DRINFELD COPRODUCT, QUANTUM FUSION TENSOR CATEGORY AND APPLICATIONS

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Abstract. The class of quantum affinizations (or quantum loop algebras, see [Dr2, CP3, GKV, VV2, Mi1, N1, Jin, H3]) includes quantum affine algebras and quantum toroidal algebras. In general they have no Hopf algebra structure, but have a "coproduct" (the Drinfeld coproduct) which does not produce tensor products of modules in the usual way because it is defined in a completion. In this paper we propose a new process to produce quantum fusion modules from it : for all quantum affinizations, we construct by deformation and renormalization a new (non semi-simple) tensor category Mod. For quantum affine algebras this process is new and different from the usual tensor product. For general quantum affinizations, for example for toroidal algebras, so far, no process to produce fusion modules was known. We derive several applications from it : we construct the fusion of (finitely many) arbitrary l-highest weight modules, and prove that it is always cyclic. We establish exact sequences involving fusion of Kirillov-Reshetikhin modules related to new T-systems extending results of [N4, N3, H5]. Eventually for a large class of quantum affinizations we prove that the subcategory of finite length modules of Mod is stable under the new monoidal bifunctor.

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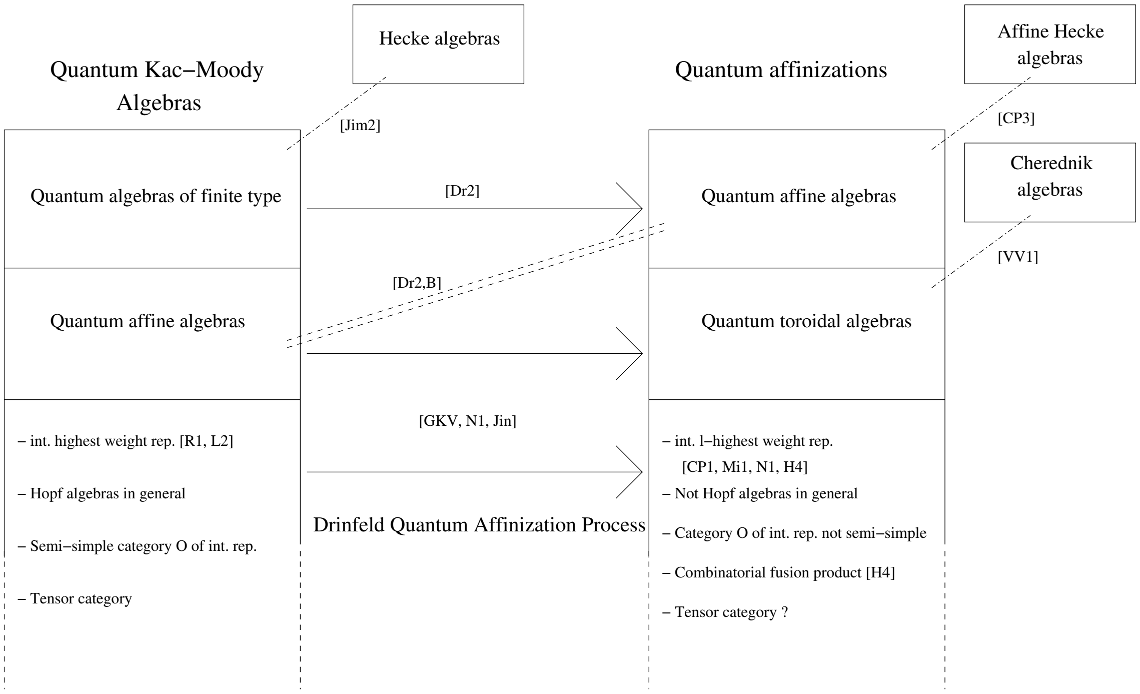
Contents

1. Introduction

In this paper $q \in \mathbb{C}^*$ is not a root of unity.

