DOUBLE AFFINE LIE ALGEBRAS AND FINITE GROUPS

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Abstract. We begin to study the Lie theoretical analogs of symplectic reflection algebras for Γ a finite cyclic group, which we call "cyclic double affine Lie algebra". We focus on type A : in the finite (resp. affine, double affine) case, we prove that these structures are finite (resp. affine, toroidal) type Lie algebras, but the gradings differ. The case which is essentially new is $\mathfrak{sl}_n(\mathbb{C}[u,v] \rtimes \Gamma)$. We describe its universal central extensions and start the study of its representation theory, in particular of its highest weight integrable modules and Weyl modules. We also consider the first Weyl algebra A_1 instead of the polynomial ring $\mathbb{C}[u,v]$, and, more generally, a rank one rational Cherednik algebra. We study quasi-finite highest weight representations of these Lie algebras.

Contents

1. Introduction

Double affine Hecke algebras have been well studied for more than fifteen years now, although they are still very mysterious, and symplectic reflection algebras appeared over seven years ago [EG] as generalizations of double affine Hecke algebras of rational type. Even more mysterious are the double affine Lie algebras and their quantized version introduced in [GKV], studied for instance in [H1, H2, Nag, Nak2, S, VV1, VV2] and the references in the survey [H3].

In this paper, we study candidates for Lie theoretical analogs of symplectic reflection algebras, which we call "cyclic double affine Lie algebras" : we look at a family of Lie algebras which have a lot of similarities with affine and double affine Lie algebras, but whose structure depends on a finite cyclic group Γ.

More precisely, we will be interested in the Lie algebras $\mathfrak{sl}_n(\mathbb{C}[u] \rtimes \Gamma)$, $\mathfrak{sl}_n(\mathbb{C}[u^{\pm 1}] \rtimes \Gamma)$, $\mathfrak{sl}_n(\mathbb{C}[u,v] \rtimes \Gamma)$, $\mathfrak{sl}_n(\mathbb{C}[u,v]^\Gamma)$, $\mathfrak{sl}_n(\mathbb{C}[u^{\pm 1},v] \rtimes \Gamma)$, $\mathfrak{sl}_n(\mathbb{C}[u^{\pm 1},v^{\pm 1}] \rtimes \Gamma)$, $\mathfrak{sl}_n(A_1 \rtimes \Gamma)$ (where A_1 is the first Weyl algebra), $\mathfrak{sl}_n(\mathsf{H}_{t,\mathbf{c}}(\Gamma))$ (where $\mathsf{H}_{t,\mathbf{c}}(\Gamma)$ is a rank one rational Cherednik algebra) and their universal central extensions. This is motivated by the recent work [Gu3] in which

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deformations of the enveloping algebras of some Lie algebras closely related to these were constructed and connected to symplectic reflection algebras for wreath products via a functor of Schur-Weyl type. When Γ is trivial, such deformations in the case of $\mathbb{C}[u^{\pm 1}]$ are the affine quantum groups, whereas the case $\mathbb{C}[u]$ corresponds to Yangians. In the double affine setup, the quantum algebras attached to $\mathbb{C}[u^{\pm 1}, v^{\pm 1}], \mathbb{C}[u^{\pm 1}, v]$ and $\mathbb{C}[u, v]$ for $\mathfrak{sl}_n$ are the quantum toroidal algebras, the affine Yangians and the deformed double current algebras [Gu1, Gu2].

In this article, we want to study more the structure and representation theory for the Lie algebras above, hoping that, in a future work, we will be able to extend some of our results to the deformed setup. We consider the central extensions for a number of reasons: in the affine case, the full extent of the representation theory comes into life when the centre acts not necessarily trivially; certain presentations of those Lie algebras are actually simpler to state for central extensions since they involve fewer relations; some of the results can be extended without much difficulty to those central extensions, etc. As vector spaces, the centers of the universal extensions are given by certain first cyclic homology groups.

At first sight, one may be tempted to think that introducing the group Γ leads to Lie algebras which are different from those that have interested Lie theorists since the advent of Kac-Moody Lie algebras (it was our first motivation), but this is not entirely the case. Indeed, in the one variable case, when we consider not only ordinary polynomials but Laurent polynomials, we prove that we get back affine Lie algebras (proposition 3.8); this is in accordance with conjectures of V. Kac [K1] and the classification obtained by V. Kac and O. Mathieu [K1, K2, Ma1, Ma2]. In the case of Laurent polynomials in two variables, we recover toroidal Lie algebras (proposition 4.1). (The mixed case $\mathbb{C}[u^{\pm 1}, v]$ also does not yield new Lie algebras.) However, when we consider only polynomials in non-negative powers of the variables, we obtain distinctly new Lie algebras (see proposition 5.3).

Another motivation comes from geometry. The loop algebra $\mathfrak{sl}_n(\mathbb{C}[u^{\pm 1}])$ can be viewed as the space of polynomial maps $\mathbb{C}^\times \longrightarrow \mathfrak{sl}_n$. One can also consider the affine line instead of the torus $\mathbb{C}^\times$. More generally, one can consider the space of regular maps $X \longrightarrow \mathfrak{sl}_n$ where X is an arbitrary affine algebraic variety [FeLo]. When X is two-dimensional, the most natural candidate is the torus $\mathbb{C}^\times \times \mathbb{C}^\times$, although a simpler case is the plane $\mathbb{C}^2$. The variety X does not necessarily have to be smooth and one interesting singular two-dimensional case is provided by the Kleinian singularities $\mathbb{C}^2/G$ where G is a finite subgroup of $SL_2(\mathbb{C})$. We are thus led to the problem of studying the Lie algebras $\mathfrak{sl}_n(\mathbb{C}[u, v]^G)$ where $\mathbb{C}[u, v]^G$ is the ring of invariant elements for the action of G. However, following one of the main ideas explained in the introduction of [EG], it may be interesting to replace $\mathbb{C}[u, v]^G$ by the smash product $\mathbb{C}[u, v] \rtimes G$. Moreover, we can expect the full representation theory to come to life when we consider the universal central extensions of $\mathfrak{sl}_n(\mathbb{C}[u, v]^G)$ and of $\mathfrak{sl}_n(\mathbb{C}[u, v] \rtimes G)$. In [FeLo], the authors showed that the dimension of the local Weyl modules at a point p in the case of a smooth affine variety X does not depend on p. One goal is to understand Weyl modules supported at a Kleinian singularity.

This paper is organized as follows. We will denote by Γ the group $\mathbb{Z}/d\mathbb{Z}$, whereas G will be a more general finite group. After general reminders on matrix Lie algebra over rings (in particular with the example of $\mathfrak{sl}_n(\mathbb{C}[G])$ in Section 2, we start with the affine case in Section 3. We study the structure of $\mathfrak{sl}_n(\mathbb{C}[u] \rtimes \Gamma)$ and $\mathfrak{sl}_n(\mathbb{C}[u^{\pm 1}] \rtimes \Gamma)$, obtain different types of decomposition and give presentations in terms of generators and relations. We prove that $\mathfrak{sl}_n(\mathbb{C}[u^{\pm 1}] \rtimes \Gamma)$ is simply the usual loop algebra $\mathfrak{sl}_{nd}(\mathbb{C}[t^{\pm 1}])$, but with a non-standard grading. Guided by the affine setup, we prove analogous results for the double affine cases in Section 4, the representations being studied in Section 5. We consider certain highest weight modules for $\mathfrak{sl}_n(\mathbb{C}[u, v] \rtimes \Gamma), \mathfrak{sl}_n(\mathbb{C}[u^{\pm 1}, v] \rtimes \Gamma), \mathfrak{sl}_n(\mathbb{C}[u^{\pm 1}, v^{\pm 1}] \rtimes \Gamma)$ and state a criterion for the integrability of their unique irreducible quotients. We also study some Weyl modules for $\mathfrak{sl}_n(\mathbb{C}[u,v] \rtimes G)$ and $\mathfrak{sl}_n(\mathbb{C}[u,v]^\Gamma)$. In the first case, we show that, contrary to what might be expected at first sight, Weyl modules are rather trivial; in the second case, we can apply results of Feigin and Loktev to derive formulas for the dimension some of the local Weyl modules and we establish a lower bound on their dimensions. In Section 6, assuming usually that $t \neq 0$, we study parabolic subalgebras of $\mathfrak{gl}_n(\mathsf{H}_{t,\mathbf{c}}(\Gamma))$ and construct an embedding of this Lie algebra into a Lie algebra of infinite matrices. This is useful in Section 7 to construct quasi-finite highest weight modules. The main result of this section is a criterion for the quasi-finiteness of the irreducible quotients of Verma modules. Further possible directions of research are discussed in Section 8.

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2. Matrix Lie algebras over rings

2.1. **General results.** In this section, we present general definitions and results which will be useful later. All algebras and tensor products are over $\mathbb{C}$, unless specified otherwise.

Definition 2.1. *Let A be an arbitrary associative algebra. The Lie algebra $\mathfrak{sl}_n(A)$ is defined as the derived Lie algebra $[\mathfrak{gl}_n(A), \mathfrak{gl}_n(A)]$ where $\mathfrak{gl}_n(A) = \mathfrak{gl}_n \otimes A$.*

In other words, the Lie subalgebra $\mathfrak{sl}_n(A) \subset \mathfrak{gl}_n(A)$ is the sum of $\mathfrak{sl}_n(\mathbb{C}) \otimes A$ and of the space of all scalar matrices with entries in $[A,A]$. Thus the cyclic homology group $HC_0(A) = \frac{A}{[A,A]}$ accounts for the discrepancy between $\mathfrak{sl}_n(A)$ and $\mathfrak{gl}_n(A)$.

Since $\mathfrak{sl}_n(A)$ is a perfect Lie algebra (that is, $[\mathfrak{sl}_n(A), \mathfrak{sl}_n(A)] = \mathfrak{sl}_n(A)$), it possesses a universal central extension $\widehat{\mathfrak{sl}}_n(A)$ unique up to isomorphism. The following theorem gives a simple presentation of $\widehat{\mathfrak{sl}}_n(A)$ in terms of generators and relations.

Theorem 2.2. [KL] *Assume that $n \geq 3$. $\widehat{\mathfrak{sl}}_n(A)$ is isomorphic to the Lie algebra generated by elements $F_{ij}(a)$ ($1 \leq i,j \leq n$, $a \in A$) which satisfy the following relations :*

$$[F_{ij}(a_1), F_{jk}(a_2)] = F_{ik}(a_1 a_2), \quad \textit{for } i \neq j \neq k \neq i,$$

$$[F_{ij}(a_1), F_{kl}(a_2)] = 0, \quad \textit{for } i \neq j \neq k \neq l \neq i.$$

Here $i \neq j \neq k \neq i$ means ($i \neq j$ and $j \neq k$ and $k \neq i$). We will use this convention in this paper.

When $n = 2$, one has to add generators $H_{12}(a_1, a_2)$ given bt $H_{12}(a_1, a_2) = [F_{12}(a_1), F_{21}(a_2)]$ for $a_1, a_2 \in A$, and the relations

$$[H_{12}(a_1,a_2), F_{12}(a_3)] = F_{12}(a_1a_2a_3 + a_3a_2a_1), \ \ [H_{12}(a_1,a_2), F_{21}(a_3)] = -F_{21}(a_3a_1a_2 + a_2a_1a_3).$$

It is also proved in [KL] that the center of $\widehat{\mathfrak{sl}}_n(A)$ is isomorphic, as a vector space, to the first cyclic homology group $HC_1(A)$. For G a finite group and $A = \mathbb{C}[G]$, $HC_1(A) = 0$, but in the double affine case below, the center will be infinite dimensional.

The following formulas taken from [VV2] can help understand better the bracket on $\widehat{\mathfrak{sl}}_n(A)$. The problem of computing explicitly the bracket with respect to the decomposition $\widehat{\mathfrak{sl}}_n(A) \cong \mathfrak{sl}_n(A) \oplus HC_1(A)$ is, in general, a difficult one, but it is possible to obtain some nice formulas by using a different splitting of $\widehat{\mathfrak{sl}}_n(A)$. Let $\langle A, A\rangle$ be the quotient of $A \otimes A$ by the two-sided ideal generated by $a_1 \otimes a_2 - a_2 \otimes a_1$ and $a_1 a_2 \otimes a_3 - a_1 \otimes a_2 a_3 - a_2 \otimes a_3 a_1$. The first cyclic homology group $HC_1(A)$ is, by definition, the kernel of the map $\langle A, A\rangle \twoheadrightarrow [A, A]$, $a_1 \otimes a_2 \mapsto [a_1, a_2]$.

For $m_1, m_2 \in \mathfrak{sl}_n$, $a_1, a_2 \in A$, and $(\cdot, \cdot)$ the Killing form on $\mathfrak{sl}_n$, set :

$$[m_1, m_2]_+ = m_1 m_2 + m_2 m_1 - \frac{2}{n}(m_1, m_2)I \ , \ [a_1, a_2]_+ = a_1 a_2 + a_2 a_1.$$

Proposition 2.3. [VV2] *The Lie algebra $\widehat{\mathfrak{sl}}_n(A)$ is isomorphic to the vector space $\mathfrak{sl}_n \otimes A \oplus \langle A, A\rangle$ endowed with the bracket:*

$$[m_1 \otimes a_1, m_2 \otimes a_2] = \frac{1}{n}(m_1, m_2)\langle a_1, a_2\rangle + \frac{1}{2}[m_1, m_2] \otimes [a_1, a_2]_+ + \frac{1}{2}[m_1, m_2]_+ \otimes [a_1, a_2],$$

$$[\langle a_1, a_2\rangle, \langle b_1, b_2\rangle] = \langle [a_1, a_2], [b_1, b_2]\rangle,$$

$$[\langle a_1, a_2\rangle, m_1 \otimes a_3] = m_1 \otimes [[a_1, a_2], a_3].$$

2.2. **Example : special linear Lie algebras over group rings.** Let G be a finite group. One interesting case for us is the group algebra $A = \mathbb{C}[G]$, in which case $HC_0(A) \cong \mathbb{C}^{\oplus cl(G)}$ where $cl(G)$ is the number of conjugacy classes of G.

Lemma 2.4. *The Lie algebra $\mathfrak{sl}_n(\mathbb{C}[G])$ is semi-simple of Dynkin type A.*

Proof: Recall that $cl(G)$ is also the number of irreducible representations of G. Enumerate the irreducible representations of G by $\rho_1, \ldots, \rho_{cl(G)}$ and let $d(j)$ be the dimension of ρ_j. Wedderburn's theorem states that the group algebra $\mathbb{C}[G]$ is isomorphic to $\bigoplus_{j=1}^{cl(G)} M_{d(j)}$ as algebras where $M_{d(j)}$ is the associative algebra of $d(j) \times d(j)$-matrices. Therefore, $\mathfrak{sl}_n(\mathbb{C}[G]) \cong \bigoplus_{j=1}^{cl(G)} \mathfrak{sl}_{nd(j)}$. □

Remark 2.5. *The direct sum above is a direct sum of Lie algebras, that is, two different copies of $\mathfrak{sl}_n$ commute. A non-degenerate symmetric invariant bilinear form κ on the semisimple Lie algebra $\mathfrak{sl}_n(\mathbb{C}[G])$ is given by the formula ($m_1, m_2 \in \mathfrak{sl}_n$, $\gamma_1, \gamma_2 \in G$) :*

$$\kappa(m_1\gamma_1, m_2\gamma_2) = Tr(m_1 \cdot m_2)\delta_{\gamma_1 = \gamma_2^{-1}}.$$

3. Cyclic affine Lie algebras

For an arbitrary ring R with action of a finite group G, $R \rtimes G$ is the ring spanned by the elements $ag, a \in R, g \in G$ with the relations $(a_1 g_1) \cdot (a_2 g_2) = a_1 g_1(a_2) g_1 g_2$. In the previous section, just by considering group rings (over $\mathbb{C}$), we ended up with semi-simple Lie algebras. Here, when R is a Laurent polynomial ring, one can expect to obtain affine Kac-Moody algebras, which is indeed what happens.

3.1. **Definition and decomposition.** Let ξ be a generator of Γ and ζ a primitive d^{th}-root of unity. Let $A = \mathbb{C}[u^{\pm 1}] \rtimes \Gamma$ and $B = \mathbb{C}[u] \rtimes \Gamma$. The action of Γ is defined by $\xi(u) = \zeta u$. We will be interested in the structure of the Lie algebra $\mathfrak{sl}_n(A)$ and of its universal central extension $\widehat{\mathfrak{sl}}_n(A)$. We will also say a few words about $\mathfrak{sl}_n(B)$. We will show that $\mathfrak{sl}_n(A)$ is a graded simple Lie algebra and explain how it is related to the classification of such Lie algebras obtained by V. Kac and O. Mathieu [K1, K2, Ma1, Ma2].

In the following, $\equiv$ is the equivalence mod d.

Lemma 3.1. *We have :*

$$[A,A]=\bigoplus_{i=1}^{d-1}\mathbb{C}[u^{\pm 1}]\xi^i \oplus \bigoplus_{j\in\mathbb{Z}, j\not\equiv 0} \mathbb{C}\cdot u^j \text{ and } [B,B]=\bigoplus_{i=1}^{d-1} u\mathbb{C}[u]\xi^i \oplus \bigoplus_{j\mathbb{Z}_{\geq 1}, j\not\equiv 0}\mathbb{C}\cdot u^j.$$

Proof. If $1\leq i\leq d-1$, then $u^j\xi^i=\frac{1}{1-\zeta^i}[u,u^{j-1}\xi^i]$. If $j\in\mathbb{Z}$ and $j\not\equiv 0$, then $u^j=\frac{1}{\zeta^j-1}[\xi,u^j\xi^{-1}]$. This proves $\supseteq$. Consider $[u^k\xi^a,u^m\xi^b]=(\zeta^{am}-\zeta^{bk})u^{k+m}\xi^{a+b}$ and suppose that the right-hand side is in $\mathbb{C}[u^{\pm 1}]^\Gamma$. Then $k+l\equiv 0$ and $a+b\equiv 0 \bmod (d)$, so $\zeta^{am}-\zeta^{bk}=0$. This proves $\subseteq$. □

Corollary 3.2. $HC_0(B)\cong \mathbb{C}[u]^\Gamma\oplus\mathbb{C}[\Gamma]$.

The Lie algebra $\mathfrak{sl}_n(A)$ admits different vector space decompositions, similar to the two standard triangular decompositions of affine Lie algebras. Let $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$) be the Lie algebra of strictly upper (resp. lower) triangular matrices in $\mathfrak{sl}_n$ (over $\mathbb{C}$) and let $\mathfrak{h}$ be the usual Cartan subalgebra of $\mathfrak{sl}_n$. The elementary matrices in $\mathfrak{gl}_n$ will be denoted E_{ij} and I will stand for the identity matrix. In the following, by abuse of notation, an element $g\otimes a$ will be denoted ga for $g\in\mathfrak{gl}_n$, $a\in A$. We have the following vector space isomorphisms (triangular decompositions) :

$$\mathfrak{sl}_n(A)\cong(\mathfrak{n}^-A)\oplus(\mathfrak{h}A\oplus I[A,A])\oplus(\mathfrak{n}^+A),$$

and $\mathfrak{sl}_n(A)$ is also isomorphic to the sum

$$\begin{aligned}
&\left(\mathfrak{sl}_n\ u^{-1}\mathbb{C}[u^{-1}]\rtimes\Gamma\oplus\left(\bigoplus_{\substack{j\leq -1, j\not\equiv 0\\ 0\leq i\leq d-1}}\mathbb{C}Iu^j\xi^i\oplus\bigoplus_{1\leq i\leq d-1}Iu^{-d}\mathbb{C}[u^{-d}]\xi^i\right)\oplus\mathfrak{n}^-\mathbb{C}[\Gamma]\right)\\
&\oplus\left(\mathfrak{h}\mathbb{C}[\Gamma]\oplus\bigoplus_{1\leq i\leq d-1}\mathbb{C}I\xi^i\right)\\
&\oplus\left(\mathfrak{sl}_n u\mathbb{C}[u]\rtimes\Gamma\oplus\left(\bigoplus_{\substack{j\geq 1, j\not\equiv 0\\ 0\leq i\leq d-1}}\mathbb{C}Iu^j\xi^i\oplus\bigoplus_{1\leq i\leq d-1}Iu^d\mathbb{C}[u^d]\xi^i\right)\oplus\mathfrak{n}^+\mathbb{C}[\Gamma]\right).
\end{aligned}$$

These lead to similar decompositions for $\mathfrak{sl}_n(B)$. These triangular decompositions are similar to those considered, for instance, in [Kh].

