Cluster algebras and quantum affine algebras

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Abstract

Let $\mathscr{C}$ be the category of finite-dimensional representations of a quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ of simply-laced type. We introduce certain monoidal subcategories $\mathscr{C}_\ell$ ($\ell \in \mathbb{N}$) of $\mathscr{C}$ and we study their Grothendieck rings using cluster algebras.

Contents

1 Introduction

1.1 Let $\mathfrak{g}$ be a simple Lie algebra of type A_n, D_n or E_n, and let $U_q(\widehat{\mathfrak{g}})$ denote the corresponding quantum affine algebra, with parameter $q \in \mathbb{C}^*$ not a root of unity. The monoidal category $\mathscr{C}$ of finite-dimensional $U_q(\widehat{\mathfrak{g}})$-modules has been studied by many authors from different perspectives (see *e.g.* [**AK**, **CP1**, **FR**, **GV**, **KS**, **N1**]). In particular its simple objects have been classified by Chari and Pressley, and Nakajima has calculated their character in terms of the cohomology of certain quiver varieties.

In spite of these remarkable results many basic questions remain open, and in particular little is known about the tensor structure of $\mathscr{C}$.

When $\mathfrak{g} = \mathfrak{sl}_2$, Chari and Pressley [**CP2**] have shown that every simple object is isomorphic to a tensor product of simple objects of a special type called Kirillov-Reshetikhin modules. Conversely, they have shown that a tensor product $S_1 \otimes \cdots \otimes S_k$ of Kirillov-Reshetikhin modules is simple if and only if $S_i \otimes S_j$ is simple for every $1 \leqslant i < j \leqslant k$. Moreover, $S_i \otimes S_j$ is simple if and only if S_i and S_j are "in general position" (a combinatorial condition on the roots of the Drinfeld polynomials of S_i and S_j). Hence, the Kirillov-Reshetikhin modules can be regarded as the *prime* simple objects of $\mathscr{C}$ [**CP6**], and one knows which products of primes are simple. As an easy corollary, one can see that the tensor powers of any simple object of $\mathscr{C}$ are simple.

For $\mathfrak{g} \neq \mathfrak{sl}_2$, the situation is far more complicated. Thus, already for $\mathfrak{g} = \mathfrak{sl}_3$, we do not know a general factorization theorem for simple objects (see [**CP6**], where a tentative list of prime simple objects is conjectured). In fact, it was shown in [**L**] that the tensor square of a simple object of $\mathscr{C}$ is not necessarily simple in general, so one should not expect results similar to the $\mathfrak{sl}_2$ case for other Lie algebras $\mathfrak{g}$.

Because of these difficulties, we decide in this paper to focus on some smaller subcategories. We introduce a sequence

$$\mathscr{C}_0 \subset \mathscr{C}_1 \subset \cdots \subset \mathscr{C}_\ell \subset \cdots, \qquad (\ell \in \mathbb{N}),$$

of full monoidal subcategories of $\mathscr{C}$, whose objects are characterized by certain strong restrictions on the roots of the Drinfeld polynomials of their composition factors. By construction, the Grothendieck ring R_ℓ of $\mathscr{C}_\ell$ is a polynomial ring in $n(\ell+1)$ variables, where n is the rank of $\mathfrak{g}$. Our starting point is that R_ℓ is naturally equipped with the structure of a cluster algebra.

Recall that cluster algebras were introduced by Fomin and Zelevinsky [**FZ1**] as a combinatorial device for studying canonical bases and total positivity. They found immediately lots of applications, including a proof of a conjecture of Zamolodchikov concerning certain discrete dynamical systems arising from the thermodynamic Bethe ansatz, called Y-systems [**FZ2**]. As observed by Kuniba, Nakanishi and Suzuki [**KNS**], Y-systems are strongly related with the representation theory of $U_q(\widehat{\mathfrak{g}})$ via some other systems of functional relations called T-systems. It was conjectured in [**KNS**] that the characters of the Kirillov-Reshetikhin modules are solutions of a T-system, and this was later proved by Nakajima [**N2**] in the simply-laced case, and by Hernandez in the general case [**H3**]. Now it is easy to notice that in the simply-laced case the equations of a T-system are exactly of the same form as the exchange relations in a cluster algebra. This led us to introduce a cluster algebra structure on R_ℓ by using an initial seed consisting of a choice of $n(\ell+1)$ Kirillov-Reshetikhin modules in $\mathscr{C}_\ell$. The exchange matrix of this seed encodes $n\ell$ equations of the T-system satisfied by these Kirillov-Reshetikhin modules. (Note that the seed contains n frozen variables – or coefficients – in the sense of [**FZ1**].) By definition of a cluster algebra, one can obtain new seeds by applying sequences of mutations to the initial seed. Then one of our main conjectures is that all the new cluster variables produced in this way are classes of simple objects of $\mathscr{C}_\ell$. (Note that in general, these simple objects are no longer Kirillov-Reshetikhin modules.)

1.2 For $\ell = 0$, the cluster structure of R_0 is trivial: there is a unique cluster consisting entirely of frozen variables.

The case $\ell = 1$ is already very interesting, and most of this paper will be devoted to it. Recall that Fomin and Zelevinsky have classified the cluster algebras with finitely many cluster variables in terms of finite root systems [**FZ3**]. It turns out that for every $\mathfrak{g}$ the ring R_1 has finitely many cluster variables, and that its cluster type coincides with the root system of $\mathfrak{g}$. Therefore, one may expect that the tensor structure of the simple objects of the category $\mathscr{C}_1$ can be described in "a finite way". In fact we conjecture that for every $\mathfrak{g}$ the category $\mathscr{C}_1$ behaves as nicely as the category $\mathscr{C}$ for $\mathfrak{sl}_2$, and we prove it for $\mathfrak{g}$ of type A_n and D_4.

More precisely, we single out a finite set of simple objects of $\mathscr{C}_1$ whose Drinfeld polynomials are naturally labeled by the set of almost positive roots of $\mathfrak{g}$ (*i.e.*, positive roots and negative simple roots). Recall that the almost positive roots are in one-to-one correspondence with the cluster variables [**FZ3**], so we shall call these objects the *cluster simple objects*. To these objects we add n distinguished simple objects which we call *frozen simple objects*. Our first claim is that the classes of these objects in R_1 coincide with the cluster variables and frozen variables.

Recall also that the cluster variables are grouped into overlapping subsets of cardinality n called clusters [**FZ1**]. The number of clusters is a generalized Catalan number, and they can be identified with the faces of the dual of a generalized associahedron [**FZ2**]. Our second claim is that a tensor product of cluster simple objects is simple if and only if all the objects belong to a common cluster. Moreover, the tensor product of a frozen simple object with any simple object is again simple. It follows that every simple object of $\mathscr{C}_1$ is a tensor product of cluster simple objects and frozen simple objects. As a consequence, the tensor powers of any simple object of $\mathscr{C}_1$ are simple.

To prove this, we show that a tensor product $S_1 \otimes \cdots \otimes S_k$ of simple objects of $\mathscr{C}_1$ is simple if and only if $S_i \otimes S_j$ is simple for every $i \neq j$. This result is proved uniformly for all types.

Note that only the frozen objects and the cluster objects attached to positive simple roots and negative simple roots are Kirillov-Reshetikhin modules. The remaining cluster objects (labelled by the positive non simple roots) probably deserve to be studied more closely.

1.3 When $\ell > 1$ the ring R_ℓ has in general infinitely many cluster variables, grouped into infinitely many clusters. A notable exception is the case $\mathfrak{g} = \mathfrak{sl}_2$, for which R_ℓ is a cluster algebra of finite type A_ℓ in the classification of [**FZ3**]. In this special case it follows from [**CP2**] that, again, the classes in R_ℓ of the simple objects of $\mathscr{C}_\ell$ are precisely the cluster monomials of R_ℓ.

We conjecture that for arbitrary $\mathfrak{g}$ and ℓ, every cluster monomial of R_ℓ is the class of a simple object. We also conjecture that, conversely, the class of a simple object S in $\mathscr{C}_\ell$ is a cluster monomial if and only if $S \otimes S$ is simple. In this case, following [**L**], we call S a *real simple object*. We believe that real simple objects form an interesting class of irreducible $U_q(\widehat{\mathfrak{g}})$-modules, and the meaning of our partial results and conjectures is that their characters are governed by the combinatorics of cluster algebras.

1.4 Let us now describe the contents of the paper in more detail.

In Section 2 we recall the definition of a cluster algebra, and we introduce the new notion of monoidal categorification of a cluster algebra (Definition 2.1). We show (Proposition 2.2) that the existence of a monoidal categorification gives an immediate answer to some important open problems in the theory of cluster algebras, like the linear independence of cluster monomials, or the positivity of the coefficients of their expansion with respect to an arbitrary cluster.

In Section 3 we briefly review the theory of finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$ and we introduce the categories $\mathscr{C}_\ell$. We also recall the definition of the Kirillov-Reshetikhin modules and we review the T-system of equations that they satisfy.

In Section 4 we introduce some simple objects $S(\alpha)$ of $\mathscr{C}_1$ attached to the almost positive roots α, and we formulate our conjecture (Conjecture 4.6) for the category $\mathscr{C}_1$. It states that $\mathscr{C}_1$ is a monoidal categorification of a cluster algebra $\mathscr{A}$ with the same Dynkin type as $\mathfrak{g}$, and that the $S(\alpha)$ are the cluster simple objects. We illustrate the conjecture in type A_3.

In Section 5 we review the definition and main properties of the q-characters of Frenkel-Reshetikhin. One of the main tools to calculate them is the Frenkel-Mukhin algorithm which we recall and illustrate with examples.

In Section 6, we introduce some truncated versions of the q-characters for $\mathscr{C}_1$. These new truncated characters are much easier to calculate and they contain all the information to determine the composition factors of an object of $\mathscr{C}_1$. The main result of this section (Proposition 6.7) is an explicit formula for the truncated q-character of $S(\alpha)$ when α is a multiplicity-free positive root.

In Section 7, we review following [**FZ2**, **FZ5**] the F-polynomials of the cluster algebra $\mathscr{A}$. These are variants of the Fibonacci polynomials of [**FZ2**], which are the building blocks of the general solution of a Y-system. They satisfy a functional equation similar to a T-system and each cluster variable can be expressed in terms of its F-polynomial in a simple way (Equation (32)). We show that Conjecture 4.6 (i) is equivalent to the fact that the (normalized) truncated q-characters of the cluster simple objects are equal to the F-polynomials, and we prove it for the multiplicity-free roots (Theorem 7.8).

In Section 8, we prove an important tensor product theorem for the category $\mathscr{C}_1$ (Theorem 8.1): if $S_1, \ldots, S_k$ are simple objects of $\mathscr{C}_1$, then $S_1 \otimes \cdots \otimes S_k$ is simple if and only if $S_i \otimes S_j$ is simple for every $i \neq j$.

In Section 9, we introduce following [**FZ2**] the notions of compatible roots and cluster expansion. Because of Theorem 8.1 and of the existence and uniqueness of a cluster expansion [**FZ2**], we reduce Conjecture 4.6 (ii) for a given $\mathfrak{g}$ to a finite check: one has to verify that $S(\alpha) \otimes S(\beta)$ is simple for every pair (α, β) of compatible roots.

In Section 10 and Section 11 we prove Conjecture 4.6 in type A_n and D_4. Conjecture 4.6 (i) is proved more generally in type D_n, by showing that the truncated q-characters of cluster simple objects are given by the explicit combinatorial formula of [**FZ2**] for the Fibonacci polynomials of 2-restricted roots.

In Section 12 we present some applications of our results for $\mathscr{C}_1$. First we show that the q-characters of the simple objects of $\mathscr{C}_1$ are solutions of a system of functional equations similar to a periodic T-system. Secondly, we explain that the l-weight multiplicities appearing in the truncated q-characters of the cluster simple objects are equal to some tensor product multiplicities. This is reminiscent of the Kostka duality for representations of $\mathfrak{sl}_n$, but in our case it is not limited to type A. Thirdly, we exploit some known geometric formulas for F-polynomials due to Fu and Keller [**FK**] to express the coefficients of the truncated q-characters of the simple objects of $\mathscr{C}_1$ as Euler characteristics of some quiver grassmannians. This is similar to the Nakajima character formula for standard modules, but our formula works for simple modules.

Finally in Section 13 we state our conjectures for $\mathscr{C}_\ell$ (arbitrary ℓ) and illustrate them for $\mathfrak{g} = \mathfrak{sl}_2$ (arbitrary ℓ) where they follow from [**CP2**], and $\mathfrak{g} = \mathfrak{sl}_3$ ($\ell = 2$). We also explain how our conjecture for $\mathfrak{g} = \mathfrak{sl}_r$ (arbitrary ℓ) would essentially follow from a general conjecture of [**GLS2**] about the relation between Lusztig's dual canonical and dual semicanonical bases.

1.5 Kedem [**Ke**] and Di Francesco [**DFK**] have studied another connection between quantum affine algebras and cluster algebras, based on other types of functional equations (Q-systems and generalized T-systems). Keller [**Kel2**] has obtained a proof of the periodicity conjecture for Y-systems attached to pairs of simply-laced Dynkin diagrams using 2-Calabi-Yau categorifications of cluster algebras. More recently, Inoue, Iyama, Kuniba, Nakanishi and Suzuki [**IIKNS**] have also studied the connection between Y-systems, T-systems, Grothendieck rings of $U_q(\widehat{\mathfrak{g}})$ and cluster algebras, motivated by periodicity problems. These papers do not study the relations between cluster monomials and irreducible $U_q(\widehat{\mathfrak{g}})$-modules.

After this paper was submitted for publication, Nakajima [**N5**] gave a geometric proof of Conjecture 4.6 for all types A, D, E, using a tensor category of perverse sheaves on quiver varieties. This category also makes sense for non Dynkin quivers and Nakajima showed that its Grothendieck ring has a cluster algebra structure and that all cluster monomials are classes of simple objects. Thanks to Proposition 2.2 below, this yields strong positivity results for every acyclic cluster algebra with a bipartite seed. To the best of our knowledge, our conjecture for

$\ell > 1$ remains open.

We have presented our main results in several seminars and conferences in 2008 and 2009 (IHP Paris (BL), MSRI Berkeley (BL), NTUA Athens (BL), ETH Zurich (DH), UNAM Mexico (DH), Math. Institute Oxford (DH), MIO Oberwolfach (BL)). We thank these institutions for their kind invitations. Special thanks are due to Arun Ram and MSRI for organizing in spring 2008 a program on combinatorial representation theory where a large part of this work was done. We also thank Keller for his preliminary Oberwolfach report on this work [**Kel1**]. Finally we thank Nakajima for helpful comments and stimulating discussions.

2 Cluster algebras and their monoidal categorifications

2.1 We refer to [**FZ4**] for an excellent survey on cluster algebras. Here we only recall the main definitions and results.

2.1.1 Let $0 \leqslant n < r$ be some fixed integers. If $\widetilde{B} = (b_{ij})$ is an $r \times (r-n)$-matrix with integer entries, then the *principal part* B of $\widetilde{B}$ is the square matrix obtained from $\widetilde{B}$ by deleting the last n rows. Given some $k \in [1, r-n]$ define a new $r \times (r-n)$-matrix $\mu_k(\widetilde{B}) = (b'_{ij})$ by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \dfrac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{otherwise,} \end{cases} \tag{1}$$

where $i \in [1, r]$ and $j \in [1, r-n]$. One calls $\mu_k(\widetilde{B})$ the *mutation* of the matrix $\widetilde{B}$ in direction k. If $\widetilde{B}$ is an integer matrix whose principal part is skew-symmetric, then it is easy to check that $\mu_k(\widetilde{B})$ is also an integer matrix with skew-symmetric principal part. We will assume from now on that $\widetilde{B}$ has skew-symmetric principal part. In this case, one can equivalently encode $\widetilde{B}$ by a quiver Γ with vertex set $\{1, \ldots, r\}$ and with b_{ij} arrows from j to i if $b_{ij} > 0$ and $-b_{ij}$ arrows from i to j if $b_{ij} < 0$. Note that Γ has no loop nor 2-cycle.

Now Fomin and Zelevinsky define a cluster algebra $\mathscr{A}(\widetilde{B})$ as follows. Let $\mathscr{F} = \mathbb{Q}(x_1, \ldots, x_r)$ be the field of rational functions in r commuting indeterminates $\mathbf{x} = (x_1, \ldots, x_r)$. One calls $(\mathbf{x}, \widetilde{B})$ the *initial seed* of $\mathscr{A}(\widetilde{B})$. For $1 \leqslant k \leqslant r-n$ define

$$x_k^* = \frac{\prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik}<0} x_i^{-b_{ik}}}{x_k}. \tag{2}$$

The pair $(\mu_k(\mathbf{x}), \mu_k(\widetilde{B}))$, where $\mu_k(\mathbf{x})$ is obtained from $\mathbf{x}$ by replacing x_k by x_k^*, is the *mutation* of the seed $(\mathbf{x}, \widetilde{B})$ in direction k. One can iterate this procedure and obtain new seeds by mutating $(\mu_k(\mathbf{x}), \mu_k(\widetilde{B}))$ in any direction $l \in [1, r-n]$. Let $\mathscr{S}$ denote the set of all seeds obtained from $(\mathbf{x}, \widetilde{B})$ by any finite sequence of mutations. Each seed of $\mathscr{S}$ consists of an r-tuple of elements of $\mathscr{F}$ called a *cluster*, and of a matrix. The elements of a cluster are its *cluster variables*. Every seed has $r-n$ neighbours obtained by a single mutation in direction $1 \leqslant k \leqslant r-n$. One does not mutate the last n elements of a cluster; they are called *frozen variables* and belong to every cluster. We then define the *cluster algebra* $\mathscr{A}(\widetilde{B})$ as the subring of $\mathscr{F}$ generated by all the cluster variables of the seeds of $\mathscr{S}$. The integer $r-n$ is called the *rank* of $\mathscr{A}(\widetilde{B})$.

A *cluster monomial* is a monomial in the cluster variables of a *single* cluster. Note that the exchange relation (2) is of the form

$$x_k x_k^* = m_+ + m_- \tag{3}$$

where m_+ and m_- are two cluster monomials.

2.1.2 The first important result of the theory is that every cluster variable z of $\mathscr{A}(\widetilde{B})$ is a Laurent polynomial in $\mathbf{x}$ with coefficients in $\mathbb{Z}$. It is conjectured that the coefficients are positive. Note that because of (2), every cluster variable can be written as a subtraction free rational expression in $\mathbf{x}$, but this is not enough to ensure the positivity of its Laurent expansion.

The second main result is the classification of *cluster algebras of finite type*, *i.e.* with finitely many different cluster variables. Fomin and Zelevinsky proved that this happens if and only if there exists a seed $(\mathbf{z}, \widetilde{C})$ such that the quiver attached to the principal part of $\widetilde{C}$ is a Dynkin quiver (that is, an arbitrary orientation of a Dynkin diagram of type A, D, E). In this case, the cluster monomials form a distinguished subset of $\mathscr{A}(\widetilde{B})$, which is conjectured to be a $\mathbb{Z}$-basis [**FZ5**, §11].

If $\mathscr{A}(\widetilde{B})$ is not of finite cluster type, the cluster monomials do not span it, but it is conjectured that they are linearly independent. It is an interesting open problem to specify a "canonical" $\mathbb{Z}$-basis of $\mathscr{A}(\widetilde{B})$ containing the cluster monomials.

2.2 We now propose a natural framework which would yield positive answers to the above questions. We say that a simple object S of a monoidal category is *prime* if there exists no non trivial factorization $S \cong S_1 \otimes S_2$. We say that S is *real* if $S \otimes S$ is simple.

Definition 2.1 *Let $\mathscr{A}$ be a cluster algebra and let $\mathscr{M}$ be an abelian monoidal category. We say that $\mathscr{M}$ is a monoidal categorification of $\mathscr{A}$ if the Grothendieck ring of $\mathscr{M}$ is isomorphic to $\mathscr{A}$, and if*

(i) *the cluster monomials of $\mathscr{A}$ are the classes of all the real simple objects of $\mathscr{M}$;*

(ii) *the cluster variables of $\mathscr{A}$ (including the frozen ones) are the classes of all the real prime simple objects of $\mathscr{M}$.*

The existence of a monoidal categorification of a cluster algebra is a very strong property, as shown by the following result.

Proposition 2.2 *Suppose that the cluster algebra $\mathscr{A}$ has a monoidal categorification $\mathscr{M}$. Then*

(i) *every cluster variable of $\mathscr{A}$ has a Laurent expansion with positive coefficients with respect to any cluster;*

(ii) *the cluster monomials of $\mathscr{A}$ are linearly independent.*

Proof — If m is a cluster monomial, we denote by $S(m)$ the simple object with class $[S(m)] = m$. Let z be a cluster variable, and let

$$z = \frac{N(x_1, \ldots, x_n)}{x_1^{d_1} \cdots x_r^{d_r}}$$

denote its cluster expansion with respect to the cluster $\mathbf{x} = (x_1, \ldots, x_r)$. Here the numerator $N(x_1, \ldots, x_r)$ is a polynomial with coefficients in $\mathbb{Z}$. Multiplying both sides by the denominator, we see that $N(x_1, \ldots, x_r)$ is the class of the tensor product

$$P := S(z) \otimes S(x_1)^{\otimes d_1} \otimes \cdots \otimes S(x_r)^{\otimes d_r}.$$

Moreover, since $\mathbf{x}$ is a cluster, every monomial $m = x_1^{k_1} \cdots x_r^{k_r}$ is the class of a simple object

$$\Sigma = S(x_1)^{\otimes k_1} \otimes \cdots \otimes S(x_r)^{\otimes k_r}$$

of $\mathscr{M}$. Hence the coefficient of m in $N(x_1,\ldots,x_r)$ is equal to the multiplicity of Σ as a composition factor of P, thus it is nonnegative. This proves (i).

By definition of a monoidal categorification, the cluster monomials form a subset of the set of classes of all simple objects of $\mathscr{M}$, which is a $\mathbb{Z}$-basis of the Grothendieck group. This proves (ii). $\Box$

Remark 2.3 (i) In recent years, many examples of categorifications of cluster algebras have been constructed (see *e.g.* [**MRZ**, **BMRRT**, **CC**, **GLS2**, **BIRS**, **CK**, **GLS4**]). They are quite different from the monoidal categorifications introduced in this paper. Indeed, these categories are only additive and have no tensor operation. The multiplication of the cluster algebra reflects the direct sum operation of the category. We shall call these categorifications *additive*. Note that there is no analogue of Proposition 2.2 for additive categorifications. Although additive categorifications have been helpful for proving positivity of cluster expansions or linear independence of cluster monomials in some cases, this always requires some additional work, for example to show the positivity of some Euler characteristics. Finally, to recover the cluster algebra from its additive categorification one does not consider the Grothendieck group (which would be too small) but a kind of "dual Hall algebra" constructed via a *cluster character*. This is in general a complicated procedure.

(ii) In view of the strength and simplicity of Proposition 2.2, one might wonder whether there exist any examples of monoidal categorifications of cluster algebras. One of the aims of this paper is to produce some examples using representations of quantum affine algebras.

(iii) Let $\mathscr{M}$ be an abelian monoidal category. If $\mathscr{M}$ is a monoidal categorification of a cluster algebra $\mathscr{A}$, then we get new combinatorial insights about the tensor structure of $\mathscr{M}$. In particular if $\mathscr{A}$ has finite cluster type, we can express any simple object of $\mathscr{M}$ as a tensor product of finitely many prime objects, and this yields a combinatorial algorithm to calculate the composition factors of a tensor product of simple objects of $\mathscr{M}$. So this can be a fruitful approach to study certain interesting monoidal categories $\mathscr{M}$. This is the point of view we adopt in this paper.

3 Finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$

In this section we briefly review some known results in the representation theory of quantum affine algebras. For more detailed surveys we refer the reader to the monograph [**CP1**, chap. 12] and the recent paper [**CH**].

3.1 Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ of type A_n, D_n or E_n. We denote by $I = [1,n]$ the set of vertices of the Dynkin diagram, by $A = [a_{ij}]_{i,j\in I}$ the Cartan matrix of $\mathfrak{g}$, by h the Coxeter number, by $\Pi = \{\alpha_i \mid i \in I\}$ the set of simple roots, by W the Weyl group, with longest element w_0.

Let $U_q(\widehat{\mathfrak{g}})$ denote the corresponding quantum affine algebra, with parameter $q \in \mathbb{C}^*$ not a root of unity. $U_q(\widehat{\mathfrak{g}})$ has a Drinfeld-Jimbo presentation, which is a q-analogue of the usual presentation of the Kac-Moody algebra $\widehat{\mathfrak{g}}$. It also has a second presentation, due to Drinfeld, which is better suited to study finite-dimensional representations. There are infinitely many generators

$$x_{i,r}^{+},\ x_{i,r}^{-},\ h_{i,s},\ k_i,\ k_i^{-1},\ c,\ c^{-1}, \qquad (i\in I,\ r\in\mathbb{Z},\ s\in\mathbb{Z}\setminus\{0\}),$$

and a list of relations which we will not repeat (see *e.g.* [**FR**]). Remember that $\widehat{\mathfrak{g}}$ can be realized as a central extension of the loop algebra $\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]$. If x_i^+, x_i^-, h_i $(i \in I)$ denote the Chevalley generators of $\mathfrak{g}$ then $x_{i,r}^{\pm}$ is a q-analogue of $x_i^{\pm}\otimes t^r$, $h_{i,s}$ is a q-analogue of $h_i\otimes t^s$, k_i stands for the q-exponential of $h_i \equiv h_i\otimes 1$, and c for the q-exponential of the central element.

For every $a \in \mathbb{C}^*$ there exists an automorphism τ_a of $U_q(\widehat{\mathfrak{g}})$ given by

$$\tau_a(x_{i,r}^{\pm}) = a^r x_{i,r}^{\pm}, \quad \tau_a(h_{i,s}) = a^s h_{i,s}, \quad \tau_a(k_i^{\pm 1}) = k_i^{\pm 1}, \quad \tau_a(c^{\pm 1}) = c^{\pm 1}.$$

There also exists an involutive automorphism σ given by

$$\sigma(x_{i,r}^{\pm}) = x_{i,-r}^{\mp}, \quad \sigma(h_{i,s}) = -h_{i,-s}, \quad \sigma(k_i^{\pm 1}) = k_i^{\mp 1}, \quad \sigma(c^{\pm 1}) = c^{\mp 1}.$$

3.2 We consider the category $\mathscr{C}$ of finite-dimensional $U_q(\widehat{\mathfrak{g}})$-modules (of type 1). It is easy to see that if V is an object of $\mathscr{C}$, then c acts on V as the identity, and the generators $h_{i,s}$ act by pairwise commuting endomorphisms of V.

Since $U_q(\widehat{\mathfrak{g}})$ is a Hopf algebra, $\mathscr{C}$ is an abelian monoidal category. It is well-known that $\mathscr{C}$ is not semisimple.

For an object V in $\mathscr{C}$ and $a \in \mathbb{C}^*$, we denote by $V(a)$ the pull-back of V under τ_a, and by V^σ the pull-back of V under σ. The maps $V \mapsto V(a)$ and $V \mapsto V^\sigma$ give auto-equivalences of $\mathscr{C}$.

3.3 Let I denote the set of vertices of the Dynkin diagram of $\mathfrak{g}$. It was proved by Chari and Pressley that the simple objects S of $\mathscr{C}$ are parametrized by I-tuples of polynomials $\pi_S = (\pi_{i,S}(u);\ i \in I)$ in one indeterminate u with coefficients in $\mathbb{C}$ and constant term 1, called the *Drinfeld polynomials* of S. In particular, for $a \in \mathbb{C}^*$ and $i \in I$ we have a *fundamental module* $V_{i,a}$ which is the simple object with Drinfeld polynomials:

$$\pi_{j,V_{i,a}}(u) = \begin{cases} 1 - au & \text{if } j = i, \\ 1 & \text{if } j \neq i. \end{cases}$$

3.4 Let S^* denote the dual of S. The Drinfeld polynomials of S^* are easily deduced from those of S, namely

$$\pi_{i,S^*}(u) = \pi_{i^\vee,S}(q^{-h}u), \qquad (i \in I),$$

where $i \mapsto i^\vee$ denotes the involution on I given by $w_0(\alpha_i) = -\alpha_{i^\vee}$.