Drinfeld [Dr1] and Jimbo [Jim] associated, independently, to any symmetrizable Kac-Moody algebra $\mathfrak{g}$ and $q \in \mathbb{C}^*$ a Hopf algebra $\mathcal{U}_q(\mathfrak{g})$ called quantum Kac-Moody algebra. The quantum algebras of finite type $\mathcal{U}_q(\mathfrak{g})$ ($\mathfrak{g}$ of finite type) and their representations have been intensively studied (see for example [CP3, L2, R2] and references therein). The quantum affine algebras $\mathcal{U}_q(\hat{\mathfrak{g}})$ ($\hat{\mathfrak{g}}$ affine algebra) are also of particular interest : they have two realizations, the usual Drinfeld-Jimbo realization and a new realization (see [Dr2, B]) as a quantum affinization of a quantum algebra of finite type $\mathcal{U}_q(\mathfrak{g})$. Quantum affine algebras and their representations have also been intensively studied (see among others [AK, CP1, CP2, CP3, EM, FR, FM, H5, Kas2, N1, N4, VV3] and references therein).

The quantum affinization process (that Drinfeld [Dr2] described for constructing the second realization of a quantum affine algebra) can be extended to all symmetrizable quantum Kac-Moody algebras $\mathcal{U}_q(\mathfrak{g})$ (see [Jin, N1]). One obtains a new class of algebras called quantum affinizations : the quantum affinization of $\mathcal{U}_q(\mathfrak{g})$ is denoted by $\mathcal{U}_q(\hat{\mathfrak{g}})$ and contains $\mathcal{U}_q(\mathfrak{g})$ as a subalgebra. The quantum affine algebras are the simplest examples and have the singular property of being also quantum Kac-Moody algebras. The quantum toroidal algebras (affinizations of quantum affine algebras) are also of particular importance, see for example [GKV, Mi1, Mi2, N1, N2, Sa, Sc, STU, TU, VV2]. In analogy to the Frobenius-Schur-Weyl duality between quantum groups of finite type and Hecke algebras [Jim2], and between quantum affine algebras and affine Hecke algebras [CP4], quantum toroidal algebras have a close relation [VV1] to double affine Hecke algebras and their degenerations (Cherednik's algebras [Che2]) which have been recently intensively studied (for example see [BE, Che1, GGOR, GS, V, VV4] and references therein). In general a quantum affinization (for example a quantum toroidal algebra) is not isomorphic to a quantum Kac-Moody algebra and has no Hopf algebra structure. For convenience of the reader, we "describe" in the following diagram the relations between these different algebras (in particular with the two main classes of algebras considered in this paper, quantum Kac-Moody algebras on the left and quantum affinizations on the right).



We denote by $\mathfrak{h}$ (resp. $\mathcal{U}_q(\mathfrak{h})$) the Cartan subalgebra of $\mathfrak{g}$ (resp. $\mathcal{U}_q(\mathfrak{g})$). $\mathcal{U}_q(\hat{\mathfrak{h}}) \subset \mathcal{U}_q(\hat{\mathfrak{g}})$ is the quantum affine analog of the Cartan subalgebra.

In analogy to the generalization of the representation theory of quantum algebras of finite type to general quantum Kac-Moody algebras (see [L2]), a natural question is to extend the representation theory of quantum affine algebras to general quantum affinizations. By extending the results of [CP3] (quantum affine algebras), [Mi1] (quantum toroidal algebras of type A) and [N1] (general simply-laced quantum affinizations), we established in [H3] a triangular decomposition of $\mathcal{U}_q(\hat{\mathfrak{g}})$ which allowed us to begin the study of the representation theory of general quantum affinizations : one can define the notion of l-highest weight representations of $\mathcal{U}_q(\hat{\mathfrak{g}})$ (analog to the usual notion of highest weight module corresponding to $\mathcal{U}_q(\hat{\mathfrak{h}})$). One says that a $\mathcal{U}_q(\hat{\mathfrak{g}})$-module is integrable (resp. in the category $\mathcal{O}$) if it is integrable (resp. in the category $\mathcal{O}$) as a $\mathcal{U}_q(\mathfrak{g})$-module. Let $\mathrm{Mod}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ be the category of integrable $\mathcal{U}_q(\hat{\mathfrak{g}})$-module which are in the category $\mathcal{O}$. It appears that $\mathcal{U}_q(\hat{\mathfrak{g}})$ has numerous l-highest weight representations in $\mathrm{Mod}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ and that one can classify the simple ones. The category $\mathrm{Mod}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ is not semi-simple.