The first triangular decomposition is analogous to the loop triangular decomposition of affine Lie algebras, but in our situation the middle Lie algebra is not commutative. The second triangular decomposition is similar to the decomposition of affine Lie algebras adapted to Chevalley-Kac generators, and it is of particular importance as the middle term $\mathfrak{H}$ in commutative. So the role of the Cartan subalgebra will be played by the commutative Lie algebra $\mathfrak{H}$ and our immediate aim is to obtain a corresponding appropriate root space decomposition of $\mathfrak{sl}_n(A)$.

It will be convenient to work with the primitive idempotents of Γ, so let us set $\mathbf{e}_j=\frac{1}{d}\sum_{i=0}^{d-1}\zeta^{-ij}\xi^i$. A vector space basis of $\mathfrak{H}$ is given by the following elements:

$$H_{i,j}=\begin{cases}(E_{i,i}-E_{i+1,i+1})\mathbf{e}_j & \text{for } 1\leq i\leq n-1, 0\leq j\leq d-1,\\ E_{n,n}\mathbf{e}_j-E_{1,1}\mathbf{e}_{j+1} & \text{for } i=0,\ 0\leq j\leq d-2.\end{cases}$$

Remark : we could define $H_{0,d-1}$ in the same way, but then we would get $\sum_{i=0}^{n-1}\sum_{j=0}^{d-1}H_{i,j}=0$ in $\mathfrak{sl}_n(A)$ (but lifts to a non-zero central element in $\widehat{\mathfrak{sl}}_n(A)$).

Lemma 3.3. *A basis of the eigenspaces for non-zero eigenvalues for the adjoint action of $\mathfrak{H}$ on $\mathfrak{sl}_n(A)$ (except for $\mathfrak{H}$ itself) is given by the following vectors:*

$$E_{ij}u^k\mathbf{e}_l \;\; 1 \leq i \neq j \leq n, k \in \mathbb{Z}, 0 \leq l \leq d-1,$$
$$E_{ii}u^k\mathbf{e}_l \;\; 1 \leq i \leq n, k \not\equiv 0, 0 \leq l \leq d-1.$$

Proof. This is a consequence of the following simple computations: For $1 \leq i \neq j \leq n, 0 \leq a \leq n-1, k \in \mathbb{Z}, 0 \leq l \leq d-1, 0 \leq b \leq d-1$,

$$[H_{a,b}, E_{ij}u^k\mathbf{e}_l] = \begin{cases} \big(\delta_{b\equiv l}(\delta_{a+1\equiv j} - \delta_{a\equiv j}) + \delta_{b-k\equiv l}(\delta_{a\equiv i} - \delta_{a+1\equiv i})\big)E_{ij}u^k\mathbf{e}_l & \text{if } a \neq 0, \\ (\delta_{n\equiv i}\delta_{b-k\equiv l} - \delta_{n\equiv j}\delta_{b\equiv l} - \delta_{1\equiv i}\delta_{b+1-k\equiv l} - \delta_{j\equiv 1}\delta_{b+1\equiv l})E_{ij}u^k\mathbf{e}_l & \text{if } a = 0. \end{cases}$$

For $1 \leq i \leq n, k \not\equiv 0, 0 \leq l \leq d-1$,

$$[H_{a,b}, E_{ii}u^k\mathbf{e}_l] = \begin{cases} \big(\delta_{a\equiv i} - \delta_{a+1\equiv i}\big)\big(\delta_{b-k\equiv l} - \delta_{b\equiv l}\big)E_{ii}u^k\mathbf{e}_l & \text{if } a \neq 0, \\ \big(\delta_{n\equiv i}\big(\delta_{b-k\equiv l} - \delta_{b\equiv l}\big) - \delta_{i\equiv 1}(\delta_{b+1-k\equiv l} - \delta_{b+1\equiv l})\big)E_{ii}u^k\mathbf{e}_l & \text{if } a = 0. \end{cases}$$

□

3.2. **Derivation element and imaginary roots.** We will introduce the real, imaginary roots and root spaces later after adding a derivation d to $\widehat{\mathfrak{sl}}_n(A)$.

The center of the universal central extension $\widehat{\mathfrak{sl}}_n(A)$ of $\mathfrak{sl}_n(A)$ is isomorphic to $HC_1(A)$ [KL]. It is known that

$$HC_1(A) \cong \frac{(\mathbb{C}[u^{\pm 1}]du)^{\Gamma}}{d(\mathbb{C}[u^{\pm 1}]^{\Gamma})},$$

the quotient of the space of Γ-invariant 1-forms on $\mathbb{C}^{\times}$ by the space of exact 1-forms coming from Γ-invariant Laurent polynomials. This can be deduced from the isomorphism $A \cong M_d(\mathbb{C}[t^{\pm 1}])$ - see proposition 3.8. Thus this cyclic homology group is one dimensional, with basis given by $u^{-1}du$, which we denote as usual by c.

Definition 3.4. *The cyclic affine Lie algebra $\overline{\mathfrak{sl}}_n(A)$ is obtained from $\widehat{\mathfrak{sl}}_n(A)$ by adding a derivation d that satisfies the relations $[\mathsf{d}, E_{ij}u^k\mathbf{e}_l] = kE_{ij}u^k\mathbf{e}_l$.*

Set $\overline{\mathfrak{H}} = \mathfrak{H} \oplus \mathbb{C}\cdot\mathsf{c} \oplus \mathbb{C}\cdot\mathsf{d}$. We can now introduce the roots as appropriate elements of

$$\overline{\mathfrak{H}}_0^* = \{\lambda \in \overline{\mathfrak{H}}^* | \lambda(\mathsf{c}) = 0\}.$$

The real root spaces are spanned by the root vectors $E_{ij}u^k\mathbf{e}_l$ where $0 \leq l \leq d-1$, $1 \leq i, j \leq n$, $k \in \mathbb{Z}$, with the condition $k \not\equiv 0$ if $i = j$.
The imaginary root spaces are spanned by the following root vectors:

$$H_i u^{kd}\mathbf{e}_l \text{ for } 1 \leq i \leq n-1, k \neq 0, 0 \leq l \leq d-1,$$
$$E_{nn}u^{kd}\mathbf{e}_l - E_{11}u^{kd}\mathbf{e}_{l+1} \text{ for } k \neq 0, 0 \leq l \leq d-2.$$

We want to identify the root lattice as a lattice in $\overline{\mathfrak{H}}_0^*$. Let us introduce the elements $\epsilon_{i,l} \in \overline{\mathfrak{H}}_0^*, 1 \leq i \leq n, 0 \leq l \leq d-1$ by setting

$$\epsilon_{i,l}(H_{a,b}) = (\delta_{a=i} - \delta_{a+1=i})\delta_{b\equiv l},\ \epsilon_{i,l}(H_{0,b}) = \delta_{n=i}\delta_{b\equiv l} - \delta_{i=1}\delta_{b+1\equiv l},$$

and $\epsilon_{i,l}(\mathsf{d}) = 0$.

Definition 3.5. *The real roots are $\epsilon_{i,k+l} - \epsilon_{j,l} + k\delta$ for $1 \leq i, j \leq n, 0 \leq l \leq d-1, k \in \mathbb{Z}$ with $i \neq j$ or, if $i = j$, then $k \equiv 0$; the imaginary ones are $kd\delta$ where $\delta(\mathsf{d}) = 1, \delta(H_{a,b}) = 0$ and $k \neq 0$.*

They generate a lattice - the root lattice - in $\overline{\mathfrak{H}}_0^*$. As one can verify, the real root spaces all have dimension one.

Lemma 3.6. *The root lattice is freely generated by the following roots, which we will call the positive simple roots:* $\epsilon_{i,l} - \epsilon_{i+1,l}$ *for* $1 \leq i \leq n-1, 0 \leq l \leq d-1$, $\epsilon_{n,l} - \epsilon_{1,l+1}$ *for* $0 \leq l \leq d-2$, *and* $(\epsilon_{n,d-1} - \epsilon_{1,0}) + \delta$.

Proof. Indeed, note that

$$\delta = \big((\epsilon_{n,d-1} - \epsilon_{1,0}) + \delta\big) + \sum_{l=0}^{d-1}\sum_{i=1}^{n-1}(\epsilon_{i,l} - \epsilon_{i+1,l}) + \sum_{l=0}^{d-2}(\epsilon_{n,l} - \epsilon_{1,l+1})$$

The set of simple roots contains nd elements, which is also the dimension of $\overline{\mathfrak{H}}_0^*$. □

3.3. **Cyclic affine Lie algebras and affine Lie algebras.**

Proposition 3.7. *The Lie algebra* $\mathfrak{sl}_n(A)$ *is graded simple (i.e. it contains no non trivial graded ideal).*

Proof. Suppose that $\widetilde{I} = \sum_{m\in\mathbb{Z}} \widetilde{I}_m$ is a non-zero graded ideal of $\mathfrak{sl}_n(\mathbb{C}[u^{\pm 1}] \rtimes \Gamma)$. Then $\widetilde{I}$ is stable under the adjoint action of $\mathfrak{H}$, hence each graded piece $\widetilde{I}_m$ must decompose into the direct sum of all the root spaces contained in $\widetilde{I}_m$. It can be checked that the ideal generated by any real root vector is the whole $\mathfrak{sl}_n(A)$, so if $\widetilde{I}$ contains a real root vector, then $\widetilde{I}$ is the whole Lie algebra. Moreover, if $\widetilde{I}$ contains an imaginary root vector, then it contains also a real one. □

However, the Lie algebra $\mathfrak{sl}_n(A)$ is not simple if we do not take the grading into account, as can be seen from proposition 3.8 below. A conjecture of V. Kac [K1], proved in general by O. Mathieu [Ma1, Ma2], gives a classification of graded simple Lie algebras of polynomial growth according to which, following proposition 3.7, $\mathfrak{sl}_n(A)$ must be isomorphic to a (perhaps twisted) loop algebra. This is indeed the case, although the isomorphism does not respect the natural grading on the loop algebra $\mathfrak{sl}_{nd} \otimes \mathbb{C}[t^{\pm 1}]$.

Proposition 3.8. *The Lie algebras* $\mathfrak{sl}_n(A)$ *and* $\mathfrak{sl}_{nd}(\mathbb{C}[t^{\pm 1}])$ *are isomorphic.*

Proof. An isomorphism is given explicitly by the following formulas: If $0 \leq l \leq d-1$, $k \in \mathbb{Z}$, $-l \leq r \leq d-l-1$, $1 \leq i \neq j \leq n$,

$$\begin{aligned}
E_{n(l+r)+i,nl+j}t^k &\leftrightarrow E_{ij}u^{kd+r}\mathbf{e}_l,\\
E_{n(l+r)+i,nl+i}t^k &\leftrightarrow E_{ii}u^{kd+r}\mathbf{e}_l \text{ for } r \neq 0,\\
(E_{nl+i,nl+i} - E_{nl+i+1,nl+i+1})t^k &\leftrightarrow H_i u^{kd}\mathbf{e}_l \text{ for } i \neq n,\\
(E_{nl,nl} - E_{nl+1,nl+1})t^k &\leftrightarrow E_{nn}u^{kd}\mathbf{e}_l - E_{11}u^{kd}\mathbf{e}_{l+1} \text{ for } l \neq 0.
\end{aligned}$$

These formulas can be obtained via the algebra isomorphism $A \cong M_d(\mathbb{C}[t^{\pm 1}])$ which is given by $E_{l+r,l}t^k \leftrightarrow u^{kd+r}\mathbf{e}_l$. □

Let $\phi : \mathfrak{sl}_{nd}(\mathbb{C}[t^{\pm 1}]) \xrightarrow{\sim} \mathfrak{sl}_n(\mathbb{C}[u^{\pm 1}] \rtimes \Gamma)$ be the isomorphism in the proof of proposition 3.8. We can put a grading on $\mathfrak{sl}_{nd}$ by giving E_{ij} degree $j - i$. This induces another grading on $\mathfrak{sl}_{nd}(\mathbb{C}[t^{\pm 1}])$, besides the one coming from the powers of t: the total of these two is the grading given by $deg(E_{ij}(t^r)) = j - i + r$.

Proposition 3.9. ϕ *is an isomorphism of graded Lie algebras when* $\mathfrak{sl}_{nd}(\mathbb{C}[t^{\pm 1}])$ *is endowed with the total grading and* $\mathfrak{sl}_n(\mathbb{C}[u^{\pm 1}] \rtimes \Gamma)$ *is graded in terms of powers of* u.

Let $\mathfrak{p}$ be the parabolic subalgebra of $\mathfrak{sl}_{nd}$ consisting of all the lower triangular matrices and the matrices with d blocks of $n \times n$ matrices along the diagonal. Let $\mathfrak{n}$ be the nilpotent subalgebra which consists of all the upper triangular matrices except those in these blocks along the diagonal, so that $\mathfrak{sl}_{nd} \cong \mathfrak{p} \oplus \mathfrak{n}$. The isomorphism given in proposition 3.8 shows that $\mathfrak{sl}_n(B)$ is isomorphic to $\mathfrak{p} \otimes \mathbb{C}[t] \oplus \mathfrak{n} \otimes t\mathbb{C}[t]$.

We gave in the previous subsection two triangular decompositions of $\mathfrak{sl}_n(A)$. The Lie algebra $\mathfrak{sl}_{nd}(\mathbb{C}[t^{\pm 1}])$ admits similar decompositions, namely:

$$\mathfrak{sl}_{nd}(\mathbb{C}[t^{\pm 1}]) \cong \mathfrak{n}_{nd}^- \otimes \mathbb{C}[t^{\pm 1}] \bigoplus \mathfrak{h}_{nd} \otimes \mathbb{C}[t^{\pm 1}] \bigoplus \mathfrak{n}_{nd}^+ \otimes \mathbb{C}[t^{\pm 1}],$$

$$\mathfrak{sl}_{nd}(\mathbb{C}[t^{\pm 1}]) \cong \left(\mathfrak{sl}_{nd} \otimes t^{-1}\mathbb{C}[t^{-1}] \oplus \mathfrak{n}_{nd}^-\right) \bigoplus \mathfrak{h}_{nd} \bigoplus \left(\mathfrak{n}_{nd}^+ \oplus \mathfrak{sl}_{nd} \otimes t\mathbb{C}[t]\right).$$

The isomorphism given in proposition 3.8 preserves the second decomposition, but not the first one.

We conclude that a lot is already known about the representation theory of cyclic affine algebras and simple finite dimensional representations are classified by tuples of $nd-1$ polynomials (see [C]). We state this more explicitly for toroidal Lie algebras in section 5.1 below.

4. Cyclic double affine Lie algebras

In this section, we set $A = \mathbb{C}[u^{\pm 1}, v^{\pm 1}] \rtimes \Gamma, B = \mathbb{C}[u^{\pm 1}, v] \rtimes \Gamma, C = \mathbb{C}[u, v] \rtimes \Gamma$. Here, ξ acts on u, v by $\xi(u) = \zeta u, \xi(v) = \zeta^{-1} v$. Note that, setting $w = uv$, we deduce that A is isomorphic to $\mathbb{C}[u^{\pm 1}, w^{\pm 1}] \rtimes \Gamma$ where Γ acts trivially on w. The same remark applies to B.

We will be interested in the structure of the Lie algebras $\mathfrak{sl}_n(A)$, $\mathfrak{sl}_n(B)$, $\mathfrak{sl}_n(C)$ and of their universal central extensions.

4.1. **Structure.** As in the previous section, we need to know certain cyclic homology groups.

$$HC_0(A) = A^\Gamma \ , \ HC_0(B) = B^\Gamma \ , \ HC_0(C) = C^\Gamma \oplus \mathbb{C}^{\oplus(|\Gamma|-1)},$$

$HC_1(A) \cong \frac{\Omega^1(A)^\Gamma}{d(A^\Gamma)}$ and similarly for B, C (see corollary 4.3 below.). As vector spaces, when $\Gamma \neq \{\mathrm{id}\}$, we have

$$HC_1(A) = v\mathbb{C}[u^{\pm 1}, v^{\pm 1}]^\Gamma du \oplus \mathbb{C}[u^{\pm 1}]^\Gamma v^{-1} dv = (\mathbb{C}[u^{\pm 1}]^\Gamma \otimes_{\mathbb{C}} \mathbb{C}[w^{\pm 1}]) u^{-1} du \oplus \mathbb{C}[u^{\pm 1}]^\Gamma w^{-1} dw,$$

$$HC_1(B) = v\mathbb{C}[u^{\pm 1}, v]^\Gamma du \oplus \mathbb{C} u^{-1} du \cong (w\mathbb{C}[w] \otimes_{\mathbb{C}} \mathbb{C}[u^{\pm 1}]^\Gamma) u^{-1} du \oplus \mathbb{C} u^{-1} du,$$

$$HC_1(C) = v\mathbb{C}[u, v]^\Gamma du.$$

These can be obtained from the computations of Hochschild homology in [Fa], which are valid for more general groups than Γ. For A and B, they can also be deduced from the proof of the following extension of proposition 3.8.

Proposition 4.1. *The Lie algebra $\mathfrak{sl}_n(A)$ (resp. $\mathfrak{sl}_n(B)$) is isomorphic to the toroidal Lie algebra $\mathfrak{sl}_{nd}(\mathbb{C}[s^{\pm 1}, t^{\pm 1}])$ (resp. $\mathfrak{sl}_{nd}(\mathbb{C}[s^{\pm 1}, t])$).*

Proof. We write A as $A = \mathbb{C}[u^{\pm 1}, w^{\pm 1}]$. Since $\mathbb{C}[u^{\pm 1}] \rtimes \Gamma \cong M_d(\mathbb{C}[s^{\pm 1}])$ (with $s = u^d$, see the proof of proposition 3.8) and $A \cong (\mathbb{C}[u^{\pm}] \rtimes \Gamma) \otimes_{\mathbb{C}} \mathbb{C}[w^{\pm 1}]$, we immediately deduce this proposition (setting $t = w$). The same argument applies to B. □

Remark 4.2. *The isomorphism in this proposition is reminiscent of lemma 4.1 in* [BGT]. *Furthermore, in that article, using vertex operator techniques, the authors constructed representations of a certain affinization of $\mathfrak{gl}_N(\mathbb{C}[G][t, t^{-1}])$, where G is an admissible subgroup of $\mathbb{C}^\times$ (in the sense defined in that article).*

Explicitly, the isomorphism in proposition 4.1 sends $E_{ab} u^i w^j \mathbf{e}_k$ with $i = m + dl, -k \leq m \leq d-1-k, 0 \leq k \leq d-1$ to $E_{a+(m+k)n, b+kn} s^l t^j$; in terms of u, v instead of u, w, it maps $E_{ab} u^i v^j \mathbf{e}_k$ with $i - j = m + dl, -k \leq m \leq d - 1 - k$ to $E_{a+(m+k)n, b+kn} s^l t^j$. In particular, if $i = 0$, then this map restrict to the same isomorphism as in proposition 3.8 for $\mathbb{C}[v^{\pm 1}]$ with v playing the role of u^{-1}. The Lie subalgebra $\mathfrak{sl}_n(\mathbb{C}[w^{\pm 1}] \rtimes \Gamma)$ gets identified with the direct sum of d copies of $\mathfrak{sl}_n(\mathbb{C}[t^{\pm 1}])$, which agrees with $\mathbb{C}[w^{\pm 1}] \rtimes \Gamma \cong \mathbb{C}[w^{\pm 1}] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] \cong \mathbb{C}[w^{\pm 1}]^{\oplus d}$.

Corollary 4.3. *The cyclic homology groups $HC_1(A)$ and $HC_1(B)$ are given by*

$$HC_1(A) \cong \frac{\Omega^1(\mathbb{C}[s^{\pm 1}, t^{\pm 1}])}{d(\mathbb{C}[s^{\pm 1}, t^{\pm 1}])}\ ,\ HC_1(B) \cong \frac{\Omega^1(\mathbb{C}[s^{\pm 1}, t])}{d(\mathbb{C}[s^{\pm 1}, t])}.$$

Proof. This is a corollary of the algebra isomorphism $A \cong M_d(\mathbb{C}[s^{\pm 1}, t^{\pm 1}]))$ in the proof of proposition 4.1. (A similar isomorphism holds for B.) $\square$

When we restrict the isomorphism of proposition 4.1 to $\mathfrak{sl}_n(C)$, we obtain an injective map from $\mathfrak{sl}_n(C)$ into $\mathfrak{sl}_{nd}(\mathbb{C}[s,t])$. It comes from an injection of C into $M_d(\mathbb{C}[s,t])$. This latter map is a special case ($n = 1, x, y, \nu = 0$) of the homomorphism introduced in subsection 6.1 in [Go2] - see also [CB]. It is not an epimorphism from C to $M_d(\mathbb{C}[s,t])$ in the set theoretical sense, but it is an epimorphism in the following categorical sense: any two ring homomorphisms $M_d(\mathbb{C}[s,t]) \rightrightarrows D$ whose composites with $C \hookrightarrow M_d(\mathbb{C}[s,t])$ agree on C must be equal.