The Drinfeld polynomials behave simply under the action of the automorphisms τ_a, namely, for a simple object S of $\mathscr{C}$ we have

$$\pi_{i,S(a)}(u) = \pi_{i,S}(au), \qquad (i \in I).$$

They also behave simply under the action of the involution σ [**CP3**, Prop. 5.1] [**H4**, Cor. 4.11], namely, if

$$\pi_{i,S}(u) = \prod_k (1 - a_k u),$$

then

$$\pi_{i^\vee,S^\sigma}(u) = \prod_k (1 - a_k^{-1} q^{-h} u).$$

We also have the following compatibility with tensor products. If S_1 and S_2 are simple, and if S is the simple object with Drinfeld polynomials $\pi_{i,S} := \pi_{i,S_1}\pi_{i,S_2}$ $(i \in I)$, then S is a subquotient of the tensor product $S_1 \otimes S_2$.

3.5 Let R denote the Grothendieck ring of $\mathscr{C}$. It is known [**FR**, Cor. 2] that R is the polynomial ring over $\mathbb{Z}$ in the classes $[V_{i,a}]$ $(i \in I,\ a \in \mathbb{C}^*)$ of the fundamental modules.

3.6 Since the Dynkin diagram of $\mathfrak{g}$ is a bipartite graph, we have a partition $I = I_0 \sqcup I_1$ such that every edge connects a vertex of I_0 with a vertex of I_1. The following notation will be very convenient and used in many places. For $i \in I$ we set

$$\xi_i = \begin{cases} 0 & \text{if } i \in I_0, \\ 1 & \text{if } i \in I_1, \end{cases} \qquad \varepsilon_i = (-1)^{\xi_i}. \tag{4}$$

Clearly, the map $i \mapsto \xi_i$ is completely determined by the choice of $\xi_{i_0} \in \{0,1\}$ for a single vertex i_0, hence there are only two possible such maps.

3.7 Let $\mathscr{C}_{\mathbb{Z}}$ be the full subcategory of $\mathscr{C}$ whose objects V satisfy:

> for every composition factor S of V and every $i \in I$, the roots of the Drinfeld polynomial $\pi_{i,S}(u)$ belong to $q^{2\mathbb{Z}+\xi_i}$.

The Grothendieck ring $R_{\mathbb{Z}}$ of $\mathscr{C}_{\mathbb{Z}}$ is the subring of R generated by the classes $[V_{i,q^{2k+\xi_i}}]$ $(i \in I,\ k \in \mathbb{Z})$. It is known that every simple object S of $\mathscr{C}$ can be written as a tensor product $S_1(a_1) \otimes \cdots \otimes S_k(a_k)$ for some simple objects $S_1, \ldots, S_k$ of $\mathscr{C}_{\mathbb{Z}}$ and some complex numbers $a_1, \ldots, a_k$ such that

$$\frac{a_i}{a_j} \notin q^{2\mathbb{Z}}, \qquad (1 \leqslant i < j \leqslant k).$$

(This follows, for example, from the fact that such a tensor product satisfies the irreducibility criterion in [**Cha2**].) Therefore, the description of the simple objects of $\mathscr{C}$ essentially reduces to the description of the simple objects of $\mathscr{C}_{\mathbb{Z}}$.

3.8 We will now introduce an increasing sequence of subcategories of $\mathscr{C}_{\mathbb{Z}}$. Let $\ell \in \mathbb{N}$. (Note that in this paper $\mathbb{N}$ denotes the set of nonnegative integers, that is, $0 \in \mathbb{N}$.)

Definition 3.1 *The category $\mathscr{C}_\ell$ is the full subcategory of $\mathscr{C}_{\mathbb{Z}}$ consisting of those objects V which satisfy:*

> *for every composition factor S of V and every $i \in I$, the roots of the Drinfeld polynomial $\pi_{i,S}(u)$ belong to $\{q^{-2k-\xi_i} \mid 0 \leqslant k \leqslant \ell\}$.*

Note that for every simple object S of $\mathscr{C}_{\mathbb{Z}}$, there exists $k \in \mathbb{Z}$ such that $S(q^k)$ belongs to $\mathscr{C}_\ell$ for some $\ell \in \mathbb{N}$.

Proposition 3.2 *$\mathscr{C}_\ell$ is an abelian monoidal category, with Grothendieck ring the polynomial ring*

$$R_\ell = \mathbb{Z}\left[[V_{i,q^{2k+\xi_i}}];\ 0 \leqslant k \leqslant \ell\right].$$

The proof of Proposition 3.2 will be given in 5.2.4. It is an easy consequence of the theory of q-characters.

Example 3.3 The simple objects of the category $\mathscr{C}_0$ have a simple description. Indeed, it follows from [**FM**, Prop. 6.15] that all tensor products of fundamental modules of $\mathscr{C}_0$ are simple, hence every simple object of $\mathscr{C}_0$ is isomorphic to the tensor product of fundamental modules corresponding to the irreducible factors of its Drinfeld polynomials. Moreover any tensor product of simple objects of $\mathscr{C}_0$ is simple.

3.9 The definition of $\mathscr{C}_\ell$ depends on the map $i \mapsto \xi_i$ which can be chosen in two different ways. However, if $\mathfrak{g}$ is not of type A_{2n}, the image of $\mathscr{C}_\ell$ under the auto-equivalence $V \mapsto V^\sigma(q^{h+2\ell+1})$ is the category defined like $\mathscr{C}_\ell$ but with the opposite choice of ξ_i. If $\mathfrak{g}$ is of type A_{2n}, the image of $\mathscr{C}_\ell$ under $V \mapsto V^*(q^h)$ is the category defined like $\mathscr{C}_\ell$ but with the opposite choice of ξ_i. This shows that the choice of ξ_i is in fact irrelevant.

3.10 For $i \in I$, $k \in \mathbb{N}^*$ and $a \in \mathbb{C}^*$, the simple object $W^{(i)}_{k,a}$ with Drinfeld polynomials

$$\pi_{j,W^{(i)}_{k,a}} = \begin{cases} (1-au)(1-aq^2u)\cdots(1-aq^{2k-2}u) & \text{if } j = i, \\ 1 & \text{if } j \neq i, \end{cases}$$

is called a *Kirillov-Reshetikhin module*. In particular for $k = 1$, $W^{(i)}_{1,a}$ coincides with the fundamental module $V_{i,a}$. By convention, $W^{(i)}_{0,a}$ is the trivial representation for every i and a.

The classes $[W^{(i)}_{k,a}]$ in R satisfy the following system of equations indexed by $i \in I$, $k \in \mathbb{N}^*$, and $a \in \mathbb{C}^*$, called *the T-system*:

$$[W^{(i)}_{k,a}][W^{(i)}_{k,aq^2}] = [W^{(i)}_{k+1,a}][W^{(i)}_{k-1,aq^2}] + \prod_j [W^{(j)}_{k,aq}], \tag{5}$$

where in the right-hand side, the product runs over all vertices j adjacent to i in the Dynkin diagram. This was conjectured in [**KNS**] and proved in [**N2**, **H3**]. Using these equations, one can calculate inductively the expression of any $[W^{(i)}_{k,a}]$ as a polynomial in the classes $[W^{(i)}_{1,a}]$ of the fundamental modules.

Example 3.4 For $\mathfrak{g} = \mathfrak{sl}_2$, we have $I = \{1\}$, and we may drop the index i in $[W^{(i)}_{k,a}]$. The T-system reads

$$[W_{k,a}][W_{k,aq^2}] = [W_{k+1,a}][W_{k-1,aq^2}] + 1, \qquad (a \in \mathbb{C}^*,\ k \in \mathbb{N}^*).$$

This easily implies that $[W_{k,a}]$ is given by the $k \times k$ determinant:

$$[W_{k,a}] = \begin{vmatrix} [W_{1,a}] & 1 & 0 & \cdots & 0 \\ 1 & [W_{1,aq^2}] & 1 & \ddots & \vdots \\ 0 & 1 & [W_{1,aq^4}] & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & [W_{1,aq^{2k-2}}] \end{vmatrix}. \tag{6}$$

4 The case $\ell = 1$: statement of results and conjectures

We now focus on the subcategory $\mathscr{C}_1$.

4.1 Let $Q = \mathbb{Z}\Pi$ be the root lattice of $\mathfrak{g}$. Let $\Phi \subset Q$ be the root system. Following [**FZ2**] we denote by $\Phi_{\geqslant -1}$ the subset of almost positive roots, that is,

$$\Phi_{\geqslant -1} = \Phi_{>0} \cup (-\Pi)$$

consists of the positive roots together with the negatives of the simple roots. In this section, we attach to each $\beta \in \Phi_{\geqslant -1}$ a simple object $S(\beta)$ of $\mathscr{C}_1$.

4.1.1 To the negative simple root $-\alpha_i$ we attach the module $S(-\alpha_i)$ whose Drinfeld polynomials are all equal to 1, except $P_{i,S(-\alpha_i)}$ which is equal to

$$P_{i,S(-\alpha_i)}(u) = \begin{cases} 1-uq^2 & \text{if} \quad i \in I_0, \\ 1-uq & \text{if} \quad i \in I_1. \end{cases} \tag{7}$$

In other words, $S(-\alpha_i)$ is equal to the fundamental module V_{i,q^2} if $i \in I_0$, and to the fundamental module $V_{i,q}$ if $i \in I_1$.

4.1.2 To the simple root α_i we attach the module $S(\alpha_i)$ whose Drinfeld polynomials are all equal to 1, except $P_{i,S(\alpha_i)}$ which is equal to

$$P_{i,S(\alpha_i)}(u) = \begin{cases} 1-u & \text{if} \quad i \in I_0, \\ 1-uq^3 & \text{if} \quad i \in I_1. \end{cases} \tag{8}$$

In other words, $S(\alpha_i)$ is equal to the fundamental module $V_{i,1}$ if $i \in I_0$, and to the fundamental module V_{i,q^3} if $i \in I_1$.

4.1.3 To $\beta = \sum_{i\in I} b_i\alpha_i \in \Phi_{>0}$ we attach the module $S(\gamma)$ whose Drinfeld polynomials are

$$P_{i,S(\beta)} = \left(P_{i,S(\alpha_i)}\right)^{b_i}, \qquad (i \in I). \tag{9}$$

If β is not a simple root, this is *not* a Kirillov-Reshetikhin module.

4.1.4 For $i \in I$, we denote by F_i the simple module whose Drinfeld polynomials are all equal to 1, except P_{i,F_i} which is equal to

$$P_{i,F_i}(u) = \begin{cases} (1-uq^0)(1-uq^2) & \text{if} \quad i \in I_0, \\ (1-uq)(1-uq^3) & \text{if} \quad i \in I_1. \end{cases} \tag{10}$$

This is a Kirillov-Reshetikhin module. The classes $[F_i]$ will play the rôle of frozen cluster variables in the Grothendieck ring R_1.

4.2 We define a quiver Γ with $2n$ vertices as follows. We start from a copy of the Dynkin diagram oriented in such a way that every vertex of I_0 is a source, and every vertex of I_1 is a sink. For every $i \in I$ we then add a new vertex i' and an arrow $i \leftarrow i'$ if $i \in I_0$ (*resp.* $i \to i'$ if $i \in I_1$).

Example 4.1 Let $\mathfrak{g}$ be of type A_3. We choose $I_0 = \{1,3\}$. The quiver Γ is

$$\begin{array}{ccc} 1 & \leftarrow & 1' \\ \downarrow & & \\ 2 & \rightarrow & 2' \\ \uparrow & & \\ 3 & \leftarrow & 3' \end{array}$$

Put $I' = \{i' \mid i \in I\}$. It is often convenient to identify I with $[1,n]$, I' with $[n+1,2n]$, and i' with $i+n$. This will be done freely in the sequel. As explained in 2.1.1 we can attach a matrix $\widetilde{B} = (b_{ij})$ to Γ. This is a $2n \times n$-matrix with set of column indices I, and set of row indices $I \cup I'$, the vertex set of Γ. The entry b_{ij} is equal to 1 if there is an arrow from j to i in Γ, to -1 if there is an arrow from i to j in Γ, and to 0 otherwise.

Example 4.2 We continue the previous example. We have

$$\widetilde{B} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

4.3 Let $\mathscr{A} = \mathscr{A}(\widetilde{B})$ be the cluster algebra attached as in 2.1.1 to the initial seed $(\mathbf{x}, \widetilde{B})$, where $\mathbf{x} = (x_1, \ldots, x_{2n})$. This is a cluster algebra of rank n with n frozen variables. By construction, the principal part of $\widetilde{B}$ is a skew-symmetric matrix encoded by a Dynkin quiver (with sink-source orientation) of the same Dynkin type as $\mathfrak{g}$. As recalled in 2.1.2, it follows from [**FZ3**] that $\mathscr{A}$ has finitely many cluster variables. Moreover, the (non-frozen) cluster variables are naturally labelled by $\Phi_{\geqslant -1}$ via their denominator [**FZ3**, Th.1.9]. Let $x[\beta]$ denote the cluster variable attached to $\beta \in \Phi_{\geqslant -1}$, and let $f_i = x_{i+n}$ denote the frozen variable attached to $i' \equiv i+n \in I'$.

Example 4.3 We continue the previous examples. We have

$$\Phi_{\geqslant -1} = \{-\alpha_1, -\alpha_2, -\alpha_3, \alpha_1, \alpha_2, \alpha_3, \alpha_1+\alpha_2, \alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3\}.$$

The cluster algebra $\mathscr{A}$ is the subring of $\mathscr{F} = \mathbb{Q}(x_1, x_2, x_3, f_1, f_2, f_3)$ generated by

$$x[-\alpha_1] = x_1,\ x[-\alpha_2] = x_2,\ x[-\alpha_3] = x_3,\ f_1,\ f_2,\ f_3,$$

and the following cluster variables

$$\begin{aligned} x[\alpha_1] &= \frac{x_2+f_1}{x_1}, \\ x[\alpha_2] &= \frac{x_1x_3+f_2}{x_2}, \\ x[\alpha_3] &= \frac{x_2+f_3}{x_3}, \\ x[\alpha_1+\alpha_2] &= \frac{f_2x_2+f_1x_1x_3+f_1f_2}{x_1x_2}, \\ x[\alpha_2+\alpha_3] &= \frac{f_2x_2+f_3x_1x_3+f_2f_3}{x_2x_3}, \\ x[\alpha_1+\alpha_2+\alpha_3] &= \frac{f_2x_2^2+f_1f_3x_1x_3+f_1f_2x_2+f_2f_3x_2+f_1f_2f_3}{x_1x_2x_3}. \end{aligned}$$

Lemma 4.4 *The cluster algebra $\mathscr{A}$ is equal to the polynomial ring in the $2n$ variables*

$$x[-\alpha_i],\ x[\alpha_i], \quad (i \in I).$$

Proof — This follows from [**BFZ**]. Indeed, $\mathscr{A}$ is an acyclic cluster algebra, and $x[-\alpha_i]$, $x[\alpha_i]$ are the generators denoted by x_i, x_i' in [**BFZ**]. The monomials in $x_1, x_1', \ldots, x_n, x_n'$ which contain no product of the form x_jx_j' are called *standard*, and by [**BFZ**, Cor. 1.21] they form a basis of $\mathscr{A}$ over the ring $\mathbb{Z}[f_i \mid i \in I]$. It then follows from the relations $f_i = x_ix_i' - \prod_{j\neq i} x_j^{-a_{ij}}$ that the set of all monomials in $x_1, x_1', \ldots, x_n, x_n'$ is a basis of $\mathscr{A}$ over $\mathbb{Z}$. $\square$

Example 4.5 We continue Example 4.3. The generators of $\mathscr{A}$ can be expressed as polynomials in

$$x[-\alpha_1],\, x[-\alpha_2],\, x[-\alpha_3],\, x[\alpha_1],\, x[\alpha_2],\, x[\alpha_3],$$

as follows:

$$\begin{aligned}
f_1 &= x[-\alpha_1]x[\alpha_1] - x[-\alpha_2],\\
f_2 &= x[-\alpha_2]x[\alpha_2] - x[-\alpha_1]x[-\alpha_3],\\
f_3 &= x[-\alpha_3]x[\alpha_3] - x[-\alpha_2],\\
x[\alpha_1+\alpha_2] &= x[\alpha_1]x[\alpha_2] - x[-\alpha_3],\\
x[\alpha_2+\alpha_3] &= x[\alpha_2]x[\alpha_3] - x[-\alpha_1],\\
x[\alpha_1+\alpha_2+\alpha_3] &= x[\alpha_1]x[\alpha_2]x[\alpha_3] - x[-\alpha_1]x[\alpha_1] - x[-\alpha_3]x[\alpha_3] + x[-\alpha_2].
\end{aligned}$$

4.4 We have seen in 3.8 that the Grothendieck ring R_1 is the polynomial ring in the classes of the $2n$ fundamental modules of $\mathscr{C}_1$:

$$S(-\alpha_i),\, S(\alpha_i), \qquad (i \in I).$$

By Lemma 4.4, the assignment

$$x[-\alpha_i] \mapsto [S(-\alpha_i)], \quad x[\alpha_i] \mapsto [S(\alpha_i)], \qquad (i \in I),$$

extends to a ring isomorphism ι from $\mathscr{A}$ to R_1.

We can now state the main theorem-conjecture of this section.

Conjecture 4.6 (i) *We have*

$$\iota(x[\beta]) = [S(\beta)], \quad \iota(f_i) = [F_i], \qquad (\beta \in \Phi_{\geqslant -1},\, i \in I).$$

(ii) *If we identify $\mathscr{A}$ to R_1 via ι, $\mathscr{C}_1$ becomes a monoidal categorification of $\mathscr{A}$. Moreover, the class in R_1 of any simple object of $\mathscr{C}_1$ is a cluster monomial (in other words, every simple object of $\mathscr{C}_1$ is real).*

The conjecture will be proved in Section 10 for $\mathfrak{g}$ of type A_n. In Section 11 we prove it for $\mathfrak{g}$ of type D_4, and we prove (i) for $\mathfrak{g}$ of type D_n.

Conjecture 4.6 (ii) immediately implies the following

Corollary 4.7 (i) *The category $\mathscr{C}_1$ has finitely many prime simple objects, namely, the cluster simple objects $S(\beta)$ $(\beta \in \Phi_{\geqslant -1})$, and the frozen simple objects F_i $(i \in I)$.*

(ii) *Each simple object of $\mathscr{C}_1$ is a tensor product of primes whose classes belong to one of the clusters of $\mathscr{A}$.*

(iii) *All the tensor powers of a simple object of $\mathscr{C}_1$ are simple.*

(iv) *The cluster monomials form a $\mathbb{Z}$-basis of $\mathscr{A}$.* □

Example 4.8 We continue the previous examples. By Corollary 4.7, for $\mathfrak{g}$ of type A_3 the category $\mathscr{C}_1$ has 12 prime simple objects:

$$S(-\alpha_1),\ S(-\alpha_2),\ S(-\alpha_3),\ S(\alpha_1),\ S(\alpha_2),\ S(\alpha_3),\ S(\alpha_1+\alpha_2),\ S(\alpha_2+\alpha_3),\ S(\alpha_1+\alpha_2+\alpha_3),$$
$$F_1,\ F_2,\ F_3.$$

The first 6 modules are fundamental representations, F_1, F_2, F_3 are Kirillov-Reshetikhin modules, $S(\alpha_1+\alpha_2)$, $S(\alpha_2+\alpha_3)$ are minimal affinizations (in the sense of [**Cha1**]), but $S(\alpha_1+\alpha_2+\alpha_3)$ is *not* a minimal affinization. Its underlying $U_q(\mathfrak{g})$-module is isomorphic to $V(\varpi_1+\varpi_2+\varpi_3)\oplus V(\varpi_2)$. (Here we denote by ϖ_i $(i\in I)$ the fundamental weights of $\mathfrak{g}$, and by $V(\lambda)$ the irreducible $U_q(\mathfrak{g})$-module with highest weight λ.)

Using the known dimensions of the fundamental modules and the formulas of Example 4.5, we can easily calculate their dimensions, namely (in the same order):

$$4,\ 6,\ 4,\ 4,\ 6,\ 4,\ 20,\ 20,\ 70,\ 10,\ 20,\ 10.$$

The cluster algebra $\mathscr{A}$ has 14 clusters:

$$\{-\alpha_1, -\alpha_2, -\alpha_3\},\ \{\alpha_1, -\alpha_2, -\alpha_3\},\ \{-\alpha_1, \alpha_2, -\alpha_3\},\ \{-\alpha_1, -\alpha_2, \alpha_3\},\ \{\alpha_1, -\alpha_2, \alpha_3\},$$
$$\{-\alpha_1, \alpha_2, \alpha_2+\alpha_3\},\ \{-\alpha_1, \alpha_3, \alpha_2+\alpha_3\},\ \{-\alpha_3, \alpha_2, \alpha_1+\alpha_2\},\ \{-\alpha_3, \alpha_1, \alpha_1+\alpha_2\},$$
$$\{\alpha_1+\alpha_2, \alpha_2, \alpha_2+\alpha_3\},\ \{\alpha_1, \alpha_3, \alpha_1+\alpha_2+\alpha_3\},\ \{\alpha_1, \alpha_1+\alpha_2, \alpha_1+\alpha_2+\alpha_3\},$$
$$\{\alpha_3, \alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3\},\ \{\alpha_1+\alpha_2, \alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3\}.$$

Here we have written for short β instead of $x[\beta]$ and we have omitted the three frozen variables which belong to every cluster.

The simple objects of $\mathscr{C}_1$ are exactly all tensor products of the form

$$S(\beta_1)^{\otimes k_1}\otimes S(\beta_2)^{\otimes k_2}\otimes S(\beta_3)^{\otimes k_3}\otimes F_1^{\otimes l_1}\otimes F_2^{\otimes l_2}\otimes F_3^{\otimes l_3},\quad (k_1,k_2,k_3,l_1,l_2,l_3)\in\mathbb{N}^6,$$

in which $\{\beta_1, \beta_2, \beta_3\}$ runs over the 14 clusters listed above. Moreover, simple objects multiply (in the Grothendieck ring) as the corresponding cluster monomials in $\mathscr{A}$. For instance, the exchange formula

$$x[-\alpha_2]x[\alpha_2] = f_2 + x[-\alpha_1]x[-\alpha_3]$$

shows that the tensor product $S(-\alpha_2)\otimes S(\alpha_2)$ has two composition factors isomorphic to F_2 and $S(-\alpha_1)\otimes S(-\alpha_3)$.

5 q-characters

5.1 An essential tool for our proof of Conjecture 4.6 in type A and D is the theory of q-characters of Frenkel and Reshetikhin [**FR**]. We shall recall briefly their definition and main properties. For more details see *e.g.* [**CH**].

5.1.1 Recall from 3.2 that if V is a finite-dimensional $U_q(\widehat{\mathfrak{g}})$-module, the endomorphisms of V which represent the generators $h_{i,m}$ commute with each other. Moreover, by Drinfeld's presentation the k_i are pairwise commutative, and they commute with all the $h_{i,m}$. Hence we can write V as a finite direct sum of common generalized eigenspaces for the simultaneous action of the k_i and of the $h_{i,m}$. These common generalized eigenspaces are called the *l-weight-spaces* of V. (Here l

stands for "loop"). The *q-character of* V is a Laurent polynomial with positive integer coefficients in some indeterminates $Y_{i,a}$ $(i \in I, a \in \mathbb{C}^*)$, which encodes the decomposition of V as the direct sum of its l-weight-spaces .

More precisely, the eigenvalues of the $h_{i,m}$ $(m > 0)$ in an l-weight-space W of V are always of the form

$$\frac{q^m - q^{-m}}{m(q - q^{-1})} \left(\sum_{r=1}^{k_i} (a_{ir})^m - \sum_{s=1}^{l_i} (b_{is})^m \right) \tag{11}$$

for some nonzero complex numbers a_{ir}, b_{is}. Moreover, they completely determine the eigenvalues of the $h_{i,m}$ $(m < 0)$ and of the k_i on W. We encode this collection of eigenvalues with the Laurent monomial

$$\prod_{i \in I} \left(\prod_{r=1}^{k_i} Y_{i,a_{ir}} \prod_{s=1}^{l_i} Y_{i,b_{is}}^{-1} \right), \tag{12}$$

and the coefficient of this monomial in the q-character of V is the dimension of W [**FM**, Prop. 2.4]. The collection of eigenvalues (11) is called the *l-weight* of W. By a slight abuse, we shall often say that the l-weight of W is the monomial (12).

Let $\mathscr{Y} = \mathbb{Z}[Y_{i,a}^{\pm}; i \in I, a \in \mathbb{C}^*]$, and denote by $\chi_q(V) \in \mathscr{Y}$ the q-character of $V \in \mathscr{C}$. One shows that $\chi_q(V)$ depends only on the class of V in the Grothendieck ring R, and that the induced map $\chi_q : R \to \mathscr{Y}$ is an injective ring homomorphism.

Example 5.1 Let $\mathfrak{g} = \mathfrak{sl}_2$. Then $I = \{1\}$, and we can drop the index $i \in I$. Hence

$$\mathscr{Y} = \mathbb{Z}[Y_a^{\pm}; a \in \mathbb{C}^*].$$

One calculates easily the q-character of the fundamental module $W_{1,a}$. It is two-dimensional, and decomposes as a sum of two common eigenspaces:

$$\chi_q(W_{1,a}) = Y_a + Y_{aq^2}^{-1}. \tag{13}$$

From the identity

$$[W_{1,a}][W_{1,aq^2}] = [W_{2,a}][W_{0,aq^2}] + 1 = [W_{2,a}] + 1,$$

(see Example 3.4), one deduces that

$$\chi_q(W_{2,a}) = Y_a Y_{aq^2} + Y_a Y_{aq^4}^{-1} + Y_{aq^2}^{-1} Y_{aq^4}^{-1}.$$

One can calculate similarly the q-character of every Kirillov-Reshetikhin module for $U_q(\widehat{\mathfrak{sl}_2})$ (see below Example 5.2).

5.1.2 $U_q(\widehat{\mathfrak{g}})$ has a natural subalgebra isomorphic to $U_q(\mathfrak{g})$, hence every $V \in \mathscr{C}$ can be regarded as a $U_q(\mathfrak{g})$-module by restriction. The l-weight-space decomposition of V is a refinement of its decomposition as a direct sum of $U_q(\mathfrak{g})$-weight-spaces. Let P be the weight lattice of $\mathfrak{g}$, with basis given by the fundamental weights ϖ_i $(i \in I)$. We denote by ω the $\mathbb{Z}$-linear map from $\mathscr{Y}$ to $\mathbb{Z}[P]$ defined by

$$\omega\left(\prod_{i,a} Y_{i,a}^{u_{i,a}} \right) = \sum_i \left(\sum_a u_{i,a} \right) \varpi_i. \tag{14}$$

If W is an l-weight-space of V with l-weight the Laurent monomial $m \in \mathscr{Y}$, then W is a subspace of the $U_q(\mathfrak{g})$-weight-space with weight $\omega(m)$. Hence, the image $\omega(\chi_q(V))$ of the q-character of V is the ordinary character of the underlying $U_q(\mathfrak{g})$-module.

For $i \in I$ and $a \in \mathbb{C}^*$ define

$$A_{i,a} = Y_{i,aq} Y_{i,aq^{-1}} \prod_{j \neq i} Y_{j,a}^{a_{ij}}. \qquad (15)$$

(Note that, because of our general assumption that $\mathfrak{g}$ is simply-laced (see §3.1), for $i \neq j$ we have $a_{ij} \in \{0,-1\}$.) Thus $\omega(A_{i,a}) = \alpha_i$, and the $A_{i,a}$ $(a \in \mathbb{C}^*)$ should be viewed as affine analogues of the simple root α_i. Following [**FR**], we define a partial order on the set $\mathscr{M}$ of Laurent monomials in the $Y_{i,a}$ by setting:

$$m \leqslant m' \quad \Longleftrightarrow \quad \frac{m'}{m} \text{ is a monomial in the } A_{i,a}.$$

This is an affine analogue of the usual partial order on P, defined by $\lambda \leqslant \lambda'$ if and only if $\lambda' - \lambda$ is a sum of simple roots α_i.

