $a_i[m]$ ($i \in I, m \in \mathbb{Z} - \{0\}$), central elements c_r ($r > 0$) and relations ($i, j \in I, m, r \in \mathbb{Z} - \{0\}$):

$$[a_i[m], a_j[r]] = \delta_{m,-r}(q^m - q^{-m})B_{i,j}(q^m)c_{|m|}$$

For $j \in I, m \in \mathbb{Z}$ we set:

$$y_j[m] = \sum_{i \in I} \tilde{C}_{i,j}(q^m)a_i[m] \in Z_q$$

Consider the $\mathbb{C}$-algebra $Z_{q,h} = Z_q[[h]]$ and:

$$\tilde{Y}_{i,l} = \exp(\sum_{m>0} h^m y_i[m]q^{lm})\exp(\sum_{m>0} h^m y_i[-m]q^{-lm}) \in Z_{q,h}$$

For $R \in \mathbb{Z}((q^{-1}))$, introduce:

$$t_R = \exp(\sum_{m>0} h^{2m} R(q^m)c_m) \in Z_{q,h}$$

We note $\mathcal{Y}_u \subset Z_{q,h}$ the subalgebra generated by the $t_R, \tilde{Y}_{i,l}^{\pm}$ ($i \in I, l \in \mathbb{Z}, R \in \mathbb{Z}((q^{-1}))$).

Definition 4. *$\tilde{\mathcal{Y}}_t$ (resp. $Z_{q,t}$) is the quotient of $\mathcal{Y}_u$ (resp. $Z_{q,h}$) by the relations $t_R = t_{R_0}$.*

We note $t = t_0$. In particular in $\tilde{\mathcal{Y}}_t$ we have $t_R = t^{R_0}$ and $\tilde{\mathcal{Y}}_t$ is a $\mathbb{Z}[t^{\pm}]$-algebra. We have defined a quantization of $\mathcal{Y}$:

Proposition 1. *We have an isomorphism of $\mathbb{Z}[t^{\pm}]$-vector space $\tilde{\mathcal{Y}}_t \simeq \mathcal{Y} \otimes \mathbb{Z}[t^{\pm}]$ and $\tilde{\mathcal{Y}}_t/(t-1) \simeq \mathcal{Y}$.*

Example: In the sl_2-case, $\tilde{\mathcal{Y}}_t$ is defined by generators $t^{\pm}, \tilde{Y}_l^{\pm}$ ($l \in \mathbb{Z}$) and relations:

$$\tilde{Y}_l\tilde{Y}_k = t^s\tilde{Y}_k\tilde{Y}_l$$

where:

$s = 0$ if $l - k = 1 + 2r$, $r \in \mathbb{Z}$

$s = 2(-1)^r$ if $l - k = 2r$, $r > 0$

$s = 2(-1)^{r+1}$ if $l - k = 2r$, $r < 0$

$s = 0$ if $l = k$

7.2. **Deformed screening operators.** Frenkel-Reshetikhin-Mukhin have shown $\mathrm{Im}(\chi_q) = \bigcap_{i \in I} \mathrm{Ker}(S_i)$ where the S_i are screening operators. We will use a t-version of this property. Introduce the screening currents:

$$\tilde{S}_{i,l} = \exp(\sum_{m>0} h^m \frac{a_i[m]}{q_i^m - q_i^{-m}} q^{lm})\exp(\sum_{m>0} h^m \frac{a_i[-m]}{q_i^{-m} - q_i^m} q^{-lm}) \in Z_{q,t}$$

Definition 5. *The i^{th} t-screening operator is the map $\tilde{S}_{i,t} : \tilde{\mathcal{Y}}_t \to Z_{q,t}$ defined by:*

$$\tilde{S}_{i,t}(\lambda) = \frac{1}{t^2 - 1}\sum_{l \in \mathbb{Z}} [\tilde{S}_{i,l}, \lambda]$$

We note $\tilde{\mathfrak{K}}_t = \bigcap_{i \in I} \mathrm{Ker}(\tilde{S}_{i,t})$. It is a subalgebra of $\tilde{\mathcal{Y}}_t$.

Theorem 4. *For m a dominant monomial there a unique $\tilde{F}_t(m)$ in a completion of $\tilde{\mathfrak{K}}_t$ such that m is the unique dominant monomial of $\tilde{F}_t(m)$. Moreover it is given by a t-analog of Frenkel-Mukhin algorithm.*

The proof of the existence if proved by showing that a t-analog of the algorithm is well-defined. See examples in the annexe.

7.3. **Definition of q,t-characters.**

Definition 6. *The morphism of q,t-characters is the map $\chi_{q,t} : Rep_t \to \tilde{\mathcal{Y}}_t^{\infty}$ which is $\mathbb{Z}[t^{\pm}]$-linear and ($u_{i,l} \geq 0$):*

$$\chi_{q,t}(\prod_{i\in I, l\in\mathbb{Z}} X_{i,l}^{u_{i,l}}) = \prod_{l\in\mathbb{Z}}^{\rightarrow} \tilde{F}_t(Y_{i,l})^{u_{i,l}}$$

Example:

Theorem 5. *We have $(\chi_{q,t})_{t=1} = \chi_q$. In particular the map $\chi_{q,t}$ is injective. In the ADE-case it is the morphism of Nakajima.*

7.4. **Consequences.** As $\chi_{q,t}$ is injective, the quantization of $\mathcal{Y}$ leads to a quantization of Rep and the involution $c_r \mapsto c_r^{-1}$ of $Z_{q,h}$ leads to an involution of Rep_t.