Using proposition 4.1 and the explicit isomorphism given right after the proof, we can give a formula for the bracket on $\widehat{\mathfrak{sl}}_n(A)$. It is easier to do it first with u, w and then translate the result to u, v. Note that we identify the center of $\widehat{\mathfrak{sl}}_n(A)$ with $\frac{\Omega^1(\mathbb{C}[s^{\pm 1}, t^{\pm 1}])}{d(\mathbb{C}[s^{\pm 1}, t^{\pm 1}])}$, so that it makes sense to write $d(f)$ for some $f \in \mathbb{C}[s^{\pm 1}, t^{\pm 1}]$ (with $s = u^d, t = w$ as above). For a commutative algebra R, the bracket on $\mathfrak{sl}_n(R)$ is given by $[E_{a_1 b_1} r_1, E_{a_2 b_2} r_2] = [E_{a_1 b_1}, E_{a_2 b_2}] r_1 r_2 + \mathrm{Tr}(E_{a_1 b_1} E_{a_2 b_2}) r_1 d(r_2)$, where Tr is the usual trace functional [Ka] and $r_1 d(r_2) \in \frac{\Omega^1(R)}{dR}$. Now consider

$$\begin{aligned}
&[E_{a_1+(m_1+k_1)n, b_1+k_1 n} s^{l_1} t^{j_1}, E_{a_2+(m_2+k_2)n, b_2+k_2 n} s^{l_2} t^{j_2}] \\
=&\delta_{b_1+k_1 n = a_2+(m_2+k_2)n} E_{a_1+(m_1+k_1)n, b_2+k_2 n} s^{l_1+l_2} t^{j_1+j_2} \\
&- \delta_{b_2+k_2 n = a_1+(m_1+k_1)n} E_{a_2+(m_2+k_2)n, b_1+k_1 n} s^{l_1+l_2} t^{j_1+j_2} \\
&+ \delta_{a_1+(m_1+k_1)n = b_2+k_2 n} \delta_{b_1+k_1 n = a_2+(m_2+k_2)n} s^{l_1} t^{j_1} d(s^{l_2} t^{j_2}).
\end{aligned}$$

The isomorphism of proposition 4.1 identifies the left-hand side with

$$[E_{a_1 b_1} u^{i_1} w^{j_1} \mathbf{e}_{k_1}, E_{a_2 b_2} u^{i_2} w^{j_2} \mathbf{e}_{k_2}],\quad i_1 = dl_1 + m_1,\ i_2 = dl_2 + m_2,$$

whereas the right-hand side is identified with

$$\begin{aligned}
&\delta_{a_2=b_1} \delta_{k_1 \equiv m_2+k_2} E_{a_1, b_2} u^{i_1+i_2} w^{j_1+j_2} - \delta_{b_2=a_1} \delta_{m_1+k_1 \equiv k_2} E_{a_2 b_1} u^{i_1+i_2} w^{j_1+j_2} \\
&\quad + \delta_{a_1=b_2} \delta_{m_1+k_1 \equiv k_2} \delta_{b_1=a_2} \delta_{k_1 \equiv m_2+k_2} u^{i_1-m_1} w^{j_1} d(u^{i_2-m_2} w^{j_2}).
\end{aligned}$$

Setting $v = wu^{-1}$, we obtain a formula in terms of u and v, if we now define m_1, m_2 by $i_i - j_i = m_i + l_i d$:

$$\begin{aligned}
&[E_{a_1 b_1} u^{i_1} v^{j_1} \mathbf{e}_{k_1}, E_{a_2 b_2} u^{i_2} v^{j_2} \mathbf{e}_{k_2}] \\
=&\delta_{a_2=b_1} \delta_{k_1 \equiv m_2+k_2} E_{a_1, b_2} u^{i_1+i_2} v^{j_1+j_2} - \delta_{b_2=a_1} \delta_{m_1+k_1 \equiv k_2} E_{a_2 b_1} u^{i_1+i_2} v^{j_1+j_2} \\
&+ \delta_{a_1=b_2} \delta_{m_1+k_1 \equiv k_2} \delta_{b_1=a_2} \delta_{k_1 \equiv m_2+k_2} u^{i_1-m_1} v^{j_1} d(u^{i_2-m_2} v^{j_2}).
\end{aligned}$$

The formula for the bracket when A is replaced by B and C is exactly the same, simply obtained by restricting the values allowed for i_1, i_2, j_1, j_2. The description of $\widehat{\mathfrak{sl}}_n(A)$ given by proposition 2.3 implies that the natural maps $\widehat{\mathfrak{sl}}_n(B), \widehat{\mathfrak{sl}}_n(C) \longrightarrow \widehat{\mathfrak{sl}}_n(A)$ are embeddings.

The next proposition is an adaptation of proposition 3.5 in [MRY] to $\widehat{\mathfrak{sl}}_n(A)$, using proposition 4.1. There are a few more relations in proposition 4.4 below than in *loc. cit.*, but the proofs are the same.

Let $\mathbf{C} = (c_{ij})$ be the affine Cartan matrix of type $\widehat{A}_{nd-1}$ with rows and columns indexed from 0 to $nd-1$. Let $f : [0, n-1] \times [0, d-1] \longrightarrow [0, nd-1]$ be the function $f(i,j) = i + jn$.

Proposition 4.4. *The Lie algebra $\widehat{\mathfrak{sl}}_n(A)$ is isomorphic to the Lie algebra $\mathfrak{t}$ generated by the elements $X^{\pm}_{i,j,r}, H_{i,j,r}$ for $0 \leq i \leq n-1, 0 \leq j \leq d-1, r \in \mathbb{Z}$, and a central element c satisfying the following list of relations.*

$$[H_{i_1,j_1,r_1}, H_{i_2,j_2,r_2}] = r_1 c_{f(i_1,j_1),f(i_2,j_2)} \delta_{r_1+r_2=0} \mathsf{c},$$
$$[H_{i_1,j_1,0}, X^{\pm}_{i_2,j_2,r_2}] = \pm c_{f(i_1,j_1),f(i_2,j_2)} X^{\pm}_{i_2,j_2,r_2},$$
$$[H_{i_1,j_1,r_1+1}, X^{\pm}_{i_2,j_2,r_2}] = [H_{i_1,j_1,r_1}, X^{\pm}_{i_2,j_2,r_2+1}],$$
$$[X^{\pm}_{i_1,j_1,r_1+1}, X^{\pm}_{i_2,j_2,r_2}] = [X^{\pm}_{i_1,j_1,r_1}, X^{\pm}_{i_2,j_2,r_2+1}],$$
$$[X^{+}_{i_1,j_1,r_1}, X^{-}_{i_2,j_2,r_2}] = \delta_{i_1=i_2}\delta_{j_1=j_2}(H_{i_1,j_1,r_1+r_2} + r_1 \delta_{r_1+r_2=0}\mathsf{c}),$$
$$ad(X^{\pm}_{i_1,j_1,r_1})^{1-c_{f(i_1,j_1),f(i_2,j_2)}}(X^{\pm}_{i_2,j_2,r_2}) = 0.$$

Remark 4.5. *The elements with $r = 0$ generate a copy of $\widehat{\mathfrak{sl}}_n(\mathbb{C}[u^{\pm 1}] \rtimes \Gamma)$ and this proposition gives a set of relations describing this algebra, which is the one in terms of Chevalley-Kac generators of $\widehat{\mathfrak{sl}}_{nd}(\mathbb{C}[t^{\pm 1}])$ - see proposition 3.8. The elements with $i \neq 0$ generate a central extension of $\mathfrak{sl}_n(\mathbb{C}[w^{\pm 1}])^{\oplus d}$ and this proposition gives a presentation of $\mathfrak{sl}_n(\mathbb{C}[w^{\pm 1}])$. (This is the one obtained by considering a Cartan matrix of finite type A_{n-1} in proposition 3.5 in* [MRY]*.)*

An isomorphism $\tau : \mathfrak{t} \xrightarrow{\sim} \widehat{\mathfrak{sl}}_n(A)$ is given explicitly on the generators by the following formulas:

$$(X^{+}_{i,j,r}, X^{-}_{i,j,r}) \mapsto \begin{cases} (E_{i,i+1} w^r \mathbf{e}_j, E_{i+1,i} w^r \mathbf{e}_j) & \text{if } i \neq 0, \\ (E_{n1} u^{-1} w^r \mathbf{e}_j, E_{1n} u w^r \mathbf{e}_{j-1}) & \text{if } i = 0, j \neq 0, \\ (E_{n1} u^{2d-1} w^r \mathbf{e}_{d-1}, E_{1n} \otimes u^{-(2d-1)} w^r \mathbf{e}_0) & \text{if } i = j = 0, \end{cases}$$
$$H_{i,j,r} \mapsto \begin{cases} (E_{ii} - E_{i+1,i+1}) w^r \mathbf{e}_j & \text{if } i \neq 0, \\ E_{nn} w^r \mathbf{e}_{j-1} - E_{11} w^r \mathbf{e}_j & \text{if } i = 0, j \neq 0, \\ E_{nn} w^r \mathbf{e}_0 - E_{11} w^r \mathbf{e}_{d-1} - dw^r u^{-1} du & \text{if } i = j = 0, \end{cases}$$
$$\mathsf{c} \mapsto w^{-1} dw = u^{-1} du + v^{-1} dv.$$

It is possible to obtain similar presentations for $\widehat{\mathfrak{sl}}_n(B)$ and $\widehat{\mathfrak{sl}}_n(C)$, which are Lie subalgebras of $\widehat{\mathfrak{sl}}_n(A)$.

4.2. **Derivations in the toroidal case.** The Kac-Moody Lie algebras of affine type associated to the semisimple Lie algebra $\mathfrak{g}$ are obtained by adding a derivation to the universal central extension of $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t^{\pm 1}]$. It has been known to people who work on extended affine Lie algebras how to extend $\widehat{\mathfrak{sl}}_n(\mathbb{C}[s^{\pm 1}, t^{\pm 1}])$ by adding derivations. We interpret this in the context of $\widehat{\mathfrak{sl}_n(A)}, \mathfrak{sl}_m(B), \mathfrak{sl}_n(C)$, following the monograph [AABGP].

To $\widehat{\mathfrak{sl}}_n(A)$, we add two derivations $\mathsf{d}_u, \mathsf{d}^v_w$ which satisfy the commutation relations

$$[\mathsf{d}_u, m \otimes u^k w^l \gamma] = 0 \text{ if } k \not\equiv 0 \bmod d, \quad [\mathsf{d}_u, m \otimes u^{dk} v^j \gamma] = km \otimes u^{dk} w^j \gamma$$

and $[\mathsf{d}^v_w, m \otimes u^k w^l \gamma] = lm \otimes u^k w^l \gamma$. We observe that $[\mathsf{d}^v_w, m \otimes u^k v^l \gamma] = lm \otimes u^k v^l \gamma$ and $[\mathsf{d}_u, m \otimes u^k v^l \gamma] = \delta_{k \equiv l} \frac{k-l}{d} m \otimes u^k v^l \gamma$. We could define similarly d_v and d^u_w. We then have the relation $\mathsf{d}_u = -\mathsf{d}_v$ and $d \cdot \mathsf{d}_u = \mathsf{d}^u_w - \mathsf{d}^v_w$. Dropping the index w, we can add two derivations d^u and d^v to $\widehat{\mathfrak{sl}}_n(A)$.

Definition 4.6. *The cyclic double affine Lie algebra* $\overline{\mathfrak{sl}}_n(A)$ *is defined by adding the two derivations* d^u *and* d^v *to* $\widehat{\mathfrak{sl}}_n(A)$. *We define similarly* $\overline{\mathfrak{sl}}_n(B)$ *and* $\overline{\mathfrak{sl}}_n(C)$.

4.3. **Triangular decompositions.** We have the following triangular decompositions :

$$\mathfrak{sl}_n(A) \cong (\mathfrak{n}^- A) \oplus (\mathfrak{h}A \oplus I[A,A]) \oplus (\mathfrak{n}^+ A), \tag{1}$$

and $\mathfrak{sl}_n(A)$ is also isomorphic to

$$\begin{aligned}
&\mathfrak{sl}_n u^{-1}\mathbb{C}[u^{-1},v^{\pm 1}] \rtimes \Gamma \oplus \left(\bigoplus_{\substack{i=1\\ j\leq -1}}^{d-1} I(\mathbb{C}[v^{\pm 1}]u^j\xi^i) \oplus \bigoplus_{\substack{s\leq -1\\ r\not\equiv s}} I(\mathbb{C}u^s v^r) \right) \oplus \mathfrak{n}^-\mathbb{C}[v^{\pm 1}] \rtimes \Gamma \\
&\oplus \left(\mathfrak{h}\mathbb{C}[v^{\pm 1}] \rtimes \Gamma \oplus \left(\bigoplus_{1\leq i\leq d-1} I(\mathbb{C}[v^{\pm 1}]\xi^i) \oplus \bigoplus_{r\not\equiv 0} I(\mathbb{C}v^r) \right) \right) \oplus \\
&\mathfrak{sl}_n u\mathbb{C}[u,v^{\pm 1}] \rtimes \Gamma \oplus \left(\bigoplus_{\substack{i=1\\ j\geq 1}}^{d-1} I(\mathbb{C}[v^{\pm 1}]u^j\xi^i) \oplus \bigoplus_{\substack{s\geq 1\\ r\not\equiv s}} I(\mathbb{C}u^s v^r) \right) \oplus \mathfrak{n}^+\mathbb{C}[v^{\pm 1}] \rtimes \Gamma.
\end{aligned} \tag{2}$$

and then by, setting $w = uv$, we see that $\mathfrak{sl}_n(A)$ is moreover isomorphic to :

$$\begin{aligned}
&\mathfrak{sl}_n u^{-1}\mathbb{C}[u^{-1},w^{\pm 1}] \rtimes \Gamma \oplus \left(\bigoplus_{\substack{i=1\\ j\leq -1, r\in\mathbb{Z}}}^{d-1} I(\mathbb{C}w^r u^j\xi^i) \oplus \bigoplus_{\substack{s\leq -1\\ s\not\equiv 0, r\in\mathbb{Z}}} I(\mathbb{C}w^r u^s) \right) \oplus \mathfrak{n}^-\mathbb{C}[w^{\pm 1}] \rtimes \Gamma \\
&\oplus \left(\mathfrak{h}\mathbb{C}[w^{\pm 1}] \rtimes \Gamma \oplus \left(\bigoplus_{1\leq i\leq d-1} I(\mathbb{C}[w^{\pm 1}]\xi^i) \right) \right) \oplus \\
&\mathfrak{sl}_n u\mathbb{C}[u,w^{\pm 1}] \rtimes \Gamma \oplus \left(\bigoplus_{\substack{i=1\\ j\geq 1, r\in\mathbb{Z}}}^{d-1} I(\mathbb{C}w^r u^j\xi^i) \oplus \bigoplus_{\substack{s\geq 1\\ s\not\equiv 0, r\in\mathbb{Z}}} I(\mathbb{C}w^r u^s) \right) \oplus \mathfrak{n}^+\mathbb{C}[w^{\pm 1}] \rtimes \Gamma.
\end{aligned} \tag{3}$$

In the last two decompositions, one can exchange u and v and get other decompositions.

The universal central extensions $\widehat{\mathfrak{sl}}_n(A), \widehat{\mathfrak{sl}}_n(B)$ and $\widehat{\mathfrak{sl}}_n(C)$ also have three triangular decompositions: they are obtained by adding the center to the middle part.

It is worth looking quickly at $\mathfrak{h}A \oplus I[A,A]$. We know that $A \cong M_d(\mathbb{C}[s^{\pm 1}, t^{\pm 1}])$, so, if $d > 1$, $[A,A] = A$ and $\mathfrak{h}A \oplus I[A,A] \cong \mathfrak{gl}_d(\mathbb{C}[s^{\pm 1}, t^{\pm 1}])^{\oplus n}$.

5. Representations of cyclic double affine Lie algebras

In this section, we begin to study the representation theory of the algebras defined in the previous sections.

5.1. **Integrable and highest weight modules for cyclic double affine Lie algebras.** At the end of section 4, we gave three triangular decompositions of $\widehat{\mathfrak{sl}}_n(A)$. The first one is analogous to the one used for Weyl modules in [FeLo], but in our situation the middle Lie algebra is not commutative if $\Gamma \neq \{\mathrm{id}\}$. (Note that an analog of the first triangular decomposition is not known for quantum toroidal algebras.) We will return to Weyl modules in section 5.2 below.

The second triangular decomposition corresponds to the first triangular decomposition of the cyclic affine Lie algebras for the parameter v, and to the second triangular decomposition of the cyclic affine Lie algebra for the parameter u. Thus it is analogous to the triangular decomposition used in [Mi1, Nak1, H1] to construct l-highest weight representations of quantum toroidal algebras. Again, in our situation, the middle Lie algebra is not commutative if $\Gamma \neq \{\mathrm{id}\}$.

However, the middle term $\mathfrak{H}$ of the last triangular decomposition is a commutative Lie algebra. Actually, this last case is obtained by considering the first triangular decomposition of the cyclic affine Lie algebras for the parameter w and the second triangular decomposition of the cyclic affine Lie algebra for the parameter u. Under the isomorphism between $\widehat{\mathfrak{sl}}_n(A)$ and the toroidal Lie algebra $\widehat{\mathfrak{sl}}_{nd}(\mathbb{C}[u^{\pm 1}, w^{\pm 1}])$ given in proposition 4.1, it corresponds to the standard decomposition of $\widehat{\mathfrak{sl}}_{nd}(\mathbb{C}[u^{\pm 1}, w^{\pm 1}])$ as used in [ChLe] for a certain central extension of $\mathfrak{sl}_{nd}(\mathbb{C}[u^{\pm 1}, w^{\pm 1}])$. (It is analogous to the decomposition considered in the quantum case [Mi1, Nak1, H1]). In particular, the notions of integrable and highest weight modules for this decomposition have been studied in [ChLe, Ra, Y]; we simply reformulate their results for the benefit of the reader in the next subsection. Integrable highest weight representations are classified by tuples of $nd-1$ polynomials. In opposition to the quantum case, evaluation morphisms are available and provide a direct way to construct integrable representations.

5.1.1. *The standard highest weight structure on $\widehat{\mathfrak{sl}}_n(A)$ and $\widehat{\mathfrak{sl}}_n(B)$.* In this subsection, we include previously known results about the standard highest weight structure on $\widehat{\mathfrak{sl}}_n(A)$ and on $\widehat{\mathfrak{sl}}_n(B)$. (Actually, the results below have been proved only for $\widehat{\mathfrak{sl}}_n(A)$, but the proofs are similar for $\widehat{\mathfrak{sl}}_n(B)$.) Let $\mathfrak{g}^{\pm}$ be the positive and negative parts of the triangular decomposition (3), and let $\mathfrak{H}$ be the middle part.

Instead of highest weight vectors, we have to consider the notion of pseudo-highest weight vectors. Suppose that $\Lambda = (\lambda_{i,j,r})^{0\leq j\leq d-1}_{0\leq i\leq n-1, r\in\mathbb{Z}}$ where $\lambda_{i,j,r} \in \mathbb{C}$. We define the Verma module $M(\Lambda)$ to be the $\widehat{\mathfrak{sl}}_n(A)$-module induced from the $\mathfrak{H}\oplus\mathfrak{g}^+$ representation generated by the vector v_Λ on which $\mathfrak{g}^+$ and c act by zero and $H_{i,j,r}$ acts by multiplication by $\lambda_{i,j,r}$ (in the notation of proposition 4.4). We have a grading on the Verma module and so it has a unique simple quotient $L(\lambda)$. For $\mu \in \mathfrak{H}^*$, we define the notion of weight space V_μ of a representation V as usual.

Definition 5.1. *A module M over $\widehat{\mathfrak{sl}}_n(A)$ is called integrable if $M = \oplus_{\mu\in\mathfrak{H}^*} M_\mu$ and if the vectors $E_{ij}u^s v^r \gamma$ act locally nilpotently if $1 \leq i \neq j \leq n, r, s \in \mathbb{Z}, \gamma \in \Gamma$.*

Note that, in the quantum case, a stronger notion of integrability is used instead of local nilpotency of the operators [ChLe, Ra].