Let S be a simple object of $\mathscr{C}$ with Drinfeld polynomials

$$\pi_{i,S}(u) = \prod_{k=1}^{n_i} (1 - u a_k^{(i)}), \qquad (i \in I). \qquad (16)$$

Then the subset $\mathscr{M}(S)$ of $\mathscr{M}$ consisting of all the monomials occuring in $\chi_q(S)$ has a unique maximal element with respect to $\leqslant$, which is equal to

$$m_S = \prod_{i \in I} \prod_{k=1}^{n_i} Y_{i,a_k^{(i)}} \qquad (17)$$

and has coefficient 1 [**FM**, Th. 4.1]. This is the *highest weight monomial* of $\chi_q(S)$. The one-dimensional l-weight-space of S with l-weight m_S consists of the *highest-weight vectors* of S, that is, the l-weight vectors $v \in S$ such that $x_{i,r}^+ v = 0$ for every $i \in I$ and $r \in \mathbb{Z}$.

A monomial $m \in \mathscr{M}$ is called *dominant* if it does not contain negative powers of the variables $Y_{i,a}$. The highest weight monomial m_S of an irreducible q-character $\chi_q(S)$ is always dominant. We will denote by $\mathscr{M}_+$ the set of dominant monomials.

Conversely, we can associate to any dominant monomial as in (17) a unique set of Drinfeld polynomials given by (16). Hence, we can equivalently parametrize the isoclasses of simple objects of $\mathscr{C}$ by $\mathscr{M}_+$. In the sequel, the simple module whose q-character has highest weight monomial $m \in \mathscr{M}_+$ will be denoted by $L(m)$. To summarize, we have

$$\chi_q(L(m)) = m \left(1 + \sum_p M_p \right),$$

where all the M_p are monomials in the variables $A_{i,a}^{-1}$. It will sometimes be convenient to renormalize the q-characters and work with

$$\widetilde{\chi}_q(L(m)) := \frac{1}{m} \chi_q(L(m)) = 1 + \sum_p M_p. \qquad (18)$$

This is a polynomial with positive integer coefficients in the variables $A_{i,a}^{-1}$.

Example 5.2 We continue Example 5.1 and describe the q-characters of all the simple objects of $\mathscr{C}$ for $\mathfrak{g} = \mathfrak{sl}_2$. For $a \in \mathbb{C}^*$, we have $A_a = Y_{aq} Y_{aq^{-1}}$. For $k \in \mathbb{N}$, put $m_{k,a} = Y_a Y_{aq^2} \cdots Y_{aq^{2k-2}}$. It follows from (6) and (13) that for the Kirillov-Reshetikhin modules we have [**FR**]:

$$\chi_q(W_{k,a}) = m_{k,a} \left(1 + A_{aq^{2k-1}}^{-1} \left(1 + A_{aq^{2(k-1)-1}}^{-1} \left(1 + \cdots \left(1 + A_{aq^3}^{-1} (1 + A_{aq}^{-1}) \right) \cdots \right) \right) \right). \qquad (19)$$

We call *q-segment of origin a and length k* the string of complex numbers

$$\Sigma(k,a) = \{a,\ aq^2,\ \cdots, aq^{2k-2}\}.$$

Two q-segments are said to be in *special position* if one does not contain the other, and their union is a q-segment. Otherwise we say that there are in *general position*. It is easy to check that every finite multi-set $\{b_1,\ldots,b_s\}$ of elements of $\mathbb{C}^*$ can be written uniquely as a union of segments $\Sigma(k_i,a_i)$ in such a way that every pair $(\Sigma(k_i,a_i),\Sigma(k_j,a_j))$ is in general position. Then, Chari and Pressley [**CP2**] have proved that the simple module S with Drinfeld polynomial

$$\pi_S(u) = \prod_{m=1}^{s}(1-ub_m)$$

is isomorphic to the tensor product of Kirillov-Reshetikhin modules $\bigotimes_i W_{k_i,a_i}$. Hence $\chi_q(S)$ can be calculated using (19).

An important consequence of the existence and uniqueness of the highest l-weight of a simple module is:

Proposition 5.3 [**FM**] *Let V and W be two objects of $\mathscr{C}$. If $\chi_q(V)$ and $\chi_q(W)$ have the same dominant monomials with the same multiplicities, then $\chi_q(V) = \chi_q(W)$.*

Indeed, to express $\chi_q(V)$ as a sum of irreducible q-characters, one can use the following simple procedure. Pick a dominant monomial m in $\chi_q(V)$ which is maximal with respect to the partial order $\leqslant$. Then $\chi_q(V)-\chi_q(L(m))$ is a polynomial with nonnegative coefficients. If $\chi_q(V)-\chi_q(L(m)) \neq 0$, then pick a maximal dominant monomial m' in $\chi_q(V)-\chi_q(L(m))$ and consider $\chi_q(V)-\chi_q(L(m))-\chi_q(L(m'))$. And so forth. After a finite number of steps one gets the decomposition of $\chi_q(V)$ into irreducible characters using only its subset of dominant monomials.

5.2 We now recall an algorithm due to Frenkel and Mukhin [**FM**] which attaches to any $m \in \mathscr{M}_+$ a polynomial $\mathrm{FM}(m)$, equal to the q-character of $L(m)$ under certain conditions.

5.2.1 We first introduce some notation. Given $i \in I$, we say that $m \in \mathscr{M}$ is *i-dominant* if every variable $Y_{i,a}$ $(a \in \mathbb{C}^*)$ occurs in m with non-negative exponent, and in this case we write $m \in \mathscr{M}_{i,+}$. Using the known irreducible q-characters of $U_q(\widehat{\mathfrak{sl}}_2)$ (see Example 5.2) we define for $m \in \mathscr{M}_{i,+}$ a polynomial $\varphi_i(m)$ as follows. Let $\overline{m}$ be the monomial obtained from m by replacing $Y_{j,a}$ by Y_a if $j = i$ and by 1 if $j \neq i$. Then there is a unique irreducible q-character $\chi_q(\overline{m})$ of $U_q(\widehat{\mathfrak{sl}}_2)$ with highest weight monomial $\overline{m}$. Write $\chi_q(\overline{m}) = \overline{m}(1+\sum_p \overline{M}_p)$, where the $\overline{M}_p$ are monomials in the variables A_a^{-1} $(a \in \mathbb{C}^*)$. Then one sets $\varphi_i(m) := m(1+\sum_p M_p)$ where each M_p is obtained from the corresponding $\overline{M}_p$ by replacing each variable A_a^{-1} by $A_{i,a}^{-1}$.

Suppose now that $m \in \mathscr{M}_+$. We define the subset D_m of $\mathscr{M}$ as follows. A monomial m' belongs to D_m if there is a finite sequence $(m_0 = m, m_1, \ldots, m_t = m')$ such that for all $r = 1,\ldots,t$ there exists $i \in I$ such that $m_{r-1} \in \mathscr{M}_{i,+}$ and m_r is a monomial occuring in $\varphi_i(m_{r-1})$. Clearly, D_m is countable and every $m' \in D_m$ satisfies $m' \leqslant m$. We can therefore write

$$D_m = \{m_0 = m,\ m_1,\ m_2,\ \ldots\}$$

in such a way that if $m_t \leqslant m_r$ then $t \geqslant r$.

Finally, we define inductively some sequences of integers $(s(m_r))_{r\geq 0}$ and $(s_i(m_r))_{r\geq 0}$ $(i \in I)$ as follows. The initial condition is $s(m_0) = 1$ and $s_i(m_0) = 0$ for all $i \in I$. For $t \geq 1$, we set

$$s_i(m_t) = \sum_{r<t,\ m_r \in \mathscr{M}_{i,+}} (s(m_r) - s_i(m_r))[\varphi_i(m_r) : m_t], \quad (i \in I), \tag{20}$$

$$s(m_t) = \max\{s_i(m_t) \mid i \in I\}, \tag{21}$$

where $[\varphi_i(m_r) : m_t]$ denotes the coefficient of m_t in the polynomial $\varphi_i(m_r)$. We then put

$$\mathrm{FM}(m) := \sum_{r\geq 0} s(m_r) m_r. \tag{22}$$

Example 5.4 Take $\mathfrak{g}$ of type A_2 and $m = Y_{1,1}Y_{2,q^3}$. We have

$$\varphi_1(m) = Y_{1,1}Y_{2,q^3} + Y_{1,q^2}^{-1}Y_{2,q}Y_{2,q^3},$$
$$\varphi_2(m) = Y_{1,1}Y_{2,q^3} + Y_{1,1}Y_{1,q^4}Y_{2,q^5}^{-1},$$

so we put $m_1 = Y_{1,q^2}^{-1}Y_{2,q}Y_{2,q^3}$ and $m_2 = Y_{1,1}Y_{1,q^4}Y_{2,q^5}^{-1}$. The monomial m_1 is 2-dominant, and we have

$$\varphi_2(m_1) = Y_{1,q^2}^{-1}Y_{2,q}Y_{2,q^3} + Y_{1,q^2}^{-1}Y_{1,q^4}Y_{2,q}Y_{2,q^5}^{-1} + Y_{1,q^4}Y_{2,q^3}^{-1}Y_{2,q^5}^{-1};$$

similarly, m_2 is 1-dominant and we have

$$\varphi_1(m_2) = Y_{1,1}Y_{1,q^4}Y_{2,q^5}^{-1} + Y_{1,q^2}^{-1}Y_{1,q^4}Y_{2,q}Y_{2,q^5}^{-1} + Y_{1,1}Y_{1,q^6}^{-1} + Y_{1,q^2}^{-1}Y_{1,q^6}^{-1}Y_{2,q}.$$

We set $m_3 = Y_{1,q^2}^{-1}Y_{1,q^4}Y_{2,q}Y_{2,q^5}^{-1}$, $m_4 = Y_{1,q^4}Y_{2,q^3}^{-1}Y_{2,q^5}^{-1}$, $m_5 = Y_{1,1}Y_{1,q^6}^{-1}$, and $m_6 = Y_{1,q^2}^{-1}Y_{1,q^6}^{-1}Y_{2,q}$. We see that m_3 is neither 1-dominant nor 2-dominant. The monomial m_4 is 1-dominant and

$$\varphi_1(m_4) = Y_{1,q^4}Y_{2,q^3}^{-1}Y_{2,q^5}^{-1} + Y_{1,q^6}^{-1}Y_{2,q^3}^{-1};$$

similarly, m_5 and m_6 are 2-dominant and

$$\varphi_2(m_5) = Y_{1,1}Y_{1,q^6}^{-1}, \qquad \varphi_2(m_6) = Y_{1,q^2}^{-1}Y_{1,q^6}^{-1}Y_{2,q} + Y_{1,q^6}^{-1}Y_{2,q^3}^{-1}.$$

Finally the monomial $m_7 = Y_{1,q^6}^{-1}Y_{2,q^3}^{-1}$ is neither 1-dominant nor 2-dominant. So

$$D_m = \{m,\ m_1,\ m_2,\ m_3,\ m_4,\ m_5,\ m_6,\ m_7\}.$$

It is easy to check that $\mathrm{FM}(m)$ is the sum of all the elements of D_m with coefficient 1, and that we have

$$\mathrm{FM}(m) = \varphi_1(m) + \varphi_1(m_2) + \varphi_1(m_4) = \varphi_2(m) + \varphi_2(m_1) + \varphi_2(m_5) + \varphi_2(m_6).$$

5.2.2 Let $m \in \mathscr{M}_+$. We say that the simple module $L(m)$ is *minuscule* if m is the only dominant monomial of $\chi_q(L(m))$. (In [**N3**] these modules are called *special*.)

Theorem 5.5 [**FM**] *If $L(m)$ is minuscule then $\chi_q(L(m)) = \mathrm{FM}(m)$. Moreover, all the fundamental modules are minuscule.*

It was proved in [**N2**] that Kirillov-Reshetikhin modules are also minuscule. Unfortunately, there exist simple modules for which the Frenkel-Mukhin algorithm fails, as shown by the next example. For another example in type C_3 see [**NN**].

Example 5.6 Take $\mathfrak{g}$ of type A_2 and $m = Y_{1,1}^2 Y_{2,q^3}$. Clearly, $L(m)$ is a simple object of $\mathscr{C}_1$, and using Conjecture 4.6 (ii) (which will be proved in Section 10 in type A_n), we have in the notation of 4.1

$$L(m) \cong S(\alpha_1) \otimes S(\alpha_1 + \alpha_2) = L(Y_{1,1}) \otimes L(Y_{1,1}Y_{2,q^3}).$$

It is easy to calculate that for $\mathfrak{g}$ of type A_{n+1} and $a \in \mathbb{C}^*$, we have

$$\chi_q(L(Y_{1,a})) = Y_{1,a} + Y_{1,aq^2}^{-1} Y_{2,aq} + Y_{2,aq^3}^{-1} Y_{3,aq^2} + \cdots + Y_{n,aq^{n+1}}^{-1}.$$

Hence, for our example in type A_2, $\chi_q(L(m)) = \chi_q(L(Y_{1,1}))\chi_q(L(Y_{1,1}Y_{2,q^3}))$ contains the monomial

$$Y_{2,q^3}^{-1} Y_{1,1} Y_{2,q^3} = Y_{1,1} = mA_{1,q}^{-1} A_{2,q^2}^{-1}$$

with coefficient 1. On the other hand, $\varphi_1(m) = m(1 + 2A_{1,q}^{-1} + A_{1,q}^{-2})$ and $\varphi_2(m) = m(1 + A_{2,q^4}^{-1})$. Now $m_1 := mA_{1,q}^{-1} = Y_{1,1} Y_{1,q^2}^{-1} Y_{2,q} Y_{2,q^3}$ is 2-dominant and $\varphi_2(m_1) = m_1(1 + A_{2,q^4}^{-1} + A_{2,q^2}^{-1} A_{2,q^4}^{-1})$. Thus $\mathrm{FM}(m) - m - 2m_1$ is of the form mP where $P \in \mathbb{Z}[A_{1,a}^{-1}, A_{2,a}^{-1};\ a \in \mathbb{C}^*]$ contains only monomials divisible by $A_{1,q}^{-2}$ or A_{2,q^4}^{-1}. Hence $\mathrm{FM}(m)$ does not contain the monomial $mA_{1,q}^{-1}A_{2,q^2}^{-1}$, thus $\mathrm{FM}(m)$ is not equal to $\chi_q(L(m))$.

Remark 5.7 (i) When $\mathrm{FM}(m) = \chi_q(L(m))$, this polynomial can be written for every $i \in I$ as a positive sum of polynomials of the form $\varphi_i(m')$ for some $m' \in \mathscr{M}_{i,+}$. For $m' \in D_m \cap \mathscr{M}_{i,+}$, the integer $s(m') - s_i(m')$ is then the coefficient of $\varphi_i(m')$ in this sum.

(ii) One can slightly generalize [**FM**, Th. 5.9] as follows: a sufficient condition for having $\mathrm{FM}(m) = \chi_q(L(m))$ is that $\mathrm{FM}(m)$ contains all the dominant monomials of $\chi_q(L(m))$. The proof is essentially the same as in [**FM**, Th. 5.9].

(iii) In [**H1**], a polynomial $F(m)$ defined by formulas similar to (20), (21), (22), has been defined for any $m \in \mathscr{M}_+$. The only difference between the definitions of $F(m)$ and $\mathrm{FM}(m)$ is the formula giving $s(m_t)$. If $L(m)$ is minuscule, then $F(m) = \mathrm{FM}(m) = \chi_q(L(m))$. Otherwise $F(m)$ may not be equal to $\mathrm{FM}(m)$. The polynomial $\mathrm{FM}(m)$ has nonnegative coefficients, may contain several dominant monomials, and it may not belong to the image of the map $\chi_q : R \to \mathscr{Y}$. On the other hand, $F(m)$ belongs to this image, has a unique dominant monomial (equal to m), but may have negative coefficients.

This is well illustrated by Example 5.6. In that case $L(m)$ has dimension 24, and we have

$$\mathrm{FM}(m) = \chi_q(L(m)) - Y_{1,1}, \qquad F(m) = \chi_q(L(m)) - Y_{1,1} - Y_{1,q^2}^{-1} Y_{2,q} - Y_{2,q^3}^{-1},$$

and $F(m)$ contains the monomial Y_{2,q^3}^{-1} with coefficient -1.

In this paper when we refer to the Frenkel-Mukhin algorithm we will always mean the polynomial $\mathrm{FM}(m)$.

5.2.3 Let us note a useful consequence of the Frenkel-Mukhin algorithm.

Proposition 5.8 *Let $m = \prod_{k=1}^r Y_{i_k,a_k} \in \mathscr{M}_+$ and let mM be a monomial occuring in $\chi_q(L(m))$, where M is a monomial in the variables $A_{i,a}^{-1}$ ($i \in I$, $a \in \mathbb{C}^*$). If $A_{j,b}^{-1}$ occurs in M there exist $k \in \{1, \ldots, r\}$ and $l \in \mathbb{N} \setminus \{0\}$ such that $b = a_k q^l$. Moreover, there also exist finite sequences*

$$(j_1 = i_k, j_2, \ldots, j_s = j) \in I^s, \quad (l_1 = 1 \leqslant l_2 \leqslant \cdots \leqslant l_s = l) \in \mathbb{N}^s$$

such that $a_{j_t,j_{t+1}} = -1$ and $A_{j_t,a_k q^{l_t}}^{-1}$ occurs in M for every $t = 1, \ldots, s-1$. Finally, l_t is odd if $\varepsilon_{j_t} = \varepsilon_{i_k}$ and even otherwise.

Proof — If $m = Y_{i,a}$ is the highest weight monomial of a fundamental module, then $\chi_q(L(m)) = \mathrm{FM}(m)$, and the proposition holds by definition of the algorithm FM. In the general case, by 3.4 $L(m)$ is a subquotient of the tensor product $\otimes_{k=1}^{r} L(Y_{i_k,a_k})$, and therefore its q-character is contained in the product of the q-characters of the factors, which proves the proposition. $\square$

5.2.4 We can now prove Proposition 3.2. Let $L(m)$ and $L(m')$ be in $\mathscr{C}_\ell$. This means that m and m' are monomials in the variables $Y_{i,q^{\xi_i+2k}}$ $(0 \leqslant k \leqslant \ell)$. If $L(m'')$ is a composition factor of $L(m) \otimes L(m')$ then m'' is a product of monomials of $\chi_q(L(m))$ and $\chi_q(L(m'))$. So we have $m'' = mm'M$ where M is a monomial in the $A_{i,a}^{-1}$. We claim that, for m'' to be dominant, the spectral parameters a have to be of the form $a = q^{\xi_i+2k+1}$ with $0 \leqslant k \leqslant \ell - 1$. Indeed, by Proposition 5.8 we know that these parameters belong to $q^{\xi_i+2\mathbb{N}+1}$. Let

$$s = \max\{r \mid A^{-1}_{i,q^{\xi_i+2r+1}} \text{occurs in } M \text{ for some } i\}.$$

If $s \geqslant \ell$ then all variables $Y_{i,q^{\xi_i+2s+2}}$ occuring in m'' have a negative exponent, and m'' cannot be dominant. Hence $a = q^{\xi_i+2k+1}$ with $0 \leqslant k \leqslant \ell - 1$. It follows that m'' depends only on the variables $Y_{i,q^{\xi_i+2k}}$ $(0 \leqslant k \leqslant \ell)$. Thus $L(m'')$ is in $\mathscr{C}_\ell$ and $\mathscr{C}_\ell$ is closed under tensor products. The description of the Grothendieck ring R_ℓ immediately follows, and this finishes the proof of Proposition 3.2.

5.2.5 As a consequence of Proposition 5.8, all simple modules in the category $\mathscr{C}_0$ are minuscule. Indeed consider such a module S with highest monomial m. A monomial m' occuring in $\chi_q(S) - m$ contains at least one A^{-1}_{j,q^r} with $r \geq 1$ and $j \in I$. As $m \in \mathbb{Z}[Y_{i,q^{\xi_i}}]_{i \in I}$, we can conclude as in Section 5.2.4 that m' is not dominant.

This property also holds for other subcategories equivalent to $\mathscr{C}_0$ obtained by shifting the spectral parameter by $a \in \mathbb{C}^*$. More explicitly, consider the category of finite-dimensional representations V which satisfy : for every composition factor S of V and every $i \in I$, the Drinfeld polynomial $\pi_{i,S}(u)$ belongs to $\mathbb{Z}[(1 - a^{-1}q^{-\xi_i}u)]$. Then one proves exactly as above that in this category any simple object is minuscule, and every tensor product of simple objects is simple.

5.3 The next proposition is often helpful to prove that certain monomials belong to the q-character of a module.

Proposition 5.9 [**H2**, Prop. 3.1] *Let V be an object of $\mathscr{C}$ and fix $i \in I$. Then there is a unique decomposition of $\chi_q(V)$ as a finite sum*

$$\chi_q(V) = \sum_{m \in \mathscr{M}_{i,+}} \lambda_m \varphi_i(m),$$

and the λ_m are nonnegative integers.

Suppose that we know that $m \in \mathscr{M}_{i,+}$ occurs in $\chi_q(V)$. Assume also that if $m' \in \mathscr{M}_{i,+} \setminus \{m\}$ is such that $\varphi_i(m')$ contains m then m' does not occur in $\chi_q(V)$. Then the proposition implies that all the monomials of $\varphi_i(m)$ occur in $\chi_q(V)$. In particular, let $m \in \mathscr{M}_+$ and let mM be a monomial of $\chi_q(L(m))$, where M is a monomial in the $A_{j,a}^{-1}$ $(j \in I)$. If M contains no variable $A_{i,a}^{-1}$ then $mM \in \mathscr{M}_{i,+}$ and $\varphi_i(mM)$ is contained in $\chi_q(L(m))$.

5.4 We will sometimes need a natural generalization of 5.3 in which the singleton $\{i\}$ is replaced by an arbitrary subset J of I. To formulate it we first need to imitate the definition of $\varphi_i(m)$ and introduce some polynomials $\varphi_J(m)$. We say that $m \in \mathscr{M}$ is *J-dominant*, and we write $m \in \mathscr{M}_{J,+}$, if m does not contain negative powers of $Y_{j,a}$ $(j \in J,\ a \in \mathbb{C}^*)$. For $m \in \mathscr{M}$ we denote by $\overline{m}$ the monomial obtained from m by replacing $Y_{i,a}^{\pm}$ by 1 if $i \notin J$. If $m \in \mathscr{M}_{J,+}$ then $\overline{m}$ can be regarded as a dominant monomial for the subalgebra $U_q(\widehat{\mathfrak{g}}_J)$ of $U_q(\widehat{\mathfrak{g}})$ generated by the Drinfeld generators attached to the vertices of J. Denote by $\chi_q(\overline{m})$ the q-character of the unique irreducible q-character of $U_q(\widehat{\mathfrak{g}}_J)$ with highest weight monomial $\overline{m}$. Write $\chi_q(\overline{m}) = \overline{m}(1+\sum_p \overline{M}_p)$, where the $\overline{M}_p$ are monomials in the variables $\overline{A_{j,a}^{-1}}$ $(j \in J,\ a \in \mathbb{C}^*)$. Then one sets $\varphi_J(m) := m(1+\sum_p M_p)$ where each M_p is obtained from the corresponding $\overline{M}_p$ by replacing each variable $\overline{A_{j,a}^{-1}}$ by $A_{j,a}^{-1}$. In particular, if $m \in \mathscr{M}_+$ then $\varphi_J(m)$ is the sum of all the monomials of $\chi_q(L(m))$ of the form mM where M is a monomial in the $A_{j,a}^{-1}$ $(j \in J)$. We can now state

Proposition 5.10 [**H2**, Prop. 3.1] *Let V be an object of $\mathscr{C}$ and let J be an arbitrary subset of I. Then there is a unique decomposition of $\chi_q(V)$ as a finite sum*

$$\chi_q(V) = \sum_{m \in \mathscr{M}_{J,+}} \lambda_m \varphi_J(m),$$

and the λ_m are nonnegative integers.

In particular, let $m \in \mathscr{M}_+$ and let mM be a monomial of $\chi_q(L(m))$, where M is a monomial in the $A_{i,a}^{-1}$ $(i \in I)$. If M contains no variable $A_{j,a}^{-1}$ with $j \in J$ then $mM \in \mathscr{M}_{J,+}$ and $\varphi_J(mM)$ is contained in $\chi_q(L(m))$.

6 Truncated q-characters

The Frenkel-Mukhin algorithm is an important tool because there is no general formula (like the Weyl character formula) for calculating an irreducible q-character of $\mathscr{C}$. Unfortunately, even when the Frenkel-Mukhin algorithm is successful, the full expansion of the irreducible q-character is in general impossible to handle because it contains too many monomials. For example, the q-character of the 5th fundamental representation for $\mathfrak{g}$ of type E_8 has 6899079264 monomials (counted with their multiplicities) [**N4**]. However, when dealing with the subcategory $\mathscr{C}_1$, we can work with certain truncations of the q-characters, as we shall explain in this section. Thus, we will see that in the category $\mathscr{C}_1$ in type D_4, the q-character $\chi_q(L(Y_{1,q^3}Y_{2,1}^2Y_{3,q^3}Y_{4,q^3}))$ which contains 167237 monomials (counted with their multiplicities) can be controlled by its truncated q-character which has only 14 monomials.

6.1 From now on we will work in the subcategory $\mathscr{C}_{\mathbb{Z}}$. It follows from Proposition 5.8 that the q-characters of objects of $\mathscr{C}_{\mathbb{Z}}$ involve only monomials in the variables Y_{i,q^r} $(i \in I, r \in \mathbb{Z})$. To simplify notation, *we will henceforth write $Y_{i,r}$ instead of Y_{i,q^r}*. Similarly, *we will write $A_{i,r}$ instead of A_{i,q^r}*.

6.2 Let V be an object of $\mathscr{C}_1$. We can write

$$\chi_q(V) = \sum_k m_k(1+\sum_p M_p^{(k)})$$

where the m_k are dominant monomials in the variables $Y_{i,\xi_i}, Y_{i,\xi_i+2}$ $(i \in I)$, and the $M_p^{(k)}$ are certain monomials in the $A_{i,r}^{-1}$. The factorization of a monomial $m = m_kM_p^{(k)}$ is in general not unique. For example, in type A_2 we have $Y_{1,0}Y_{1,2}A_{1,1}^{-1} = Y_{2,1}$. However, if $m_k M_p^{(k)}$ is such that $M_p^{(k)}$ contains a negative power of $A_{i,r}$ for some $i \in I$ and some $r \geqslant 3$, because of the restriction on the variables Y occuring in m_k and of the formula

$$A_{i,r}^{-1} = Y_{i,r+1}^{-1} Y_{i,r-1}^{-1} \prod_{j:\ a_{ij}=-1} Y_{j,r},$$

for any other expression $m_k M_p^{(k)} = \widetilde{m}_k \widetilde{M}_p^{(k)}$ the monomial $\widetilde{M}_p^{(k)}$ also contains a negative power of $A_{i,r}$ for some $i \in I$ and some $r \geqslant 3$. We define the *truncated q-character of V* to be the Laurent polynomial obtained from $\chi_q(V)$ by keeping only the monomials $M_p^{(k)}$ which do not contain any $A_{i,r}^{-1}$ with $r \geqslant 3$. We denote this truncated polynomial by $\chi_q(V)_{\leqslant 2}$. Our motivation for introducing this truncation is the following

Proposition 6.1 *The map $V \mapsto \chi_q(V)_{\leqslant 2}$ is an injective homomorphism from the Grothendieck ring R_1 of $\mathscr{C}_1$ to $\mathscr{Y}$.*

Proof — It is clear from the definition that our truncated q-character is additive and multiplicative, hence induces a homomorphism from R_1 to $\mathscr{Y}$. Let us prove injectivity. Let S be a simple object of $\mathscr{C}_1$, and $\chi_q(S) = m(1 + \sum_k M_k)$ where the M_k are monomials in the $A_{i,r}^{-1}$. If M_k contains a variable $A_{i,r}^{-1}$ with $r \geqslant 3$ then mM_k can not be dominant. Indeed, if

$$s = \max\{r \mid A_{i,r}^{-1} \text{ occurs in } M_k \text{ for some } i\ \},$$

then all variables $Y_{i,s+1}$ occuring in mM_k have a negative exponent, and therefore mM_k is not dominant. (In the terminology of [**FM**, §6.1], mM_k is a *right negative* monomial.) Hence, the dominant monomials of $\chi_q(S)$ are all contained in its truncated version $\chi_q(S)_{\leqslant 2}$. Thus, if for two objects V and W of $\mathscr{C}_1$ we have $\chi_q(V)_{\leqslant 2} = \chi_q(W)_{\leqslant 2}$ then $\chi_q(V)$ and $\chi_q(W)$ have the same dominant monomials, and the claim follows from Proposition 5.3. □

Remark 6.2 One might consider a different truncated q-character, obtained by keeping only the dominant monomials. By Proposition 5.3, this truncation is injective on $R_{\mathbb{Z}}$, but it is difficult to use because it is not multiplicative.