No Hopf algebra structure is known for $\mathcal{U}_q(\hat{\mathfrak{g}})$ except in the case of quantum affine algebras. However in [H3] we proved via Frenkel-Reshetikhin q-characters the existence of a combinatorial fusion product in the Grothendieck group of $\mathrm{Mod}(\mathcal{U}_q(\hat{\mathfrak{g}}))$, that it to say a product in the semi-simplified category. As $\mathrm{Mod}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ is not semi-simple, this fusion product can not a priori be directly translated in terms of modules.

Each quantum affinization has a "coproduct" (the Drinfeld coproduct) which is defined in a completion of $\mathcal{U}_q(\hat{\mathfrak{g}}) \otimes \mathcal{U}_q(\hat{\mathfrak{g}})$. Although it can not be directly used to construct tensor products of integrable representations (it involves infinite sums), we propose in the present paper a new process to produce quantum fusion modules from it : for all quantum affinizations we construct by deformation of the Drinfeld coproduct and renormalization a new (non semi-simple) tensor category Mod. In particular it gives a representation theoretical interpretation of the combinatorial fusion product and we get several applications from it. For quantum affine algebras this process is new and different from the usual tensor product (we prove that it has different properties), and for general quantum affinizations (as general quantum toroidal algebras) no process to produce fusion of modules was known so far.

For this construction, the first technical point that we solve is a rationality problem for the dependence in the deformation parameter u of the Drinfeld coproduct. Let $\mathcal{U}_q'(\hat{\mathfrak{g}}) = \mathcal{U}_q(\hat{\mathfrak{g}}) \otimes \mathbb{C}(u)$. We consider a "good" category $\mathrm{Mod}(\mathcal{U}_q'(\hat{\mathfrak{g}}))$ of $\mathcal{U}_q'(\hat{\mathfrak{g}})$-modules V with an integrable $\mathcal{U}_q(\mathfrak{h})$-submodule $W \subset V$ satisfying $V \simeq (W \otimes_{\mathbb{C}} \mathbb{C}(u))$, $\mathcal{U}_q(\hat{\mathfrak{h}})(W \otimes \mathbb{C}[u^{\pm}]) \subset (W \otimes \mathbb{C}[u^{\pm}])$, and some additional technical properties. It appears that the action of $\mathcal{U}_q(\hat{\mathfrak{g}})$ has a nice regularity property which is compatible with the Drinfeld coproduct and leads to this rationality, that is to say that $\mathrm{Mod}(\mathcal{U}_q'(\hat{\mathfrak{g}}))$ is stable under a tensor product. The second point is the associativity : the Drinfeld coproduct is coassociative, but the deformation parameter u breaks the symmetry and the u-deformed Drinfeld coproduct is not coassociative in the usual sense. However there is a "twisted" coassociative property which allows us to get the associativity of a new monoidal structure on the category $\mathrm{Mod} = \mathrm{Mod}^0 \oplus (\mathrm{Mod}(\mathcal{U}_q'(\hat{\mathfrak{g}})) \oplus \mathrm{Mod}(\mathcal{U}_q'(\hat{\mathfrak{g}})) \oplus \cdots)$ (here Mod^0 is an abelian semi-simple category with a unique simple object corresponding to a neutral object of the category).

Besides we prove that for a large class of quantum affinizations (including quantum affine algebras and most quantum toroidal algebras) the subcategory of modules with a finite composition series is stable under the monoidal bifunctor. This proof uses the specialization process described below and an investigation of the compatibility property between generalizations of Frenkel-Reshetikhin q-characters and the monoidal bifunctor. We get in particular that all l-highest weight modules of the category $\mathrm{Mod}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ have a finite composition series.