Proposition 5.2. [ChLe] *The irreducible module $L(\Lambda)$ is integrable if and only if, for any $0 \leq i \leq n-1, 0 \leq j \leq d-1$, we have $\lambda_{i,j,0} \in \mathbb{Z}_{\geq 0}$ and there exist monic polynomials $P_{i,j}(z)$ with non-zero constant terms and of degree $\lambda_{i,j,0}$ such that $\sum_{r\geq 1} \lambda_{i,j,r} z^{r-1} = -\frac{P'_{i,j}(z)}{P_{i,j}(z)}$ and $\sum_{r\geq 1} \lambda_{i,j,-r} z^{r-1} = -\lambda_{i,j,0} z^{-1} + z^{-2}\frac{P'_{i,j}(z^{-1})}{P_{i,j}(z^{-1})}$ as formal power series.*

The proof of the necessary condition in this proposition reduces to the case of the loop Lie algebra of $\mathfrak{sl}_2$, which is why it extends automatically from the affine to the double affine setup. The sufficiency is proved as in [ChLe] using tensor products of evaluation modules (the formulas for the power series are a bit different from those in [Proposition 3.1, [ChLe]] as we use different variables; see also [Mi2]). In the quantum context, the polynomials $P_{i,j}(z)$ are called Drinfeld polynomials. A similar criterion for integrability exists for quantum toroidal algebras: this is explained in [H1] after proving that certain subalgebras of a quantum toroidal algebras are isomorphic to the quantized enveloping algebra of $\mathfrak{sl}_2(\mathbb{C}[t^{\pm 1}])$. Affine Yangians for $\mathfrak{sl}_n$ are built from copies of $Y(\mathfrak{sl}_2)$, the Yangian for $\mathfrak{sl}_2$: this follows from the PBW property of affine Yangians proved in [Gu2]; therefore, a similar integrability condition holds for them also.

5.1.2. *The standard highest weight structure on $\widehat{\mathfrak{sl}}_n(C)$.* The case which interests us more and presents some novelty is $\widehat{\mathfrak{sl}}_n(C)$, because the Lie algebra $\mathfrak{sl}_n(C)$ is not isomorphic to $\mathfrak{sl}_{nd}(\mathbb{C}[s,t])$. The three triangular decompositions of $\widehat{\mathfrak{sl}}_n(A)$ given at the end of the section 4 yield such decompositions for $\widehat{\mathfrak{sl}}_n(C)$ and we consider the one coming from (3). More precisely, $\mathfrak{sl}_n(C)$ can be decomposed into the direct sum of the following three subalgebras.

$$\begin{aligned} &\left(\mathfrak{sl}_n v\mathbb{C}[v,w]\rtimes\Gamma\oplus\left(\bigoplus_{j\geq 1,1\leq i\leq d-1} I(\mathbb{C}[w]v^j\xi^i)\oplus\bigoplus_{r\geq 1, r\not\equiv 0} I(\mathbb{C}[w]v^r)\right)\oplus\mathfrak{n}^-\mathbb{C}[w]\rtimes\Gamma\right)\\ \oplus&\left(\mathfrak{h}\mathbb{C}[w]\rtimes\Gamma\oplus\Big(\bigoplus_{1\leq i\leq d-1} I(w\mathbb{C}[w]\xi^i)\Big)\right)\\ \oplus&\left(\mathfrak{sl}_n u\mathbb{C}[u,w]\rtimes\Gamma\oplus\left(\bigoplus_{j\geq 1,1\leq i\leq d-1} I(\mathbb{C}[w]u^j\xi^i)\oplus\bigoplus_{r\geq 1, r\not\equiv 0} I(\mathbb{C}[w]u^r)\right)\oplus\mathfrak{n}^+\mathbb{C}[w]\rtimes\Gamma\right). \end{aligned}\tag{4}$$

We have an embedding $\widehat{\mathfrak{sl}}_n(C)\hookrightarrow\widehat{\mathfrak{sl}}_{nd}(\mathbb{C}[u^{\pm 1},v])$, but in order to classify integrable, highest weight representations of $\widehat{\mathfrak{sl}}_n(C)$, we will instead use the following presentation.

Proposition 5.3. *The Lie algebra $\widehat{\mathfrak{sl}}_n(C)$ is isomorphic to the Lie algebra $\mathfrak{k}$ generated by the elements $X^{\pm}_{i,j,r}, H_{i,j,r}, X^+_{0,j,r}, X^-_{0,j,r+1}, H_{0,j,r+1}$ for $1\leq i\leq n-1, 0\leq j\leq d-1, r\in\mathbb{Z}_{\geq 0}$, satisfying the following list of relations.*

$$[H_{i_1,j_1,r_1},H_{i_2,j_2,r_2}]=0,\;\; [H_{i_1,j_1,0},X^{\pm}_{i_2,j_2,r_2}]=\pm c_{f(i_1,j_1),f(i_2,j_2)}X^{\pm}_{i_2,j_2,r_2},$$

$$[H_{i_1,j_1,r_1+1},X^{\pm}_{i_2,j_2,r_2}]=[H_{i_1,j_1,r_1},X^{\pm}_{i_2,j_2,r_2+1}],\;\; [X^{\pm}_{i_1,j_1,r_1+1},X^{\pm}_{i_2,j_2,r_2}]=[X^{\pm}_{i_1,j_1,r_1},X^{\pm}_{i_2,j_2,r_2+1}],$$

$$[X^+_{i_1,j_1,r_1},X^-_{i_2,j_2,r_2}]=\delta_{i_1=i_2}\delta_{j_1=j_2}H_{i_1,j_1,r_1+r_2},\;\; ad(X^{\pm}_{i_1,j_1,r_1})^{1-c_{f(i_1,j_1),f(i_2,j_2)}}(X^{\pm}_{i_2,j_2,r_2})=0.$$

The proof of proposition 4.4 in [MRY] works also for $\widehat{\mathfrak{sl}}_n(B)$ and it is possible to deduce from it proposition 5.3. See [Gu3] for more details.

The elements with $i\neq 0$ generate a Lie subalgebra isomorphic to $\mathfrak{sl}_n(\mathbb{C}[w]\rtimes\Gamma)$, the $X^+_{i,j,r}, 0\leq i\leq n-1, 0\leq j\leq d-1, r\in\mathbb{Z}_{\geq 0}$ generate the positive part $\widehat{\mathfrak{sl}}_n(C)^+$ of the decomposition (4) and the $X^-_{i,j,r}, X^-_{0,j,r+1}, 0\leq i\leq n-1, 0\leq j\leq d-1, r\in\mathbb{Z}_{\geq 0}$ generate the negative part $\widehat{\mathfrak{sl}}_n(C)^-$. Note that $\widehat{\mathfrak{sl}}_n(C)^+\cong\widehat{\mathfrak{sl}}_n(C)^-$ via $X^+_{i,j,r}\mapsto X^-_{i,j,r}$ for $1\leq i\leq n-1$ and $X^+_{0,j,r}\mapsto X^-_{0,j,r+1}$. The elements with $r=0$ generate a copy of $\mathfrak{sl}_n(\mathbb{C}[u]\rtimes\Gamma)$, whereas the elements $X^{\pm}_{i,j,0}, X^-_{0,j,1}$ with $1\leq i\leq n-1, 0\leq j\leq d-1$ generate a Lie subalgebra isomorphic to $\mathfrak{sl}_n(\mathbb{C}[v]\rtimes\Gamma)$. For a fixed $0\leq j\leq d-1$, the elements $X^+_{0,j,r}, X^-_{0,j,r+1}, H_{0,j,r+1}$ for all $r\in\mathbb{Z}_{\geq 0}$ generate a subalgebra of $\mathfrak{sl}_2(\mathbb{C}[w])$ which, as a vector space, is $\mathfrak{n}_2^- w\mathbb{C}[w]\oplus\mathfrak{h}_2 w\mathbb{C}[w]\oplus\mathfrak{n}_2^+\mathbb{C}[w]$ where the index $_2$ indicates the corresponding subalgebra of $\mathfrak{sl}_2$. Let us denote this subalgebra of $\mathfrak{sl}_2(\mathbb{C}[w])$ by $\check{\mathfrak{sl}}_2(\mathbb{C}[w])$.

Integrability of representations of $\widehat{\mathfrak{sl}}_n(C)$ has the same meaning as in definition 5.1. As for $\widehat{\mathfrak{sl}}_n(A)$ and $\widehat{\mathfrak{sl}}_n(B)$, we have Verma modules $M(\Lambda)$ and their irreducible quotients $L(\Lambda)$ for each pseudo-weight $\Lambda=(\lambda_{i,j,r}\in\mathbb{C})$ with $0\leq i\leq n-1, r\in\mathbb{Z}, 0\leq j\leq d-1$ but $r\geq 1$ if $i=0$; the highest weight cyclic generator is again denoted v_Λ.

Proposition 5.4. *The irreducible module $L(\Lambda)$ is integrable if and only if $\lambda_{i,j,0} \in \mathbb{Z}_{\geq 0}$ for $1 \leq i \leq n-1, 0 \leq j \leq d-1$, and there exist monic polynomials $P_{i,j}(z)$ for $0 \leq i \leq n-1, 0 \leq j \leq d-1$, such that $\sum_{r\geq 1} \lambda_{i,j,r} z^{-r-1} = -\deg(P_{i,j}) z^{-1} + \frac{P'_{i,j}(z)}{P_{i,j}(z)}$ as formal power series and $P_{i,j}(z)$ is of degree $\lambda_{i,j,0}$ if $1 \leq i \leq n-1, 0 \leq j \leq d-1$.*

Let us say a few words about the proof. The main difference with the cases $\widehat{\mathfrak{sl}}_n(A)$ and $\widehat{\mathfrak{sl}}_n(B)$ is that we do not have the elements $X^-_{0,j,0}$, $0 \leq j \leq d-1$. However, it is still possible to apply the same argument as in the affine $\mathfrak{sl}_2$-case [C], modulo some small differences. For instance, proposition 1.1 in [C] is fundamental for the rest of that article, but it cannot be applied in our case when $i = 0$: what we need instead is an expression for $(X_{0,j,0})^r (X^-_{0,j,1})^r$. Proposition 1.1 in [C] is a consequence of lemma 7.5 in [Ga]. To obtain an expression for $(X_{0,j,0})^r (X^-_{0,j,1})^r$, we just have to apply the automorphism of $\check{\mathfrak{sl}}_2(\mathbb{C}[w])$ given by $E_{21} w^{r+1} \mapsto E_{12} w^r$, $E_{12} w^r \mapsto E_{21} t^{r+1}$ and $(E_{11} - E_{22}) t^{r+1} \mapsto (E_{22} - E_{11}) t^{r+1}$ for $r \in \mathbb{Z}_{\geq 0}$. We lose the condition that the degree of $P_{0,j}(z)$ is $\lambda_{0,j}$ because $\widehat{\mathfrak{sl}}_n(C)$ does not contain the $\mathfrak{sl}_2$-copies generated by $X^{\pm}_{0,j,0}, H_{0,j,0}$. The degree of $P_{0,j}(z)$ is the smallest integer r such that $(X^-_{0,j,1})^{r+1} v_\Lambda = 0$. The proof of the sufficiency of the condition in the proposition consists in the construction of an integrable quotient of the Verma module $M(\Lambda)$ using tensor products of evaluation modules, as in [ChLe]. Note that also in the case of $\widehat{\mathfrak{sl}}_n(C)$, the degree of $P_{0,j}(z)$ is the smallest integer r such that $(X^-_{0,j,1})^{r+1}$ acts by zero on the cyclic highest weight vector.

5.2. **Weyl modules for $\mathfrak{sl}_n(\mathbb{C}[u,v] \rtimes \Gamma)$.** For a Lie algebra $\mathfrak{g}$ and a commutative $\mathbb{C}$-algebra $\mathcal{A}$, we will often use the notation $\mathfrak{g}(\mathcal{A})$ or $\mathfrak{g}\mathcal{A}$ to denote $\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{A}$. If $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{H} \oplus \mathfrak{n}^-$ the Weyl modules [FeLo] are certain representations of $\mathfrak{g}(\mathcal{A})$ generated by a weight vector v satisfying $\big(\mathfrak{n}^+(\mathcal{A})\big).v = 0$. (In this subsection, we consider only the local Weyl modules, not the global ones.) The motivation to study Weyl modules is that they should be simpler to understand than the finite dimensional irreducible modules. This is what happens in the quantum affine setup where the Weyl modules for the affine Lie algebras are closely related to finite dimensional irreducible modules of the corresponding affine quantum group when $q \mapsto 1$ [ChPr, ChLo]. The definition of Weyl modules depends on the choice of a triangular decomposition, but only the first of our triangular decompositions for cyclic double affine Lie algebras seems appropriate. It should be noted that we cannot use proposition 4.1 to deduce results about Weyl modules for $\mathfrak{sl}_n(A)$ in our context because, when Γ is non-trivial, the isomorphism in that proposition does not map the triangular decomposition (1) of $\mathfrak{sl}_n(A)$ to the decomposition of $\mathfrak{sl}_{nd}(\mathbb{C}[s^{\pm 1}, t^{\pm 1}])$ considered in [FeLo], which is $\mathfrak{sl}_{nd}(\mathbb{C}[s^{\pm 1}, t^{\pm 1}]) = \mathfrak{n}^-_{nd}\mathbb{C}[s^{\pm 1}, t^{\pm 1}] \oplus \mathfrak{h}_{nd}\mathbb{C}[s^{\pm 1}, t^{\pm 1}] \oplus \mathfrak{n}^+_{nd}\mathbb{C}[s^{\pm 1}, t^{\pm 1}]$.

In this subsection, we need a stronger definition of integrability than the one presented in 5.1. It is the same as the one used in [FeLo].

Definition 5.5. *A module M over a Lie algebra of the type $\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{A}$ is said to be integrable if M_μ is non-zero for only finitely many $\mu \in P$.*

When $\mathcal{A}$ is the coordinate ring of an affine algebraic variety X, Weyl modules are associated to multisets of points of X. In the simplest case of a (closed) point, we have an augmentation $\mathcal{A} \longrightarrow \mathbb{C}$. However, when it comes to the triangular decomposition (1), the middle term is isomorphic to $\mathfrak{d} \otimes_{\mathbb{C}} M_d(\mathbb{C}[s^{\pm 1}, t^{\pm 1}])$, where $\mathfrak{d}$ is the abelian Lie algebra of the diagonal matrices in $\mathfrak{gl}_n$. When $d = 1$, we are exactly in the same situation as in [FeLo] (with X the two-dimensional torus $\mathbb{C}^\times \times \mathbb{C}^\times$), but when $d > 1$, the Lie algebra is non-commutative. On new possibility is to consider maximal two-sided ideals in $A \rtimes \Gamma$, or, equivalently, augmentation maps. We are thus led to the following definition, which we can formulate in a more general setting.

Definition 5.6. *Let $\mathcal{A}$ be a commutative, finitely generated algebra with a unit and let G be a finite group acting on $\mathcal{A}$ by algebra automorphisms. Consider an augmentation ϵ of $\mathcal{A} \rtimes G$, that is, an algebra homomorphism $\mathcal{A} \rtimes G \longrightarrow \mathbb{C}$, and let $\lambda \in \mathfrak{h}^*$ be a dominant integral weight. We define the Weyl module $W^{\epsilon}_{\mathcal{A}\rtimes G}$ to be the maximal integrable cyclic $\mathfrak{sl}_n(\mathcal{A} \rtimes G)$-module generated by a vector v_λ such that, for $a \in \mathcal{A} \rtimes G$:*

$$(ha)(v_\lambda) = \lambda(h)\epsilon(a)v_\lambda, \ \ \mathfrak{n}^+(\mathcal{A} \rtimes G)(v_\lambda) = 0.$$

The existence of such a maximal module can be proved as in [FeLo] using the notion of global Weyl module. This definition agrees with the one used in the paper [FeLo] in the case $G = \{\mathrm{id}\}$. We note that, by $\mathfrak{sl}_2$-theory, we have that $f_i^{\lambda(h_i)+1}v_\lambda = 0$ where, as usual, we denote by $f_i, h_i, e_i, 1 \le i \le n-1$ the standard Chevalley generators of $\mathfrak{sl}_n$.

It turns out that Weyl modules for the smash product $\mathcal{A} \rtimes G$ are related to Weyl modules for a much smaller ring. We have a decomposition of G-modules $\mathcal{A} = \mathcal{A}^G \oplus \mathcal{A}'$, where $\mathcal{A}'$ is the subrepresentation without invariants. Let us denote by $\overline{\mathcal{A}}$ the quotient of $\mathcal{A}$ by the two-sided ideal generated by $\mathcal{A}'$. Note that it may be much smaller than $\mathcal{A}^G$, it can even reduce to $\mathbb{C}$, for instance, when G is $\mathbb{Z}/d\mathbb{Z}$ and $\mathcal{A} = \mathbb{C}[u]$ or even $\mathcal{A} = \mathbb{C}[u, v]$.

Consider an augmentation ϵ of $\mathcal{A} \rtimes G$. Note that $\mathcal{A}' \subset [\mathcal{A} \rtimes G, \mathcal{A} \rtimes G]$, so $\epsilon(\mathcal{A}') = 0$ and ϵ descends to an augmentation $\overline{\epsilon}$ of $\overline{\mathcal{A}}$.

Theorem 5.7. *Let $\lambda \in \mathfrak{h}^*$ be a dominant integral weight. We have an isomorphism of modules over $\mathfrak{sl}_n(\mathcal{A} \rtimes G)$:*

$$W^{\epsilon}_{\mathcal{A}\rtimes\mathbb{C}[G]}(\lambda) \cong W^{\overline{\epsilon}}_{\overline{\mathcal{A}}}(\lambda).$$

We have a surjective map $\mathcal{A} \rtimes G \to \overline{\mathcal{A}}$, hence $\mathfrak{sl}_n(\mathcal{A} \rtimes G)$ acts on $W^{\overline{\epsilon}}_{\overline{\mathcal{A}}}(\lambda)$ and this yields a surjective map $W^{\epsilon}_{\mathcal{A}\rtimes G}(\lambda) \to W^{\overline{\epsilon}}_{\overline{\mathcal{A}}}(\lambda)$ of modules over $\mathfrak{sl}_n(\mathcal{A} \rtimes G)$. The ring $\overline{\mathcal{A}}$ is the quotient of $\mathcal{A} \rtimes G$ by the ideal J_ϵ generated by $\mathcal{A}'$ and elements $\gamma - \epsilon(\gamma)$ in $\mathbb{C}[G]$ for $\gamma \in G$. The only thing we need to show is that $\mathfrak{sl}_n \otimes J_\epsilon$ acts on $W^{\epsilon}_{\mathcal{A}\rtimes G}(\lambda)$ by zero.

Actually, since $\mathfrak{sl}_n \otimes J_\epsilon$ is an ideal, it is enough to show that it acts by zero on the highest weight vector v_λ. As $\epsilon(J_\epsilon) = 0$, this is true for $\mathfrak{b} \otimes J_\epsilon$, so it remains to prove our claim for $E_{ij} \otimes \epsilon$ with $i > j$. Now the question is reduced to $\mathfrak{sl}_2$-case, so, to simplify the notation, let us set $f = E_{ij}$, $e = E_{ji}$ and $h = E_{ii} - E_{jj}$.

Lemma 5.8. *Let $P \in \mathcal{A} \rtimes \Gamma$. If $\epsilon(P) = 0$, then $(f \otimes P^{\lambda(h)})v_\lambda = 0$.*

Proof. We already know that $f^{\lambda(h)+1}v_\lambda = 0$. Applying $e \otimes P$ j times and using the assumption $\epsilon(P) = 0$ yields $(f^{\lambda(h)+1-j} \otimes P^j)v_\lambda = 0$, so taking $j = \lambda(h)$ proves the lemma. □

Lemma 5.9. *If $(f \otimes P)v_\lambda = 0$ and $Q \in \mathcal{A}$, then $\big(f \otimes (PQ + QP)\big)v_\lambda = 0$. Moreover, if Q belongs to the commutator $[\mathfrak{sl}_n(\mathcal{A} \rtimes G), \mathfrak{sl}_n(\mathcal{A} \rtimes G)]$, then $(f \otimes PQ)v_\lambda = (f \otimes QP)v_\lambda = 0$.*

Proof. Applying $f \otimes P$ to both sides of $(h \otimes Q)v_\lambda = \lambda(h)\epsilon(Q)v_\lambda$ yields the first equality. If now, say, $Q = Q_1Q_2 - Q_2Q_1$, then $(E_{ii} + E_{jj}) \otimes Q = [h \otimes Q_1, h \otimes Q_2]$; starting from $(f \otimes P)v_\lambda = 0$ and applying $E_{ii} \otimes Q, E_{jj} \otimes Q$, we obtain the second equality. (Note that $E_{ii} \otimes Q$ belongs to $\mathfrak{sl}_2(\mathcal{A} \rtimes G)$ since, by assumption, $Q \in [\mathfrak{sl}_n(\mathcal{A} \rtimes G), \mathfrak{sl}_n(\mathcal{A} \rtimes G)]$.) □

Proof of theorem 5.7. First let us show that

$$(f \otimes (x - \epsilon(x)))v_\lambda = 0 \quad \text{for } x \in \mathbb{C}[G]. \tag{5}$$

Note that $\mathbb{C}[G] = \mathbb{C}c_\epsilon \oplus I_\epsilon$, where $c_\epsilon^2 = c_\epsilon$, $\epsilon(c_\epsilon) = 1$, and I_ϵ is the kernel of $\epsilon_{|\mathbb{C}[G]}$. We have $I_\epsilon^{l+1} = I_\epsilon$, so lemma 5.8 yields equation (5) for $x \in I_\epsilon$. Moreover, note that $c_\epsilon - 1$ belongs to I_ϵ, so we have (5) also for $x = c_\epsilon$ and, therefore, for any x.