Example 6.3 We continue Example 5.4 in type A_2. With our new simplified notation, we have $m = Y_{1,0}Y_{2,3}$, and one checks easily that

$$\chi_q(L(m))_{\leqslant 2} = m + m_1 = Y_{1,0}Y_{2,3} + Y_{1,2}^{-1}Y_{2,1}Y_{2,3} = Y_{1,0}Y_{2,3}\left(1 + A_{1,1}^{-1}\right).$$

On the other hand, it follows from Example 5.6 that

$$\begin{aligned}
\chi_q(L(Y_{1,0}^2Y_{2,3}))_{\leqslant 2} &= \chi_q(L(Y_{1,0}))_{\leqslant 2}\,\chi_q(L(Y_{1,0}Y_{2,3}))_{\leqslant 2} \\
&= Y_{1,0}\left(1 + A_{1,1}^{-1} + A_{1,1}^{-1}A_{2,2}^{-1}\right) Y_{1,0}Y_{2,3}\left(1 + A_{1,1}^{-1}\right) \\
&= Y_{1,0}^2Y_{2,3}\left(1 + 2A_{1,1}^{-1} + A_{1,1}^{-2} + A_{1,1}^{-1}A_{2,2}^{-1} + A_{1,1}^{-2}A_{2,2}^{-1}\right).
\end{aligned}$$

Example 6.4 Let $\mathfrak{g}$ be of type A, D, E. Then for $i \in I$

$$\chi_q(L(Y_{i,2}))_{\leqslant 2} = Y_{i,2}, \quad \chi_q(L(Y_{i,1}))_{\leqslant 2} = Y_{i,1}\left(1 + A_{i,2}^{-1}\right),$$

$$\chi_q(L(Y_{i,0}))_{\leqslant 2} = Y_{i,0}\left(1 + A_{i,1}^{-1}\prod_{j\in I, a_{ij}=-1}(1+A_{j,2}^{-1})\right),$$

$$\chi_q(L(Y_{i,\xi_i}Y_{i,\xi_i+2}))_{\leqslant 2} = Y_{i,\xi_i}Y_{i,\xi_i+2}.$$

Indeed, for all these modules the q-character is given by the Frenkel-Mukhin algorithm (the first three are fundamental modules, and the last one is a Kirillov-Reshetikhin module.)

Let us introduce for $P,Q\in\mathbb{Z}[Y_{i,r}^{\pm}; i\in I, r\in\mathbb{Z}]$ the notation

$$P \leqslant Q \quad\Longleftrightarrow\quad Q-P\in\mathbb{N}[Y_{i,r}^{\pm}; i\in I, r\in\mathbb{Z}].$$

Corollary 6.5 *Let $S = L(m)$ be a simple object of $\mathscr{C}_1$. Suppose that* $\mathrm{FM}(m) \geqslant \chi_q(S)_{\leqslant 2}$. *Then* $\mathrm{FM}(m) = \chi_q(S)$.

Proof — By the proof of Proposition 6.1, $\chi_q(S)_{\leqslant 2}$ contains all the dominant monomials of $\chi_q(S)$, and the claim follows from Remark 5.7 (ii). □

6.3 Let $\gamma = \sum_{i\in I} c_i\alpha_i$ be an element of the root lattice with nonnegative coordinates c_i. We denote by $J = \{j\in I \mid c_j \neq 0\}$ the *support of* γ, and we assume that J is connected. Let

$$m = \prod_{i\in I_0} Y_{i,0}^{c_i}\prod_{i\in I_1} Y_{i,3}^{c_i}.$$

This is the highest l-weight of a simple object $L(m)$ of $\mathscr{C}_1$. The next proposition shows how to calculate the truncated q-character of $L(m)$ in terms of the truncation of $\varphi_J(m)$.

Let $K = \{k\in I-J \mid a_{kj} = -1 \text{ for some } j\in J\}$ be the subset of vertices adjacent to J. For $k\in K$ denote by j_k the unique $j\in J$ such that $a_{kj}=-1$. (The uniqueness of j_k follows from the fact that the Dynkin graph has no cycle and J is connected.) Write

$$\varphi_J(m)_{\leqslant 2} = m\left(1+\sum_p M_p\right),$$

where the M_p are monomials in $A_{j,1}^{-1}, A_{j,2}^{-1}$ $(j\in J)$. In fact, by Proposition 5.8, it is easy to see that each M_p is of the form $M_p = \prod_{j\in J} A_{j,1+\xi_j}^{-\mu_{j,p}}$ for some $\mu_{j,p}\in\mathbb{N}$.

Proposition 6.6 *We have*

$$\chi_q(L(m))_{\leqslant 2} = m\left(1+\sum_p M_p\prod_{k\in K\cap I_1}(1+A_{k,2}^{-1})^{\mu_{j_k,p}}\right).$$

Proof — From 5.4, we have that $\varphi_J(m)\leqslant\chi_q(L(m))$. If $k\in K\cap I_1$ then $j_k\in J\cap I_0$, and if a monomial M_p contains $A_{j_k,1}^{-1}$ with exponent $\mu_{j_k,p}>0$, then mM_p contains $Y_{k,1}^{\mu_{j_k,p}}$ and no other $Y_{k,r}$ for $r\neq 1$. Therefore mM_p is k-dominant. Using 5.3 we deduce that

$$\varphi_k(mM_p) = M_p(1+A_{k,2}^{-1})^{\mu_{j_k,p}} \leqslant \chi_q(L(m)),$$

and this implies that

$$m\left(1+\sum_p M_p\prod_{k\in K\cap I_1}(1+A_{k,2}^{-1})^{\mu_{j_k,p}}\right)\leqslant\chi_q(L(m))_{\leqslant 2}. \tag{23}$$

Note that if $k\in K\cap I_0$ then $j_k\in J\cap I_1$, and if mM_p is k-dominant then it contains $Y_{k,2}^{\mu_{j_k,p}}$. Hence, in this case $\varphi_k(mM_p)_{\leqslant 2}=mM_p$ does not contribute anything new to the truncated q-character.

Let us now show that (23) is an equality. Suppose on the contrary that it is a strict inequality, and let mM be a monomial in $\chi_q(L(m))_{\leqslant 2}$ which appears in the left-hand side of (23) with a strictly smaller multiplicity. Here M is a monomial in the variables A_{i,ξ_i+1}^{-1}. Moreover, by definition of $\varphi_J(m)$ all the monomials of $\chi_q(L(m))$ obtained from m by multiplying by variables A_{j,ξ_j+1}^{-1} with $j\in J$ appear in $\varphi_J(m)$ with the same multiplicity. Hence M has to contain at least one variable A_{k,ξ_k+1}^{-1} with $k\notin J$. By Proposition 5.8 we can assume that $k\in K\cap I_1$. We will also assume that mM is maximal with these properties. Denote by r the multiplicity of mM in $\chi_q(L(m))$, and by v the exponent of $A_{k,2}^{-1}$ in M. Since mM belongs to a truncated q-character, M does not contain any variable $A_{i,3}^{-1}$ with $a_{ik}=-1$, hence mM has to contain $Y_{k,3}^{-v}$ and therefore it can not be k-dominant. Hence, by 5.3, there exist k-dominant monomials $m_1,\ldots,m_s$, in $\chi_q(L(m))$ (counted with their multiplicities) such that each $\varphi_k(m_i)$ contains mM with multiplicity r_i and $r_1+\cdots+r_s=r$. Since $m_i>mM$ for every i, the multiplicities of m_i in both sides of (23) are equal. Moreover, by our construction, the left-hand side of (23) also contains $\sum_i\varphi_k(m_i)$, hence the multiplicity of mM in this left-hand side is at least r, which contradicts our assumption. Thus (23) is an equality. $\Box$

6.4 We now calculate $\chi_q(S(\alpha))_{\leqslant 2}$ for a *multiplicity-free positive root* α, *i.e.* a root of the form $\alpha=\alpha_J=\sum_{j\in J}\alpha_j$ for some connected subset J of I. If $\mathfrak{g}$ is not of type A, we assume that the trivalent node of the Dynkin diagram belongs to I_0 (this is no loss of generality, see 3.9).

Proposition 6.7 *Let J be a connected subset of I, and $\alpha=\alpha_J$ the corresponding multiplicity-free positive root. Let $\eta\in\{0,1\}^I$ be the characteristic function of J,* i.e. *$\eta_i=1$ if $i\in J$ and $\eta_i=0$ otherwise. Let m be the highest weight monomial of $\chi_q(S(\alpha))$. We have*

$$\chi_q(S(\alpha))_{\leqslant 2} \;=\; m\sum_{\nu\in\mathscr{S}}\prod_{i\in I}A_{i,1+\xi_i}^{-\nu_i},$$

where $\mathscr{S}$ denotes the set of all finite sequences $\nu\in\{0,1\}^I$ such that $\nu_i\leqslant\eta_i$ for every $i\in I_0$, and for every $i\in I_1$

$$\nu_i\leqslant\max\left(0,\ -\eta_i+\sum_{j\,:a_{ij}=-1}\nu_j\right). \tag{24}$$

Proof — We first prove the result for $J=I=\{1,\ldots,n\}$. In this case $\eta_i\equiv 1$ and the condition $\nu_i\leqslant\eta_i$ is always satisfied. We use induction on n. For $n=1$ we have $\chi_q(S(\alpha))_{\leqslant 2}=Y_{1,0}(1+A_{1,1}^{-1})$ if $I=\{1\}=I_0$, and $\chi_q(S(\alpha))_{\leqslant 2}=Y_{1,3}$ if $I=I_1$. Assume now that $n\geqslant 2$, and let m_1 be a monomial in the truncated q-character of $S(\alpha)$. We can suppose that the vertex labelled 1 is monovalent and adjacent to the vertex labelled 2. Put $I'=\{2,\ldots,n\}$. There are two cases.

(a) If $1\in I_1$ then $m=Y_{1,3}Y_{2,0}Y_{3,3}\cdots$. Put $m'=mY_{1,3}^{-1}$. As $L(m)$ appears as a subquotient of $L(Y_{1,3})\otimes L(m')$ and

$$\chi_q(L(Y_{1,3}))_{\leqslant 2}\chi_q(L(m'))_{\leqslant 2}=Y_{1,3}\chi_q(L(m'))_{\leqslant 2},$$

we have $m_1=Y_{1,3}m_1'$ where m_1' is a monomial in $\chi_q(L(m'))_{\leqslant 2}$. By Proposition 6.6, $\chi_q(L(m'))_{\leqslant 2}$ can be calculated from $\varphi_{I'}(m')_{\leqslant 2}$, and by induction on n we may assume that

$$\varphi_{I'}(m')_{\leqslant 2}= \;m'\sum_{\nu\in\mathscr{S}'}\prod_{i\in I'}A_{i,1+\xi_i}^{-\nu_i},$$

where $\mathscr{S}'$ is defined like $\mathscr{S}$ replacing I by I'. By Proposition 6.6, $\chi_q(L(m'))_{\leqslant 2}$ can only differ from $\varphi_{I'}(m')_{\leqslant 2}$ by certain summands $M_p(1+A_{1,2}^{-1})^{\mu_{2,p}}$ so this shows that the exponent v_i of $A_{i,1+\xi_i}^{-1}$ in m_1/m satisfies the condition (24) for $i \in I'$. It remains to check that $v_1 = 0$. For this we can use the fact that $\chi_q(L(m)) \leqslant \chi_q(L(Y_{1,3}Y_{2,0}))\chi_q(L(m'Y_{2,0}^{-1}))$. Now by Proposition 6.6 and the first formula in Example 6.3 (with the nodes 1 and 2 of the A_2 Dynkin diagram exchanged), we get

$$\chi_q(L(Y_{1,3}Y_{2,0}))_{\leqslant 2} = Y_{1,3}Y_{2,0}(1+A_{2,1}^{-1}\prod_{j\neq 1, a_{j2}=-1}(1+A_{j,2}^{-1})).$$

This does not contain $A_{1,2}^{-1}$, neither $\chi_q(L(m'Y_{2,0}^{-1}))_{\leqslant 2}$.

(b) If $1 \in I_0$ then $2 \in I_1$ is not trivalent, so $v_2 \leqslant 1$. We can assume that the vertex 3 is adjacent to 2. We thus have $m = Y_{1,0}Y_{2,3}Y_{3,0}\cdots$. Put $m' = mY_{1,0}^{-1}$. Then $m_1 = m_0 m_1'$ where m_0 is a monomial in $\chi_q(L(Y_{1,0}))_{\leqslant 2}$ and m_1' is a monomial in $\chi_q(L(m'))_{\leqslant 2}$. By Proposition 6.6 the monomial m_0 belongs to $\{Y_{1,0},\, Y_{1,0}A_{1,1}^{-1},\, Y_{1,0}A_{1,1}^{-1}A_{2,2}^{-1}\}$. On the other hand, by Proposition 6.6 again, $\chi_q(L(m'))_{\leqslant 2} = \varphi_{I'}(m')_{\leqslant 2}$ in this case, which is known by induction. If $m_0 \neq Y_{1,0}A_{1,1}^{-1}A_{2,2}^{-1}$ we see that the exponent v_i of $A_{i,1+\xi_i}^{-1}$ in m_1/m satisfies condition (24) for every $i \in I$. If $m_0 = Y_{1,0}A_{1,1}^{-1}A_{2,2}^{-1}$ condition (24) would be violated if m_1'/m' had $v_3 = 0$. But in this case m_0m_1' could not be a monomial of $\chi_q(L(m))_{\leqslant 2}$. Indeed let us prove that if $A_{2,2}^{-1}$ appears, then $A_{3,1}^{-1}$ must appear. We have the inequality $\chi_q(L(m))_{\leqslant 2} \leqslant \chi_q(L(Y_{1,0}Y_{2,3}Y_{3,0}))_{\leqslant 2}\chi_q(L(m'Y_{2,3}^{-1}Y_{3,0}^{-1}))_{\leqslant 2}$. For type A_3, again by using the first formula in Example 6.3 and Proposition 6.6, we have

$$\chi_q(L(Y_{2,3}Y_{3,0}))_{\leqslant 2} = Y_{2,3}Y_{3,0}(1+A_{3,1}^{-1})\ ,\ \chi_q(L(Y_{1,0}Y_{2,3}))_{\leqslant 2} = Y_{1,0}Y_{2,3}(1+A_{1,1}^{-1}).$$

Then we have the two inequalities $\chi_q(L(Y_{1,0}Y_{2,3}Y_{3,0}))_{\leqslant 2} \leqslant \chi_q(L(Y_{1,0}))_{\leqslant 2}\chi_q(L(Y_{2,3}Y_{3,0}))_{\leqslant 2}$ and $\chi_q(L(Y_{1,0}Y_{2,3}Y_{3,0})) \leqslant \chi_q(L(Y_{1,0}Y_{2,3}))_{\leqslant 2}\chi_q(Y_{3,0}))_{\leqslant 2}$ which follow as above from a subquotient argument. As a consequence we get

$$\chi_q(L(Y_{1,0}Y_{2,3}Y_{3,0}))_{\leqslant 2} \leqslant Y_{1,0}Y_{2,3}Y_{3,0}\left(1+A_{1,1}^{-1}+A_{3,1}^{-1}+A_{1,1}^{-1}A_{3,1}^{-1}(1+A_{2,2}^{-1})\right). \tag{25}$$

Now we get the result by Proposition 6.6.

Finally, the general case of a root $\alpha = \alpha_J$ for a subinterval J of I follows from the case $J = I$. Indeed what we have already proved gives us $\varphi_J(m)_{\leqslant 2}$, and Proposition 6.6 then yields the value of $\chi_q(S(\alpha))_{\leqslant 2}$. □

Corollary 6.8 *Assume that the trivalent node is in I_0 if $\mathfrak{g}$ is of type D or E. Let α_J be a multiplicity-free root. Then $\chi_q(S(\alpha_J))$ is given by the Frenkel-Mukhin algorithm.*

Proof — Let m be the highest l-weight of $S(\alpha)$. It follows easily from the explicit formula of Proposition 6.7 that $\mathrm{FM}(m) \geqslant \chi_q(S(\alpha))_{\leqslant 2}$. The only point which needs to be checked reduces to type A_3 for $m = Y_{1,0}Y_{2,3}Y_{3,0}$. By (25), it suffices to check that

$$m(1+A_{1,1}^{-1}+A_{3,1}^{-1}+A_{1,1}^{-1}A_{3,1}^{-1}(1+A_{2,2}^{-1})) \leqslant \mathrm{FM}(m).$$

We have clearly $m(1+A_{1,1}^{-1}+A_{3,1}^{-1}+A_{1,1}^{-1}A_{3,1}^{-1}) \leqslant \mathrm{FM}(m)$. Then

$$\varphi_2(mA_{1,1}^{-1}A_{3,1}^{-1}) = \varphi_2(Y_{2,1}^2Y_{2,3}) = \varphi_2(Y_{2,1})\varphi_2(Y_{2,1}Y_{2,3})$$

where $mA_{1,1}^{-1}A_{3,1}^{-1}A_{2,2}^{-1}$ appears. Now the claim follows from Corollary 6.5. □

If $\mathfrak{g}$ is of type A, all its positive roots are multiplicity-free, thus Proposition 6.7 and Example 6.4 give explicit formulae for all the truncated q-characters $\chi_q(S(\alpha))_{\leqslant 2}$ in that case.

Example 6.9 We take $\mathfrak{g}$ of type A_3. We assume that $I_0 = \{1,3\}$ and $I_1 = \{2\}$. The truncated q-characters of the modules $S(\alpha)$ ($\alpha \in \Phi_{\geqslant -1}$) are

$$
\begin{array}{rclrcl}
\chi_q(S(-\alpha_1))_{\leqslant 2} & = & Y_{1,2}, & \chi_q(S(\alpha_1))_{\leqslant 2} & = & Y_{1,0}(1+A_{1,1}^{-1}+A_{1,1}^{-1}A_{2,2}^{-1}),\\
\chi_q(S(-\alpha_2))_{\leqslant 2} & = & Y_{2,1}(1+A_{2,2}^{-1}), & \chi_q(S(\alpha_2))_{\leqslant 2} & = & Y_{2,3},\\
\chi_q(S(-\alpha_3))_{\leqslant 2} & = & Y_{3,2}, & \chi_q(S(\alpha_3))_{\leqslant 2} & = & Y_{3,0}(1+A_{3,1}^{-1}+A_{2,2}^{-1}A_{3,1}^{-1}),
\end{array}
$$

$$
\begin{array}{rcl}
\chi_q(S(\alpha_1+\alpha_2))_{\leqslant 2} & = & Y_{1,0}Y_{2,3}(1+A_{1,1}^{-1}),\\
\chi_q(S(\alpha_2+\alpha_3))_{\leqslant 2} & = & Y_{2,3}Y_{3,0}(1+A_{3,1}^{-1}),\\
\chi_q(S(\alpha_1+\alpha_2+\alpha_3))_{\leqslant 2} & = & Y_{1,0}Y_{2,3}Y_{3,0}(1+A_{1,1}^{-1}+A_{3,1}^{-1}+A_{1,1}^{-1}A_{3,1}^{-1}+A_{1,1}^{-1}A_{2,2}^{-1}A_{3,1}^{-1}).
\end{array}
$$

Moreover, by Example 6.4, the frozen simple objects F_i have the following truncated q-characters

$$\chi_q(F_1)_{\leqslant 2} = Y_{1,0}Y_{1,2}, \quad \chi_q(F_2)_{\leqslant 2} = Y_{2,1}Y_{2,3}, \quad \chi_q(F_3)_{\leqslant 2} = Y_{3,0}Y_{3,2}.$$

Corollary 6.10 *Let α_J be a multiplicity-free root. Then $S(\alpha_J)$ is a prime simple object.*

Proof — Let m be the highest l-weight of $S(\alpha_J)$. We have $m = \prod_{i\in J\cap I_0} Y_{i,0} \prod_{i\in J\cap I_1} Y_{i,3}$. Suppose that $S(\alpha_J)$ is not prime. Then

$$S(\alpha_J) \cong L(m_1)\otimes\cdots\otimes L(m_k), \tag{26}$$

where $m_1\cdots m_k = m$. Clearly, there must exist $i \in J\cap I_0$ and $j \in J\cap I_1$ with $a_{ij} = -1$ and such that $Y_{i,0}$ and $Y_{j,3}$ do not occur in the same monomial m_r $(1 \leqslant r \leqslant k)$. Let m_s be the monomial containing $Y_{i,0}$. Then $\chi_q(L(m_s))$ contains $m_sA_{i,1}^{-1}A_{j,2}^{-1}$ by Proposition 6.6, hence $mA_{i,1}^{-1}A_{j,2}^{-1}$ occurs in $\chi_q(L(m_1)\otimes\cdots\otimes L(m_k))$. On the other hand, it follows from Proposition 6.7 that $mA_{i,1}^{-1}A_{j,2}^{-1}$ is not a monomial of $\chi_q(S(\alpha_J))$, which contradicts (26). □

7 F-polynomials

In [**FZ5**], Fomin and Zelevinsky have shown that the cluster variables of $\mathscr{A}$ have a nice expression in terms of certain polynomials called the F-polynomials, which are closely related to the Fibonacci polynomials of [**FZ2**]. Moreover, [**FZ2**] gives some explicit formulae for the Fibonacci polynomials in type A and D. We will recall these results in a form suitable to our present purpose.

7.1 In Section 4, we have used $\mathbf{x} = \{x[-\alpha_i] \mid i \in I\}$ as our reference cluster. It will be convenient here to work with a slightly different cluster denoted by $\mathbf{z} = \{z_i \mid i \in I\}$, and given by

$$z_i = \begin{cases} x[-\alpha_i] & \text{if } i \in I_0,\\ x[\alpha_i] & \text{if } i \in I_1.\end{cases}$$

It is easy to see that one passes from $\mathbf{x}$ to $\mathbf{z}$ by applying the sequence of mutations $\prod_{k\in I_1}\mu_k$ (it does not matter in which order since they pairwise commute). A straightforward calculation shows that the exchange matrix $\widetilde{B}_{\mathbf{z}} = [b_{ij}^{\mathbf{z}}]$ at $\mathbf{z}$ is given by

$$b_{ij}^{\mathbf{z}} = \begin{cases} \varepsilon_j a_{ij} & \text{if } i,j \in I \text{ and } i \neq j,\\ -1 & \text{if } j \in I \text{ and } i = j+n \in I',\\ -a_{kj} & \text{if } j \in I_0, \text{ and } i = k+n \in I' \text{ with } k \neq j,\\ 0 & \text{otherwise.}\end{cases}$$

Here the column set is indexed by I and the row set is indexed by $I \cup I' \equiv [1,n] \cup [n+1,2n]$. Following [**FZ5**, §6] we define the following elements of $\mathscr{F}$:

$$y_j = \prod_{i \in I} f_i^{b^{\mathbf{z}}_{i+n,j}}, \qquad \widehat{y}_j = y_j \prod_{i \in I} z_i^{b^{\mathbf{z}}_{ij}}. \tag{27}$$

Example 7.1 We take $\mathfrak{g}$ of type A_3 and $I_0 = \{1,3\}$. We have

$$\widetilde{B}_{\mathbf{z}} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

We have

$$y_1 = f_1^{-1} f_2, \quad y_2 = f_2^{-1}, \quad y_3 = f_2 f_3^{-1}, \quad \widehat{y}_1 = z_2^{-1} f_1^{-1} f_2, \quad \widehat{y}_2 = z_1 z_3 f_2^{-1}, \quad \widehat{y}_3 = z_2^{-1} f_2 f_3^{-1}.$$

Following [**FZ5**, §10] we denote by E the linear automorphism of the root lattice Q given by

$$E(\alpha_i) = -\varepsilon_i \alpha_i, \qquad (i \in I).$$

We also define piecewise-linear involutions τ_ε $(\varepsilon = \pm 1)$ of Q by

$$[\tau_\varepsilon(\gamma) : \alpha_i] = \begin{cases} -[\gamma : \alpha_i] - \sum_{j \neq i} a_{ij} \max(0, [\gamma : \alpha_j]) & \text{if } \varepsilon_i = \varepsilon, \\ [\gamma : \alpha_i] & \text{if } \varepsilon_i \neq \varepsilon. \end{cases} \tag{28}$$

Here, for $\gamma \in Q$, we denote by $[\gamma : \alpha_i]$ the coefficient of α_i in the expansion of γ on the basis of simple roots. It is easy to see that τ_ε preserves $\Phi_{\geqslant -1}$. For $\alpha \in \Phi_{\geqslant -1}$, one then defines the *g-vector*

$$\mathbf{g}(\alpha) = E\tau_-(\alpha). \tag{29}$$

The involution τ_- relates the natural labellings of the cluster variables with respect to $\mathbf{x}$ and $\mathbf{z}$. We shall write $z[\alpha] = x[\tau_-(\alpha)]$. In particular, since $\tau_-(-\alpha_i) = -\varepsilon_i \alpha_i$, we have $z_i = z[-\alpha_i]$ $(i \in I)$.

7.2 Consider the multiplicative group $\mathbb{P}$ of all Laurent monomials in the variables f_i $(i \in I)$. As in [**FZ5**, Def. 2.2], we introduce the addition $\oplus$ given by

$$\prod_i f_i^{a_i} \oplus \prod_i f_i^{b_i} = \prod_i f_i^{\min(a_i, b_i)}.$$

Endowed with this operation and its ordinary multiplication and division, $\mathbb{P}$ becomes a semifield, called the *tropical semifield*. If $F(t_1, \ldots, t_n)$ is a subtraction-free rational expression with integer coefficients in some variables t_i, then we can evaluate it in $\mathbb{P}$ by specializing the t_i to some elements p_i of $\mathbb{P}$. This will be denoted by $F|_{\mathbb{P}}(p_1, \ldots, p_n)$.

7.3 To define the F-polynomials, we need to introduce a variant of $\mathscr{A}$ called the *cluster algebra with principal coefficients*. We shall denote it by $\mathscr{A}_{\rm pr}$. It is given by the initial seed $(\mathbf{u}, \widetilde{B}_{\rm pr})$, where $\mathbf{u} = (u_1,\ldots,u_n,v_1,\ldots,v_n)$, and $\widetilde{B}_{\rm pr}$ is the $2n\times n$ matrix with the same principal part as $\widetilde{B}_{\mathbf{z}}$ and with lower part equal to the $n\times n$ identity matrix. Thus $(v_1,\ldots,v_n)$ are the frozen variables of $\mathscr{A}_{\rm pr}$. By [**FZ3**] every cluster variable of $\mathscr{A}_{\rm pr}$ is of the form

$$u[\alpha] = \frac{N_\alpha(u_1,\ldots,u_n,v_1,\ldots,v_n)}{u_1^{a_1}\cdots u_n^{a_n}} \tag{30}$$

for some $\alpha = \sum_{i\in I} a_i\alpha_i \in \Phi_{\geqslant -1}$. Here N_α is a polynomial, and $N_{-\alpha_i}\equiv 1$ $(i\in I)$. Following [**FZ5**, §3], we can now define the *F-polynomials* by specializing all the u_i to 1:

$$F_\alpha(v_1,\ldots,v_n) = N_\alpha(1,\ldots,1,v_1,\ldots,v_n), \qquad (\alpha\in\Phi_{\geqslant -1}). \tag{31}$$

For example, for the simple root α_i we have $F_{\alpha_i}(v_1,\ldots,v_n) = 1+v_i$. It is known that F_α is a polynomial with positive integer coefficients [**FZ5**, Cor. 11.7]. The main formula is then [**FZ5**, Cor. 6.3]:

$$z[\alpha] = \frac{F_\alpha(\widehat{y}_1,\ldots,\widehat{y}_n)}{F_\alpha|_{\mathbb{P}}(y_1,\ldots,y_n)}\mathbf{z}^{\mathbf{g}(\alpha)}, \tag{32}$$

where, if $\mathbf{g}(\alpha) = (g_1,\ldots,g_n)$, we write for short $\mathbf{z}^{\mathbf{g}(\alpha)} = z_1^{g_1}\cdots z_n^{g_n}$. This means that the cluster variable $z[\alpha]$ is completely determined by the corresponding F-polynomial F_α and g-vector $\mathbf{g}(\alpha)$.