Theorem 6. *For m a dominant monomial there is a unique $\tilde{L}_t(m) \in Im(\chi_{q,t})$ such that:*

$$\overline{\tilde{L}_t(m)} = \tilde{L}_t(m)$$

$$\chi_{q,t}(M_m) = \tilde{L}_t(m) + \sum_{m'<m, m' \, dominant} P_{m',m}(t)\tilde{L}_t(m')$$

where $P_{m',m}(t) \in t^{-1}\mathbb{Z}[t^{-1}]$.

Conjecture 1. *For m a dominant monomial, the image $(\tilde{L}_t(m))_{t=1}$ of $\tilde{L}_t(m)$ in $\mathcal{Y}$ is $\chi_q(V_m)$ where V_m is the simple modume of Drinfeld-polynomials associated to m.*

In the ADE-case the conjecture 1 is a consequence of Nakajima's geometric theory.

8. Annexe

8.1. We suppose $\mathfrak{g} = G_2$. The Cartan matrix is $C = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$ and $r_1 = 1$, $r_2 = 3$. The tree of the second fundamental representation is:

$$
\begin{tikzcd}[column sep=small]
 & & Y_{2,0} \arrow[d, "{2,3}"] & & \\
 & & Y_{2,6}^{-1}Y_{1,5}Y_{1,3}Y_{1,1} \arrow[d, "{1,6}"] & & \\
 & & Y_{1,7}^{-1}Y_{1,3}Y_{1,1} \arrow[d, "{1,4}"] & & \\
 & & Y_{2,4}Y_{1,7}^{-1}Y_{1,5}^{-1}Y_{1,1} \arrow[dll, "{1,2}"'] \arrow[drr, "{2,7}"] & & \\
Y_{1,7}^{-1}Y_{1,5}^{-1}Y_{1,3}^{-1}Y_{2,4}Y_{2,2} \arrow[d, "{2,5}"] \arrow[drr, "{2,7}"] & & & & Y_{2,10}^{-1}Y_{1,9}Y_{1,1} \arrow[dll, "{1,2}"'] \arrow[d, "{1,10}"] \\
Y_{2,4}Y_{2,8}^{-1} \arrow[d, "{2,7}"] & & Y_{2,2}Y_{2,10}^{-1}Y_{1,9}Y_{1,3}^{-1} \arrow[dll, "{2,5}"'] \arrow[drr, "{1,10}"] & & Y_{1,11}^{-1}Y_{1,1} \arrow[d, "{1,2}"] \\
Y_{2,8}^{-1}Y_{2,10}^{-1}Y_{1,9}Y_{1,7}Y_{1,5} \arrow[drr, "{1,10}"] & & & & Y_{2,2}Y_{1,11}^{-1}Y_{1,3}^{-1} \arrow[dll, "{2,5}"'] \\
 & & Y_{2,8}^{-1}Y_{1,11}^{-1}Y_{1,7}Y_{1,5} \arrow[d, "{1,8}"] & & \\
 & & Y_{1,11}^{-1}Y_{1,9}^{-1}Y_{1,5} \arrow[d, "{1,6}"] & & \\
 & & Y_{1,11}^{-1}Y_{1,9}^{-1}Y_{1,7}^{-1}Y_{2,6} \arrow[d, "{2,9}"] & & \\
 & & Y_{2,12}^{-1} & &
\end{tikzcd}
$$

8.2. We suppose $\mathfrak{g} = A_2$. The Cartan matrix is $C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ and $r_1 = 1$, $r_2 = 1$. The tree of $\tilde{F}_t(Y_{1,0}Y_{1,2}^2)$ is:

$$\begin{tikzcd}
 & Y_{1,0}Y_{1,2}^2 \arrow[d, "1{,}3"] & \\
 & \mathbf{(1+t^2)}Y_{2,6}^{-1}Y_{1,5}Y_{1,3}Y_{1,1} \arrow[dl, "1{,}1"'] \arrow[d, "1{,}3"] \arrow[dr, "2{,}4"] & \\
\mathbf{t^2}Y_{1,4}^{-1}Y_{2,3}Y_{2,1} \arrow[d, "2{,}4"] \arrow[dr, "1{,}3"] & Y_{1,0}Y_{1,4}^{-2}Y_{2,3}^2 \arrow[d, "1{,}1"] \arrow[dr, "2{,}4"] & \mathbf{(1+t^2)}Y_{1,0}Y_{1,2}Y_{2,5}^{-1} \arrow[d, "1{,}3"] \\
\mathbf{t^2}Y_{2,5}^{-1}Y_{2,1} \arrow[d, "2{,}2"] & Y_{1,2}^{-1}Y_{1,4}^{-2}Y_{2,3}^2Y_{2,1} \arrow[d, "2{,}4"] & \mathbf{(1+t^2)}Y_{1,0}Y_{1,4}^{-1}Y_{2,5}^{-1}Y_{2,3} \arrow[dl, "1{,}1"'] \arrow[d, "2{,}4"] \\
\mathbf{t^2}Y_{2,5}^{-1}Y_{2,3}^{-1}Y_{1,2} \arrow[dr, "1{,}3"] & \mathbf{(1+t^2)}Y_{1,2}^{-1}Y_{2,3}Y_{2,5}^{-1}Y_{2,1}Y_{1,4}^{-1} \arrow[d, "2{,}2"] \arrow[dr, "2{,}4"] & Y_{1,1}Y_{2,5}^{-2} \arrow[d, "1{,}1"] \\
 & \mathbf{t^2}Y_{2,5}^{-1}Y_{1,4}^{-1} \arrow[d, "2{,}4"] & Y_{1,2}^{-1}Y_{2,5}^{-2}Y_{2,1} \arrow[dl, "2{,}2"] \\
 & Y_{2,5}^{-2}Y_{2,3}^{-1} &
\end{tikzcd}$$

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