Since $\mathcal{A}' \in [\mathfrak{sl}_n(\mathcal{A} \rtimes G), \mathfrak{sl}_n(\mathcal{A} \rtimes G)]$ and $\mathcal{A}^G$ commutes with $\mathbb{C}[G]$, lemma 5.9 implies that $(f \otimes \mathcal{A}(x - \epsilon(x)))v_\lambda = 0$.

It remains to show that $(f \otimes \mathcal{A}')v_\lambda = 0$. By lemma 5.9, for any $a \in \mathcal{A}'$, $\gamma \in G$, we have

$$(f \otimes \gamma a)v_\lambda = (f \otimes a\gamma)v_\lambda = (f \otimes a\epsilon(\gamma))v_\lambda.$$

So

$$\big(f \otimes (a - \gamma^{-1}(a))\big)v_\lambda = \frac{1}{\epsilon(\gamma)}\big(f \otimes \big(\gamma a - a\gamma\big)\big)v_\lambda = 0.$$

At last, note that the elements $(a - \gamma^{-1}(a))$ for $a \in \mathcal{A}'$, $\gamma \in G$, span $\mathcal{A}'$. □

5.3. **Weyl modules associated to rings of invariants.** When $\mathcal{A}$ is a commutative, unital, finitely generated $\mathbb{C}$-algebra, Weyl modules for $\mathfrak{sl}_n \otimes_{\mathbb{C}} \mathcal{A}$ can be attached to multisets of points in $\mathrm{Spec}(\mathcal{A})$ or, more generally, to any ideal in $\mathcal{A}$. In this subsection, we first apply an approach due to Feigin-Loktev [FeLo] and Chari-Pressley [ChPr] to describe certain local Weyl modules for $\mathfrak{sl}_n(\mathbb{C}[u,v])$ and $\mathfrak{sl}_n(\mathbb{C}[u,v]^\Gamma)$: this approach realizes them as the Schur-Weyl duals of certain modules of co-invariants. A natural question to ask is: what is the dimension of these local Weyl modules? For loop algebras $\mathfrak{sl}_n(\mathbb{C}[u,u^{-1}])$, this question was fully answered in [ChLo]. For $\mathfrak{sl}_n(\mathbb{C}[u,v])$ and a multiple of the fundamental weight of the natural representation of $\mathfrak{sl}_n$ on $\mathbb{C}^n$, this problem was solved in [FeLo], but the answer relies on the difficult theorem of M. Haiman on the dimension of diagonal harmonics [Ha2]. To compute the dimension of the Weyl modules that we introduce below, we would need an extension of Haiman's theorem to certain rings of coinvariants attached to wreath products of the cyclic group $\mathbb{Z}/d\mathbb{Z}$, but this is still an open problem as far as we know. At least, we are able to provide a lower bound for the dimension of some local Weyl modules by using a theorem of R. Vale [Va], which generalizes an earlier result of I. Gordon [Go1].

Definition 5.10. *Let U be a representation of $\mathfrak{sl}_n \otimes \mathcal{A}$ and $\mu \in \mathfrak{h}^*$. Suppose that we have an augmentation map $\epsilon : \mathcal{A} \longrightarrow \mathbb{C}$. A vector $v_\mu \in U$ is called a highest weight vector if $(g \otimes a)v_\mu = 0$ when $g \in \mathfrak{n}^+, a \in \mathcal{A}$ and $(h \otimes a)v_\mu = \mu(h)\epsilon(a)v_\mu$ for all $h \in \mathfrak{h}, a \in \mathcal{A}$.*

Theorem 5.11. [FeLo] *Suppose that $\mu \in \mathfrak{h}^*$ is a dominant integral weight. There exists a universal finite dimensional module $W_\epsilon^{\mathcal{A}}(\mu)$ such that any finite dimensional representation of $\mathfrak{sl}_n \otimes \mathcal{A}$ generated by a highest weight vector v_μ is a quotient of $W_\epsilon^{\mathcal{A}}(\mu)$.*

Let us recall a general result of [FeLo], see also [Lo]. The symmetric group S_l acts on $\mathcal{A}^{\otimes l}$ and we can form the quotient $\mathsf{DH}_l(\mathcal{A}) = \mathcal{A}^{\otimes l}/(\mathrm{Sym}^l(\mathcal{A})_\epsilon)$ of $\mathcal{A}^{\otimes l}$ by the ideal generated by the tensors invariant under the action of S_l and which are in the kernel of ϵ (extended as an augmentation $\mathrm{Sym}(\mathcal{A})$). When $\mathcal{A} = \mathbb{C}[u,v]$, this quotient is called the space of diagonal co-invariants.

If E is a representation of S_l, denote by $\mathsf{SW}_l^n(E)$ the representation of $\mathfrak{sl}_n$ given by $\big((\mathbb{C}^n)^{\otimes l} \otimes E\big)^{S_l}$. This is the classical Schur-Weyl construction. Note that $\mathsf{DH}_l(\mathcal{A})$ is a representation of S_l. More generally, as is observed in [Lo], if E is a representation of the smash product $(\mathcal{A}^{\otimes l}) \rtimes S_l$, then $\mathsf{SW}_l^n(E)$ is a representation of $\mathfrak{sl}_n(\mathcal{A})$.

Let ω_1 be the fundamental weight of $\mathfrak{sl}_n$ which is the highest weight of its natural representation $\mathbb{C}^n$.

Theorem 5.12. [FeLo] *The Weyl module $W_\epsilon^{\mathcal{A}}(l\omega_1)$ of $\mathfrak{sl}_n(\mathcal{A})$ is isomorphic to $\mathsf{SW}_l^n(\mathsf{DH}_l(\mathcal{A}))$.*

When $\mathcal{A} = \mathbb{C}[u,v]$, the dimension of the ring of diagonal harmonics $\mathsf{DH}_l(\mathcal{A})$ is a difficult result proved by M. Haiman in [Ha2]. Until after proposition 5.15, we will assume that Γ is an arbitrary finite subgroup of $SL_2(\mathbb{C})$. The preceding theorem can be applied to $\mathcal{A} = \mathbb{C}[u,v]^G$ and $\epsilon : \mathbb{C}[u,v]^G \longrightarrow \mathbb{C}$ the homomorphism given by the maximal ideal $\mathbb{C}[u,v]^G_+$ corresponding to the singularity. (Here, G is an arbitrary finite subgroup of $SL_2(\mathbb{C})$.) It gives a nice description of the Weyl module for multiples of ω_1, but, to compute its dimension, we would have to know more about the structure of $\mathsf{DH}_l(\mathbb{C}[u,v]^G)$ as a module for S_l: as far as we know, this is still an open problem when $G \neq \{1\}$. We are, however, able to obtain a partial result by considering the ring $\mathcal{A} = \mathbb{C}[u,v]$ but with highest weight conditions on $\mathfrak{h} \otimes \mathbb{C}[u,v]^\Gamma$, and from it we can deduce a lower bound when $\mathcal{A} = \mathbb{C}[u,v]^\Gamma$.

By $H_{G,l}$, denote the quotient of $\mathbb{C}[u_1,\dots,u_l,v_1,\dots,v_l]$ by the ideal generated by $S_l \ltimes G^{\times l}$-invariants with zero at the origin. This is a module for $S_l \ltimes G^{\times l}$, and, moreover, for $(S_l \ltimes G^{\times l}) \ltimes \mathbb{C}[u,v]^{\otimes l}$.

Definition 5.13. *Let $\mu \in \mathfrak{h}^*$ be a dominant integral weight and let $W_G(\mu)$ be the maximal finite dimensional module over $\mathfrak{sl}_n(\mathbb{C}[u,v])$, generated by a vector v_μ such that*

$$(\mathfrak{n}^+ \otimes \mathbb{C}[u,v])v_\mu = 0, \qquad (h \otimes P)v_\mu = \mu(h)P(0,0)v_\mu \quad \text{for } h \in \mathfrak{h},\ P \in \mathbb{C}[u,v]^G.$$

We say that $W_G(\mu)$ is the Weyl module for $\mathfrak{sl}_n(\mathbb{C}[u,v])$ associated to the ideal $(\mathbb{C}[u,v]^G_+)$.

Remark 5.14. *The existence of a maximal finite dimensional module with this property can be proved as in* [FeLo].

Proposition 5.15. *The Weyl module $W_G(l\omega_1)$ is Schur-Weyl dual to $H_{G,l}$, i.e. $W_G(l\omega_1) = \mathsf{SW}^n_l(H_{G,l})$.*

Proof. The argument is the same as the one used in [FeLo], so we just sketch it for the benefit of the reader. The Weyl module $W_G(l\omega_1)$ is the quotient of the global Weyl module of $\mathfrak{sl}_n(\mathbb{C}[u,v])$ for the weigh $l\omega_1$ by the submodule generated by $\left(h \otimes \mathbb{C}[u,v]^G_+\right) v_{l\omega_1}$. The global Weyl module for this weight is $\mathrm{Sym}^l(\mathbb{C}^n \otimes \mathbb{C}[u,v])$ and we have to quotient by the submodule generated by the action of $\mathrm{Sym}^l(\mathbb{C}[u,v]^G_+)$. Thus the Weyl module $W_G(l\omega_1)$ is obtained by applying the Schur-Weyl construction to the quotient of $\mathbb{C}[u_1,\dots,u_l,v_1,\dots,v_l]$ by the ideal generated by the $S_l \ltimes G^{\times l}$-invariant polynomials. $\square$

The following theorem of R. Vale is a generalization of a theorem of I. Gordon for S_l [Go1].

Theorem 5.16. [Va] *The representation $H_{\Gamma,l}$ has a quotient $H^0_{\Gamma,l}$ such that the trace on $H^0_{\Gamma,l} \otimes$* Sign *of a permutation $\sigma \in S_l$ consisting of s cycles is equal to $(dl+1)^s$.*

Now let us apply Theorem 5.16 to the character calculation for $W_\Gamma(l\omega_1)$. Let $F(l,k)$ be the set of functions from $\{1,\dots,l\}$ to $\{1,\dots,k\}$. This set admits an action of S_l by permutation of the arguments. Denote by $\mathbb{C}F(l,k)$ the corresponding complex representation of S_l.

Lemma 5.17. *Suppose that $\sigma \in S_l$ is a product of s cycles. Then the trace of σ on $\mathbb{C}F(l,k)$ is equal to k^s.*

Proof. The trace of σ is equal to the number of functions stable under the action of σ. A function is stable if it has the same value on all the elements of each cycle, so it is determined by s elements of $\{1,\dots,k\}$. $\square$

Lemma 5.18. *The $\mathfrak{sl}_n$-module $\mathsf{SW}^n_l(\mathbb{C}F(l,k) \otimes \mathrm{Sign})$ is isomorphic to $\bigwedge^l\left((\mathbb{C}^n)^{\oplus k}\right)$.*

Proof. Note that $(\mathbb{C}^n)^{\otimes l} \otimes \mathbb{C}F(l,k)$ is isomorphic to $\left((\mathbb{C}^n)^{\oplus k}\right)^{\otimes l}$ as $SL_n \times S_l$-module. The isomorphism can be constructed as the map sending $(v_1 \otimes \dots \otimes v_l) \otimes f$ to $v_1^{(f(1))} \otimes \dots \otimes v_l^{(f(l))}$, where $v^{(i)}$ belongs to the i-th summand of $(\mathbb{C}^n)^{\oplus k}$. Then the lemma follows by restricting this isomorphism to the Sign component. $\square$

Theorem 5.19. *The Weyl module $W_\Gamma(l\omega_1)$ has a quotient which, as a representation of $\mathfrak{sl}_n$, is isomorphic to $\bigwedge^l\left((\mathbb{C}^n)^{\oplus(dl+1)}\right)$.*

Proof. Since $H^0_{\Gamma,l}$ is a quotient of $H_{\Gamma,l}$, we have that $\mathsf{SW}^n_l(H^0_{\Gamma,l})$ is a quotient of $W_\Gamma(l\omega_1)$ by proposition 5.15. Lemma 5.17 and theorem 5.16 imply that $H^0_{\Gamma,l}$ is isomorphic to $\mathbb{C}F(l,k)\otimes\mathrm{Sign}$ with $k=dl+1$. By lemma 5.18, $\mathsf{SW}^n_l(H^0_{\Gamma,l})$ is thus equal to $\bigwedge^l\left((\mathbb{C}^n)^{\oplus(dl+1)}\right)$. $\square$

Corollary 5.20. *The dimension of $W_\Gamma(l\omega_1)$ is bounded below by $\begin{pmatrix} n(dl+1) \\ l \end{pmatrix}$.*

By modifying slightly the argument in the previous paragraphs, we can give a lower bound also for some local Weyl modules when $\mathcal{A}=\mathbb{C}[u,v]^\Gamma$. Let us introduce the following action of $\Gamma^{\times l}$ on $\mathbb{C}F(l,k)$: if we fix a generator ξ_i of the i^{th} copy of Γ in $\Gamma^{\times l}$ and if $f\in\mathbb{C}F(l,k)$, then $\xi_i(f)=\zeta^{j-1}f$ if $f(i)=j$. This action can be combined with the one associated to S_l to obtain an action of $\Gamma^{\times l}\rtimes S_l$. The trace of ξ_i on $\mathbb{C}F(l,k)$ is $\left(\sum_{j=0}^{k-1}\zeta^j\right)k^{l-1}$, which equals $\left(\sum_{j=0}^{\overline{k}}\zeta^j\right)k^{l-1}$ where $0\le\overline{k}\le d-1$ and $k-1\equiv\overline{k}\bmod(d)$. In particular, if $k\equiv 1\bmod(d)$, then this trace is k^{l-1}.

Theorem 5.16 (see [Va]) also states that the trace of ξ_i on $H^0_{\Gamma,l}\otimes\mathrm{Sign}$ is equal to $(dl+1)^{l-1}$. That theorem actually applies to any element in $\Gamma^{\times l}\rtimes S_l$, and from it we can deduce that we have an isomorphism of $\Gamma^{\times l}\rtimes S_l$-modules between $H^0_{\Gamma,l}$ and $\mathbb{C}F(l,ld+1)\otimes\mathrm{Sign}$. ($\Gamma^{\times l}$ acts trivially on Sign.) Therefore, $(H^0_{\Gamma,l})^{\Gamma^{\times l}}\cong\mathbb{C}F(l,l+1)\otimes\mathrm{Sign}$ since the functions in $\mathbb{C}F(l,ld+1)$ which are invariants under $\Gamma^{\times l}$ can be identified with $\mathbb{C}F(l,l+1)$.

Now, we can repeat the argument we used above. We have that $\mathsf{SW}^n_l((H^0_{\Gamma,l})^{\Gamma^{\times l}})$ is a quotient of $\mathsf{SW}^n_l(H^{\Gamma^{\times l}}_{\Gamma,l})$ isomorphic to $\bigwedge^l\left((\mathbb{C}^n)^{\oplus(l+1)}\right)$. Let $W^\Gamma(\mu)$ be defined as $W_\Gamma(\mu)$ in definition 5.13, but with $\mathbb{C}[u,v]$ replaced by $\mathbb{C}[u,v]^\Gamma$. The Weyl module $W^\Gamma(l\omega_1)$ is isomorphic to $\mathsf{SW}^n_l(\mathsf{DH}_l(\mathbb{C}[u,v]^\Gamma))$. Since $\mathsf{DH}_l(\mathbb{C}[u,v]^\Gamma)\cong(H_{\Gamma,l})^{\Gamma^{\times l}}$, we conclude that $W^\Gamma(l\omega_1)$ has a quotient isomorphic to $\bigwedge^l\left((\mathbb{C}^n)^{\oplus(l+1)}\right)$ as $\mathfrak{sl}_n$-module, whence the following corollary.

Corollary 5.21. *The dimension of $W^\Gamma(l\omega_1)$ is bounded below by $\begin{pmatrix} n(l+1) \\ l \end{pmatrix}$.*

There is no reason to expect that the lower bound in corollary 5.20 is the best possible. Indeed, we can show that, when $d=2, l=4$, it is too low, following ideas of I. Gordon. In this case, $\Gamma^{\times l}\rtimes S_l$ is isomorphic to the Weyl group W of type B_4. In [Ha1], M. Haiman explains that the ring of diagonal coinvariants in this case has dimension 9^4+1, which is one more than the dimension of a certain quotient introduced in conjectures 7.1.2,7.1.3 and 7.2.3 in *op. cit.* These conjectures were proved by I. Gordon in [Go1] and we denoted above this quotient by $H^0_{\Gamma,l}$.

This means that, in $H_{\mathbb{Z}/2\mathbb{Z},4}$, there is a one-dimensional subspace E which carries a non-trivial representation of W. There is an action of $\mathfrak{sl}_2(\mathbb{C})$ on $H_{\mathbb{Z}/2\mathbb{Z},4}$ commuting with the action of S_4 (this is actually true in $\mathbb{C}[u_1,v_1,\ldots,u_l,v_l]$ for any $l\in\mathbb{Z}_{\ge1}$), so E is also a representation of $\mathfrak{sl}_2(\mathbb{C})$ and must thus be trivial. The standard diagonal element $h\in\mathfrak{sl}_2(\mathbb{C})$ acts by $\sum_{i=1}^4\left(u_i\frac{d}{du_i}-v_i\frac{d}{dv_i}\right)$, so this operators acts trivially on E, which implies that the monomials which appear in E have their u-degree equal to their v-degree.

The Weyl module $W_{\mathbb{Z}/2\mathbb{Z}}(4\omega_1)$ is obtained by applying the Schur-Weyl functor to $H_{\mathbb{Z}/2\mathbb{Z},4}$, so, if $n\ge4$, $W_{\mathbb{Z}/2\mathbb{Z}}(4\omega_1)$ has dimension greater than $\begin{pmatrix} 9n \\ 4 \end{pmatrix}$, which shows that the lower bound in corollary 5.20 is too low.

6. Matrix Lie algebras over rational Cherednik algebras of rank one

The polynomial ring $\mathbb{C}[u,v]$ can be deformed into the first Weyl algebra $A_1 = \mathbb{C}\langle u,v\rangle/(vu - uv - 1)$, which can be viewed as the ring $\mathcal{D}(\mathbb{C})$ of algebraic differential operators on the affine line $\mathbb{A}^1_{\mathbb{C}}$. Such differential operators play an important role in the representation theory of Cherednik algebras and the G-DDCA of [Gu1, Gu2, Gu3] are also deformations of the enveloping algebra of $\mathfrak{gl}_n(A_1 \rtimes \Gamma)$ when G is a finite subgroup of $SL_2(\mathbb{C})$.

More generally, the rational Cherednik algebra $\mathsf{H}_{t,\mathbf{c}}(G_l)$ for the wreath product $G_l = G^{\times l} \rtimes S_l$ admits two specializations of particular interest: $\mathsf{H}_{t=0,\mathbf{c}=\mathbf{0}}(G_l) \cong \mathbb{C}[x_1, y_1, \ldots, x_l, y_l] \rtimes G_l$ and $\mathsf{H}_{t=1,\mathbf{c}=\mathbf{0}}(G_l) \cong A_l \rtimes G_l$ where A_l is the l^{th} Weyl algebra. The representation theories of these two algebras differ greatly: for instance, in the first case, $\mathsf{H}_{t=0,\mathbf{c}=\mathbf{0}}(G_l)$ has infinitely many irreducible finite dimensional representations, whereas $\mathsf{H}_{t=1,\mathbf{c}=\mathbf{0}}(G_l)$ has none. Actually, $\mathsf{H}_{t=1,\mathbf{c}}(G_l)$ does not have any finite dimensional representations for generic values of $\mathbf{c}$. The Γ-DDCA also admit two such specializations and it is reasonable to expect that their representation theories will thus differ noticeably. In this article, we want to start investigating the categories of modules for these two specializations, so, in this section we will study matrix Lie algebras over rational Cherednik algebras of rank one with $t \neq 0$.