7.4 Recall from §4.4 the ring isomorphism $\iota : \mathscr{A}\to R_1$. Note that by comparing the relation

$$x[\alpha_i]x[-\alpha_i] = f_i + \prod_{j:a_{ij}=-1} x[-\alpha_j]$$

with the T-system equation (5) satisfied by the Kirillov-Reshetikhin modules $S(\alpha_i)$ and $S(-\alpha_i)$, we obtain immediately

$$\iota(f_i) = [F_i], \qquad (i\in I).$$

Taking into account Proposition 6.1, we can regard ι as an isomorphism from $\mathscr{A}$ to the subring of $\mathscr{Y}$ generated by the truncated q-characters of objects of $\mathscr{C}_1$. We then have (*cf.* Example 6.4)

$$\iota(z_i) = Y_{i,\xi_i+2}, \qquad \iota(f_i) = Y_{i,\xi_i}Y_{i,\xi_i+2}, \qquad (i\in I). \tag{33}$$

Moreover, extending ι to a homomorphism from $\mathscr{F}$ to the fraction field of $\mathscr{Y}$, we can consider the elements $\iota(\widehat{y}_j)$.

Lemma 7.2 *For $j\in I$, we have*

$$\iota(\widehat{y}_j) = A^{-1}_{j,\xi_j+1}.$$

Proof — If $j\in I_1$, we have

$$\iota(\widehat{y}_j) = \iota(f_j)^{-1}\prod_{i\neq j}\iota(z_i)^{-a_{ij}} = Y^{-1}_{j,\xi_j}Y^{-1}_{j,\xi_j+2}\prod_{i\neq j}Y^{-a_{ij}}_{i,\xi_i+2} = A^{-1}_{j,\xi_j+1},$$

since $\xi_i+2 = \xi_j+1$ if $j\in I_1$ and $i\in I_0$. On the other hand, if $j\in I_0$, we have

$$\iota(\widehat{y}_j) = \iota(f_j)^{-1}\prod_{i\neq j}\iota\left(\frac{z_i}{f_i}\right)^{a_{ij}} = Y^{-1}_{j,\xi_j}Y^{-1}_{j,\xi_j+2}\prod_{i\neq j}Y^{-a_{ij}}_{i,\xi_i}Y^{-a_{ij}}_{i,\xi_i+2}Y^{a_{ij}}_{i,\xi_i+2} = Y^{-1}_{j,\xi_j}Y^{-1}_{j,\xi_j+2}\prod_{i\neq j}Y^{-a_{ij}}_{i,\xi_i} = A^{-1}_{j,\xi_j+1},$$

since $\xi_i = \xi_j+1$ if $j\in I_0$ and $i\in I_1$. □

Lemma 7.3 *Let $\alpha \in \Phi_{\geqslant -1}$ and set $\beta = \tau_- \alpha = \sum_i b_i \alpha_i$. We have*

$$\iota\left(\frac{\mathbf{z}^{\mathbf{g}(\alpha)}}{F_\alpha|_{\mathbb{P}}(y_1,\ldots,y_n)}\right) = \begin{cases} \prod_{i\in I_0} Y_{i,0}^{b_i} \prod_{i\in I_1} Y_{i,3}^{b_i} & \text{if } \beta > 0, \\ Y_{i,2-\xi_i} & \text{if } \beta = -\alpha_i. \end{cases}$$

Proof — Write $\alpha = \sum_i a_i \alpha_i$. If $\alpha = -\alpha_i$ then $F_\alpha = 1$. Moreover, $\beta = -\varepsilon_i \alpha_i$, $\mathbf{g}(\alpha) = \alpha_i$, so

$$\iota\left(\frac{\mathbf{z}^{\mathbf{g}(\alpha)}}{F_\alpha|_{\mathbb{P}}(y_1,\ldots,y_n)}\right) = \iota(z_i) = Y_{i,\xi_i+2},$$

which proves the formula in this case.

Otherwise if $\alpha > 0$, by [**FZ5**, Cor. 10.10] the polynomial F_α has a unique monomial of maximal degree, which is divisible by all the other occuring monomials and has coefficient 1. This monomial is $m = \prod_i v_i^{a_i}$. When we evaluate m at $v_i = y_i = \prod_{k\in I} f_k^{b^{\mathbf{z}}_{k+n,i}}$ we obtain

$$\prod_{i\in I} f_i^{-a_i} \prod_{i\in I_1} f_i^{-\sum_{j\neq i} a_j a_{ij}} = \prod_{i\in I_0} f_i^{-a_i} \prod_{i\in I_1} f_i^{-a_i - \sum_{j\neq i} a_j a_{ij}} = \prod_{i\in I_0} f_i^{-b_i} \prod_{i\in I_1} f_i^{b_i}.$$

Moreover F_α has constant term 1. Therefore, if $\beta > 0$, then $b_i > 0$ for every $i \in I$ and

$$F_\alpha|_{\mathbb{P}}(y_1,\ldots,y_n) = \prod_{i\in I_0} f_i^{-b_i}.$$

Otherwise, if $\beta = -\alpha_i$ with $i \in I_1$ we have $F_\alpha|_{\mathbb{P}}(y_1,\ldots,y_n) = f_i^{-1}$ (the case $\beta = -\alpha_i$ with $i \in I_0$ has already been dealt with). On the other hand, $\mathbf{g}(\alpha) = E(\beta) = -\sum_{i\in I} b_i \varepsilon_i \alpha_i$. Hence, if $\beta > 0$ then

$$\iota\left(\frac{\mathbf{z}^{\mathbf{g}(\alpha)}}{F_\alpha|_{\mathbb{P}}(y_1,\ldots,y_n)}\right) = \prod_{i\in I_0} \iota(f_i)^{b_i} \prod_{i\in I_0} \iota(z_i)^{-b_i} \prod_{i\in I_1} \iota(z_i)^{b_i} = \prod_{i\in I_0} Y_{i,0}^{b_i} \prod_{i\in I_1} Y_{i,3}^{b_i}.$$

If $\beta = -\alpha_i$ and $i \in I_1$, we have

$$\iota\left(\frac{\mathbf{z}^{\mathbf{g}(\alpha)}}{F_\alpha|_{\mathbb{P}}(y_1,\ldots,y_n)}\right) = \iota(f_i)\iota(z_i^{-1}) = Y_{i,1}.$$

□

Corollary 7.4 *Let $\beta \in \Phi_{\geqslant -1}$ and set $\alpha = \tau_- \beta$. Let Y^β denote the highest weight monomial of $\chi_q(S(\beta))$. We have*

$$\iota(x[\beta]) = Y^\beta F_\alpha\left(A^{-1}_{1,\xi_1+1},\ldots,A^{-1}_{n,\xi_n+1}\right).$$

Proof — This follows immediately from Lemma 7.2, Lemma 7.3, and the relation $x[\beta] = z[\tau_-\beta]$. □

Using the notation of (18), we see that the proof of Conjecture 4.6 (i) is now reduced to establishing the following polynomial identity in $\mathbb{N}[A^{-1}_{i,\xi_i+1}]$:

$$\widetilde{\chi}_q(S(\beta))_{\leqslant 2} = F_{\tau_-(\beta)}, \qquad (\beta \in \Phi_{>0}), \tag{34}$$

that is, *the normalized truncated q-character of $S(\beta)$ should coincide with the F-polynomial attached to the root $\tau_-(\beta)$.*

7.5 We now recall an explicit formula of Fomin and Zelevinsky for the F-polynomials. This is obtained by combining [**FZ2**, Prop. 2.10] with [**FZ5**, Th. 11.6]. It covers all the F-polynomials in type A and D.

7.5.1 Let $\alpha = \sum_i a_i\alpha_i \in \Phi_{>0}$. Assume that $a_i \leqslant 2$ for every $i \in I$. Let $\gamma = \sum_i c_i\alpha_i$. We say that γ is *α-acceptable* if

(i) $0 \leqslant c_i \leqslant a_i$ for every $i \in I$;

(ii) if $i \in I_1$ and $j \in I_0$ are adjacent then $c_i \leqslant (2-a_j)+c_j$;

(iii) there is no simple path $(i_0,\ldots,i_m)$ of length $m \geqslant 1$ contained in the support of α such that $a_{i_0} = a_{i_m} = 1$ and for $k = 0,\ldots,m$

$$c_{i_k} = \begin{cases} 1 & \text{if } i_k \in I_1, \\ a_{i_k}-1 & \text{if } i_k \in I_0. \end{cases}$$

In condition (iii) above, by a simple path we mean any path in the Dynkin diagram whose vertices are pairwise distinct. Finally, let $e(\gamma,\alpha)$ be the number of connected components of the set

$$\{i \in I \mid c_i = 1 \text{ if } i \in I_1 \text{ and } c_i = a_i - 1 \text{ if } i \in I_0\}$$

that are contained in $\{i \in I \mid a_i = 2\}$. Then

Proposition 7.5 [FZ2, FZ5]

$$F_\alpha(v_1,\ldots,v_n) = \sum_\gamma 2^{e(\gamma,\alpha)}\mathbf{v}^\gamma, \tag{35}$$

where the sum is over all α-acceptable $\gamma \in Q$, and we write $\mathbf{v}^\gamma = \prod_{i\in I} v_i^{c_i}$.

Example 7.6 Take $\mathfrak{g}$ of type D_4 and choose $I_0 = \{2\}$, where 2 labels the trivalent node. Let $\alpha = \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$ be the highest root. Then

$$F_\alpha = 1+2v_2+v_2^2+v_1v_2+v_2v_3+v_2v_4+v_1v_2^2+v_2^2v_3+v_2^2v_4+v_1v_2^2v_3+v_1v_2^2v_4+v_2^2v_3v_4+v_1v_2^2v_3v_4.$$

7.5.2 If α is a *multiplicity-free root*, that is, if $a_i \leqslant 1$ for all roots, (35) simplifies greatly. Indeed, in the definition of an α-acceptable vector γ condition (ii) is a consequence of condition (i), and condition (iii) reduces to

(iv) for every $i \in I_1$, $c_i \leqslant \min\{c_j \mid j \in \mathrm{supp}(\alpha) \text{ and } j \text{ adjacent to } i\}$.

Example 7.7 Take $\mathfrak{g}$ of type A_3 and choose $I_0 = \{1,3\}$. Then

$$\begin{aligned}
F_{\alpha_1} &= 1+v_1,\\
F_{\alpha_2} &= 1+v_2,\\
F_{\alpha_3} &= 1+v_3,\\
F_{\alpha_1+\alpha_2} &= 1+v_1+v_1v_2,\\
F_{\alpha_2+\alpha_3} &= 1+v_3+v_2v_3,\\
F_{\alpha_1+\alpha_2+\alpha_3} &= 1+v_1+v_3+v_1v_3+v_1v_2v_3.
\end{aligned}$$

7.6 We can now prove the following particular case of Conjecture 4.6 (i) valid for all Lie types (see Eq. (34)). We assume (as we may, *cf.* 3.9) that the trivalent node is in I_0 if $\mathfrak{g}$ is of type D or E.

Theorem 7.8 *Let β be a multiplicity-free positive root. Then*

$$\widetilde{\chi}_q(S(\beta))_{\leqslant 2} = F_{\tau_-(\beta)}.$$

Proof — Suppose first that $\beta = \alpha_i$ with $i \in I_1$. Then $\chi_q(S(\beta))_{\leqslant 2} = Y_{i,3}$, and $\tau_-(\beta) = -\alpha_i$ so that $F_{\tau_-(\beta)} = 1$. Hence the claim is verified in this case.

So we can assume that $\beta = \alpha_J$ is the multiplicity-free root supported on a connected subset J of I which is not reduced to a single element of I_1. Set

$$\begin{aligned} J' &= \{j \in J \mid j \in I_1 \text{ and } a_{ij} = -1 \text{ for some } i \in I \setminus J\}, \\ K' &= \{i \in I \setminus J \mid i \in I_1 \text{ and } a_{ij} = -1 \text{ for some } j \in J\}, \end{aligned}$$

and define $K = (J \setminus J') \sqcup K'$. It follows immediately from the definition of τ_- that the multiplicity-free root α_K supported on K is equal to $\tau_-(\alpha_J)$.

Let us now compare the sequences $\nu \in \mathscr{S}$ of Proposition 6.7 for α_J, with the α_K-acceptable vectors $\gamma = \sum_i c_i \alpha_i$ of 7.5.2. First we note that if $i \notin K \cup J = K' \sqcup (J \setminus J') \sqcup J'$ then both ν_i and c_i are equal to 0 (this is a straightforward consequence of the definitions of $\mathscr{S}$ and of an α_K-acceptable vector). For $j \in J \cap I_0 = K \cap I_0$ the only condition satisfied by ν_j and c_j is that they should belong to $\{0,1\}$. If $j \in (J \setminus J') \cap I_1$ then j must have two neighbours j' and j'' in $J \cap I_0$, and the condition on ν_j is $0 \leqslant \nu_j \leqslant \max\{0, \nu_{j'} + \nu_{j''} - 1\}$, while the condition on c_j is $0 \leqslant c_j \leqslant \min\{c_{j'}, c_{j''}\}$. Clearly, since $c_{j'}$ and $c_{j''}$ belong to $\{0,1\}$, the second condition can be rewritten $0 \leqslant c_j \leqslant \max\{0, c_{j'} + c_{j''} - 1\}$. If $j \in K'$ then j has a unique neighbour $j' \in J$, and the condition on ν_j is $0 \leqslant \nu_j \leqslant \nu_{j'}$ while the condition on c_j is $0 \leqslant c_j \leqslant c_{j'}$. Finally, if $j \in J'$ then j has a unique neighbour $j' \in J$, and the condition on ν_j is $0 \leqslant \nu_j \leqslant \max\{0, -1 + \nu_{j'}\}$ while the condition on c_j is $c_j = 0$. Clearly the condition on ν_j forces $\nu_j = 0$. In conclusion, the exponents ν and the vectors γ must satisfy the same conditions, and the theorem follows from Proposition 6.7 and 7.5.2. ☐

8 A tensor product theorem

8.1 In a cluster algebra $\mathscr{A}$, a product $y_1 \cdots y_k$ of cluster variables is a cluster monomial if and only if for every $1 \leqslant i < j \leqslant k$ the product $y_i y_j$ is a cluster monomial. In this section we show the following theorem, which is consistent with our categorification Conjecture 4.6 (ii), and will be needed to prove it.

Theorem 8.1 *Let $S_1, \ldots, S_k$, be simple objects of $\mathscr{C}_1$. Suppose that $S_i \otimes S_j$ is simple for every $1 \leqslant i < j \leqslant k$. Then $S_1 \otimes \cdots \otimes S_k$ is simple.*

Note that we may have $S_i \cong S_j$ for some $i < j$, in which case our assumption includes the simplicity of $S_i^{\otimes 2}$. The proof will be given in 8.5.

A similar result for a special class of modules of the Yangian of $\mathfrak{gl}_n$ attached to skew Young diagrams was given by Nazarov and Tarasov [**NT**].

8.2 We need to recall some results about tensor products of objects of $\mathscr{C}$. Let Δ be the comultiplication of $U_q(\widehat{\mathfrak{g}})$. In general one does not know explicit formulae for calculating Δ in terms of the Drinfeld generators. However, the next proposition contains partial information which will be sufficient for our purposes. We have a natural grading of $U_q(\widehat{\mathfrak{g}})$ by the root lattice Q of $\mathfrak{g}$ given by

$$\deg(x_{i,r}^{\pm}) = \pm\alpha_i, \quad \deg(h_{i,s}) = \deg(k_i^{\pm}) = \deg(c^{\pm}) = 0, \qquad (i \in I,\ r \in \mathbb{Z},\ s \in \mathbb{Z}\setminus\{0\}).$$

Let U^+ (*resp.* U^-) be the subalgebra of $U_q(\widehat{\mathfrak{g}})$ consisting of elements of positive (*resp.* negative) Q-degree.

Proposition 8.2 [**D**, Prop. 7.1] *For $i \in I$ and $r > 0$ we have*

$$\Delta(h_{i,r}) - h_{i,r} \otimes 1 - 1 \otimes h_{i,r} \in U^- \otimes U^+.$$

The next result is about tensor products of fundamental modules. For a fundamental module $L(Y_{i,r}) = V_{i,q^r}$ in $\mathscr{C}_{\mathbb{Z}}$ we denote by $u_{i,r}$ a highest weight vector (it is unique up to rescaling by a non-zero complex number).

Theorem 8.3 [**Cha2**, **K**, **VV**] *Let $r_1, \ldots, r_s$ be integers with $r_1 \leqslant \cdots \leqslant r_s$. Then, for any $i_1, \ldots, i_s$, the tensor product of fundamental modules $L(Y_{i_s,r_s}) \otimes \cdots \otimes L(Y_{i_1,r_1})$ is a cyclic module generated by the tensor product of highest weight vectors $u_{i_s,r_s} \otimes \cdots \otimes u_{i_1,r_1}$. Moreover, there is a unique homomorphism*

$$\phi : L(Y_{i_s,r_s}) \otimes \cdots \otimes L(Y_{i_1,r_1}) \to L(Y_{i_1,r_1}) \otimes \cdots \otimes L(Y_{i_s,r_s})$$

with $\phi(u_{i_s,r_s} \otimes \cdots \otimes u_{i_1,r_1}) = u_{i_1,r_1} \otimes \cdots \otimes u_{i_s,r_s}$, and its image is the simple module $L(Y_{i_1,r_1} \cdots Y_{i_s,r_s})$.

Example 8.4 (*cf.* Example 5.1.) Take $\mathfrak{g} = \mathfrak{sl}_2$ and consider the tensor product $L(Y_0) \otimes L(Y_2)$. We have

$$L(Y_0) = \mathbb{C}u_0 \oplus \mathbb{C}v_0, \qquad L(Y_2) = \mathbb{C}u_2 \oplus \mathbb{C}v_2,$$

where u_0 and u_2 are the highest weight vectors, and $v_0 = x_0^- u_0,\ v_2 = x_0^- u_2$. Each of these four vectors spans a one-dimensional $U_q(\mathfrak{sl}_2)$-weight-space, hence also an l-weight-space, and we have

$$h_r u_0 = \frac{[r]_q}{r} u_0, \quad h_r u_2 = \frac{[r]_q}{r} q^{2r} u_2, \quad h_r v_0 = -\frac{[r]_q}{r} q^{2r} v_0, \quad h_r v_2 = -\frac{[r]_q}{r} q^{4r} v_2,$$

where $[r]_q = (q^r - q^{-r})/(q - q^{-1})$. By Theorem 8.3, the submodule of $L(Y_0) \otimes L(Y_2)$ generated by $u_0 \otimes u_2$ is the three-dimensional simple module $L(Y_0 Y_2)$, with basis

$$u_0 \otimes u_2, \quad x_0^-(u_0 \otimes u_2) = u_0 \otimes v_2 + q v_0 \otimes u_2, \quad v_0 \otimes v_2.$$

The $U_q(\mathfrak{sl}_2)$-weight-spaces of $L(Y_0 Y_2)$ are also one-dimensional and coincide with its l-weight-spaces. By Proposition 8.2 we have

$$h_r(x_0^-(u_0 \otimes u_2)) = \frac{[r]_q}{r}(1 - q^{4r}) u_0 \otimes v_2 + \lambda v_0 \otimes u_2$$

for some $\lambda \in \mathbb{C}$. Hence $u_0 \otimes v_2 + q v_0 \otimes u_2$ has l-weight $Y_0 Y_4^{-1}$, which is the product of the l-weights of u_0 and v_2. On the other hand, again by Proposition 8.2, we have $h_r(v_0 \otimes u_2) = 0$ for every $r > 0$, hence $v_0 \otimes u_2$ is an l-weight-vector with l-weight 1. This shows that $u_0 \otimes v_2$ is *not* an l-weight vector.

8.3 To an object V of $\mathscr{C}_1$ which is generated by a highest weight vector v, we attach as in 6.2 a truncated *below* q-character $\chi_q(V)_{\geqslant 3}$ as follows. We have $\chi_q(V) = m(1+\sum_p M^{(p)})$, where m is the l-weight of v, and the $M^{(p)}$ are monomials in the $A_{i,r}^{-1}$ $(i \in I,\ r \in \mathbb{Z}_{>0})$. Define

$$\chi_q(V)_{\geqslant 3} := m\left(1+{\sum_p}^{*} M^{(p)}\right)$$

where $\sum^*$ means the sum restricted to the $M^{(p)}$ which have no variable $A_{i,1}^{-1}$, $A_{i,2}^{-1}$ $(i \in I)$. We shall also denote by $V_{\geqslant 3}$ the subspace of V obtained by taking the direct sum of the corresponding l-weight-spaces.

8.4 Let S be a simple object in $\mathscr{C}_1$ and let $m = \prod_{i\in I} Y_{i,\xi_i}^{a_i} Y_{i,\xi_i+2}^{b_i}$ be its highest weight monomial. We define

$$m^- = \prod_{i\in I} Y_{i,\xi_i}^{a_i}, \quad S^- = L(m^-), \quad m^+ = \prod_{i\in I} Y_{i,\xi_i+2}^{b_i}, \quad S^+ = L(m^+).$$

Then S^- is a simple object of the subcategory $\mathscr{C}_0$, and therefore $S^- \cong \bigotimes_{i\in I} L(Y_{i,\xi_i})^{\otimes a_i}$ is a tensor product of fundamental modules (*cf.* Example 3.3). Similarly, $S^+ \cong \bigotimes_{i\in I} L(Y_{i,\xi_i+2})^{\otimes b_i}$. Applying Theorem 8.3, we get a surjective homomorphism $\phi_S : S^+ \otimes S^- \twoheadrightarrow S \subset S^- \otimes S^+$.

Lemma 8.5 *Let u^- be a highest weight vector of S^-. We have $\phi_S^{-1}(S_{\geqslant 3}) = S^+ \otimes u^-$ and ϕ_S restricts to a bijection from $S^+ \otimes u^-$ to $S_{\geqslant 3}$.*

Proof — First we have $\chi_q(S^+ \otimes S^-)_{\geqslant 3} = \chi_q(S^+)_{\geqslant 3}\, \chi_q(S^-)_{\geqslant 3}$. Then by Proposition 5.8 we have $\chi_q(S^+)_{\geqslant 3} = \chi_q(S^+)$ and $\chi_q(S^-)_{\geqslant 3} = m^-$. So $\chi_q(S^+ \otimes S^-)_{\geqslant 3} = \chi_q(S^+)m^-$. Then we note that $\chi_q(S)_{\geqslant 3} = m^-\chi_q(S^+)$. Indeed, the inequality $\chi_q(S)_{\geqslant 3} \leqslant m^-\chi_q(S^+)$ follows from the factorization $m = m^-m^+$. On the other hand, S^+ is minuscule by § 5.2.5, so $\chi_q(S^+) = \mathrm{FM}(m^+)$. By using inductively 5.4 for $\chi_q(S)$ we get the converse inequality. Hence we get that $\chi_q(S^+ \otimes S^-)_{\geqslant 3} = \chi_q(S)_{\geqslant 3}$. Since ϕ_S is a homomorphism of $U_q(\widehat{\mathfrak{g}})$-modules, it preserves l-weight-spaces, hence it follows that it restricts to a vector space isomorphism from $(S^+ \otimes S^-)_{\geqslant 3}$ to $S_{\geqslant 3}$. Now since u^- is a highest-weight vector of S^-, Proposition 8.2 shows that for every l-weight vector $v \in S^+$, $v \otimes u^-$ is an l-weight vector of $S^+ \otimes S^-$ with l-weight equal to the product of the l-weight of v by m^-. This implies that $S^+ \otimes u^- \subset (S^+ \otimes S^-)_{\geqslant 3}$. Hence, since these two vector spaces have the same q-character, they are equal and the lemma is proved. □

8.5 We can now give the proof of Theorem 8.1. For $i = 1, \ldots, k$, define $m_i^{\pm}$, $S_i^{\pm}$, $u_i^{\pm}$, ϕ_{S_i} as above. First consider a pair $1 \leqslant i < j \leqslant k$. Note that by Example 3.3 and Section 5.2.5, $S_i^{\pm} \otimes S_j^{\pm}$ is minuscule and simple. Now, since $S_i \otimes S_j$ is simple, and $S_j^+ \otimes S_i^+ \otimes S_i^- \otimes S_j^-$ can be written as a product of fundamental modules with non-increasing spectral parameters, by Theorem 8.3, we have a surjective homomorphism $\phi : S_j^+ \otimes S_i^+ \otimes S_i^- \otimes S_j^- \twoheadrightarrow S_i \otimes S_j$, which is unique up to a scalar multiple. Using three times Theorem 8.3, this homomorphism, composed with the inclusion $S_i \otimes S_j \subset S_i^- \otimes S_i^+ \otimes S_j^- \otimes S_j^+$, can be factored as follows

$$S_j^+ \otimes (S_i^+ \otimes S_i^-) \otimes S_j^- \to (S_j^+ \otimes S_i^-) \otimes S_i^+ \otimes S_j^- \to S_i^- \otimes (S_j^+ \otimes S_i^+) \otimes S_j^- \cong$$

$$\cong S_i^- \otimes S_i^+ \otimes (S_j^+ \otimes S_j^-) \to S_i^- \otimes S_i^+ \otimes S_j^- \otimes S_j^+.$$

The map from the second module to the fourth one can be written as $\alpha \otimes \mathrm{id}_{S_j^-}$, where

$$\alpha : S_j^+ \otimes S_i^- \otimes S_i^+ \to S_i^- \otimes S_i^+ \otimes S_j^+$$

restricts to a homomorphism $\overline{\alpha}$ from $S_j^+ \otimes S_i$ to $S_i \otimes S_j^+$. Since ϕ is surjective, by applying Lemma 8.5 to $S = S_j$, we get that $\mathrm{im}(\overline{\alpha}) \otimes u_j^-$ contains $S_i \otimes S_j^+ \otimes u_j^-$, hence $\overline{\alpha}$ is surjective. To summarize, we have obtained the existence of a unique homomorphism $S_j^+ \otimes S_i \to S_i \otimes S_j^+$ mapping $u_j^+ \otimes u_i$ to $u_i \otimes u_j^+$, which is surjective.

Now we proceed by induction on $k \geqslant 2$ to prove Theorem 8.1. For $k = 2$ there is nothing to prove, so take $k \geqslant 3$ and assume that $S_1 \otimes \cdots \otimes S_{k-1}$ is simple. By Theorem 8.3 we obtain a surjective homomorphism

$$S_k^+ \otimes (S_1^+ \otimes \cdots \otimes S_{k-1}^+) \otimes (S_1^- \otimes \cdots \otimes S_{k-1}^-) \otimes S_k^- \twoheadrightarrow S_k^+ \otimes (S_1 \otimes \cdots \otimes S_{k-1}) \otimes S_k^-.$$

Using the above result, we then have a sequence of surjective homomorphisms

$$S_k^+ \otimes (S_1 \otimes \cdots \otimes S_{k-1}) \twoheadrightarrow S_1 \otimes S_k^+ \otimes S_2 \otimes \cdots \otimes S_{k-1} \twoheadrightarrow \cdots \twoheadrightarrow (S_1 \otimes \cdots \otimes S_{k-1}) \otimes S_k^+,$$

hence

$$S_k^+ \otimes (S_1^+ \otimes \cdots \otimes S_{k-1}^+) \otimes (S_1^- \otimes \cdots \otimes S_{k-1}^-) \otimes S_k^- \twoheadrightarrow (S_1 \otimes \cdots \otimes S_{k-1}) \otimes S_k^+ \otimes S_k^- \twoheadrightarrow S_1 \otimes \cdots \otimes S_k.$$

Thus we have shown that $V := S_1 \otimes \cdots \otimes S_k$ is a quotient of a tensor product of fundamental modules, which by Theorem 8.3 is a cyclic module generated by its highest weight vector. Hence V has no proper submodule containing its highest weight-space.

