Simple tensor products of representations of quantum affine algebras

David Hernandez

Let $q \in \mathbb{C}^*$ which is not a root of unity and let $\mathcal{U}_q(\mathfrak{g})$ be a quantum affine algebra (not necessarily simply-laced or untwisted). Let $\mathcal{F}$ be the tensor category of finite-dimensional representations of $\mathcal{U}_q(\mathfrak{g})$.
In my talk at the Oberwolfach Workshop, I presented the main result of [7], expected in various papers of the vast literature about $\mathcal{F}$.

Theorem 1 [7] *Let $S_1, \cdots, S_N$ be objects of $\mathcal{F}$. The tensor product $S_1 \otimes \cdots \otimes S_N$ is simple if and only if $S_i \otimes S_j$ is simple for any $i < j$.*

The "only if" part of the statement is known : it is an immediate consequence of the commutativity of the Grothendieck ring $\mathrm{Rep}(\mathcal{U}_q(\mathfrak{g}))$ of $\mathcal{F}$ proved in [5] (see [6] for the twisted types). The "if" part of the statement is proved in [7].

The following is an extended version of the introduction of [7].

If the reader is not familiar with the representation theory of quantum affine algebras, he may wonder why such a result is non trivial. Indeed, in tensor categories associated to "classical" representation theory, there are "few" non trivial tensor products of representations which are simple. For instance, let V, V' be non-zero simple finite-dimensional modules of a simple algebraic group in characteristic 0. Then, it is well-known that $V \otimes V'$ is simple if and only if V or V' is of dimension 1. But in positive characteristic there are examples of non trivial simple tensor products given by the Steinberg theorem. And in $\mathcal{F}$ there are "many" simple tensor products of non trivial simple representations. For instance, it is proved in [3] that for $\mathfrak{g} = \hat{sl}_2$ an arbitrary simple object V of $\mathcal{F}$ is *real*, i.e. $V \otimes V$ is simple. Although it is known [10] that there are non real simple objects in $\mathcal{F}$ in general, many other examples of non trivial simple tensor products can be found in [8].

The statement of Theorem 1 has been conjectured and proved by several authors in various special cases. The result is proved for $\mathfrak{g} = \hat{sl}_2$ in [3], for a special class of modules of the Yangian of gl_n attached to skew Young diagrams in [12], for tensor products of fundamental representations in [1, 4], for a special class of tensor products satisfying an irreducibility criterion in [2], for a certain "small" subtensor category $\mathcal{C}_1$ of $\mathcal{F}$ when $\mathfrak{g}$ is simply-laced in [8].
So, even in the case $\mathfrak{g} = \hat{sl}_3$, Theorem 1 had not been established. Our complete proof is valid for arbitrary simple objects of $\mathcal{F}$ and for arbitrary $\mathfrak{g}$.

Let us give a few first comments. Theorem 1 allows to produce simple tensor products $V \otimes V'$ where $V = S_1 \otimes \cdots \otimes S_k$ and $V' = S_{k+1} \otimes \cdots \otimes S_N$. Besides it implies that $S_1 \otimes \cdots \otimes S_N$ is real if we assume that the S_i are real in addition to the assumptions of Theorem 1.

The main ingredients of the proof are the following : the parametrization of simple objects of $\mathcal{F}$ [3], a cyclicity property of tensor product of fundamental representations [2, 9, 14], the theory of Frenkel-Reshetikhin q-characters [5, 4], a "filtration" of $\mathcal{F}$ by tensor subcategories [8], the notion of truncated q-characters

[8], a certain property of tensor products of l-weight vectors (analogs of weight vectors for q-characters) that we establish [7], a compatibility property of intertwining operators with a decomposition of q-characters that we establish [7].

Our result is stated in terms of the tensor structure of $\mathcal{F}$. Thus, it is purely representation theoretical. But we have three additional motivations, related respectively to physics, topology, combinatorics, and also to other structures of $\mathcal{F}$.

First, although the category $\mathcal{F}$ is not braided (in general $V \otimes V'$ is not isomorphic to $V' \otimes V$), $\mathcal{U}_q(\mathfrak{g})$ has a *universal R-matrix* in a completion of the tensor product $\mathcal{U}_q(\mathfrak{g}) \otimes \mathcal{U}_q(\mathfrak{g})$. In general the universal R-matrix can not be specialized to finite-dimensional representations, but it gives rise to $V(z) \otimes V' \to V' \otimes V(z)$ which depend meromorphically on a formal parameter z (here the representation $V(z)$ is obtained by homothety of spectral parameter). From the physical point of view, it is an important question to localize the zeros and poles of these operators. The reducibility of tensor products of objects in $\mathcal{F}$ is known to have strong relations with this question. This is the first motivation to study irreducibility of tensor products in terms of irreducibility of tensor products of pairs of constituents [1].

Secondly, if $V \otimes V'$ is simple the universal R-matrix can be specialized and we get a well-defined intertwining operator $V \otimes V' \to V' \otimes V$. In general the action of the R-matrix is not trivial. As the R-matrix satisfies the Yang-Baxter equation, when V is real we can define an action of the braid group $\mathcal{B}_N$ on $V^{\otimes N}$ (as for representations of quantum groups of finite type). It is known [13] that such situations are important to construct *topological invariants*.