Definition 6.1. *Let $\mathbf{c} = (c_1, \ldots, c_{d-1}) \in \mathbb{C}^{d-1}$. The rational Cherednik algebra $\mathsf{H}_{t,\mathbf{c}}(\Gamma)$ of rank one is the algebra generated by elements u, v, γ with $\gamma \in \Gamma = \mathbb{Z}/d\mathbb{Z}$ and the following relations:*

$$\gamma u \gamma^{-1} = \zeta u, \quad \gamma v \gamma^{-1} = \zeta^{-1} v$$

$$vu - uv = t + \sum_{i=1}^{d-1} c_i \xi^i \text{ where } \xi \text{ is a generator of } \Gamma. \tag{6}$$

It will also be convenient to rewrite equation (6) in the form $vu - uv = t + \sum_{i=0}^{d-1} \widetilde{c}_i(\mathbf{e}_i - \mathbf{e}_{i+1})$ for some $\widetilde{c}_i \in \mathbb{C}$. We will need to use later the element ω which can be written in the following three equivalent ways:

$$\omega = -uv + \sum_{i=0}^{d-1} \widetilde{c}_i \mathbf{e}_{i+1} = -vu + t + \sum_{i=0}^{d-1} \widetilde{c}_i \mathbf{e}_i = -\frac{uv + vu}{2} + \frac{t}{2} + \frac{1}{2}\sum_{i=0}^{d-1} \widetilde{c}_i(\mathbf{e}_i + \mathbf{e}_{i+1}).$$

Then one can check that $[\omega, u] = -tu, \ [\omega, v] = tv$.

Definition 6.2. *Let $\mathbf{c} = (c_1, \ldots, c_{d-1}) \in \mathbb{C}^{d-1}$. We will call trigonometric Cherednik algebra of rank one the algebra $\mathbf{H}_{t,\mathbf{c}}(\Gamma) = \mathbb{C}[u^{\pm 1}] \otimes_{\mathbb{C}[u]} \mathsf{H}_{t,\mathbf{c}}(\Gamma)$.*

Remark 6.3. *Trigonometric Cherednik algebras exist only for Weyl groups (real Coxeter groups), but we propose to use the terminology in the previous definition because it is convenient. Moreover, as is explained in* [Gu3]*, $\mathbf{H}_{t,\mathbf{c}}(\Gamma)$ depends actually only on the t parameter, that is, $\mathbf{H}_{t,\mathbf{c}}(\Gamma) \cong \mathbf{H}_{t,\mathbf{c}=\mathbf{0}}(\Gamma)$, but this is not true for $\mathsf{H}_{t,\mathbf{c}}(\Gamma)$. An explicit isomorphism $\mathbf{H}_{t,\mathbf{c}}(\Gamma) \xrightarrow{\sim} \mathbf{H}_{t,\mathbf{c}=\mathbf{0}}(\Gamma)$ is given by*

$$v \mapsto v + \left(\sum_{i=1}^{d-1} \frac{c_i}{1 - \zeta^{-i}} \xi^i\right) u^{-1}.$$

One can also note that $\mathbf{H}_{t,\mathbf{c}}(\Gamma)$ is generated by $\omega, u, u^{-1}, \Gamma$ and that $[\omega, u^{-1}] = tu^{-1}$.

The associative algebras $\mathbf{H}_{t,\mathbf{c}}(\Gamma)$ and $\mathsf{H}_{t,\mathbf{c}}(\Gamma)$ can be turned into Lie algebras in the usual way and the representation theory of a central extension of the Lie algebra $\mathfrak{gl}_n(\mathbf{A}_1)$ was studied in [BKLY, KaRa]. (Here, $\mathbf{A}_1$ is the algebra of differential operators on $\mathbb{C}^\times$.) The case of the quantum torus $\mathcal{D}_q(\mathbb{C}^\times) = \mathbb{C}\langle u^{\pm 1}, v^{\pm 1}\rangle/(vu = quv)$ was considered in [BoLi]. In this section, we present some results about the structure of the Lie algebras $\mathfrak{sl}_n(\mathsf{H}_{t,\mathbf{c}}(\Gamma)), \mathfrak{sl}_n(\mathbf{H}_{t,\mathbf{c}}(\Gamma))$ mostly when $t \neq 0$.

The (Lie) algebras $\mathbf{H}_{t,\mathbf{c}}(\Gamma)$ and $\mathsf{H}_{t,\mathbf{c}}(\Gamma)$ are graded: $\deg(u) = -1, \deg(v) = 1, \deg(\gamma) = 0$. This induces gradings on the associative algebras $M_n(\mathbf{H}_{t,\mathbf{c}}(\Gamma)), M_n(\mathsf{H}_{t,\mathbf{c}}(\Gamma))$ and on the Lie algebras $\mathfrak{gl}_n(\mathbf{H}_{t,\mathbf{c}}(\Gamma)), \mathfrak{gl}_n(\mathsf{H}_{t,\mathbf{c}}(\Gamma))$. However, we will consider instead the following grading:

$$\deg(E_{ij} v^r u^s \gamma) = (r-s)n + j - i.$$

In the case $\mathbf{H}_{t=1,\mathbf{c}=\mathbf{0}}(\Gamma = \{1\})$, this is the opposite of the principal $\mathbb{Z}$-gradation considered in [BKLY]. The graded pieces of degree k will be denoted $\mathfrak{gl}_n(\mathbf{H}_{t,\mathbf{c}}(\Gamma))[k]$ and $\mathfrak{gl}_n(\mathsf{H}_{t,\mathbf{c}}(\Gamma))[k]$.

6.1. **Central extensions.** It was computed in [AFLS] that $HH_1(A_1 \rtimes \Gamma) = 0$, hence $HC_1(A_1 \rtimes \Gamma) = 0$. Furthermore, it is proved in [EG] that, for any $\mathbf{c}$, $HH_1(\mathsf{H}_{t,\mathbf{c}}(\Gamma)) = 0$ for all $t \in \mathbb{C}^\times$ except in a countable set. For all such values of $t, \mathbf{c}$, the Lie algebra $\mathfrak{sl}_n(\mathsf{H}_{t,\mathbf{c}}(\Gamma))$ has no non-trivial central extension. For this reason, contrary to [BKLY], we will not consider central extensions.

6.2. **Parabolic subalgebras.** In this subsection, we will assume that $t \neq 0$, so, without loss of generality, let us set $t = 1$. For a Lie algebra with triangular decomposition, one usually wants to construct representations by induction from its non-negative Lie subalgebra (a sort of Borel subalgebra) or, more generally, from a bigger subalgebra which contains this one. This suggests that the following definition may be relevant.

Definition 6.4. [BKLY] *A parabolic subalgebra $\mathfrak{q}$ of the Lie algebra $\mathfrak{gl}_n(\mathsf{H}_{t=1,\mathbf{c}}(\Gamma))$ is a graded Lie subalgebra of the form*

$$\mathfrak{q} = \oplus_{\mathbb{Z}} \mathfrak{q}[k],\ \mathfrak{q}[k] = \mathfrak{gl}_n(\mathsf{H}_{t=1,\mathbf{c}}(\Gamma))[k] \text{ if } k \geq 0, \mathfrak{q}[k] \subset \mathfrak{gl}_n(\mathsf{H}_{t=1,\mathbf{c}}(\Gamma))[k] \text{ if } k < 0).$$

For $k < 0$, we can decompose $\mathfrak{q}[k]$ as

$$\mathfrak{q}[k] = \bigoplus_{\substack{r,l,i,j \\ -rn+j-i=k}} E_{ij} u^r I_k^{i,l} \mathbf{e}_l$$

for some subspace $I_k^{i,l} \subset \mathbb{C}[\omega]$.

Lemma 6.5. *The subspace $I_k^{i,l}$ is an ideal of $\mathbb{C}[\omega]$.*

Proof. Suppose that $E_{ij} u^r p(\omega) \mathbf{e}_l \in \mathfrak{q}[k]$ with $p(\omega) \in I_k^{i,l}$, and choose $f(\omega) \in \mathbb{C}[\omega]$. If $r = 0$, then, since $E_{ii} f(\omega) \mathbf{e}_l \in \mathfrak{q}[0]$, we deduce that, if $1 \leq i \neq j \leq n$, $[E_{ii} f(\omega) \mathbf{e}_l, E_{ij} p(\omega) \mathbf{e}_l] = E_{ij} f(\omega) p(\omega) \mathbf{e}_l \in \mathfrak{q}[k]$, hence $f(\omega) p(\omega) \in I_k^{i,l}$.

Now suppose that $r > 0$. We want to prove by induction on $a \in \mathbb{Z}_{\geq 0}$ that $\omega^a p(\omega) \in I_k^{i,l}$. We note that

$$[I\omega^{a+1}, E_{ij}(u^r p(\omega) \mathbf{e}_l)] = E_{ij}\left(u^r((\omega - r)^{a+1} - \omega^{a+1}) p(\omega) \mathbf{e}_l\right) \in \mathfrak{q}[k]$$

and that the term of highest power in $(\omega - r)^{a+1} - \omega^{a+1}$ is $-r(a+1)\omega^a$, so that we can apply induction. □

Following the ideas of [BKLY, KaRa], we chose a monic generator $b_k^{i,l}(\omega)$ of the principal ideal $I_k^{i,l}$ if this ideal is non-zero; otherwise, we set $b_k^{i,l}(\omega) = 0$. These are called the characteristic polynomials of $\mathfrak{q}$.

Definition 6.6. *The parabolic subalgebra $\mathfrak{q}$ is called non-degenerate if $\mathfrak{q}[k] \neq 0\, \forall k \in \mathbb{Z}$.*

Proposition 6.7. *A parabolic subalgebra $\mathfrak{q}$ is non-degenerate if and only if the polynomials $b_{-1}^{i,l}(\omega), 1 \leq i \leq n, 0 \leq l \leq d-1$ are all non-zero.*

Proof. The parabolic subalgebra $\mathfrak{q}$ is non-degenerate if and only if the polynomials $b_k^{i,l}(\omega), 1 \leq i \leq n, 0 \leq l \leq d-1$ are all non-zero for all $k \in \mathbb{Z}_{\leq -1}$, so it is enough to prove the following for $i = 1, \ldots, n$: if $b_k^{i,l}(\omega) \neq 0$ and $b_{-1}^{i-1,l}(\omega) \neq 0$, $b_{k-1}^{i,l}(\omega)$ is also non-zero and divides $b_k^{i-1,l}(\omega) b_{-1}^{i,l+r}(\omega - r)$. Here, if $2 \leq i \leq n$, r is determined by $k + i - 1 = -rn + j$ for some $1 \leq j \leq n$. (We set $b_k^{0,l}(\omega) = b_k^{n,l}(\omega)$.)

If $i \neq 1$, we have

$$\begin{aligned}&[E_{i,i-1} b_{-1}^{i,l+r}(\omega)\mathbf{e}_{l+r}, E_{i-1,j} u^r b_k^{i-1,l}(\omega)\mathbf{e}_l]\\ &= E_{ij} u^r b_{-1}^{i,l+r}(\omega - r) b_k^{i-1,l}(\omega)\mathbf{e}_l - \delta_{j,i}\delta_{r0} E_{i-1,i-1} u^r b_{-1}^{i,l+r}(\omega) b_k^{i-1,l}(\omega)\mathbf{e}_{l+r} \in \big[\mathfrak{q}[-1], \mathfrak{q}[k]\big]\end{aligned}$$

and $\big[\mathfrak{q}[-1], \mathfrak{q}[k]\big] \subset \mathfrak{q}[k-1]$, so $b_{-1}^{i,l+r}(\omega - r) b_k^{i-1,l}(\omega) \in I_{k-1}^{i,l}$ and the claim is true if $i \neq n$.

To prove the claim when $i = 1$ (and with r determined by $k = -(r+1)n + j$), we consider the following commutator:

$$\begin{aligned}&[E_{1n} u b_{-1}^{1,l+r}(\omega)\mathbf{e}_{l+r}, E_{nj} u^r b_k^{n,l}(\omega)\mathbf{e}_l]\\ &= E_{1j} u^{r+1} b_{-1}^{1,l+r}(\omega - r) b_k^{n,l}(\omega)\mathbf{e}_l - \delta_{j1}\delta_{r,-1} E_{nn} u^{r+1} b_k^{n,l}(\omega - 1) b_{-1}^{1,l+r}(\omega)\mathbf{e}_{l+r}\end{aligned}$$

which belongs to $\big[\mathfrak{q}[-1], \mathfrak{q}[k]\big] \subset \mathfrak{q}[k-1]$, so $b_{-1}^{1,l+r}(\omega - r) b_k^{n,l}(\omega) \in I_{k-1}^{1,l}$.

Using similar computations, one can prove the following for $i = 1, \ldots, n$: if $b_{k-1}^{i,l}(\omega) \neq 0$, then $b_k^{i-1,l}(\omega) \neq 0$ and it divides $b_{k-1}^{i,l}(\omega)$. $\square$

The characteristic polynomials $b_{-1}^{i,l}(\omega), 1 \leq i \leq n, 0 \leq l \leq d-1$ can help us describe the derived Lie subalgebra $[\mathfrak{q}, \mathfrak{q}]$. Set

$$\begin{aligned}\mathfrak{gl}_n(\mathsf{H}_{t=1,\mathbf{c}}(\Gamma))[0, \mathbf{b}] = &\operatorname{span}\{H_i \omega^r b_{-1}^{i+1,l}(\omega)\mathbf{e}_l | 1 \leq i \leq n-1, r \in \mathbb{Z}_{\geq 0}, 0 \leq l \leq d-1\}\\ &\oplus \operatorname{span}\{E_{11} uv(\omega+1)^r b_{-1}^{1,l}(\omega+1)\mathbf{e}_{l+1} - E_{nn} vu\omega^r b_{-1}^{1,l}(\omega)\mathbf{e}_l | r \in \mathbb{Z}_{\geq 0}\}\end{aligned}$$

Proposition 6.8. *Let* $\mathbf{b} = (b_{-1}^{i,l}(\omega))_{1\leq i\leq n}^{0\leq l\leq d-1}$ *be the first nd characteristic polynomials of the parabolic subalgebra* $\mathfrak{q}$. *Then*

$$[\mathfrak{q}, \mathfrak{q}] = \left(\bigoplus_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \mathfrak{q}[k]\right) \oplus \mathfrak{gl}_n(\mathsf{H}_{t=1,\mathbf{c}}(\Gamma))[0, \mathbf{b}].$$

Proof. Since $\mathfrak{q}[k+1] = [\mathfrak{q}[k], \mathfrak{q}[1]]$ if $k \in \mathbb{Z}_{\geq 0}$, it is enough to show that $[\mathfrak{q}[1], \mathfrak{q}[-1]] = \mathfrak{gl}_n(\mathsf{H}_{t=1,\mathbf{c}}(\Gamma))[0, \mathbf{b}]$.

$$[E_{i,i+1}\omega^r \mathbf{e}_{l_1}, E_{i+1,i} b_{-1}^{i+1,l_2}(\omega)\mathbf{e}_{l_2}] = \delta_{l_1 l_2}(E_{ii} - E_{i+1,i+1})\omega^r b_{-1}^{i+1,l_2}(\omega)\mathbf{e}_{l_1}$$

$$\begin{aligned}[E_{1n} u b_{-1}^{1,l_1}(\omega)\mathbf{e}_{l_1}, E_{n1} v(\omega+1)^r \mathbf{e}_{l_2}] =&\delta_{l_1+1,l_2}\big(E_{11} uv(\omega+1)^r b_{-1}^{1,l_1}(\omega+1)\mathbf{e}_{l_2}\\ &- E_{nn} vu\omega^r b_{-1}^{1,l_1}(\omega)\mathbf{e}_{l_1}\big)\end{aligned}$$

$$[E_{i,i+1}\omega^r \mathbf{e}_{l_1}, E_{1n} u b_{-1}^{1,l_2}(\omega)\mathbf{e}_{l_2}] = 0, \ \ [E_{i+1,i} b_{-1}^{i+1,l_1}(\omega)\mathbf{e}_{l_1}, E_{n1} v\omega^r \mathbf{e}_{l_2}] = 0 \text{ if } 1 \leq i \leq n-1$$

$\square$

6.3. **Embedding into $\overline{\mathfrak{gl}}_\infty$.** One of the main ingredients in the study of the representation theory of $\mathfrak{gl}_n(\mathbf{A}_1)$ in [BKLY, KaRa] is an embedding of the algebra $M_n(\mathbf{A}_1)$ into the algebra $\overline{M}_\infty$ of infinite matrices with only finitely many non-zero diagonals. This induces an embedding of the Lie algebra $\mathfrak{gl}_n(\mathbf{A}_1)$ into $\overline{\mathfrak{gl}}_\infty$. It comes from the action of $\mathfrak{gl}_n(\mathbf{A}_1)$ on $\mathbb{C}^n \otimes \mathbb{C}[u, u^{-1}]$. In this subsection, we obtain similar embeddings for $\mathfrak{gl}_n(\mathsf{H}_{t=1,\mathbf{c}}(\Gamma))$ and $\mathfrak{gl}_n(\mathbf{A}_1 \rtimes \Gamma)$ when Γ is cyclic. The embedding $\mathfrak{gl}_n(\mathbf{A}_1 \rtimes \Gamma) \hookrightarrow \overline{\mathfrak{gl}}_\infty$ is the same as the one considered in [BKLY, KaRa] when Γ is trivial, and $\mathfrak{gl}_n(\mathsf{H}_{t=1,\mathbf{c}}(\Gamma)) \hookrightarrow \overline{\mathfrak{gl}}_\infty$ comes also from the action of $\mathfrak{gl}_n(\mathsf{H}_{t=1,\mathbf{c}}(\Gamma))$ on $\mathbb{C}^n \otimes \mathbb{C}[u, u^{-1}]$ via the Dunkl embedding of $\mathsf{H}_{t=1,\mathbf{c}}(\Gamma)$. (We will reserve the notation M_∞ and $\mathfrak{gl}_\infty$ for the algebra and the Lie algebra of infinite matrices with finitely many non-zero entries.)

The space M_∞ has a linear basis given by elementary matrices E_{ij} with $(i,j) \in \mathbb{Z} \times \mathbb{Z}$. The embedding of associative algebras $\iota : M_n(\mathbf{A}_1 \rtimes \Gamma) \hookrightarrow \overline{M}_\infty$ is given explicitly by the formula

$$\iota(E_{ij} u^s \omega^r \mathbf{e}_k) = \sum_{l \in \mathbb{Z}} (-ld - k)^r E_{(ld+k+s)n+i-1,(ld+k)n+j-1}, \quad s \in \mathbb{Z}, 0 \leq k \leq d-1.$$

This restricts to an embedding $\iota : M_n(\mathsf{H}_{t=1,\mathbf{c}}(\Gamma)) \hookrightarrow \overline{M}_\infty$ by pulling back via $\mathsf{H}_{t=1,\mathbf{c}}(\Gamma) \hookrightarrow \mathbf{H}_{t=1,\mathbf{c}}(\Gamma) \cong \mathbf{A}_1 \rtimes \Gamma$. Explicitly, since

$$v = -u^{-1}(\omega - \sum_{l=0}^{d-1} \widetilde{c}_{l-1} \mathbf{e}_l),$$

we get

$$\iota(E_{ij} v \mathbf{e}_k) = \sum_{l \in \mathbb{Z}} (ld + k + \widetilde{c}_{k-1}) E_{(ld+k-1)n+i-1,(ld+k)n+j-1}.$$

This can be extended to

$$\iota(E_{ij} v^s \omega^r \mathbf{e}_k) = \sum_{l \in \mathbb{Z}} \left(\prod_{p=0}^{s-1} (ld + k - p + \widetilde{c}_{k-p-1}) \right) (-ld - k)^r E_{(ld+k-s)n+i-1,(ld+k)n+j-1}. \tag{7}$$

The principal grading on the algebra $\overline{M}_\infty$ and on the Lie algebra $\overline{\mathfrak{gl}}_\infty$ is given by $\deg(E_{ij}) = j - i$ and the embedding ι respects all the gradings.