We can now conclude, as in [**CP2**, §4.10], by considering the dual module $V^* = S_k^* \otimes \cdots \otimes S_1^*$. By our assumption, $S_j^* \otimes S_i^* \cong (S_i \otimes S_j)^*$ is simple for every $1 \leqslant i < j \leqslant k$. Moreover, by 3.4, the modules S_i^* belong to a category defined like $\mathscr{C}_1$ except for a shift of all spectral parameters by q^{-h}. Thus, by using the same proof, V^* also has no proper submodule containing its highest weight-space. But if W was a proper submodule of V not containing its highest weight-space, then the annihilator W° of W in V^* would be a proper submodule of V^* containing its lowest weight-space. Let λ_i (resp. μ_i) be the lowest (resp. highest) weight of S_i^* considered as a $U_q(\mathfrak{g})$-module. Then $\lambda = \lambda_1 + \cdots + \lambda_n$ (resp. $\mu = \mu_1 + \cdots + \mu_n$) is the lowest (resp. highest) weight of V^*. In the direct sum decomposition of V^* as a $U_q(\mathfrak{g})$-module, there is a unique simple module L with lowest weight λ, and by our assumption, L is contained in W°. By [**CP4**, Proposition 5.1 (b)] (see also [**FM**, Theorem 1.3 (3)]), μ is the highest weight of L, and therefore W° must also contain the highest weight-space of V^*, which is impossible. This finishes the proof of Theorem 8.1. □

9 Cluster expansions

9.1 Following [**FZ2**, **FZ3**], we say that two roots $\alpha, \beta \in \Phi_{\geqslant -1}$ are *compatible* if the cluster variables $x[\alpha]$ and $x[\beta]$ belong to a common cluster of $\mathscr{A}$. Let γ be an element of the root lattice Q. A *cluster expansion* of γ is a way to express γ as

$$\gamma = \sum_{\alpha \in \Phi_{\geqslant -1}} n_\alpha \alpha, \tag{36}$$

where all n_α are nonnegative integers, and $n_\alpha n_\beta = 0$ whenever α and β are not compatible. In plain words, a cluster expansion is an expansion into a sum of pairwise compatible roots in $\Phi_{\geqslant -1}$.

Theorem 9.1 [**FZ2**, Th. 3.11] *Every element of the root lattice has a unique cluster expansion.*

9.2 For $\gamma = \sum_i c_i\alpha_i \in Q$, define

$$Y^{\gamma} := \prod_{c_i<0} Y_{i,2-\xi_i}^{|c_i|} \prod_{c_i>0} Y_{i,\xi_i-\varepsilon_i+1}^{c_i} \in \mathscr{M}_+.$$

This definition is such that for $\alpha \in \Phi_{\geqslant -1}$, the monomial Y^{α} is the highest l-weight of $S(\alpha)$.

Let S be a simple object in $\mathscr{C}_1$. Its highest l-weight is of the form

$$m = \prod_{i\in I} Y_{i,\xi_i}^{a_i} Y_{i,\xi_i+2}^{b_i}$$

for some nonnegative integers a_i, b_i. Clearly we can write m as

$$m = \left(\prod_{i\in I} (Y_{i,\xi_i} Y_{i,\xi_i+2})^{\min(a_i,b_i)}\right) Y^{\gamma},$$

for some unique $\gamma \in Q$. Hence using Theorem 9.1 and (36), we have a unique factorization

$$m = \prod_{i\in I} (Y_{i,\xi_i} Y_{i,\xi_i+2})^{\min(a_i,b_i)} \prod_{\alpha\in\Phi_{\geqslant -1}} (Y^{\alpha})^{n_\alpha},$$

where the α for which $n_\alpha > 0$ are pairwise compatible. Suppose that Conjecture 4.6 (i) is established. Since $S(\alpha) = L(Y^{\alpha})$ and $F_i = L(Y_{i,\xi_i} Y_{i,\xi_i+2})$, to prove Conjecture 4.6 (ii) we then need to prove that

$$L(m) \cong \bigotimes_{i\in I} F_i^{\otimes \min(a_i,b_i)} \bigotimes_{\alpha\in\Phi_{\geqslant -1}} S(\alpha)^{\otimes n_\alpha}.$$

Given that the highest l-weights of the two sides coincide, this amounts to prove that the tensor product of the right-hand side is a simple module. Because of Theorem 8.1, this will be the case if and only if every pair of factors has a simple tensor product. Thus, assuming Conjecture 4.6 (i), the proof of Conjecture 4.6 (ii) reduces to check that

(i) $F_i \otimes S(\alpha)$ and $F_i \otimes F_j$ are simple for every $\alpha \in \Phi_{\geqslant -1}$ and $i, j \in I$;

(ii) if $\alpha, \beta \in \Phi_{\geqslant -1}$ are compatible, then $S(\alpha) \otimes S(\beta)$ is simple;

(iii) for every $\alpha \in \Phi_{>0}$, $S(\alpha)$ is prime.

Note that the simple objects F_i and $S(-\alpha_i)$ are clearly prime. Indeed, if F_i was not prime we could only have $F_i \cong S(\alpha_i) \otimes S(-\alpha_i)$, in contradiction with the T-system

$$[S(\alpha_i) \otimes S(-\alpha_i)] = [F_i] + \prod_{j\,:\,a_{ij}=-1} [S(-\alpha_j)].$$

10 Type A

In this section we assume that $\mathfrak{g}$ is of type A_n. The vertices of the Dynkin diagram are labelled by the interval $I = [1, n]$ in linear order.

10.1 The positive roots are all multiplicity-free, labelled by the subintervals of I. For $[i, j] \subset I$, we denote by $\alpha_{[i,j]} := \sum_{k=i}^{j} \alpha_k$ the corresponding positive root. By Theorem 7.8, Eq. (34) is verified for all positive roots. This and (33) proves Conjecture 4.6 (i) in type A.

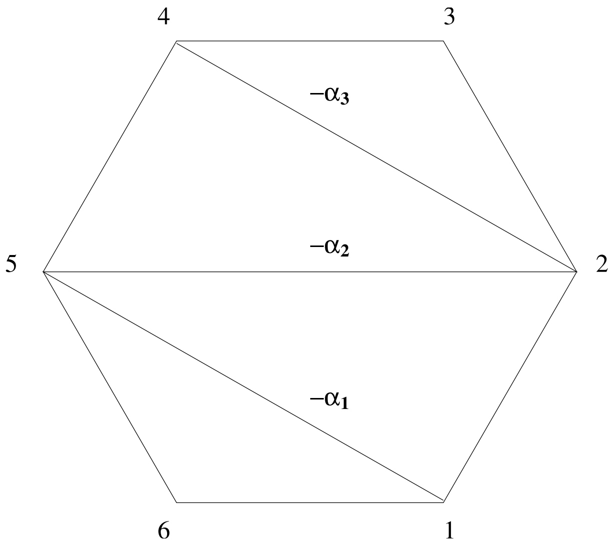

Figure 1: *The snake in type A_3.*

10.2 We will now prove 9.2 (i) and (ii). Note that 9.2 (iii) follows from Corollary 6.10.

10.2.1 We first dispose of (i), which is easy. Indeed, by Example 6.4, $\chi_q(F_i \otimes F_j)_{\leqslant 2}$ is equal to a single dominant monomial hence $F_i \otimes F_j$ is simple. Next, we use the fact that $\chi_q(S(\alpha))_{\leqslant 2}$ is given by the Frenkel-Mukhin algorithm (see Example 6.4 and Corollary 6.8). Thus, if m is the highest l-weight of $S(\alpha)$, we have

$$\mathrm{FM}(Y_{i,\xi_i}Y_{i,\xi_i+2}m)_{\leqslant 2} = Y_{i,\xi_i}Y_{i,\xi_i+2}\mathrm{FM}(m)_{\leqslant 2} = Y_{i,\xi_i}Y_{i,\xi_i+2}\chi_q(L(m))_{\leqslant 2}.$$

Therefore $\chi_q(L(Y_{i,\xi_i}Y_{i,\xi_i+2}m))$ contains $\chi_q(F_i \otimes S(\alpha))_{\leqslant 2}$, and $F_i \otimes S(\alpha)$ is simple.

10.2.2 To describe explicitly the pairs of compatible roots we are going to use the geometric model of [**FZ2**, §3.5]. The set $\Phi_{\geqslant -1}$ has cardinality $n(n+1)/2 + n = n(n+3)/2$. We identify the elements of $\Phi_{\geqslant -1}$ with the diagonals of a regular convex $(n+3)$-gon $\mathscr{P}_{n+3}$ as follows. Let $1, \ldots, n+3$ be the vertices of $\mathscr{P}_{n+3}$, labelled counterclockwise. The negative simple roots are identified with the following diagonals:

$$-\alpha_k \equiv \begin{cases} [i,\ n+3-i] & \text{if } k = 2i-1, \\ [i+1,\ n+3-i] & \text{if } k = 2i. \end{cases}$$

These diagonals form a "snake", as shown in Figure 1. To identify the remaining diagonals (not belonging to the snake) with positive roots, we associate each $\alpha_{[i,j]}$ with the unique diagonal that crosses the diagonals $-\alpha_i, -\alpha_{i+1}, \ldots, -\alpha_j$ and does not cross any other diagonal $-\alpha_k$ of the snake. Under this identification the roots α and β are compatible if and only if they are represented by two non-crossing diagonals. Hence the clusters are in one-to-one correspondence with the triangulations of $\mathscr{P}_{n+3}$.

Example 10.1 Take $n = 3$. The identification of the negative simple roots with the snake diagonals of a hexagon is shown in Figure 1. The positive roots are represented by the remaining diagonals as follows:

$$\alpha_1 \equiv [2,6],\ \alpha_2 \equiv [1,4],\ \alpha_3 \equiv [3,5],\ \alpha_1+\alpha_2 \equiv [4,6],\ \alpha_2+\alpha_3 \equiv [1,3],\ \alpha_1+\alpha_2+\alpha_3 \equiv [3,6].$$

The 14 clusters listed in Example 4.8 correspond to the 14 triangulations of the hexagon.

10.2.3 It follows from the geometric model that all pairs of compatible roots (α,β) of $\Phi_{\geqslant -1}$ are of one of the following forms:

(a) $\alpha = -\alpha_i$, $\beta = -\alpha_j$;

(b) $\alpha = -\alpha_i$, $\beta \in \Phi_{>0}$ with $[\beta : \alpha_i] = 0$;

(c) $\alpha = \alpha_{[i,j]}$, $\beta = \alpha_{[k,l]}$ with $k > j+1$ (two disjoint intervals);

(d) $\alpha = \alpha_{[i,j]}$, $\beta = \alpha_{[i,l]}$ or $\alpha = \alpha_{[i,k]}$, $\beta = \alpha_{[j,k]}$ (two intervals with a common end);

(e) $\alpha = \alpha_{[i,k]}$, $\beta = \alpha_{[j,l]}$ with $i < j \leqslant k < l$ and $k-j$ even (two overlapping intervals);

(f) $\alpha = \alpha_{[i,l]}$, $\beta = \alpha_{[j,k]}$ with $i < j < k < l$ and $k-j$ odd (one interval strictly contained into the other).

Let us prove that $S(\alpha)\otimes S(\beta)$ is simple in all these cases.

(a) This follows from Example 3.3. Indeed, $S(-\alpha_i)$ and $S(-\alpha_j)$ are fundamental modules in a shift of the subcategory $\mathscr{C}_0$.

(b) Write $\beta = \alpha_J$. Let $m = Y_{i,\xi_i+1}\prod_{j\in J\cap I_0} Y_{j,0}\prod_{j\in J\cap I_1} Y_{j,3}$ be the highest weight monomial of $\chi_q(S(-\alpha_i)\otimes S(\beta))$. Since by assumption $i \notin J$, it is easy to check that $\mathrm{FM}(m)$ contains the product

$$\chi_q(L(Y_{i,\xi_i+1}))_{\leqslant 2}\,\chi_q(S(\beta))_{\leqslant 2} \ = \ \chi_q(S(-\alpha_i)\otimes S(\beta))_{\leqslant 2}$$

given by Example 6.4 and Proposition 6.7. Hence, by 3.4, it contains in particular $\chi_q(L(m))_{\leqslant 2}$. By Corollary 6.5, it follows that $\mathrm{FM}(m) = \chi_q(L(m))$. Thus, $\chi_q(S(-\alpha_i)\otimes S(\beta)) = \chi_q(L(m))$, and $S(-\alpha_i)\otimes S(\beta)$ is irreducible.

(c) Same reasoning as (b).

(d) We assume that $i \leqslant j \leqslant l$ and argue by induction on $s = l-i$. If $s = 0$, we have to check that $S(\alpha_i)\otimes S(\alpha_i)$ is simple. This is clear, since if we choose $\xi_i = 1$ then $\chi_q(S(\alpha_i))_{\leqslant 2} = Y_{i,3}$ and its square contains a single dominant monomial. Suppose the claim is proved when $l-i = s$, and take $l-i = s+1$. Let $m_{[i,j]}$ and $m_{[i,l]}$ denote the highest weight monomials of $S(\alpha_{[i,j]})$ and $S(\alpha_{[i,l]})$. Choose again $\xi_i = 1$. Then, by Proposition 6.7, the truncated q-characters of $S(\alpha_{[i,j]})$ and $S(\alpha_{[i,l]})$ do not contain any A^{-1}_{k,ξ_k+1} with $k \leqslant i$. So we have

$$\chi_q(S(\alpha_{[i,j]}))_{\leqslant 2} = \varphi_{[i+1,j+1]}(m_{[i,j]})_{\leqslant 2}, \qquad \chi_q(S(\alpha_{[i,l]}))_{\leqslant 2} = \varphi_{[i+1,l+1]}(m_{[i,l]})_{\leqslant 2}.$$

Hence $\chi_q(S(\alpha_{[i,j]})\otimes S(\alpha_{[i,l]}))_{\leqslant 2} = \varphi_{[i+1,l+1]}(m_{[i,j]}m_{[i,l]})_{\leqslant 2}$. So by using the induction hypothesis and 5.4, we get

$$\chi_q(S(\alpha_{[i,j]})\otimes S(\alpha_{[i,l]}))_{\leqslant 2} \leqslant \chi_q(L(m_{[i,j]}m_{[i,l]}))_{\leqslant 2}.$$

This shows that $S(\alpha_{[i,j]})\otimes S(\alpha_{[i,l]}) \cong L(m_{[i,j]}m_{[i,l]})$ is simple. The case of $S(\alpha_{[i,j]})\otimes S(\alpha_{[k,j]})$ with $1 \leqslant k \leqslant j$ is similar.

(e) (f) Let $m = m_{[i,k]}m_{[j,l]} = m_{[i,l]}m_{[j,k]}$. We want to show that

$$\chi_q(L(m))_{\leqslant 2} = \begin{cases} \chi_q(L(m_{[i,k]}))_{\leqslant 2}\ \chi_q(L(m_{[j,l]}))_{\leqslant 2} & \text{if } k-j \text{ is odd,} \\ \chi_q(L(m_{[i,l]}))_{\leqslant 2}\ \chi_q(L(m_{[j,k]}))_{\leqslant 2} & \text{if } k-j \text{ is even.} \end{cases} \tag{37}$$

To do this, it is enough to show that all the monomials in the right-hand side occur in the left-hand side with the same multiplicity. We argue by induction on $l-i \geqslant 2$. For $l-i=2$ we are necessarily in the first case with $j=k=i+1=l-1$. Let us choose $\xi_i = 1$. Then $\xi_l = 1$, and by Proposition 6.6

$$\chi_q(L(m_{[i,k]}))_{\leqslant 2} = \varphi_{[i,l]}(m_{[i,k]})_{\leqslant 2}, \qquad \chi_q(L(m_{[j,l]}))_{\leqslant 2} = \varphi_{[i,l]}(m_{[i,k]})_{\leqslant 2}.$$

Thus we can assume for simplicity of notation that $I=[i,l]=[1,3]$. Writing for short $v_m = A^{-1}_{m,\xi_m+1}$, we have

$$\chi_q(L(m_{[1,2]}))_{\leqslant 2}\ \chi_q(L(m_{[2,3]}))_{\leqslant 2} = m(1+v_2+v_2v_3)(1+v_2+v_1v_2).$$

If a monomial in the right-hand side does not contain v_1, then it belongs to

$$\varphi_{[2,3]}(m_{[1,2]})_{\leqslant 2}\ \varphi_{[2,3]}(m_{[2,3]})_{\leqslant 2} \ = \ \varphi_{[2,3]}(m)_{\leqslant 2} \ \leqslant \ \chi_q(L(m))$$

(the equality follows from (d)). For the same reason, if a monomial does not contain v_3, it belongs to $\chi_q(L(m))$. The only monomial in the right-hand side which contains both v_1 and v_3 is $m' = mv_1v_2^2v_3$. Now $m'' := mv_2^2v_3 = Y_{1,1}^2Y_{1,3}Y_{2,2}^{-1}Y_{3,1}$ belongs to $\chi_q(L(m))$ by what we have just said, and it is 1-dominant. We have $\varphi_1(m'')_{\leqslant 2} = m'' + m'$, thus, by 5.3, m' also belongs to $\chi_q(L(m))$. This proves (37) when $l-i=2$.

Assume now that (37) holds when $l-i=s$ and take $l-i=s+1$. Choose $\xi_i=1$. Using again Proposition 6.6, we can also assume without loss of generality that $I=[i,l]=[1,s+1]$. Arguing as in the case $l-i=2$ and using (c) or (d) or the induction hypothesis, we see that every monomial m' occuring in the right-hand side of (37) occurs in $\chi_q(L(m))$ with the same multiplicity if $m'm^{-1}$ does not contain $v_1v_2\cdots v_{s+1}$. So we only need to consider the monomials $m' = mv_1^{c_1}\cdots v_{s+1}^{c_{s+1}}$ with $c_k > 0$ for every $k \in I$. For such a monomial m', it can be checked using Proposition 6.7 that there exists k with $c_k = 2$, and that the smallest such k belongs to I_0. Let us denote it by $k_{\min}$. Then Proposition 6.7 shows that $m'' := m'/v_{k_{\min}-1}$ appears in the right-hand side of (37), and since m'' does not contain $v_{k_{\min}-1}$, by what we said above, m'' also appears in $\chi_q(L(m))$ with the same multiplicity. Now m'' is $(k_{\min}-1)$-dominant and $\varphi_{k_{\min}-1}(m'') = m''+m'$ is contained in $\chi_q(L(m))$ by 5.3. Moreover, the multiplicities of m' and m'' in $\varphi_{k_{\min}-1}(m'')$ are the same. Hence, their multiplicities are also the same on both sides of (37). This finishes the proof of (e) and (f).

This finishes the proof of 9.2 (ii) in type A. Hence Conjecture 4.6 (ii) is proved in type A.

10.3 The truth of Conjecture 4.6 in type A has the following interesting consequence, which will be used to validate Conjecture 4.6 (i) in type D.

Let $\gamma = \sum_i c_i\alpha_i$, with $c_i \geqslant 0$. We define $Y^\gamma \in \mathscr{M}_+$ as in 9.2. Write $\tau_-(\gamma) = \delta = \sum_i d_i\alpha_i$.

Corollary 10.2 *If $0 \leqslant d_i \leqslant 2$ for every $i \in I$, the truncated q-character of $L(Y^\gamma)$ is equal to*

$$\chi_q(L(Y^\gamma))_{\leqslant 2} = Y^\gamma F_\delta\left(A^{-1}_{1,\xi_1+1},\ldots,A^{-1}_{n,\xi_n+1}\right),$$

where the F-polynomial F_δ is given by the explicit formula (35).

Proof — Let $\gamma = \sum_{\alpha\in\Phi_{>0}} n_\alpha \alpha$ be the cluster expansion of γ. By 10.2, we have

$$\chi_q(L(Y^\gamma))_{\leqslant 2} = \prod_{\alpha\in\Phi_{>0}} \chi_q(S(\alpha))_{\leqslant 2}^{n_\alpha} = Y^\gamma \prod_{\alpha\in\Phi_{>0}} F_{\tau_-(\alpha)}(A_{i,\xi_i+1}^{-1})^{n_\alpha}.$$

Since τ_- is linear on $\oplus_i \mathbb{N}\alpha_i$, we have $\tau_-(\gamma) = \sum_{\alpha\in\Phi_{>0}} n_\alpha \tau_-(\alpha) = \delta$, and this is the cluster expansion of δ. Now if we define $F_\delta(v_1,\ldots,v_n) := \prod_{\alpha\in\Phi_{>0}} F_{\tau_-(\alpha)}(v_1,\ldots,v_n)^{n_\alpha}$, the proof of Proposition 7.5 given in [**FZ2**] shows that, since δ is 2-restricted, F_δ can still be calculated by formula (35). Indeed, the two main steps (Lemmas 2.11 and 2.12) are proved for a 2-restricted vector which is not necessarily a root. □

Example 10.3 Let $\mathfrak{g}$ be of type A_3 and take $I_0 = \{2\}$ and $I_1 = \{1,3\}$. Choose $\gamma = \alpha_1 + 2\alpha_2 + \alpha_3$. Then $\tau_-(\gamma) = \gamma$. Writing $v_i = A_{i,\xi_i+1}^{-1}$, we have

$$\chi_q(L(Y_{1,3}Y_{2,0}^2Y_{3,3}))_{\leqslant 2} \;=\; Y_{1,3}Y_{2,0}^2Y_{3,3}(1+2v_2+v_2^2+v_1v_2+v_2v_3+v_1v_2^2+v_2^2v_3+v_1v_2^2v_3),$$

where the monomials of the right-hand side are given by the combinatorial rule of Proposition 7.5.

11 Type D

In this section we take $\mathfrak{g}$ of type D_n, and we label the Dynkin diagram as in [**B**].

11.1 Let $\beta \in \Phi_{>0}$ and let $\alpha = \tau_-(\beta)$. We want to prove

Theorem 11.1 *The normalized truncated q-character of $S(\beta)$ is equal to*

$$\widetilde{\chi}_q(S(\beta))_{\leqslant 2} \;=\; F_\alpha(v_1,\ldots,v_n),$$

where we write for short $v_i = A_{i,\xi_i+1}^{-1}$ $(i\in I)$, and the F-polynomial F_α is given by the combinatorial formula of Proposition 7.5.

We assume that the trivalent node $n-2$ belongs to I_0 (by 3.9, this is no loss of generality). We first establish some formulas in type D_4.

Lemma 11.2 *Let $\mathfrak{g}$ be of type D_4. We have*

$$\begin{aligned}
\chi_q(L(Y_{1,3}Y_{2,0}))_{\leqslant 2} &= Y_{1,3}Y_{2,0}\left(1+v_2(1+v_3)(1+v_4)\right),\\
\chi_q(L(Y_{2,0}Y_{3,3}Y_{4,3}))_{\leqslant 2} &= Y_{2,0}Y_{3,3}Y_{4,3}\left(1+v_2(1+v_1)\right),\\
\chi_q(L(Y_{1,3}Y_{2,0}^2Y_{3,3}Y_{4,3}))_{\leqslant 2} &= Y_{1,3}Y_{2,0}^2Y_{3,3}Y_{4,3}\big(1+v_2(2+v_1+v_3+v_4)\\
&\quad + v_2^2(1+v_1)(1+v_3)(1+v_4)\big).
\end{aligned}$$

Proof — The first two formulas concern multiplicity-free roots, and thus follow from Proposition 6.7. For the third one, taking $J = \{1,2\}$ and setting $m = Y_{1,0}Y_{2,3}^2Y_{3,0}Y_{4,0}$, we have by Example 6.3 (or Corollary 10.2)

$$\varphi_J(m)_{\leqslant 2} \;= m\left(1+v_2(2+v_1)+v_2^2(1+v_1)\right),$$

and by 5.4 we know that $\varphi_J(m) \leqslant \chi_q(L(m))$. Replacing J by $\{2,3\}$ and by $\{2,4\}$ we get that

$$m\left(1+v_2(2+v_1+v_3+v_4)+v_2^2(1+v_1+v_3+v_4)\right) \leqslant \chi_q(L(m))_{\leqslant 2}.$$

Now $m_1 := mv_2^2v_1 = Y_{1,1}Y_{2,2}^{-1}Y_{3,1}^2Y_{3,3}Y_{4,1}^2Y_{4,3}$ is J-dominant for $J = \{3,4\}$, and

$$\varphi_J(m_1)_{\leqslant 2} = m_1(1+v_3)(1+v_4),$$

so, by 5.4, we see that $m_1(1+v_3)(1+v_4) \leqslant \chi_q(L(m))$. Arguing similarly with $m_2 := mv_2^2v_3$ and $m_3 := mv_2^2v_4$ we obtain that

$$m\left(1+v_2(2+v_1+v_3+v_4)+v_2^2(1+v_1)(1+v_3)(1+v_4)\right) \leqslant \chi_q(L(m))_{\leqslant 2}. \tag{38}$$

Conversely, we have $\chi_q(L(m)) \leqslant \chi_q(L(Y_{2,0}))\chi_q(L(Y_{1,3}Y_{2,0}Y_{3,3}Y_{4,3})$. Using Proposition 6.7, we get

$$\chi_q(L(m))_{\leqslant 2} \ \leqslant m\left(1+v_2(1+v_1)(1+v_3)(1+v_4)\right)(1+v_2).$$

This upper bound is equal to the lower bound of (38) plus

$$m\left(v_2v_1v_3+v_2v_1v_4+v_2v_3v_4+v_2v_1v_3v_4\right).$$

We also have $\chi_q(L(m)) \leqslant \chi_q(L(Y_{1,3}Y_{2,0}))\chi_q(Y_{2,0}Y_{3,3}Y_{4,3})$. Using the first two formulas, we get

$$\chi_q(L(m))_{\leqslant 2} \ \leqslant m\left(1+v_2(1+v_3)(1+v_4)\right)\left(1+v_2(1+v_1)\right).$$

This second upper bound does not contain the monomials $mv_2v_1v_3$ and $mv_2v_1v_3v_4$. We can rule out the two remaining monomials by using the three-fold symmetry $1 \leftrightarrow 3 \leftrightarrow 4$, and we obtain an upper bound equal to the lower bound. □

Proof of Th. 11.1 — If β is a multiplicity-free positive root, the result follows from Theorem 7.8. So we assume that $\beta = \sum_i b_i\alpha_i$ has some multiplicity. In view of the list of roots for D_n, this implies that $b_{n-2} = 2$, and $b_{n-1} = b_n = 1$. Since $n-2 \in I_0$, we also have $\alpha = \sum_i a_i\alpha_i$ with $a_{n-2} = 2$ and $a_{n-1} = a_n = 1$.

Consider the subgraph Δ' of the Dynkin diagram supported on $[1, n-1]$, of type A_{n-1}. Set $\beta' = \beta - \alpha_n$ and $\alpha' = \alpha - \alpha_n$. Then $\alpha' = \tau'_-(\beta')$, where τ'_- is defined like τ_-, but for Δ'. Let Y^β be the highest l-weight of $S(\beta)$. By Corollary 10.2,

$$\varphi_{[1,n-1]}(Y^\beta) = Y^\beta F_{\alpha'}(v_1,\ldots,v_{n-1}),$$

where $F_{\alpha'}$ is the F-polynomial for Δ', given by the explicit formula (35). Since $n \in I_1$, this formula shows that $F_{\alpha'}(v_1,\ldots,v_{n-1}) = F_\alpha(v_1,\ldots,v_{n-1},0)$. Hence the specialization at $v_n = 0$ of the polynomial $\widetilde{\chi}_q(S(\beta))_{\leqslant 2}$ is equal to $F_\alpha(v_1,\ldots,v_{n-1},0)$. Similarly, the specialization at $v_{n-1} = 0$ of $\widetilde{\chi}_q(S(\beta))_{\leqslant 2}$ is equal to $F_\alpha(v_1,\ldots,v_{n-2},0,v_n)$.