Finally, in a tensor category, there are natural important questions such as the parametrization of simple objects or the decomposition of tensor products of simple objects in the Grothendieck ring. But another problem of the same importance is the factorization of simple objects V in *prime objects*, i.e. the decomposition $V = V_1 \otimes \cdots \otimes V_N$ where the V_i can not be written as a tensor product of non trivial simple objects. This problem for $\mathcal{F}$ is one of the main motivation in [8]. When we have established that the tensor products of some pairs of prime representations are simple, Theorem 1 gives the factorization of arbitrary tensor products of these representations.

This factorization problem is related to the program of realization of *cluster algebras* in $\mathrm{Rep}(\mathcal{U}_q(\mathfrak{g}))$ initiated in [8] when $\mathfrak{g}$ is simply-laced. A cluster algebra has a distinguished set of generators called *cluster variables*. A notion of *compatibility* of cluster variables comes with the definition of cluster algebras (cluster variables are compatible if they occur in the same seed). A product of compatible cluster variables is called a *cluster monomial*. Let us recall the notion of *monoidal categorification* due to Leclerc [8]. A tensor category $\mathcal{C}$ is said to be a monoidal categorification of a cluster algebra $\mathcal{A}$, if there is a ring isomorphism $\phi : K_0(\mathcal{C}) \to \mathcal{A}$, where $K_0(\mathcal{C})$ is the Grothendieck ring of $\mathcal{C}$, such that ϕ induces bijections

$$\{\text{Classes of real simple objects of } \mathcal{C}\} \leftrightarrow \{\text{Cluster monomials of } \mathcal{A}\},$$
$$\{\text{Classes of prime real simple objects of } \mathcal{C}\} \leftrightarrow \{\text{Cluster variables of } \mathcal{A}\}.$$

If one can establish a monoidal categorification, we get results about $\mathcal{A}$ (positivity, linear independence of cluster monomials) and $\mathcal{C}$ (Clebsh-Gordan coefficients, factorization in prime modules). A cluster algebra $\mathcal{A}$ of finite type (ADE) has a monoidal categorification $\mathcal{C}_1$ which is a tensor subcategory of $\mathcal{F}$ for $\mathcal{U}_q(\hat{\mathfrak{g}})$ where $\mathfrak{g}$ has the type of $\mathcal{A}$. This was proved in [8] for types A, D_4 and in [11] for the other types. In the proof of [8], the statement of Theorem 1 for $\mathcal{C}_1$ is a crucial step (the proof of Theorem 1 in this case is drastically simplified; several new technical ingredients are used in the general case). It reduces the proof of the irreducibility of tensor products of representations corresponding to compatible cluster variables to the proof of the irreducibility of the tensor products of pairs of simple representations corresponding to compatible cluster variables. We plan to use Theorem 1 in the future to establish monoidal categorifications associated to non necessarily simply-laced quantum affine algebras, involving categories different than the small subcategories $\mathcal{C}_1$ considered in [8, 11].

References

[1] T. Akasaka and M. Kashiwara, *Finite-dimensional representations of quantum affine algebras*, Publ. Res. Inst. Math. Sci. **33** (1997), no. 5, 839-867.

[2] V. Chari, *Braid group actions and tensor products*, Int. Math. Res. Not. **2003**, no. 7, 357–382.

[3] V. Chari and A. Pressley, *Quantum affine algebras*, Comm. Math. Phys. **142** (1991), no. 2, 261–283.

[4] E. Frenkel and E. Mukhin, *Combinatorics of q-Characters of Finite-Dimensional Representations of Quantum Affine Algebras*, Comm. Math. Phys., vol **216** (2001), no. 1, 23–57.

[5] E. Frenkel and N. Reshetikhin, *The q-Characters of Representations of Quantum Affine Algebras and Deformations of W-Algebras*, Recent Developments in Quantum Affine Algebras and related topics, Cont. Math., vol. **248** (1999), 163–205.

[6] D. Hernandez, *Kirillov-Reshetikhin conjecture : the general case*, Int. Math. Res. Not. **2010**, no. 1, 149–193.

[7] D. Hernandez, *Simple tensor products*, Preprint arXiv:0907.3002.

[8] D. Hernandez and B. Leclerc, *Cluster algebras and quantum affine algebras*, To appear in Duke Math. J. (Preprint arXiv:0903.1452).

[9] M. Kashiwara, *On level-zero representations of quantized affine algebras*, Duke Math. J. **112** (2002), no. 1, 117–175.

[10] B. Leclerc, *Imaginary vectors in the dual canonical basis of $U_q(n)$*, Transform. Groups **8** (2003), no. 1, 95–104.

[11] H. Nakajima, *Quiver varieties and cluster algebras*, Preprint arXiv:0905.0002.

[12] M. Nazarov and V. Tarasov, *On irreducibility of tensor products of Yangian modules associated with skew Young diagrams*, Duke Math. J. **112** (2002), 343–378.

[13] N. Reshetikhin and V. Turaev, *Invariants of 3-manifolds via link polynomials and quantum groups*, Invent. Math. **103** (1991), no. 3, 547–597.

[14] M. Varagnolo et E. Vasserot, *Standard modules of quantum affine algebras*, Duke Math. J. **111** (2002), no. 3, 509–533.