We will need, as in [BKLY], to consider infinite matrices over the ring of truncated polynomials $R_m = \mathbb{C}[t]/(t^{m+1})$. Fixing $a \in \mathbb{C}$, we define an algebra map $\varphi_a^{[m]} : M_n(\mathsf{H}_{t=1,\mathbf{c}}(\Gamma)) \longrightarrow \overline{M}_\infty(R_m)$ by

$$\varphi_a^{[m]}(E_{ij} u \mathbf{e}_k) = \sum_{l \in \mathbb{Z}} E_{(ld+k+1)n+i-1,(ld+k)n+j-1},$$

$$\varphi_a^{[m]}(E_{ij} v \mathbf{e}_k) = \sum_{l \in \mathbb{Z}} (ld + k + a + t + \widetilde{c}_{k-1}) E_{(ld+k-1)n+i-1,(ld+k)n+j-1}.$$

This extends to a map $\varphi_a^{[m]} : M_n(\mathbf{A}_1 \rtimes \Gamma) \longrightarrow \overline{M}_\infty(R_m)$. Explicitly,

$$\varphi_a^{[m]}(E_{ij} u^s \omega^r \mathbf{e}_k) = \sum_{l \in \mathbb{Z}} (-ld - k - a - t)^r E_{(ld+k+s)n+i-1,(ld+k)n+j-1}, \quad s \in \mathbb{Z}, 0 \leq k \leq d-1.$$

This embedding when $d = 2, n = 1$ is related to the embedding considered in [Sh] from the Lie algebra $\mathfrak{gl}_\lambda$ to $\overline{\mathfrak{gl}}_{\infty,s}$ (in the notation of [Sh]) since $\mathfrak{gl}_\lambda$ is obtained by turning into a Lie algebra a certain primitive quotient of $\mathfrak{U}\mathfrak{sl}_2(\mathbb{C})$ and this primitive quotient is isomorphic to the spherical subalgebra of $\mathsf{H}_{t=1,c=\lambda}(\mathbb{Z}/2\mathbb{Z})$.

6.4. **Geometric interpretation.** It was observed in [KaRa] that the algebra of holomorphic differential operators on $\mathbb{C}^\times$ has a geometric interpretation in terms of a certain infinite dimensional vector bundle over the cylinder $\mathbb{C}/\mathbb{Z}$. The algebras $\mathbf{A}_1 \rtimes \Gamma$ and $A_1 \rtimes \Gamma$ afford similar interpretations: to explain it, we have to extend them to a holomorphic setting.

Let $\mathcal{O}(w)$ be the ring of entire functions (holomorphic on all of $\mathbb{C}$) in the variable w. Let $\mathbf{A}_1^{\mathcal{O}} \rtimes \Gamma$ to be the span of the operators of the form $u^r f(w)\gamma$ with $f \in \mathcal{O}(w)$, $r \in \mathbb{Z}$. This span has an algebra structure extending the one on $\mathbf{A}_1 \rtimes \Gamma$. Let $A_1^{\mathcal{O}} \rtimes \Gamma$ be the subalgebra of $\mathbf{A}_1^{\mathcal{O}} \rtimes \Gamma$ consisting of linear combinations of operators of the form $u^r f(w)\gamma$ and $v^s f(w)\gamma$ with $r, s \geq 0$ and f holomorphic.

For $k \in \mathbb{Z}$, we define an automorphism θ_k of $\overline{M}_\infty$ and of $\overline{\mathfrak{gl}}_\infty$ by $\theta_k(E_{ij}) = E_{i+k,j+k}$. For $0 \leq k \leq d-1$, let $\overline{M}_\infty^k \subset \overline{M}_\infty$ be the subspace of matrices such that the (i,j) entry is zero if $j \notin \cup_{l\in\mathbb{Z}}[ldn + kn, ldn + (k+1)n[$. The following definition is adapted from definition 3.4 in [KaRa].

Definition 6.9. *A (n,d)-monodromic loop is a holomorphic map $\ell : \mathbb{C} \longrightarrow \overline{M}_\infty$ such that $\ell(w) = \ell_0(w) + \cdots + \ell_{d-1}(w)$ with $\ell_k(w-d) = \theta_n^d \ell_k(w)$ and $\ell_k(w) \in \overline{M}_\infty^k$ for $0 \leq k \leq d-1$.*

When $d = 1$, the following proposition was established in [KaRa].

Proposition 6.10. *The algebra $M_n(\mathbf{A}_1^{\mathcal{O}} \rtimes \Gamma)$ is isomorphic to the algebra $\mathcal{L}_{n,d}$ of (n,d)-monodromic loops.*

Proof. We can construct a map $M_n(\mathbf{A}_1^{\mathcal{O}} \rtimes \Gamma) \longrightarrow \mathcal{L}_{n,d}$ by $E \mapsto (w \mapsto \varphi_w^{[0]}(E))$. That the formula $w \mapsto \varphi_w^{[0]}(E)$ defines an (n,d)-monodromic loop follows from the formula for $\varphi_w^{[0]}$. The inverse is given in the following way. If, given a monodromic loop ℓ, the loop ℓ_k is concentrated along the $(sn+m)^{th}$ $(0 \leq m \leq n-1)$ diagonal below the main one (so $sn+m \geq 0$; if it is above, the argument is similar), so that $\ell_k = \sum_{i=1}^n \sum_{l\in\mathbb{Z}} f_{i,l,k}(w) E_{(ld+k+s)n+m+i-1,(ld+k)n+i-1}$, then the preimage of ℓ_k is $\sum_{i=1}^{n-m} E_{m+i,i} u^s f_{i,0,k}(-\omega - k)\mathbf{e}_k + \sum_{i=1}^m E_{i,n-m+i} u^{s+1} f_{n-m+i,0,k}(-\omega - k)\mathbf{e}_k$. ∎

Since $M_n(A_1^{\mathcal{O}} \rtimes \Gamma) \hookrightarrow M_n(\mathbf{A}_1^{\mathcal{O}} \rtimes \Gamma)$, we can identify $M_n(A_1^{\mathcal{O}} \rtimes \Gamma)$ with the algebra of (n,d)-monodromic loops ℓ such that, writing $\ell(w) = \sum_{i,j\in\mathbb{Z}} \ell_{i,j}(w)E_{ij}$, we have that, if $i = l_1 n + p_1 - 1, j = l_2 n + p_2 - 1, 1 \leq p_1, p_2 \leq n$ and $l_1 < l_2$, then $\ell_{i,j}(w) = 0$ for $w = p - l_2 d$ and $p = 0, \ldots, l_2 - l_1 - 1$.

7. Highest weight representations for matrix Lie algebras over Cherednik algebras of rank one

Inspired by the papers [GGOR, Gu1], we suggest a notion of category $\mathcal{O}$ for the Lie algebra $\mathfrak{sl}_n(\mathsf{H}_{t=1,\mathbf{c}}(\Gamma))$ and we study certain modules in it which we call quasi-finite highest weight modules.

Definition 7.1. *Assume that $t \neq 0$. The category $\mathcal{O}\big(\mathfrak{sl}_n(\mathsf{H}_{t=1,\mathbf{c}}(\Gamma))\big)$ is the category of finitely generated modules M over $\mathfrak{U}\mathfrak{sl}_n(\mathsf{H}_{t=1,\mathbf{c}}(\Gamma))$ upon which $\mathfrak{sl}_n(v\mathbb{C}[v])$ acts locally nilpotently.*

This definition applies also to the Γ-deformed double current algebras $\mathsf{D}_{\lambda,\mathbf{b}}^n(\Gamma)$ of [Gu3]. One justification for it is that the Schur-Weyl functor studied in [Gu1, Gu3] sends modules in the category $\mathcal{O}$ of a rational Cherednik algebra for $\Gamma^{\times l} \rtimes S_l$ to a module in $\mathcal{O}\big(\mathsf{D}_{\lambda,\mathbf{b}}^n(\Gamma)\big)$ (for appropriate values of $\lambda, \mathbf{b}$). It is possible, using induction, to construct analogs of Verma modules in this category, and one can ask about the classification of irreducible (integrable) modules in the category $\mathcal{O}\big(\mathfrak{sl}_n(\mathsf{H}_{t=1,\mathbf{c}}(\Gamma))\big)$. We will not try to answer this question. Instead, we will study certain modules in these categories by following the ideas in [BKLY, KaRa].

Recall the grading on $\mathfrak{sl}_n(A_1 \rtimes \Gamma)$ and the embeddings $\varphi_a^{[m]} : \mathfrak{sl}_n(A_1 \rtimes \Gamma) \hookrightarrow \overline{\mathfrak{gl}}_\infty(R_m)$. The Lie algebra $\overline{\mathfrak{gl}}_\infty$ has an obvious triangular structure compatible with the grading given by $\deg(E_{ij}) = j - i$ and the embeddings $\varphi_a^{[m]}$ respect the grading on the source and target spaces. The following definition comes naturally from the triangular structure.

Definition 7.2. *Let $\lambda_{i,k,r} \in \mathbb{C}$, $1 \le i \le n$, $0 \le k \le d-1$, $r \in \mathbb{Z}_{\ge 0}$ and let $\lambda \in \mathfrak{gl}_n(\mathsf{H}_{1,\mathbf{c}}(\Gamma))[0]^*$ be given by $\lambda(E_{ii} w^r \mathbf{e}_k) = \lambda_{i,k,r}$. Extending λ to a one-dimensional representation $\mathbb{C}_\lambda$ of the Lie algebra $\mathfrak{gl}_n(\mathsf{H}_{1,\mathbf{c}}(\Gamma))[\ge 0]$ $(= \oplus_{k=0}^\infty \mathfrak{gl}_n(\mathsf{H}_{1,\mathbf{c}}(\Gamma))[k])$ by letting $\mathfrak{gl}_n(\mathsf{H}_{1,\mathbf{c}}(\Gamma))[k]$ act trivially if $k > 0$, we define the Verma module $M(\lambda)$ by*

$$M(\lambda) = \mathfrak{U}\mathfrak{gl}_n(\mathsf{H}_{1,\mathbf{c}}(\Gamma)) \otimes_{\mathfrak{U}\mathfrak{gl}_n(\mathsf{H}_{1,\mathbf{c}}(\Gamma))[\ge 0]} \mathbb{C}_\lambda.$$

The following lemma and definition are quite standard.

Lemma 7.3. *The Verma module $M(\lambda)$ has a unique irreducible quotient which we denote by $L(\lambda)$.*

Definition 7.4. *A $\mathfrak{gl}_n(\mathsf{H}_{1,\mathbf{c}}(\Gamma))$-module is called a highest weight module of highest weight $\lambda \in \mathfrak{gl}_n(\mathsf{H}_{1,\mathbf{c}}(\Gamma))[0]^*$ if it is generated by a vector v on which $h \in \mathfrak{gl}_n(\mathsf{H}_{1,\mathbf{c}}(\Gamma))[0]$ acts by multiplication by $\lambda(h)$ and $\mathfrak{gl}_n(\mathsf{H}_{1,\mathbf{c}}(\Gamma))[k]$ acts trivially if $k \in \mathbb{Z}_{>0}$. A vector with this last property is said to be singular.*

The highest weight vector which generates the Verma module $M(\lambda)$ will be denoted v_λ.

Given a parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{gl}_n(\mathsf{H}_{1,\mathbf{c}}(\Gamma))$ with $\mathbf{b}$ the set of its first nd characteristic polynomials, one can define similarly generalized Verma modules $M(\mathfrak{q}, \lambda)$ by chosing λ such that $\lambda(h) = 0$ for any $h \in \mathfrak{gl}_n(\mathsf{H}_{1,\mathbf{c}}(\Gamma))[0, \mathbf{b}]$, since, in this case, λ descends to $\mathfrak{q}/[\mathfrak{q}, \mathfrak{q}]$: see proposition 6.8.

The goal of this section is to study quasifinite irreducible highest weight modules, so we have to introduce the next definition.

Definition 7.5. *A graded highest weight module M over $\mathfrak{gl}_n(\mathsf{H}_{1,\mathbf{c}}(\Gamma))$, $M = \oplus_{k \in \mathbb{Z}} M[k]$, is said to be quasifinite if* $\dim_{\mathbb{C}} M[k] < \infty \, \forall \, k \in \mathbb{Z}$.

In order to obtain below a condition equivalent to the quasifiniteness of $L(\lambda)$, we need one more definition, as in [KaRa].

Definition 7.6. *The Verma module $M(\lambda)$ is said to be highly degenerate if there exists a singular vector $v \in M(\lambda)[-1]$ such that $v = A v_\lambda$ with $A \in \mathfrak{gl}_n(\mathsf{H}_{1,\mathbf{c}}(\Gamma))[-1]$ and $qdet(A) \neq 0$.*

A few words of explanation are in order. The space $\mathfrak{gl}_n(\mathsf{H}_{1,\mathbf{c}}(\Gamma))[-1]$ is spanned by $E_{i+1,i}\omega^r \mathbf{e}_l$ and by $E_{1n}\omega^r u \mathbf{e}_l$, so the entries of a matrix A in $\mathfrak{gl}_n(\mathsf{H}_{1,\mathbf{c}}(\Gamma))[-1]$ do not necessarily belong to a commutative ring. By *qdet*(A), we thus mean the quasi-determinant of A (which, in this case, is, up to a sign, the product of the non-zero entries of A).

Proposition 7.7. *The Verma module $M(\lambda)$ is highly degenerate if and only if λ vanishes on $\mathfrak{gl}_n(\mathsf{H}_{1,\mathbf{c}}(\Gamma))[0, \mathbf{b}]$ for some nd monic (so non-zero) polynomials $\mathbf{b} = (b^{i,l}(\omega))_{1 \le i \le n}^{0 \le l \le d-1}$.*

Proof. The same argument as in the proof of proposition 4.1 in [BKLY] applies. The polynomials $b^{i,l}(\omega)$ are related to the matrix A in the following way. Since $A \in \mathfrak{gl}_n(\mathsf{H}_{t=1,\mathbf{c}}(\Gamma))[-1]$, it can be written as a linear combination of matrices of the type $E_{i+1,i} b^{i+1,l}(w) \mathbf{e}_l$ for $1 \le i \le n-1$ and $E_{1n} b^{1,l}(w) u \mathbf{e}_l$ with $0 \le l \le d-1$. It follows from the proof of proposition 6.8 that $\mathfrak{gl}_n(\mathsf{H}_{1,\mathbf{c}}(\Gamma))[0, \mathbf{b}]$ is spanned by $[B, A]$ for all $B \in \mathfrak{gl}_n(\mathsf{H}_{1,\mathbf{c}}(\Gamma))[1]$. ☐

Proposition 7.8. [BKLY] *Given $\lambda \in \mathfrak{gl}_n(\mathsf{H}_{1,\mathbf{c}}(\Gamma))[0]^*$ as before, the following conditions are equivalent:*

(1) *$M(\lambda)$ is highly degenerate.*

(2) *$L(\lambda)$ is quasi-finite.*

(3) *$L(\lambda)$ is a quotient of a generalized Verma module $M(\mathfrak{q},\lambda)$ where all the characteristic polynomials $\mathbf{b} = (b^{i,l}(\omega))_{1\le i\le n}^{0\le l\le d-1}$ of $\mathfrak{q}$ are non-zero.*

Proof. Proposition 7.7 shows that (1) and (3) are equivalent. Let us show that, if all the polynomials $b^{i,l}(\omega)$ are non-zero, then $\dim_{\mathbb{C}}\big(\mathfrak{gl}_n(\mathsf{H}_{1,\mathbf{c}}(\Gamma))[k]/\mathfrak{q}[k]\big)$ is finite, hence $L(\lambda)$ is quasi-finite. Under this assumption, it follows from the proof of lemma 6.6 that $b_k^{i,l}(w)$ are non-zero for all $k \in \mathbb{Z}_{\le -1}, 1 \le i \le n, 0 \le l \le d-1$. Recall that, for $k<0$, we can write

$$\mathfrak{gl}_n(\mathsf{H}_{1,\mathbf{c}}(\Gamma))[k] = \sum_{\substack{s,l,i,j\\ -sn+j-i=k}} E_{ij}u^s\mathbb{C}[\omega]\mathbf{e}_l \text{ and } \mathfrak{q}[k] = \sum_{\substack{s,l,i,j\\ -sn+j-i=k}} E_{ij}u^s\mathbb{C}[\omega]b_k^{i,l}(w)\mathbf{e}_l.$$

Our claim now follows from the observation that $\dim_{\mathbb{C}}\big(\mathbb{C}[w]/(b_k^{i,l}(w))\big) < \infty$.

Now suppose that $L(\lambda)$ is quasi-finite. Then $\dim_{\mathbb{C}} L(\lambda)[-1] < \infty$, so, if $\widetilde{M}(\lambda)$ denotes the unique maximal submodule of $M(\lambda)$, then $\widetilde{M}(\lambda)[-1] \neq \{0\}$. All the vectors in $\widetilde{M}(\lambda)[-1] \neq \{0\}$ are singular and at least one satisfies the condition in definition 7.6. Therefore, $M(\lambda)$ is highly degenerate. ☐

In theorems 5.2 and 5.4, we stated a criterion in terms of certain power series for the integrability of the simple quotients of Verma modules for $\widehat{\mathfrak{sl}}_n(A), \widehat{\mathfrak{sl}}_n(B)$ and $\widehat{\mathfrak{sl}}_n(C)$. We now want to give a similar criterion for the quasi-finiteness of $L(\lambda)$. To achieve this, given $\lambda \in \mathfrak{gl}_n(\mathsf{H}_{1,\mathbf{c}}(\Gamma))[0]^*$ as before, set $d_{i,l,r} = \lambda(E_{ii}w^r\mathbf{e}_l)$ for $1\le i\le n$ and $D_{i,l}(z) = \sum_{r=0}^{\infty}\frac{d_{i,l,r}}{r!}z^r$. Recall that a quasipolynomial is a linear combination of functions of the form $p(z)e^{az}$ where $p(z)$ is a polynomial and $a\in\mathbb{C}$.

Theorem 7.9. *The module $L(\lambda)$ is quasi-finite if and only if there exist quasipolynomials $\phi_{i,l}(z), 1\le i\le n, 0\le l\le d-1$, such that*

$$D_{1,l}(z) = \frac{\phi_{1,l}(z)}{1-e^{dz}}$$

$$D_{i,l}(z) = \frac{\phi_{1,l}(z) + (1-e^{dz})\phi_{i,l}(z)}{1-e^{dz}} \text{ if } 2\le i\le n.$$

Proof. The proof is similar to the proof of theorem 4.1 in [BKLY], using the description of $\mathfrak{gl}_n(\mathsf{H}_{1,\mathbf{c}}(\Gamma))[0,\mathbf{b}]$ given just before proposition 6.8. Let us explain the differences. Writing $b^{i,l}(\omega)$ as $\omega^{m_{i,l}} + f_{i,l,m_{i,l}-1}\omega^{m_{i,l}-1} + \cdots + f_{i,l,0}$, we obtain the equations $\sum_{r=0}^{m_{i,l}} f_{i,l,r}F_{i,l,r+\widetilde{r}} = 0$ for $1\le i\le n, \widetilde{r} = 0,1,\ldots$ where $F_{i,l,r} = d_{i,l,r} - d_{i-1,l,r}$ for $2\le i\le n$ and $f_{i,l,m_{i,l}} = 1$. To express $F_{1,l,r}$ in terms of the $d_{i,l,r}$ we write

$$E_{11}uv(\omega+1)^r b_{-1}^{1,l}(\omega+1)\mathbf{e}_{l+1} - E_{nn}vu\omega^r b_{-1}^{1,l}(\omega)\mathbf{e}_l$$

as

$$\begin{aligned}-E_{11}(\omega+1)^{r+1}b_{-1}^{1,l}(\omega+1)\mathbf{e}_{l+1} + (\widetilde{c}_l+1)E_{11}(\omega+1)^r b_{-1}^{1,l}(\omega+1)\mathbf{e}_{l+1}\\ +E_{nn}\omega^{r+1}b_{-1}^{1,l}(\omega)\mathbf{e}_l - (\widetilde{c}_l+1)E_{nn}\omega^r b_{-1}^{1,l}(\omega)\mathbf{e}_l\end{aligned}$$

We thus see that

$$F_{1,l,r} = d_{n,l,r+1} - (\tilde{c}_l+1)d_{n,l,r} - \sum_{j=0}^{r+1}\begin{pmatrix} r+1\\ j\end{pmatrix} d_{1,l+1,j} + (\widetilde{c}_l+1)\sum_{j=0}^{r}\begin{pmatrix} r\\ j\end{pmatrix} d_{1,l+1,j}.$$

Setting $F_{i,l}(z) = \sum_{r=0}^{\infty} F_{i,l,r} \frac{z^r}{r!}$ for $1 \le i \le n$, we can conclude as in [BKLY] that $F_{i,l}(z)$ is a quasipolynomial. For $2 \le i \le n$, we can write $F_{i,l}(z) = D_{i,l}(z) - D_{i-1,l}(z)$, and for $i = 1$, we have

$$F_{1,l}(z) = D'_{n,l}(z) - (\widetilde{c}_l + 1)D_{n,l}(z) - (e^z D_{1,l+1})'(z) + (\widetilde{c}_l + 1)e^z D_{1,l+1}(z).$$

Here, $D'_{i,l}(z)$ is the derivative of $D_{i,l}(z)$. This implies that $D'_{i,l}(z) - D'_{i-1,l}(z)$ is also a quasi-polynomial (for $2 \le i \le n$) and hence so is

$$(D_{1,l}(z) - e^z D_{1,l+1}(z))' - (\widetilde{c}_l + 1)(D_{1,l}(z) - e^z D_{1,l+1}(z)).$$

Consequently, $e^z D_{1,l+1}(z) - D_{1,l}(z)$ is a quasi-polynomial, and thus so is $(1 - e^{dz})D_{1,l}(z)$. □

It is possible to construct quasi-finite representations of $\mathfrak{gl}_n(\mathsf{H}_{1,\mathbf{c}}(\Gamma))$ as tensor products of certain modules. This is where the embeddings $\varphi_a^{[m]}$ come into play. Unfortunately, they are not necessarily irreducible.