Thus we are reduced to prove that the monomials of $\widetilde{\chi}_q(S(\beta))_{\leqslant 2}$ and $F_\alpha(v_1,\ldots,v_n)$ containing both v_{n-1} and v_n are the same and have the same coefficients.

We first show that such a monomial is of the form $mv_{n-2}^2v_{n-1}v_n$ where m is a monomial in $v_1,\ldots,v_{n-3}$. For the polynomial $F_\alpha(v_1,\ldots,v_n)$ this follows easily from the definition of an α-acceptable vector and the values of a_{n-2}, a_{n-1}, a_n. For $\widetilde{\chi}_q(S(\beta))_{\leqslant 2}$, we have

$$\widetilde{\chi}_q(S(\beta))_{\leqslant 2} \ \leqslant\ \widetilde{\chi}_q(S(\beta-2\alpha_{n-2}-\alpha_{n-1}-\alpha_n))_{\leqslant 2}\ \widetilde{\chi}_q(S(2\alpha_{n-2}+\alpha_{n-1}+\alpha_n))_{\leqslant 2},$$

where, by Proposition 6.6, $\widetilde{\chi}_q(S(\beta-2\alpha_{n-2}-\alpha_{n-1}-\alpha_n))_{\leqslant 2}$ does not contain v_{n-2}, v_{n-1}, v_n. Moreover, $2\alpha_{n-2}+\alpha_{n-1}+\alpha_n$ is supported on a root system of type A_3, so by Example 10.3 and Proposition 6.6 all the monomials of $\widetilde{\chi}_q(S(2\alpha_{n-2}+\alpha_{n-1}+\alpha_n))_{\leqslant 2}$ containing v_{n-1} and v_n are of the form $m'v_{n-2}^2v_{n-1}v_n$ where m' is a monomial in v_{n-3}.

We now note that if $mv_{n-2}^2 v_{n-1} v_n$ appears in $F_\alpha(v_1,\dots,v_n)$ (where m is a monomial in the variables $v_1,\dots,v_{n-3}$), then mv_{n-2}^2 also occurs, and with the same multiplicity. This follows again from the definition of an α-acceptable vector γ and of the integer $e(\gamma,\alpha)$. We have seen that mv_{n-2}^2 occurs in $\widetilde{\chi}_q(S(\beta))_{\leqslant 2}$ and $F_\alpha(v_1,\dots,v_n)$ with the same multiplicity. Thus all we have to do is to show that mv_{n-2}^2 and $mv_{n-2}^2 v_{n-1} v_n$ occur in $\widetilde{\chi}_q(S(\beta))_{\leqslant 2}$ with the same multiplicity. Let us denote these multiplicities by a and b, respectively.

Since $Y^\beta mv_{n-2}^2$ is $\{n-1,n\}$-dominant, $\widetilde{\chi}_q(S(\beta))_{\leqslant 2}$ contains $mv_{n-2}^2(1+v_{n-1})(1+v_n)$ with multiplicity at least a by 5.4, so $a \leqslant b$.

Conversely, let $\beta'' = \beta - \alpha_n - \alpha_{n-1}$. The multiplicity a of mv_{n-2}^2 in $\widetilde{\chi}_q(S(\beta))_{\leqslant 2}$ is equal to its multiplicity in $\widetilde{\chi}_q(S(\beta''))_{\leqslant 2}$. This is a multiplicity in type A_{n-2}, so using Corollary 10.2, we can check that it coincides with the multiplicity of mv_{n-2}^2 in the product

$$\widetilde{\chi}_q(S(\beta'' - 2\alpha_{n-2} - \alpha_{n-3}))_{\leqslant 2}\, \widetilde{\chi}_q(S(2\alpha_{n-2} + \alpha_{n-3}))_{\leqslant 2}$$

or equivalently in the product

$$\widetilde{\chi}_q(S(\beta'' - 2\alpha_{n-2} - \alpha_{n-3}))_{\leqslant 2}\, \widetilde{\chi}_q(S(2\alpha_{n-2} + \alpha_{n-3} + \alpha_{n-1} + \alpha_n))_{\leqslant 2}.$$

Using the third formula of Lemma 11.2 and the fact that the first factor contains no variable v_{n-2}, v_{n-1}, or v_n, we obtain that in this product the multiplicity a of mv_{n-2}^2 is equal to the multiplicity of $mv_{n-2}^2 v_{n-1} v_n$. But, by 3.4, this is greater or equal to b. Hence $a \geqslant b$. This concludes the proof of Theorem 11.1. □

Hence (34) is verified, and this proves Conjecture 4.6 (i) in type D_n.

11.2 In this section we take $\mathfrak{g}$ of type D_4 and we prove that Conjecture 4.6 (ii) is verified. For this we need to prove 9.2 (i) (ii) and (iii).

11.2.1 The proof that $F_i \otimes F_j$ is simple is the same as in type A_n (*cf.* 10.2.1). We then consider $F_i \otimes S(\alpha)$ for $\alpha \in \Phi_{\geqslant -1}$. If $\alpha = -\alpha_i$ or if α is a multiplicity-free positive root, we can repeat the argument of 10.2.1. If $\alpha = \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$, we choose $I_1 = \{2\}$, so that $\tau_-(\alpha) = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$. By Theorem 11.1, we have

$$\widetilde{\chi}_q(S(\alpha))_{\leqslant 2} = 1 + v_1 + v_3 + v_4 + v_1v_3 + v_1v_4 + v_3v_4 + v_1v_3v_4 + v_1v_2v_3v_4.$$

It is easy to check that this is the result given by the Frenkel-Mukhin algorithm, so we can argue again as in 10.2.1. This proves 9.2 (i).

11.2.2 Let $\alpha, \beta \in \Phi_{\geqslant -1}$ be two compatible roots. We want to show that $S(\alpha) \otimes S(\beta)$ is simple. If α or β is negative, we can repeat the argument of 10.2.3 (a) (b). So let us suppose that $\alpha, \beta \in \Phi_{>0}$.

If the union of the supports of α and β is strictly smaller than I, we can assume by symmetry that it is contained in $\{1,2,3\}$. Take $I_1 = \{2\}$. By Proposition 6.6, we then have

$$\chi_q(S(\alpha))_{\leqslant 2} = \varphi_{\{1,2,3\}}(Y^\alpha)_{\leqslant 2}, \quad \chi_q(S(\beta))_{\leqslant 2} = \varphi_{\{1,2,3\}}(Y^\beta)_{\leqslant 2},$$

that is, our q-characters are in fact of type A_3. On the other hand, α and β are also compatible as roots of type A_3, hence the equality

$$\chi_q(S(\alpha))_{\leqslant 2}\, \chi_q(S(\beta))_{\leqslant 2} = \chi_q(S(\alpha) \otimes S(\beta))_{\leqslant 2}$$

follows from 10.2.3.

We are thus reduced to those pairs of compatible positive roots (α, β) whose union of supports is equal to I. By [**FZ2**, Prop. 3.16], these are, up to the 3-fold symmetry $1 \leftrightarrow 3 \leftrightarrow 4$,

(a) $\alpha = \alpha_1, \quad \beta = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$;

(b) $\alpha = \alpha_1 + \alpha_2, \quad \beta = \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$;

(c) $\alpha = \alpha_1 + \alpha_2 + \alpha_3, \quad \beta = \alpha_2 + \alpha_3 + \alpha_4$;

(d) $\alpha = \alpha_1 + \alpha_2 + \alpha_3, \quad \beta = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$;

(e) $\alpha = \alpha_1 + \alpha_2 + \alpha_3, \quad \beta = \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$;

(f) $\alpha = \beta = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$;

(g) $\alpha = \beta = \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$.

(a) Take $I_0 = \{2\}$. Then $\chi_q(S(\alpha))_{\leqslant 2} = Y_{1,3}$ and, since $\tau_-(\beta) = \alpha_2$ we have

$$\chi_q(S(\beta))_{\leqslant 2} = Y_{1,3}Y_{2,0}Y_{3,1}Y_{4,1}(1+v_2),$$

hence $\chi_q(S(\alpha)\otimes S(\beta))_{\leqslant 2}$ contains a unique dominant monomial and $S(\alpha)\otimes S(\beta)$ is simple.

(b) Take $I_1 = \{2\}$. Then, by Theorem 11.1,

$$\chi_q(S(\alpha)\otimes S(\beta))_{\leqslant 2} = Y^\alpha Y^\beta (1+v_1)(1+v_1+v_3+v_4+v_1v_3+v_1v_4+v_3v_4+v_1v_3v_4+v_1v_2v_3v_4).$$

The only dominant monomials in this product are $Y^\alpha Y^\beta$ and $Y^\alpha Y^\beta v_1v_2v_3v_4$, hence it is enough to show that $Y^\alpha Y^\beta v_1v_2v_3v_4$ occurs in $\chi_q(L(Y^\alpha Y^\beta))$. Now $Y^\alpha Y^\beta = Y_{1,0}^2 Y_{2,3}^3 Y_{3,0} Y_{4,0}$, and clearly $m := Y^\alpha Y^\beta v_4 = Y_{1,0}^2 Y_{2,1} Y_{2,3}^3 Y_{3,0} Y_{4,2}^{-1}$ occurs in $\chi_q(L(Y^\alpha Y^\beta))$. Since m does not contain v_1, v_2, v_3, we know by 5.4 that $\varphi_{\{1,2,3\}}(m)$ is contained in $\chi_q(L(Y^\alpha Y^\beta))$. To calculate $\varphi_{\{1,2,3\}}(m)$, we write $m = Y^\gamma (Y_{2,1}Y_{2,3})Y_{4,2}^{-1}$ where $\gamma = 2\alpha_1 + 2\alpha_2 + \alpha_3$ is in the root lattice of type A_3 and has the cluster expansion $\gamma = (\alpha_1+\alpha_2) + (\alpha_1+\alpha_2+\alpha_3)$. It then follows from Section 10 that

$$\varphi_{\{1,2,3\}}(m) = m(1+v_1)(1+v_1+v_3+v_1v_3+v_1v_2v_3),$$

hence $mv_1v_2v_3 = Y^\alpha Y^\beta v_1v_2v_3v_4$ occurs in $\chi_q(L(Y^\alpha Y^\beta))$.

(c) Take $I_0 = \{2\}$. Then $\tau_-(\alpha) = \alpha_2 + \alpha_4$ and $\tau_-(\beta) = \alpha_1 + \alpha_2$, so

$$\chi_q(S(\alpha)\otimes S(\beta))_{\leqslant 2} = Y^\alpha Y^\beta (1+v_2+v_2v_4)(1+v_2+v_1v_2).$$

But by Section 10, this is equal to $\varphi_{\{1,2,4\}}(Y^\alpha Y^\beta)$, which is contained in $\chi_q(L(Y^\alpha Y^\beta))$.

(d) Take $I_0 = \{2\}$. Then $\tau_-(\alpha) = \alpha_2 + \alpha_4$ and $\tau_-(\beta) = \alpha_2$, so one can argue as in (c).

(e) Take $I_1 = \{2\}$. Then, by Theorem 11.1,

$$\chi_q(S(\alpha)\otimes S(\beta))_{\leqslant 2} = Y^\alpha Y^\beta (1+v_1+v_3+v_1v_3+v_1v_2v_3)$$
$$\times (1+v_1+v_3+v_4+v_1v_3+v_1v_4+v_3v_4+v_1v_3v_4+v_1v_2v_3v_4).$$

The only dominant monomials other than $Y^\alpha Y^\beta$ are $Y^\alpha Y^\beta v_1v_2v_3$, $Y^\alpha Y^\beta v_1v_2v_3v_4$ which occurs with coefficient 2, and $Y^\alpha Y^\beta v_1^2v_2^2v_3^2v_4$, hence it is enough to show that they occur in $\chi_q(L(Y^\alpha Y^\beta))$. For the first one, which does not depend on v_4, one can argue as in (c) or (d). Now $Y^\alpha Y^\beta = Y_{1,0}^2 Y_{2,3}^3 Y_{3,0}^2 Y_{4,0}$, and clearly $m := Y^\alpha Y^\beta v_4 = Y_{1,0}^2 Y_{2,1} Y_{2,3}^3 Y_{3,0}^2 Y_{4,2}^{-1}$ occurs in $\chi_q(L(Y^\alpha Y^\beta))$. Since m does not contain v_1, v_2, v_3, we know by 5.4 that $\varphi_{\{1,2,3\}}(m)$ is contained in $\chi_q(L(Y^\alpha Y^\beta))$. To calculate $\varphi_{\{1,2,3\}}(m)$, we write $m = Y^{\gamma}(Y_{2,1}Y_{2,3})Y_{4,2}^{-1}$ where $\gamma = 2\alpha_1 + 2\alpha_2 + 2\alpha_3$ is in the root lattice of type A_3 and has the cluster expansion $\gamma = 2(\alpha_1 + \alpha_2 + \alpha_3)$. It then follows from Section 10 that

$$\varphi_{\{1,2,3\}}(m) = m(1 + v_1 + v_3 + v_1v_3 + v_1v_2v_3)^2,$$

hence $mv_1v_2v_3 = Y^{\alpha}Y^{\beta}v_1v_2v_3v_4$ and $mv_1^2v_2^2v_3^2 = Y^{\alpha}Y^{\beta}v_1^2v_2^2v_3^2v_4$ occur in $\chi_q(L(Y^{\alpha}Y^{\beta}))$, the first one with coefficient 2.

(f) Take $I_0 = \{2\}$. Then $\chi_q(S(\alpha))_{\leqslant 2} = Y^{\alpha}(1 + v_2)$, and its square has only one dominant monomial, namely $(Y^{\alpha})^2$.

(g) Take $I_1 = \{2\}$. Then, by Theorem 11.1,

$$\chi_q(S(\alpha)^{\otimes 2})_{\leqslant 2} = (Y^{\alpha})^2(1 + v_1 + v_3 + v_4 + v_1v_3 + v_1v_4 + v_3v_4 + v_1v_3v_4 + v_1v_2v_3v_4)^2.$$

The only dominant monomials other than $(Y^{\alpha})^2$ are $(Y^{\alpha})^2v_1v_2v_3v_4$ which occurs with coefficient 2, and $(Y^{\alpha})^2v_1^2v_2^2v_3^2v_4^2$, hence it is enough to show that they occur in $\chi_q(L((Y^{\alpha})^2))$. Now $m_1 := (Y^{\alpha})^2v_4$ and $m_2 := (Y^{\alpha})^2v_4^2$ both occur in $\chi_q(L((Y^{\alpha})^2))$, the first one with coefficient 2, and both are $\{1,2,3\}$-dominant. Considering $\varphi_{\{1,2,3\}}(m_1)$ and $\varphi_{\{1,2,3\}}(m_2)$ we can conclude as in (e).

This finishes the proof of 9.2 (ii).

11.2.3 Finally 9.2 (iii) follows from Corollary 6.10 for all multiplicity-free roots. It remains to check it for the longest root $\alpha = \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$. This can be done easily using our explicit formulas for the truncated q-characters. For example we see from Lemma 11.2 that the monomial $Y_{1,3}Y_{2,0}^2Y_{3,3}Y_{4,3}v_2v_3v_4$ occurs in $\chi_q(L(Y_{1,3}Y_{2,0}))\chi_q(L(Y_{2,0}Y_{3,3}Y_{4,3}))$ but not in $\chi_q(S(\alpha))$. All other factorizations of $S(\alpha)$ can be ruled out in a similar way, and we omit the details.

This concludes the proof of Conjecture 4.6 in type D_4. □

12 Applications

In this section we present some interesting consequences of our results concerning $\mathscr{C}_1$.

12.1 By construction, the cluster variables of a cluster algebra satisfy some algebraic identities coming from the mutation procedure. When we restrict to mutations on the *bipartite belt* [**FZ5**] these identities are similar to a T-system. In the case $\ell = 1$ this T-system is periodic and involves the classes of all the cluster simple objects of $\mathscr{C}_1$, as we shall now see.

Define $\tau = \tau_+\tau_-$ (see 7.1). For every $i \in I$, define a sequence $\gamma_i(j)$ $(j \in \mathbb{Z})$ of elements of $\Phi_{\geqslant -1}$ by

$$\gamma_i(2j) = \tau^j(-\alpha_i); \tag{39}$$

for odd j, $\gamma_i(j)$ is defined via the parity condition

$$\gamma_i(j) = \gamma_i(j+1) \quad \text{if} \quad \varepsilon_i = (-1)^{j+1}. \tag{40}$$

It follows from [**FZ2**] that every element of $\Phi_{\geqslant -1}$ is of the form $\gamma_i(j)$ for some $i \in I$ and some $j \in [0, h+2]$. For every $i \in I$, define a sequence $\beta_i(j)$ $(j \in \mathbb{Z})$ of elements of $\Phi_{\geqslant -1}$ by the initial condition

$$\beta_i(0) = \alpha_i, \tag{41}$$

and the recursion

$$\beta_i(j+1) = \tau_{(-1)^{j+1}}(\beta_i(j)), \qquad (j \in \mathbb{Z}). \tag{42}$$

Define the following monomials in the classes of the frozen simple objects F_i:

$$p_i^+(j) = \prod_{k\in I}[F_k]^{\max(0,[\beta_k(j):\alpha_i])}, \qquad p_i^-(j) = \prod_{k\in I}[F_k]^{\max(0,-[\beta_k(j):\alpha_i])}, \qquad (i \in I,\ j \in \mathbb{Z}). \tag{43}$$

The next result follows from Conjecture 4.6 (i), hence it is now proved if $\mathfrak{g}$ is of type A_n or D_n.

Theorem 12.1 *The cluster simple objects $S(\alpha)$ $(\alpha \in \Phi_{\geqslant -1})$ of $\mathscr{C}_1$ satisfy the following system of equations in the Grothendieck ring R_1*

$$[S(\gamma_i(j+1))]\,[S(\gamma_i(j-1))] = p_i^+(j) + p_i^-(j)\prod_{k\neq i}[S(\gamma_k(j))]^{-a_{ik}}, \qquad (i \in I,\ j \in \mathbb{Z}).$$

Proof — Set

$$p_i(j) = \frac{p_i^+(j)}{p_i^-(j)} = \prod_{k\in I}[F_k]^{[\beta_k(j):\alpha_i]}, \tag{44}$$

a Laurent monomial in the $[F_i]$. We will first show that

$$p_i(j+1)p_i(j-1) = \prod_{k\neq i}(p_k^+(j))^{-a_{ik}}. \tag{45}$$

To do that, set

$$q_i(j) = \frac{\prod_{k\neq i}(p_k^+(j))^{-a_{ik}}}{p_i(j+1)p_i(j-1)}.$$

For $i \in I$, define following **[FZ2]** a piecewise-linear automorphism σ_i of Q by

$$[\sigma_i(\gamma):\alpha_j] = \begin{cases} [\gamma:\alpha_j] & \text{if } j \neq i, \\[2ex] -[\gamma:\alpha_i] - \displaystyle\sum_{k\neq i} a_{ik}\max(0,[\gamma:\alpha_k]) & \text{if } j = i. \end{cases} \tag{46}$$

Then $\tau_\varepsilon = \prod_{\varepsilon_i=\varepsilon}\sigma_i$. Using the definition of $p_i^\pm(j)$ one can see that the exponent of $[F_u]$ in $q_i(j)$ is equal to

$$\begin{aligned}&[\sigma_i(\beta_u(j)) + \beta_u(j) - \beta_u(j+1) - \beta_u(j-1) : \alpha_i] \\ &\qquad = [\sigma_i(\beta_u(j)) + \beta_u(j) - \tau_{(-1)^{j+1}}(\beta_u(j)) - \tau_{(-1)^j}(\beta_u(j)) : \alpha_i].\end{aligned}$$

Suppose that $\varepsilon_i = (-1)^{j+1}$. Then σ_i appears once in $\tau_{(-1)^{j+1}}$ and does not appear in $\tau_{(-1)^j}$. It follows that

$$[\tau_{(-1)^{j+1}}(\beta_u(j)) + \tau_{(-1)^j}(\beta_u(j)) : \alpha_i] = [\sigma_i(\beta_u(j)) + \beta_u(j) : \alpha_i],$$

hence, the exponent of $[F_u]$ in $q_i(j)$ is 0. The case $\varepsilon_i = (-1)^j$ is identical. Thus, $q_i(j) = 1$ and (45) is proved.

Set $y_i(j) = 1/p_i(j)$. Then, using the tropical semifield structure on the set of Laurent monomials in the $[F_i]$ (see 7.2), one has $1 \oplus y_i(j) = 1/p_i^+(j)$, hence

$$p_i^+(j) = \frac{1}{1\oplus y_i(j)}, \qquad p_i^-(j) = \frac{y_i(j)}{1\oplus y_i(j)}.$$

With this new notation, (45) becomes

$$y_i(j+1)y_i(j-1) = \prod_{k\neq i}(1\oplus y_k(j))^{-a_{ik}}, \tag{47}$$

and the relation of Theorem 12.1 takes the form

$$[S(\gamma_i(j+1))]\,[S(\gamma_i(j-1))] = \frac{1+y_i(j)\prod_{k\neq i}[S(\gamma_k(j))]^{-a_{ik}}}{1\oplus y_i(j)}. \tag{48}$$

Equations (47) and (48) coincide with formulas (8.11) and (8.12) of **[FZ5]**, which proves the theorem. □

Example 12.2 We take $\mathfrak{g}$ of type A_3 and choose $I_0=\{1,3\}$, $I_1=\{2\}$. Hence $\tau_+=\sigma_1\sigma_3$, $\tau_-=\sigma_2$, and $\tau=\sigma_1\sigma_3\sigma_2$. We have

$$\begin{array}{lll}
\gamma_1(0)=-\alpha_1, & \gamma_2(0)=-\alpha_2, & \gamma_3(0)=-\alpha_3,\\
\gamma_1(1)=\alpha_1, & \gamma_2(1)=-\alpha_2, & \gamma_3(1)=\alpha_3,\\
\gamma_1(2)=\alpha_1, & \gamma_2(2)=\alpha_1+\alpha_2+\alpha_3, & \gamma_3(2)=\alpha_3,\\
\gamma_1(3)=\alpha_2+\alpha_3, & \gamma_2(3)=\alpha_1+\alpha_2+\alpha_3, & \gamma_3(3)=\alpha_1+\alpha_2,\\
\gamma_1(4)=\alpha_2+\alpha_3, & \gamma_2(4)=\alpha_2, & \gamma_3(4)=\alpha_1+\alpha_2,\\
\gamma_1(5)=-\alpha_3, & \gamma_2(5)=\alpha_2, & \gamma_3(5)=-\alpha_1,\\
\gamma_1(6)=-\alpha_3, & \gamma_2(6)=-\alpha_2, & \gamma_3(6)=-\alpha_1,
\end{array}$$

and

$$\begin{array}{lll}
\beta_1(0)=\alpha_1, & \beta_2(0)=\alpha_2, & \beta_3(0)=\alpha_3,\\
\beta_1(1)=\alpha_1+\alpha_2, & \beta_2(1)=-\alpha_2, & \beta_3(1)=\alpha_2+\alpha_3,\\
\beta_1(2)=\alpha_2+\alpha_3, & \beta_2(2)=-\alpha_2, & \beta_3(2)=\alpha_1+\alpha_2,\\
\beta_1(3)=\alpha_3, & \beta_2(3)=\alpha_2, & \beta_3(3)=\alpha_1,\\
\beta_1(4)=-\alpha_3, & \beta_2(4)=\alpha_1+\alpha_2+\alpha_3, & \beta_3(4)=-\alpha_1,\\
\beta_1(5)=-\alpha_3, & \beta_2(5)=\alpha_1+\alpha_2+\alpha_3, & \beta_3(5)=-\alpha_1,\\
\beta_1(6)=\alpha_3, & \beta_2(6)=\alpha_2, & \beta_3(6)=\alpha_1.
\end{array}$$

The formulas of Theorem 12.1 read for $i=1,3$:

$$\begin{aligned}
&[S(\alpha_1)]\,[S(-\alpha_1)] = [F_1]+[S(-\alpha_2)],\\
&[S(\alpha_2+\alpha_3)]\,[S(\alpha_1)] = [F_3]+[S(\alpha_1+\alpha_2+\alpha_3)],\\
&[S(-\alpha_3)]\,[S(\alpha_2+\alpha_3)] = [F_2]+[F_3]\,[S(\alpha_2)],\\
&[S(\alpha_3)]\,[S(-\alpha_3)] = [F_3]+[S(-\alpha_2)],\\
&[S(\alpha_1+\alpha_2)]\,[S(\alpha_3)] = [F_1]+[S(\alpha_1+\alpha_2+\alpha_3)],\\
&[S(-\alpha_1)]\,[S(\alpha_1+\alpha_2)] = [F_2]+[F_1]\,[S(\alpha_2)].
\end{aligned}$$

and for $i=2$:

$$\begin{aligned}
&[S(\alpha_1+\alpha_2+\alpha_3)]\,[S(-\alpha_2)] = [F_1][F_3]+[F_2]\,[S(\alpha_1)]\,[S(\alpha_3)],\\
&[S(\alpha_2)]\,[S(\alpha_1+\alpha_2+\alpha_3)] = [F_2]+[S(\alpha_1+\alpha_2)]\,[S(\alpha_2+\alpha_3)],\\
&[S(-\alpha_2)]\,[S(\alpha_2)] = [F_2]+[S(-\alpha_1)]\,[S(-\alpha_3)].
\end{aligned}$$

Remark 12.3 In type E_n, using Proposition 6.7, we can still prove identities involving only multiplicity-free roots. For example, one can show that for every $i \in I$,

$$\begin{aligned}
[S(\alpha_i)][S(-\alpha_i)] &= [F_i] + \prod_{j\neq i}[S(-\alpha_j)]^{-a_{ij}}, \\
[S(\alpha_i - \textstyle\sum_{j\neq i} a_{ij}\alpha_j)][S(-\alpha_i)] &= \prod_{j\neq i}[F_j]^{-a_{ij}} + [F_i]\prod_{j\neq i}[S(\alpha_j)]^{-a_{ij}}.
\end{aligned}$$

The first formula is a classical T-system, but not the second one.

12.2 If $\mathfrak{g}$ is of type A_n, we have the following well-known duality for the characters of the finite-dimensional irreducible $\mathfrak{g}$-modules. If $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$ are two dominant weights, identified in the standard way with partitions in at most n parts, there holds

$$\dim V(\lambda)_\mu = [V(\mu_1) \otimes \cdots \otimes V(\mu_n) \,:\, V(\lambda)],$$

where $V(\lambda)$ stands for the irreducible $\mathfrak{g}$-module with highest weight λ. In plain words, the weight multiplicities of $V(\lambda)$ coincide with the multiplicities of $V(\lambda)$ as a direct summand of certain tensor products. This is sometimes called *Kostka duality* and is specific to type A.