In order to go back to the usual category $\mathrm{Mod}(\mathcal{U}_q(\hat{\mathfrak{g}}))$, we prove the existence of certain forms for all cyclic modules ($\mathcal{A}$-forms) : such forms can be specialized at $u = 1$ and give cyclic $\mathcal{U}_q(\hat{\mathfrak{g}})$-modules; the proof uses the new rationality property. As a consequence, we can construct an l-highest weight fusion $\mathcal{U}_q(\hat{\mathfrak{g}})$-module $V_1 *_f V_2$ from two l-highest weight $\mathcal{U}_q(\hat{\mathfrak{g}})$-modules V_1, V_2. In particular a simple module of $\mathrm{Mod}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ is a quotient of a fusion module of fundamental representations (analog to a result of Chari-Pressley for the usual coproduct of quantum affine algebras). We establish a cyclicity property of the fusion of any l-highest weight modules : it is always an l-highest weight module. For quantum affine algebras this property is very different from the properties of the usual tensor product. It allows to control the "size" of the modules obtained by $*_f$; in particular we can produce "big" integrable l-highest weight modules whose existence was a priori not known. We get moreover another bifunctor $\otimes_d : \mathrm{Modf}(\mathcal{U}_q(\hat{\mathfrak{g}})) \times \mathrm{Modf}(\mathcal{U}_q(\hat{\mathfrak{g}})) \to \mathrm{Modf}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ where $\mathrm{Modf}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ is the category of finite dimensional representations (in general $\otimes_d$ does not coincide with $*_f$).

Let us describe an application. One important aspect of the representation theory of quantum affine algebras is the existence of the Kirillov-Reshetikhin modules whose characters are given by the certain explicit "fermionic" formulas. The proof of the corresponding statement (the Kirillov-Reshetikhin conjecture) can be obtained from certain T-systems originated from the theory of integrable systems. These T-systems can be interpreted in the form exact sequences involving tensor products of Kirillov-Reshetikhin modules; they were first established in [N3] for simply-laced quantum affine algebras (with the main result of [N4]) and then in [H5] for all quantum affine algebras (with a different proof). In the present paper we establish for a large class of general quantum affinizations exact sequences involving fusion of Kirillov-Reshetikhin modules : the usual tensor product can be replaced in the general situation by the fusion constructed in this paper. It is related to new generalized T-systems that we define and establish. The existence of such new T-systems is a new particular regularity property of the representation theory of general quantum affinizations.

We would like to mention other possible applications and new developments : for example the quantum fusion tensor category should lead to the description of the blocks of the category $\mathrm{Mod}(\mathcal{U}_q(\hat{\mathfrak{g}}))$. Besides the generalized T-systems established in this paper should lead to new fermionic formulas in analogy to the case of quantum affine algebras and should have a nice interpretation in terms of integrable systems. It would be interesting to relate the quantum fusion to the Feigin-Loktev fusion procedure defined for classical affine algebras [FL]. We believe that the deformation of the Drinfeld coproduct used is a particular case of a more general framework involving some deformed Hopf structure on $\mathbb{Z}$-graded algebras ("quantum Hopf vertex algebras" that we hope to describe in another paper). Eventually the tensor category constructed in this paper should be related to certain new Hopf algebras in the Tannaka-Krein reconstruction philosophy.

Let us describe the organization of this paper :

In section 2 we give the background for general quantum affinizations and their representations and we recall the main results of [H3]. In section 3 we construct the "quantum fusion" tensor category Mod (theorem 3.12) with the help of the intermediate category $\mathrm{Mod}(\mathcal{U}_q'(\hat{\mathfrak{g}}))$ (definition 3.6) and establish the rationality property in u. The case of fusion of Kirillov-Reshetikhin modules of the quantum affine algebra $\mathcal{U}_q(\hat{sl_2})$ is studied explicitly in subsection 3.4. In section 4 we define and study the notion of $\mathcal{A}$-form (definition 4.1). In particular we prove in theorem 4.6 that one can define $\mathcal{A}$-forms of cyclic modules of the category $\mathrm{Mod}(\mathcal{U}_q'(\hat{\mathfrak{g}}))$. It gives rise to the two following constructions : a fusion l-highest weight module from two l-highest weight modules (definition 4.10), and in the case of a quantum affine algebra a new bifunctor for the category of finite dimensional representations (theorem 4.13). In section 5 we prove for a large class of quantum affinizations that the subcategory of finite length modules is stable under the monoidal bifunctor (theorem 5.27) by proving the compatibility with the q-characters (theorem 5.12) and with the combinatorial fusion product introduced in [H3]; as a by-product we get a representation theoretical interpretation of this combinatorial fusion product (corollary 5.15). In section 6 we describe several applications : first we prove that a simple module of $\mathrm{Mod}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ is a quotient of the fusion of fundamental representations (proposition 6.1); we establish a cyclicity property which allows to control the "size" of this fusion module (theorem 6.2 and corollary 6.3). Then we establish an exact sequence involving fusion of Kirillov-Reshetikhin modules (theorem 6.7). It is related to new generalized T-systems (theorem 6.10). In subsection 6.3 we address complements on additional questions and further possible applications.