First, we need to construct irreducible representations of $\overline{\mathfrak{gl}}_\infty(R_m)$ using a standard procedure. An element $\lambda \in \overline{\mathfrak{gl}}_\infty(R_m)[0]^*$ is determined by $\lambda_k^{(j)} = \lambda(E_{kk}t^j)$, $k \in \mathbb{Z}, j = 0, \dots, m$, which we call its labels, following the terminology in [BKLY]. Using induction from the subalgebra of upper-triangular matrices and its one-dimensional representation determined by such a λ, we construct a Verma module for $\overline{\mathfrak{gl}}_\infty(R_m)$ and this Verma module has a unique irreducible highest weight quotient $L(m, \lambda)$.

Proposition 7.10 ([BKLY] proposition 4.4)**.** *The irreducible $\overline{\mathfrak{gl}}_\infty(R_m)$-module $L(m, \lambda)$ is quasi-finite if and only if for each $j = 0, \dots, m$ all but finitely many of the $\lambda_k^{(j)} - \lambda_{k+1}^{(j)}$ are zero.*

Let $\mathbf{m} = (m_1, \dots, m_N) \in \mathbb{Z}_{\ge 0}^{\oplus N}$ and $\lambda = (\lambda(1), \dots, \lambda(N))$ with $\lambda(i) \in \overline{\mathfrak{gl}}_\infty(R_{m_i})[0]^*$ such that $L(m_i, \lambda(i))$ is quasi-finite. We can form the tensor product $L(\mathbf{m}, \lambda) = \bigotimes_{i=1}^N L(m_i, \lambda(i))$, which is an irreducible quasi-finite representation of $\overline{\mathfrak{gl}}_\infty[\mathbf{m}] = \oplus_{i=1}^N \overline{\mathfrak{gl}}_\infty(R_{m_i})$. By pulling it back via the map $\varphi_{\mathbf{a}}^{[\mathbf{m}]} = \oplus_{i=1}^N \varphi_{a_i}^{[m_i]} : \mathfrak{gl}_n(\mathsf{H}_{t=1,\mathbf{c}}(\Gamma)) \longrightarrow \overline{\mathfrak{gl}}_\infty[\mathbf{m}]$ for $\mathbf{a} = (a_1, \dots, a_N) \in \mathbb{C}^N$, we obtain a representation of $\mathfrak{gl}_n(\mathsf{H}_{t=1,\mathbf{c}}(\Gamma))$ which we denote $\mathsf{L}_{\mathbf{a}}(\mathbf{m}, \lambda)$.

Theorem 4.2 in [BKLY] does not hold for $\mathfrak{gl}_n(\mathsf{H}_{t=1,\mathbf{c}}(\Gamma))$, so we cannot deduce that the representation $\mathsf{L}_{\mathbf{a}}(\mathbf{m}, \lambda)$ is necessarily irreducible. Let us discuss what is the difference here. Theorem 4.2 in *loc. cit.* states that pulling back a quasifinite representation of $\overline{\mathfrak{gl}}_\infty[\mathbf{m}]$ to $\mathfrak{gl}_n(\mathbf{A}_1 \rtimes \Gamma)$ via $\varphi_{\mathbf{a}}^{[\mathbf{m}]}$ gives a representation which has the same submodules. (The proof in the case $\Gamma = \{1\}$ extends to any $d > 1$.) The main ideas of the proof are the following (see also [KaRa]). First, we have to introduce a holomorphic enlargement of $\mathfrak{gl}_n(\mathbf{A}_1 \rtimes \Gamma)$, as at the end of section 6: it is the Lie algebra $\mathfrak{gl}_n(\mathbf{A}_1^{\mathcal{O}} \rtimes \Gamma)$ spanned by $E_{ij}u^s f(\omega)\mathbf{e}_l$ where $f(\omega)$ is an entire function of ω, the bracket of $\mathfrak{gl}_n(\mathbf{A}_1 \rtimes \Gamma)$ extending to $\mathfrak{gl}_n(\mathbf{A}_1^{\mathcal{O}} \rtimes \Gamma)$ naturally. Secondly, the formula for the embedding $\varphi_{\mathbf{a}}^{[\mathbf{m}]}$ (when $a_i \neq a_j$ for $i \neq j$) can be used to obtain a map $\varphi_{\mathbf{a}}^{[\mathbf{m}],\mathcal{O}} : \mathfrak{gl}_n(\mathbf{A}_1^{\mathcal{O}} \rtimes \Gamma) \longrightarrow \overline{\mathfrak{gl}}_\infty[\mathbf{m}]$, which is onto, but not necessarily into. The last step is to show that, if V is a quasi-finite module over $\mathfrak{gl}_n(\mathbf{A}_1 \rtimes \Gamma)$, then, by continuity, we can make $\mathfrak{gl}_n(\mathbf{A}_1^{\mathcal{O}} \rtimes \Gamma)[k]$ act on V if $k \neq 0$. This involves computing an upper bound on the norm of certain operators.

The first and third step work also for $\mathsf{H}_{t=1,\mathbf{c}}(\Gamma)$, but not the second one. We consider the algebra $\mathsf{H}_{t=1,\mathbf{c}}^{\mathcal{O}}(\Gamma)$ which is spanned by elements of the form $v^r f(\omega)\mathbf{e}_l$ and $u^s f(\omega)\mathbf{e}_l$ where $f(\omega)$ is an entire function of ω and the multiplication is given by (in the case $r \ge s$)

$$v^r f(\omega)\mathbf{e}_{l_1} u^s g(\omega)\mathbf{e}_{l_2} = \delta_{l_1 - s, l_2} v^{r-s} \left(\prod_{k=1}^{s} (-\omega + c_{l_2+s-k} + 1 + s - k) \right) f(\omega - s) g(\omega) \mathbf{e}_{l_2}$$

We have also a map $\varphi_{\mathbf{a}}^{[\mathbf{m}],\mathcal{O}} : \mathfrak{gl}_n(\mathsf{H}^{\mathcal{O}}_{t=1,\mathbf{c}}(\Gamma)) \longrightarrow \overline{\mathfrak{gl}}_\infty[\mathbf{m}]$, but it is not onto: for instance, if $a = 0 = m = c_l$ for $l = 0,\dots,d-1$, then $\varphi_a^{[m]}(E_{ij}vf(\omega)\mathbf{e}_k) = \sum_{l\in\mathbb{Z}}(ld+k)f(-ld-k)E_{(ld+k-1)n+i-1,(ld+k)n+j-1}$. Therefore, in the image, the coefficient of $E_{-n+i-1,j-1}$ is always zero, independently of $f(\omega)$. At least, we have the following result.

Proposition 7.11. *Assume that $a_i - a_j \notin \mathbb{Z}$ for $1 \le i \neq j \le N$ and $\widetilde{c}_k + a_i \notin \mathbb{Z}$ for all $1 \le i \le N$, $0 \le k \le d-1$. Then the map $\varphi_{\mathbf{a}}^{[\mathbf{m}],\mathcal{O}} : \mathfrak{gl}_n(\mathsf{H}^{\mathcal{O}}_{t=1,\mathbf{c}}(\Gamma)) \longrightarrow \overline{\mathfrak{gl}}_\infty[\mathbf{m}]$ is onto.*

Proof. Decompose $\overline{\mathfrak{gl}}_\infty$ as $\overline{\mathfrak{gl}}_\infty = \overline{\mathfrak{n}}_\infty^- \oplus \overline{\mathfrak{h}}_\infty \oplus \overline{\mathfrak{n}}_\infty^+$, where $\overline{\mathfrak{h}}_\infty$ is the Lie subalgebra of all the diagonal blocks of size n (with one having a corner at the $(0,0)$-entry), and $\overline{\mathfrak{n}}_\infty^\pm$ are the complements of $\overline{\mathfrak{h}}_\infty$ consisting of strickly upper and lower triangular matrices. That $\varphi_{\mathbf{a}}^{[\mathbf{m}],\mathcal{O}}$ is onto the subspace $\overline{\mathfrak{n}}_\infty^-$ when restricted to the subspace spanned by the elements $E_{ij}u^sf(\omega)\mathbf{e}_k$ with $s \in \mathbb{Z}_{\geq 0}, 1 \le i,j \le n, 0 \le k \le d-1$ follows from [BKLY, KaRa], so let us focus instead on $\mathfrak{gl}_n(\mathsf{H}^{\mathcal{O}}_{t=1,\mathbf{c}}(\Gamma))[>0]$. Explicitly, using the Taylor formula for the expansion of a funtion of t around $t=0$ and (7), $\varphi_{a_i}^{[m_i]}$ is given by

$$\varphi_{a_i}^{[m_i]}(E_{ij}v^sf(\omega)\mathbf{e}_k) = \sum_{l\in\mathbb{Z}}\sum_{b=0}^{m_i}\frac{g^{(b)}(a_i+ld)}{b!}t^bE_{(ld+k-s)n+i-1,(ld+k)n+j-1} \tag{8}$$

if we set $g(t) = \left(\prod_{p=0}^{s-1}(k-p+t+\widetilde{c}_{k-p-1})\right)f(-k-t)$. As in proposition 3.1 in [KaRa], we can use the fact that, for every discrete set of points in $\mathbb{C}$, there is a holomorphic function on $\mathbb{C}$ with prescribed values of its first m_i derivatives at each points of such a set. Combining this with our assumption that $a_i - a_j \notin \mathbb{Z}$ for $1 \le i \neq j \le N$, we deduce that, given a matrix $E = \oplus_{i=1}^N E_i$ in $\overline{\mathfrak{gl}}_\infty[\mathbf{m}]$, there exists an entire function $g(t)$ such that the right-hand side of (8) is equal to E_i for all $i = 1,\dots,N$. To complete the proof, we have to find an entire function $f(\omega)$ such that, if we set $\widetilde{g}(t) = \left(\prod_{p=0}^{s-1}(k-p+t+\widetilde{c}_{k-p-1})\right)f(-k-t)$, then $g^{(b)}(a_i+ld) = \widetilde{g}^{(b)}(a_i+ld)$ for $1 \le i \le N$, $0 \le b \le m_i$ all $l \in \mathbb{Z}$. Set $P(t) = \prod_{p=0}^{s-1}(k-p+t+\widetilde{c}_{k-p-1})$, so that

$$\widetilde{g}^{(b)}(t) = \sum_{a=0}^{b}\begin{pmatrix} b \\ a \end{pmatrix}P^{(n-a)}(t)f^{(a)}(-k-t).$$

Fix $1 \le i \le N$, $l \in \mathbb{Z}$ and consider the system of equations

$$g^{(b)}(a_i+ld) = \sum_{a=0}^{b}\begin{pmatrix} b \\ a \end{pmatrix}P^{(b-a)}(a_i+ld)z_a$$

for $b = 0,1,\dots,m_i$, $z_0,\dots,z_{m_i}$ being the unknown variables (which we would like to express in terms of $\widetilde{g}^{(b)}(a_i+ld)$). Our hypothesis that $\widetilde{c}_k + a_i \notin \mathbb{Z}$ implies that $P(a_i+ld) \neq 0$, so the matrix of this system is triangular with non-zero entries along the diagonal. We can thus solve it: let $\widetilde{z}^0_{i,l}, \widetilde{z}^1_{i,l},\dots,\widetilde{z}^{m_i}_{i,l}$ be a solution. Then we can rephrase the problem by saying that we now have to find an entire function $f(\omega)$ such that $f^{(a)}(a_i+ld) = \widetilde{z}^a_{i,l}$ for $1 \le i \le N$, $0 \le a \le m_i$ and all $l \in \mathbb{Z}$. To deduce the existence of such a function, we can now apply the same argument as the one used to deduce the existence of $g(\omega)$ above. ☐

The representation $\mathsf{L}_{\mathbf{a}}(\mathbf{m},\lambda)$ is a highest weight module, so it is interesting to calculate its associated series $\mathbf{D}_{i,k}(z)$, which is equal to $\sum_{j=1}^N D_{i,j,k}(z)$. The formulas are similar to those in [BKLY]. Set $h_l^{(p)}(j) = \lambda_l^{(p)}(j) - \lambda_{l+1}^{(p)}(j)$, $g_{j,l}(z) = \sum_{p=0}^{m_j} h_l^{(p)}(j)\frac{(-z)^p}{p!}$. We have

$$D_{i,j,k}(z) = \sum_{p=1}^{m_j} \sum_{l\in\mathbb{Z}} \lambda^{(p)}_{(ld+k)n+i-1}(j)\frac{(-z)^p}{p!}e^{-(a_j+ld+k)z}$$

and

$$\begin{aligned} D_{i,j,k}(z) = &(1-e^{dz})^{-1}\sum_{l\in\mathbb{Z}} e^{-(a_j+ld+k)z}\big(g_{j,(ld+k)n+i-1}(z) + g_{j,(ld+k)n+i}(z) + \cdots \\ &+ g_{j,(ld+k)n+dn+i-2}(z)\big) \end{aligned}$$

8. Further discussions

In this section we present further possible research directions related to the results of the present paper.

8.1. **Double affine Lie algebras and Kleinian singularities.** In this section we present further possible research directions related to the results of the present paper. G will be an arbitrary finite subgroup of $SL_2(\mathbb{C})$. Such a group G does not always act on the torus $\mathbb{C}^{\times 2}$ or on $\mathbb{C}\times\mathbb{C}^\times$, so we can consider only the algebras $\mathbb{C}[u,v]\rtimes G$ and $\mathbb{C}[u,v]^G$. Moreover, when G is not cyclic, each of these Lie algebras has only one triangular decomposition, namely:

$$\mathfrak{sl}_n(\mathbb{C}[u,v]\rtimes G) \cong \mathfrak{n}^-(\mathbb{C}[u,v]\rtimes G)\oplus\mathfrak{h}(\mathbb{C}[u,v]\rtimes G)\oplus\mathfrak{n}^+(\mathbb{C}[u,v]\rtimes G)$$
$$\mathfrak{sl}_n(\mathbb{C}[u,v]^G) \cong \mathfrak{n}^-(\mathbb{C}[u,v]^G)\oplus\mathfrak{h}(\mathbb{C}[u,v]\rtimes G)\oplus\mathfrak{n}^+(\mathbb{C}[u,v]^G)$$

These also admit universal central extensions. Since $\mathbb{C}[u,v]^G$ is commutative, we have that $HC_1(\mathbb{C}[u,v]^G) = \frac{\Omega^1(\mathbb{C}[u,v]^G)}{d(\mathbb{C}[u,v]^G)}$. We know from [Ka] that the bracket on the universal central extension $\widehat{\mathfrak{sl}}_n(\mathbb{C}[u,v]^G) = \mathfrak{sl}_n(\mathbb{C}[u,v]^G)\oplus\frac{\Omega^1(\mathbb{C}[u,v]^G)}{d(\mathbb{C}[u,v]^G)}$ of $\mathfrak{sl}_n(\mathbb{C}[u,v]^G)$ is given by

$$[m_1\otimes p_1, m_2\otimes p_2] = [m_1,m_2]\otimes(p_1p_2) + Tr(m_1m_2)p_1dp_2$$

As for $\widehat{\mathfrak{sl}}_n(\mathbb{C}[u,v]\rtimes G)$, it is known that its kernel $HC_1(\mathbb{C}[u,v]\rtimes G)$ is equal to $\frac{\Omega^1(\mathbb{C}[u,v])^G}{d(\mathbb{C}[u,v]^G)}$ (see [Fa]), but to obtain an explicit formula for its bracket, one would have to choose a splitting $\langle\mathbb{C}[u,v]\rtimes G,\mathbb{C}[u,v]\rtimes G\rangle = [\mathbb{C}[u,v]\rtimes G,\mathbb{C}[u,v]\rtimes G]\oplus HC_1(\mathbb{C}[u,v]\rtimes G)$ - see section 2.

In [KaVa], the authors proved that the derived category of coherent sheaves on the minimal resolution $\widetilde{\mathbb{C}^2/G}$ of the singularity $\mathbb{C}^2/G$ is equivalent to the derived category of modules over the skew-group ring $\mathbb{C}[u,v]\rtimes G$. It is thus natural to ask if there is a connection between the derived category of representations of $\mathfrak{sl}_n(\mathbb{C}[u,v]\rtimes G)$ and the derived category of modules over a certain sheaf of Lie algebras on $\widetilde{\mathbb{C}^2/G}$.

In the same line of thought, since $A_1\rtimes G$ and A_1^G are Morita equivalent, one can wonder about the connections between $\mathfrak{sl}_n(A_1\rtimes G)$ and $\mathfrak{sl}_n(A_1^G)$. We observe, however, that even if A and B are Morita equivalent rings, the categories of representations of $\mathfrak{sl}_n(A)$ and $\mathfrak{sl}_n(B)$ are not necessarily equivalent: as a counterexample, one can consider $A=\mathbb{C}[t^{\pm1}]$ and $B=\mathbb{C}[u^{\pm1}]\rtimes(\mathbb{Z}/d\mathbb{Z})\cong M_d(\mathbb{C}[t^{\pm1}])$, in which case $\mathfrak{sl}_n(B)=\mathfrak{sl}_{nd}(\mathbb{C}[t^{\pm1}])$.

The definitions of Weyl modules from section 5.2 can be adapted to $\mathfrak{sl}_n(\mathbb{C}[u,v]\rtimes G)$ and $\mathfrak{sl}_n(\mathbb{C}[u,v]^G)$. Studying these appear to be a reasonable way to approach the representation theory of these Lie algebras since, when G is not cyclic, we do not have triangular decompositions similar to (3) or presentations as in proposition 5.3. One interesting question is to compute the dimension of local Weyl modules at the Kleinian singularity. For smooth points on an affine variety and certain highest weights, the dimension of local Weyl modules has been computed in [FeLo], and the case of a double point was treated in [Ku]. One can expect the study of such local Weyl modules to be related to the geometry of the minimal resolution of the Kleinian singularity.

8.2. **Quiver Lie algebras.** Symplectic reflection algebras for wreath products of G are known to be Morita equivalent to certain deformed preprojective algebras of affine Dynkin quivers which are called Gan-Ginzburg algebras in the literature [GaGi]. In the rank one case, these are the usual deformed preprojective algebras $\Pi^{\lambda}(Q)$. The affine Dynkin diagram in question is associated to G via the McKay correspondence. The quantum Lie algebra analogs of these Gan-Ginzburg algebras were introduced in [Gu4] and are deformations of the enveloping algebra of a Lie algebra which is slightly larger than the universal central extension of $\mathfrak{sl}_n(\Pi(Q))$, where $\Pi(Q) = \Pi^{\lambda=0}(Q)$. The same themes as in the previous sections can be studied in the context of the Lie algebra $\mathfrak{sl}_n(\Pi(Q))$, in particular when the graph underlying Q is an affine Dynkin diagram. Actually, when Q is the cyclic quiver on d vertices, $\Pi(Q) \cong \mathbb{C}[u,v] \rtimes \Gamma$. Furthermore, if e_0 is the extending vertex of an affine Dynkin diagram, then $e_0\Pi(Q)e_0 \cong \mathbb{C}[u,v]^G$.

All these are examples of matrix Lie algebras over interesting non-commutative rings. It is possible to replace $\mathfrak{sl}_n$ by another semisimple Lie algebra: this is explained in [BeRe]. It would also be interesting to compare our work with the constructions in [HOT].

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