We find it interesting to note that Eq. (34) yields a similar duality for the truncated q-characters of the cluster simple objects $S(\beta)$ for any $\mathfrak{g}$. Indeed assume that (34) holds, namely that

$$\widetilde{\chi}_q(S(\beta))_{\leqslant 2} \;=\; F_{\tau_-(\beta)}(v_1, \ldots, v_n), \qquad (v_i = A^{-1}_{i,\xi_i+1}),$$

(this is proved in type A and D and for all multiplicity-free roots in type E). Then writing for short $\alpha = \tau_-(\beta) = \sum_i a_i\alpha_i$, $\mathbf{v}^\gamma = \prod_{i\in I} v_i^{c_i}$ for $\gamma = \sum_i c_i\alpha_i \in Q$, and

$$F_\alpha(v_1, \ldots, v_n) = \sum_\gamma n_{\beta,\gamma}\mathbf{v}^\gamma,$$

we have that $n_{\beta,\gamma}$ is the multiplicity of the l-weight $Y^\beta\mathbf{v}^\gamma$ in $S(\beta)$. On the other hand, define

$$T(\alpha) := \bigotimes_{i\in I} S(-\varepsilon_i\alpha_i)^{\otimes a_i}.$$

Note that $\chi_q(T(\alpha))_{\leqslant 2} = \prod_i Y^{a_i}_{i,\xi_i+2}$ is reduced to a single monomial, so $T(\alpha)$ is simple. (From the cluster algebra point of view, $(x[-\varepsilon_i\alpha_i]; i \in I)$ is a cluster of $\mathscr{A}$.) Put

$$d_i = \begin{cases} -\displaystyle\sum_{j\neq i} c_j a_{ij} & \text{if } i \in I_0, \\ \displaystyle\sum_{j\neq i}(c_j - a_j)a_{ij} & \text{if } i \in I_1, \end{cases} \qquad e_i = \begin{cases} a_i - c_i & \text{if } i \in I_0, \\ -c_i - \displaystyle\sum_{j\neq i} c_j a_{ij} & \text{if } i \in I_1, \end{cases}.$$

If $n_{\beta,\gamma} \neq 0$ then d_i and e_i are nonnegative and we can consider the simple module

$$U(\gamma) := \bigotimes_{i\in I} S(-\varepsilon_i\alpha_i)^{\otimes d_i} \otimes F_i^{\otimes e_i}.$$

Proposition 12.4 *Assume that Eq. (34) holds for $S(\beta)$. Then the multiplicity of $U(\gamma)$ as a composition factor of the tensor product $S(\beta) \otimes T(\alpha)$ is equal to the l-weight multiplicity $n_{\beta,\gamma}$.*

Proof — This is a direct calculation using Eq. (27) and the same idea as in the proof of Proposition 2.2. The details are left to the reader. ☐

12.3 So far we have only used some combinatorial and representation-theoretical techniques. However, our results have geometric consequences. Indeed, by work of Fu and Keller [**FK**], the F-polynomials have nice geometric descriptions in terms of quiver grassmannians. This goes as follows.

Let C be the oriented graph obtained from the Dynkin diagram of $\mathfrak{g}$ by deciding that the vertices in I_1 are sources and those in I_0 are sinks. So C is a Dynkin quiver, and we can associate to every positive root α the unique (up to isomorphism) indecomposable representation $M[\alpha]$ of C over $\mathbb{C}$ with dimension vector α. Regarding an element $\gamma = \sum_i c_i\alpha_i$ of the root lattice with nonnegative coordinates c_i as a dimension vector for C, we can consider for every representation M of C the quiver grassmannian

$$\mathrm{Gr}_\gamma(M) := \{N \mid N \text{ is a subrepresentation of } M \text{ with dimension } \gamma\}.$$

This is a closed subset of the ordinary grassmannian of subspaces of dimension $\sum_i c_i$ of the complex vector space M. So in particular, $\mathrm{Gr}_\gamma(M)$ is a projective variety. Denote by $\chi(\mathrm{Gr}_\gamma(M))$ its topological Euler characteristic. Then we have the following formula, inspired from a similar formula of Caldero and Chapoton for cluster expansions of cluster variables [**CC**].

Theorem 12.5 [**FK**, Th. 6.5] *For $\alpha \in \Phi_{>0}$ we have*

$$F_\alpha(v_1,\ldots,v_n) = \sum_\gamma \chi(\mathrm{Gr}_\gamma(M[\alpha]))\,\mathbf{v}^\gamma.$$

This yields immediately

Theorem 12.6 *If Eq. (34) holds for $\beta \in \Phi_{\geqslant -1}$, then*

$$\chi_q(S(\beta))_{\leqslant 2} = Y^\beta \sum_\gamma \chi(\mathrm{Gr}_\gamma(M[\tau_-(\beta)]))\,\mathbf{v}^\gamma, \qquad (v_i = A^{-1}_{i,\xi_i+1}). \tag{49}$$

Example 12.7 Take $\mathfrak{g}$ of type D_4 and choose $I_0 = \{2\}$, so that C is the quiver of type D_4 with its three arrows pointing to the trivalent node 2. Let $\beta = \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$ be the highest root. We have $\tau_-(\beta) = \beta$. The representation $M[\beta]$ of C is of dimension 5. There are thirteen non-empty quiver grassmannians corresponding to the dimension vectors

$$(0,0,0,0),\ (0,1,0,0),\ (0,2,0,0),\ (1,1,0,0),\ (0,1,1,0),\ (0,1,0,1),\ (1,2,0,0),$$

$$(0,2,1,0),\ (0,2,0,1),\ (1,2,1,0),\ (1,2,0,1),\ (0,2,1,1),\ (1,2,1,1).$$

The variety $\mathrm{Gr}_{(0,1,0,0)}(M[\beta])$ is a projective line, hence its Euler characteristic is equal to 2. The twelve other grassmannians are reduced to a point. Therefore we obtain that

$$\chi_q(S(\beta))_{\leqslant 2} = Y_{1,3}Y_{2,0}^2Y_{3,3}Y_{4,3}\left(1 + 2v_2 + v_2^2 + v_1v_2 + v_2v_3 + v_2v_4 + v_1v_2^2 + v_2^2v_3 + v_2^2v_4\right.$$
$$\left. + \, v_1v_2^2v_3 + v_1v_2^2v_4 + v_2^2v_3v_4 + v_1v_2^2v_3v_4\right),$$

in agreement with Lemma 11.2.

Note that if moreover Conjecture 4.6 (ii) holds, then we can write any simple module $L(m)$ in $\mathscr{C}_1$ as a tensor product of cluster simple objects. Taking into account the additivity properties of the Euler characteristics and the results of [**CK**], this gives for $\chi_q(L(m))_{\leqslant 2}$ a formula similar to (49), in which the indecomposable representation $M[\tau_-(\beta)]$ is replaced by a generic representation of C (or equivalently a representation without self-extension).

Example 12.8 Take $\mathfrak{g}$ of type A_2 and choose $I_0 = \{1\}$, so that C is the quiver $1 \longleftarrow 2$. Consider the simple module $S = L(Y_{1,0}^2 Y_{2,3})$. We have seen that $S \cong L(Y_{1,0}Y_{2,3}) \otimes L(Y_{1,0})$. This corresponds to the fact that the generic representation of C of dimension vector $2\alpha_1 + \alpha_2$ is

$$M = (\mathbb{C} \xleftarrow{\mathrm{id}} \mathbb{C}) \oplus (\mathbb{C} \xleftarrow{0} 0).$$

There are five non-empty quiver grassmannians for M corresponding to the dimension vectors

$$(0,0), \quad (1,0), \quad (2,0), \quad (1,1), \quad (2,1).$$

The variety $\mathrm{Gr}_{(1,0)}(M)$ is a projective line, hence its Euler characteristic is equal to 2. The four other grassmannians are reduced to a point. Therefore we obtain that

$$\chi_q(S)_{\leqslant 2} \ = Y_{1,0}^2 Y_{2,3} \left(1 + 2v_1 + v_1^2 + v_1 v_2 + v_1^2 v_2\right),$$

in agreement with Example 6.3.

Theorem 12.6 is very similar to a formula of Nakajima [**N1**, §13] for the q-character of a standard module. Indeed, as shown by Lusztig [**Lu1**], the lagrangian quiver varieties used in Nakajima's character formula are isomorphic to grassmannians of submodules of a projective module over a preprojective algebra. There are however two important differences. In our case the geometric formula gives only the truncated q-character (but this is enough to determine the full q-character of an object of $\mathscr{C}_1$). More importantly, Theorem 12.6 concerns *simple* modules and not *standard* modules. In Nakajima's approach, the q-characters of the simple modules are obtained as alternating sums of q-characters of standard modules using intersection cohomology methods.

13 General ℓ

We now consider the category $\mathscr{C}_\ell$ for an arbitrary integer ℓ.

13.1 We define a quiver Γ_ℓ with vertex set $\{(i,k) \mid i \in I,\ 1 \leqslant k \leqslant \ell+1\}$. The arrows of Γ_ℓ are given by the following rule. Suppose that (i,k) is such that $i \in I_0$ and k is odd, or $i \in I_1$ and k is even. Then the arrows adjacent to (i,k) are

(h) the horizontal arrows $(i,k-1) \to (i,k)$ if $k > 1$ and $(i,k+1) \to (i,k)$ if $k \leqslant \ell$;

(v) the vertical arrows $(i,k) \to (j,k)$ where $a_{ij} = -1$ and $k \leqslant \ell$.

All arrows are of this type.

Example 13.1 Take $\mathfrak{g}$ of type A_3 and choose $I_0 = \{1,3\}$ and $I_1 = \{2\}$. The quiver Γ_3 is then

$$\begin{array}{ccccccc}
(1,1) & \leftarrow & (1,2) & \rightarrow & (1,3) & \leftarrow & (1,4) \\
\downarrow & & \uparrow & & \downarrow & & \\
(2,1) & \rightarrow & (2,2) & \leftarrow & (2,3) & \rightarrow & (2,4) \\
\uparrow & & \downarrow & & \uparrow & & \\
(3,1) & \leftarrow & (3,2) & \rightarrow & (3,3) & \leftarrow & (3,4)
\end{array}$$

Type of $\mathfrak{g}$	ℓ	Type of $\mathscr{A}_\ell$
A_1	ℓ	A_ℓ
X_n	1	X_n
A_2	2	D_4
A_2	3	E_6
A_2	4	E_8
A_3	2	E_6
A_4	2	E_8

Table 1: *Algebras $\mathscr{A}_\ell$ of finite cluster type.*

13.2 Let $\widetilde{B}_\ell$ be the $n(\ell+1)\times n\ell$-matrix with set of column indices $I\times[1,\ell]$ and set of row indices $I\times[1,\ell+1]$. The entry $b_{(i,k),(j,m)}$ is equal to 1 if there is an arrow from (j,m) to (i,k) in Γ_ℓ, to -1 if there is an arrow from (i,k) to (j,m), and to 0 otherwise.

Let $\mathscr{A}_\ell = \mathscr{A}(\widetilde{B}_\ell)$ be the cluster algebra attached to the initial seed $(\mathbf{x},\widetilde{B}_\ell)$, where

$$\mathbf{x} = (x_{(i,k)} \mid i\in I,\ 1\leqslant k\leqslant \ell+1).$$

This is a cluster algebra of rank $n\ell$, with n frozen variables $f_i := x_{(i,\ell+1)}$ $(i\in I)$. It follows easily from **[FZ3]** that $\mathscr{A}_\ell$ has in general infinitely many cluster variables. The exceptional pairs $(\mathfrak{g},\ell)$ for which $\mathscr{A}_\ell$ has finite cluster type are listed in Table 1.

13.3 For $i\in I$ and $k\in[1,\ell+1]$, define

$$r(i,k) = \begin{cases} 2\left\lceil \dfrac{\ell-k+1}{2}\right\rceil & \text{if } i\in I_0,\\[2ex] 2\left\lceil \dfrac{\ell-k+2}{2}\right\rceil - 1 & \text{if } i\in I_1,\end{cases}$$

where $\lceil x\rceil$ denotes the smallest integer $\geqslant x$. These integers satisfy

(a) $r(i,k)\geqslant r(i,k+1)\geqslant r(i,k+2) = r(i,k)-2$,

(b) if $a_{ij}=-1$ then $r(j,k)$ is the unique integer strictly between $r(i,k)$ and $r(i,k)+2(-1)^k\varepsilon_i$.

Recall from 3.10 the Kirillov-Reshetikhin modules $W^{(i)}_{k,a}$ $(i\in I,\ k\in\mathbb{N},\ a\in\mathbb{C}^*)$. The Kirillov-Reshetikhin modules in $\mathscr{C}_\ell$ have spectral parameters of the form $a=q^r$ for some integer r between 0 and $\ell+1$. To simplify notation *we shall write $W^{(i)}_{k,r}$ instead of $W^{(i)}_{k,q^r}$*. We can now state our main conjecture, which generalizes Conjecture 4.6 to arbitrary ℓ.

Conjecture 13.2 *The map $x_{(i,k)}\mapsto [W^{(i)}_{k,r(i,k)}]$ extends to a ring isomorphism $\imath$ from the cluster algebra $\mathscr{A}_\ell$ to the Grothendieck ring R_ℓ of $\mathscr{C}_\ell$. If we identify $\mathscr{A}_\ell$ with R_ℓ via $\imath$, $\mathscr{C}_\ell$ becomes a monoidal categorification of $\mathscr{A}_\ell$.*

The idea to choose this initial seed for $\mathscr{A}_\ell$ comes from the T-systems. Indeed, after replacing $x_{(i,k)}$ by $[W^{(i)}_{k,r(i,k)}]$, the exchange relations (2) for the initial cluster variables become,

$$[W^{(i)}_{k,r(i,k)+2(-1)^k\varepsilon_i}] = \frac{[W^{(i)}_{k-1,r(i,k-1)}][W^{(i)}_{k+1,r(i,k+1)}] + \prod_{a_{ij}=-1}[W^{(j)}_{k,r(j,k)}]}{[W^{(i)}_{k,r(i,k)}]} \qquad (i\in I,\ k\leqslant \ell),$$

which is an instance of Eq. (5).

13.4 It follows from the work of Chari and Pressley [**CP2**] that Conjecture 13.2 holds for $\mathfrak{g} = \mathfrak{sl}_2$ and any ℓ (see Example 5.2). In this case $\mathscr{A}_\ell \equiv R_\ell$ is a cluster algebra of finite cluster type A_ℓ, with one frozen variable $[W_{\ell+1,0}]$. The cluster variables are the classes of the other Kirillov-Reshetikhin modules of $\mathscr{C}_\ell$, namely

$$[W_{k,2s}], \qquad (1 \leqslant k \leqslant \ell, \quad 0 \leqslant s \leqslant \ell - k + 1).$$

To determine the compatible pairs of cluster variables, one can use again the geometric model of [**FZ2**] and attach to each cluster variable a diagonal of the $(\ell+3)$-gon $\mathscr{P}_{\ell+3}$ as follows:

$$[W_{k,2s}] \longmapsto [s+1,\, s+k+2].$$

We then have that $W_{k,2s} \otimes W_{k',2s'}$ is simple if and only if the corresponding diagonals do not intersect in the interior of $\mathscr{P}_{\ell+3}$.

13.5 Take $\mathfrak{g}$ of type A_2 and $\ell = 2$. We choose $I_0 = \{1\}$ and $I_1 = \{2\}$. In this case Conjecture 13.2 holds (see below 13.8). The cluster algebra $\mathscr{A}_2$ has finite cluster type D_4, hence every simple object of $\mathscr{C}_2$ is isomorphic to a tensor product of cluster simple objects and frozen simple objects.

The sixteen cluster simple objects are

$$L(Y_{1,0}),\, L(Y_{1,2}),\, L(Y_{1,4}),\, L(Y_{2,1}),\, L(Y_{2,3}),\, L(Y_{2,5}),\, L(Y_{1,0}Y_{1,2}),\, L(Y_{1,2}Y_{1,4}),\, L(Y_{2,1}Y_{2,3}),\, L(Y_{2,3}Y_{2,5}),$$

$$L(Y_{1,0}Y_{2,3}),\, L(Y_{1,2}Y_{2,5}),\, L(Y_{1,4}Y_{2,1}),\, L(Y_{1,0}Y_{1,2}Y_{2,5}),\, L(Y_{1,0}Y_{2,3}Y_{2,5}),\, L(Y_{1,0}Y_{1,2}Y_{2,3}Y_{2,5}).$$

They have respective dimensions

$$3,\ 3,\ 3,\ 3,\ 3,\ 3,\ 6,\ 6,\ 6,\ 6,\ 8,\ 8,\ 8,\ 15,\ 15,\ 35.$$

Note that only the first ten are Kirillov-Reshetikhin modules. The next five are evaluation modules. The last module $L(Y_{1,0}Y_{1,2}Y_{2,3}Y_{2,5})$ is not an evaluation module. Its restriction to $U_q(\mathfrak{g})$ is isomorphic to $V(2\varpi_1 + 2\varpi_2) \oplus V(\varpi_1 + \varpi_2)$.

The two frozen modules are the Kirillov-Reshetikhin modules $L(Y_{1,0}Y_{1,2}Y_{1,4})$, $L(Y_{2,1}Y_{2,3}Y_{2,5})$, of dimension 10.

By [**FZ3**] there are fifty clusters, *i.e.* fifty factorization patterns of a simple object of $\mathscr{C}_2$ as a tensor product of cluster simple objects and frozen simple objects.

13.6 Take $\mathfrak{g}$ of type A_4 and $\ell = 3$. In this case $\mathscr{A}_3$ has infinitely many cluster variables. Choose $I_0 = \{1,3\}$ and $I_1 = \{2,4\}$. Thus the simple module $L(Y_{1,4}Y_{2,1}Y_{2,7}Y_{3,4})$ belongs to $\mathscr{C}_3$. However, it is not a real simple object [**L**, §4.3] because in the Grothendieck ring R_3 we have

$$[L(Y_{1,4}Y_{2,1}Y_{2,7}Y_{3,4})]^2 = [L(Y_{1,4}^2Y_{2,1}^2Y_{2,7}^2Y_{3,4}^2)] + [L(Y_{2,1}Y_{2,3}Y_{2,5}Y_{2,7}Y_{4,3}Y_{4,5})].$$

If $\mathscr{C}_3$ is indeed a monoidal categorification of $\mathscr{A}_3$, then $[S]$ cannot be a cluster monomial.

For $\mathfrak{g}$ of type A_3 and $\ell = 3$, there is a similar example. The simple module $L(Y_{1,4}Y_{2,1}Y_{2,7}Y_{3,4})$ belongs to $\mathscr{C}_3$, and we have

$$[L(Y_{1,4}Y_{2,1}Y_{2,7}Y_{3,4})]^2 = [L(Y_{1,4}^2Y_{2,1}^2Y_{2,7}^2Y_{3,4}^2)] + [L(Y_{2,1}Y_{2,3}Y_{2,5}Y_{2,7})].$$

We expect the existence of non real simple objects in $\mathscr{C}_\ell$ whenever $\mathscr{A}_\ell$ does not have finite cluster type.

13.7 Take $\mathfrak{g}$ of type A_n and let ℓ be arbitrary. Let us sketch why in this case Conjecture 13.2 would essentially follow from a conjecture of [**GLS2**] (see also [**GLS4**, §23.1]).

First, we can use the quantum affine analogue of the Schur-Weyl duality [**CP5**, **Che**, **GRV**] to relate the finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$ with the finite-dimensional representations of the affine Hecke algebras $\widehat{H}_m(t)$ of type A_m $(m \geqslant 1)$ with parameter $t = q^2$. More precisely, for every m we have a functor $\mathscr{F}_m$ from $\mathrm{mod}\,\widehat{H}_m(t)$ to the category $\mathscr{C}$ of finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$, which maps every simple module of $\widehat{H}_m(t)$ to a simple module of $U_q(\widehat{\mathfrak{g}})$ or to the zero module. The simple $U_q(\widehat{\mathfrak{g}})$-module with highest l-weight $\prod_{i=1}^{n}\prod_{r=1}^{k_i} Y_{i,a_i}$ is the image by $\mathscr{F}_m$ of a simple $\widehat{H}_m(t)$-module, where $m = \sum_i ik_i$.

Moreover, the functors $\mathscr{F}_m$ are multiplicative in the following sense: for M_1 in $\mathrm{mod}\,\widehat{H}_{m_1}(t)$ and M_2 in $\mathrm{mod}\,\widehat{H}_{m_2}(t)$ one has

$$\mathscr{F}_{m_1+m_2}(M_1 \odot M_2) = \mathscr{F}_{m_1}(M_1) \otimes \mathscr{F}_{m_2}(M_2)\,,$$

where $- \odot -$ denotes the induction product from $\mathrm{mod}\,\widehat{H}_{m_1}(t) \times \mathrm{mod}\,\widehat{H}_{m_2}(t)$ to $\mathrm{mod}\,\widehat{H}_{m_1+m_2}(t)$. Let $\mathscr{R}$ be the sum over m of the Grothendieck groups of the categories $\mathrm{mod}\,\widehat{H}_m(t)$ endowed with the multiplication induced by $\odot$. The functors $\mathscr{F}_m$ thus induce a surjective ring homomorphism $\Psi : \mathscr{R} \to R$, which maps classes of simples to classes of simples or to zero.

Let $\mathscr{D}_{m,\ell}$ denote the full subcategory of $\mathrm{mod}\,\widehat{H}_m(t)$ whose objects are those modules on which the generators $y_1,\ldots,y_m$ of the maximal commutative subalgebra of $\widehat{H}_m(t)$ have all their eigenvalues in

$$\left\{ t^k \mid k \in \mathbb{Z},\ \frac{1-n}{2} \leqslant k \leqslant \frac{n}{2} + \ell \right\},$$

(see [**L**]). It is easy to check that every simple object of $\mathscr{C}_\ell$ is of the form $\mathscr{F}_m(M)$ for some m and some simple object M of $\mathscr{D}_{m,\ell}$. Therefore, denoting by $\mathscr{R}_\ell$ the sum over m of the Grothendieck groups of the categories $\mathscr{D}_{m,\ell}$, we see that Ψ restricts to a surjective ring homomorphism from $\mathscr{R}_\ell$ to R_ℓ.

By a dual version of Ariki's theorem [**A**, **LNT**] the $\mathbb{Z}$-basis of $\mathscr{R}_\ell$ given by the classes of the simple objects can be identified with the dual canonical basis of the coordinate ring $\mathbb{C}[N]$ of a maximal unipotent subgroup N of $SL_{n+\ell+1}(\mathbb{C})$.

So, to summarize, for $\mathfrak{g}$ of type A_n, Conjecture 13.2 can be reformulated as a conjecture about multiplicative properties of the dual canonical basis of $\mathbb{C}[N]$. In [**GLS2**], a cluster algebra structure on $\mathbb{C}[N]$ has been studied in relation with the representation theory of preprojective algebras. It was shown that the cluster monomials belong to the dual of Lusztig's *semicanonical basis* of $\mathbb{C}[N]$ [**Lu2**]. More precisely they are the elements parametrized by the irreducible components of the nilpotent varieties with an open orbit. It was also conjectured that these elements of the dual semicanonical basis belong to the dual canonical basis, hence, by Ariki's theorem, are classes of irreducible representations of some $\widehat{H}_m$.

Finally, one can check that the initial seed of the cluster algebra $\mathscr{A}_\ell$ given by Conjecture 13.2 is the image under Ψ of a seed of $\mathbb{C}[N] \equiv \mathscr{R}_\ell$. So if the conjecture of [**GLS2**] was proved, by applying Ψ we would deduce that all cluster monomials of $\mathscr{A}_\ell$ are classes of simple objects of $\mathscr{C}_\ell$. To finish the proof of Conjecture 13.2 one would still have to explain why *all classes of real simple objects* are cluster monomials.

13.8 In [**GLS1**] it was shown that if N is of type A_r $(r \leqslant 4)$ the dual canonical and dual semicanonical basis of $\mathbb{C}[N]$ coincide. Moreover these are the only cases for which $\mathbb{C}[N]$ has finite cluster type. It then follows from 13.7 that Conjecture 13.2 holds if $n + \ell \leqslant 4$, and moreover in this case all simple objects of $\mathscr{C}_\ell$ are real. This proves the conjecture for $\mathfrak{g}$ of type A_2 and $\ell = 2$ (see 13.5).

13.9 For $\mathfrak{g}$ of type A_n, there is an interesting relation between the cluster algebra $\mathscr{A}_\ell$ and the grassmannian $\mathrm{Gr}(n+1, n+\ell+2)$ of $(n+1)$-dimensional subspaces of $\mathbb{C}^{n+\ell+2}$. Indeed, the homogeneous coordinate ring $\mathbb{C}[\mathrm{Gr}(n+1, n+\ell+2)]$ has a cluster algebra structure [**S**] with an initial seed given by a similar rectangular lattice (see also [**GSV**, **GLS3**]). More precisely, denote the Plücker coordinates of $\mathbb{C}[\mathrm{Gr}(n+1, n+\ell+2)]$ by

$$[i_1, \ldots, i_{n+1}], \qquad (1 \leqslant i_1 < \cdots < i_{n+1} \leqslant n+\ell+2).$$

The $\ell+2$ Plücker coordinates

$$[1,2,\ldots,n+1],\ [2,3,\ldots,n+2],\ \ldots,\ [\ell+2,\ell+3,\ldots,n+\ell+2],$$

belong to the subset of frozen variables of the cluster algebra $\mathbb{C}[\mathrm{Gr}(n+1, n+\ell+2)]$. Hence, the quotient ring S_ℓ of $\mathbb{C}[\mathrm{Gr}(n+1, n+\ell+2)]$ obtained by specializing these variables to 1 is also a cluster algebra, with the same principal part. By comparing the initial seed of $\mathscr{A}_\ell$ with the initial seed of S_ℓ obtained from [**S**, §4,§5], we see immediately that these two cluster algebras are isomorphic.

So we can reformulate Conjecture 13.2 for $\mathfrak{g} = \mathfrak{sl}_{n+1}$ by stating that *$\mathscr{C}_\ell$ should be a monoidal categorification of the quotient ring S_ℓ of $\mathbb{C}[\mathrm{Gr}(n+1, n+\ell+2)]$ by the relations*

$$[1,2,\ldots,n+1] = [2,3,\ldots,n+2] = \cdots = [\ell+2,\ell+3,\ldots,n+\ell+2] = 1.$$

Note that for $\mathfrak{g} = \mathfrak{sl}_2$ we recover the situation of §13.4. Note also that [**GLS3**] provides an *additive* categorification of the cluster algebra $\mathbb{C}[\mathrm{Gr}(n+1, n+\ell+2)]$, as a Frobenius subcategory of the module category of a preprojective algebra of type $A_{n+\ell+1}$.

Example 13.3 Take $n = 2$ and $\ell = 2$ (see §13.5). In this case S_2 is the ring obtained from $\mathbb{C}[\mathrm{Gr}(3,6)]$ by quotienting the following relations

$$[1,2,3] = [2,3,4] = [3,4,5] = [4,5,6] = 1.$$

For simplicity, we denote again by $[i,j,k]$ the image of the Plücker coordinate in the quotient S_2. Then the identification of the Grothendieck ring R_2 with S_2 gives the following identities of cluster (and frozen) variables:

$$\begin{array}{lll}
[L(Y_{1,0})] = [3,4,6], & [L(Y_{1,2})] = [2,3,5], & [L(Y_{1,4})] = [1,2,4], \\
[L(Y_{2,1})] = [3,5,6], & [L(Y_{2,3})] = [2,4,5], & [L(Y_{2,5})] = [1,3,4], \\
[L(Y_{1,0}Y_{1,2})] = [2,3,6], & [L(Y_{1,2}Y_{1,4})] = [1,2,5], & [L(Y_{1,0}Y_{1,2}Y_{1,4})] = [1,2,6], \\
[L(Y_{2,1}Y_{2,3})] = [2,5,6], & [L(Y_{2,3}Y_{2,5})] = [1,4,5], & [L(Y_{2,1}Y_{2,3}Y_{2,5})] = [1,5,6], \\
[L(Y_{1,0}Y_{2,3})] = [2,4,6], & [L(Y_{1,2}Y_{2,5})] = [1,3,5], & [L(Y_{1,4}Y_{2,1})] = [1,3,4][2,5,6] - [1,5,6], \\
[L(Y_{1,0}Y_{1,2}Y_{2,5})] = [1,3,6], & [L(Y_{1,0}Y_{2,3}Y_{2,5})] = [1,4,6], & [L(Y_{1,0}Y_{1,2}Y_{2,3}Y_{2,5})] = [2,3,6][1,4,5] - 1.
\end{array}$$

Moreover, as is easily checked, the dimension of a simple module in $\mathscr{C}_2$ is obtained by evaluating the corresponding cluster monomial in S_2 on the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 3 & 6 & 10 \end{bmatrix}$$

Thus,

$$\dim L(Y_{10}Y_{12}Y_{23}Y_{25}) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 0 & 1 & 10 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ 0 & 3 & 4 \\ 0 & 3 & 6 \end{vmatrix} - 1 = 35.$$

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