The main results of this paper were first announced in the "séminaire quantique" in Strasbourg and at the conference "Representations of Kac-Moody Algebras and Combinatorics" in Banff respectively in January and March 2005.

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2. Background

In this section we give backgrounds on general quantum affinizations and their representations (we also remind results of [H3]). In the following for a formal variable u, $\mathbb{C}[u^{\pm}], \mathbb{C}(u), \mathbb{C}[[u]], \mathbb{C}((u))$ are the standard notations.

2.1. **Cartan matrix.**

A generalized Cartan matrix is a matrix $C = (C_{i,j})_{1\leq i,j\leq n}$ satisfying $C_{i,j} \in \mathbb{Z}$, $C_{i,i} = 2$, $(i \neq j \Rightarrow C_{i,j} \leq 0)$ and $(C_{i,j} = 0 \Leftrightarrow C_{j,i} = 0)$. We denote $I = \{1, ..., n\}$ and $l = \mathrm{rank}(C)$. For $i, j \in I$, we put $\delta_{i,j} = 0$ if $i \neq j$, and $\delta_{i,j} = 1$ if $i = j$.

In the following we suppose that C is symmetrizable, that means that there is a matrix $D = \mathrm{diag}(r_1, ..., r_n)$ ($r_i \in \mathbb{N}^*$) such that $B = DC$ is symmetric. In particular if C is symmetric then it is symmetrizable with $D = I_n$. In the following B, C and D are fixed.

$q \in \mathbb{C}^*$ is not a root of unity and is fixed. We put $q_i = q^{r_i}$ and for $l \in \mathbb{Z}$, we set $[l]_q = \frac{q^l - q^{-l}}{q - q^{-1}} \in \mathbb{Z}[q^{\pm}]$. Let $C(q)$ be the quantized Cartan matrix defined by ($i \neq j \in I$):

$$C_{i,i}(q) = q_i + q_i^{-1} \ , \ C_{i,j}(q) = [C_{i,j}]_q.$$

We denote by $D(q)$ the diagonal matrix such that for $i, j \in I$, $D_{i,j}(q) = \delta_{i,j}[r_i]_q$. If $\det(C(q)) \neq 0$, we denote by $\tilde{C}(q)$ the inverse matrix of $C(z)$.

We fix a realization $(\mathfrak{h}, \Pi, \Pi^\vee)$ of C (see [Kac]): $\mathfrak{h}$ is a $2n - l$ dimensional $\mathbb{Q}$-vector space, $\Pi = \{\alpha_1, ..., \alpha_n\} \subset \mathfrak{h}^*$ (set of the simple roots), $\Pi^\vee = \{\alpha_1^\vee, ..., \alpha_n^\vee\} \subset \mathfrak{h}$ (set of simple coroots) and for $1 \leq i, j \leq n$: $\alpha_j(\alpha_i^\vee) = C_{i,j}$. Denote by $\Lambda_1, ..., \Lambda_n \in \mathfrak{h}^*$ (resp. the $\Lambda_1^\vee, ..., \Lambda_n^\vee \in \mathfrak{h}$) the fundamental weights (resp. coweights). By definition $\alpha_i(\Lambda_j^\vee) = \Lambda_i(\alpha_j^\vee) = \delta_{i,j}$.

Consider a symmetric bilinear form $(,) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{Q}$ such that for $i \in I$, $v \in \mathfrak{h}^*$: $(\alpha_i, v) = v(r_i\alpha_i^\vee)$. It is non degenerate and gives an isomorphism $\nu : \mathfrak{h}^* \to \mathfrak{h}$. In particular for $i \in I$ we have $\nu(\alpha_i) = r_i\alpha_i^\vee$, and for $\lambda, \mu \in \mathfrak{h}^*$ $\lambda(\nu(\mu)) = \mu(\nu(\lambda))$.

Denote $P = \{\lambda \in \mathfrak{h}^* | \forall i \in I, \lambda(\alpha_i^\vee) \in \mathbb{Z}\}$ the set of integral weights and $P^+ = \{\lambda \in P | \forall i \in I, \lambda(\alpha_i^\vee) \geq 0\}$ the set of dominant weights. For example we have $\alpha_1, ..., \alpha_n \in P$ and $\Lambda_1, ..., \Lambda_n \in P^+$.

Denote $Q=\bigoplus_{i\in I}\mathbb{Z}\alpha_i\subset P$ the root lattice and $Q^+=\sum_{i\in I}\mathbb{N}\alpha_i\subset Q$. Define $h:Q^+\to\mathbb{N}$ such that $h(l_1\alpha_1+...+l_n\alpha_n)=l_1+...+l_n$.

For $\lambda,\mu\in\mathfrak{h}^*$, write $\lambda\geq\mu$ if $\lambda-\mu\in Q^+$. For $\lambda\in\mathfrak{h}^*$, denote $\mathcal{S}(\lambda)=\{\mu\in\mathfrak{h}^*|\mu\leq\lambda\}$.

2.2. **Quantum Kac-Moody algebra.** In the following $\mathfrak{g}$ is the symmetrizable Kac-Moody algebra associated with the data $(\mathfrak{h},\Pi,\Pi^\vee)$ fixed in section 2.1 (see [Kac]).

Definition 2.1. *The quantum Kac-Moody algebra* $\mathcal{U}_q(\mathfrak{g})$ *is the* $\mathbb{C}$*-algebra with generators* k_h *(*$h\in\mathfrak{h}$*),* $x_i^\pm$ *(*$i\in I$*) and relations:*

$$k_hk_{h'}=k_{h+h'}\ ,\ k_0=1\ ,\ k_hx_j^{\pm}k_{-h}=q^{\pm\alpha_j(h)}x_j^{\pm},$$

$$x_i^+x_j^--x_j^-x_i^+=\delta_{i,j}\frac{k_{r_i\alpha_i^\vee}-k_{-r_i\alpha_i^\vee}}{q_i-q_i^{-1}},$$

$$\sum\nolimits_{r=0...1-C_{i,j}}(-1)^r\begin{bmatrix}1-C_{i,j}\\ r\end{bmatrix}_{q_i}(x_i^\pm)^{1-C_{i,j}-r}x_j^\pm(x_i^\pm)^r=0\ \textit{(for } i\neq j\textit{)}.$$

This algebra was introduced in [Dr1, Jim]. It is remarkable that one can define a Hopf algebra structure on $\mathcal{U}_q(\mathfrak{g})$. In this paper we use the standard coproduct given by

$$\Delta(k_h)=k_h\otimes k_h\ ,\ \Delta(x_i^+)=x_i^+\otimes 1+k_i^+\otimes x_i^+\ ,\ \Delta(x_i^-)=x_i^-\otimes k_i^-+1\otimes x_i^-,$$

where we denote $k_i^{\pm1}=k_{\pm r_i\alpha_i^\vee}$. In particular we have a tensor structure on the category of $\mathcal{U}_q(\mathfrak{g})$-modules.

2.3. **Quantum affinizations.** For $\mathfrak{g}$ a semi-simple Lie algebra, the affine Kac-Moody algebra associated to $\mathfrak{g}$ is usually denoted by $\hat{\mathfrak{g}}$ (see [Kac]). That is why the quantum affine algebra is usually denoted by $\mathcal{U}_q(\hat{\mathfrak{g}})$. We will use this notation for general quantum affinizations. In the following $\mathfrak{g}$ is the symmetrizable Kac-Moody algebra fixed in section 2.1 with datas C,B,D. We will define $\mathcal{U}_q(\hat{\mathfrak{g}})$ the affinization of $\mathcal{U}_q(\mathfrak{g})$ as an algebra with generators $x_{i,r}^\pm$ ($i\in I,r\in\mathbb{Z}$), k_h ($h\in\mathfrak{h}$), $h_{i,m}$ ($i\in I,m\in\mathbb{Z}-\{0\}$). Let z be a formal variable and $\delta(z)=\sum_{r\in\mathbb{Z}}z^r$. The relations are given in terms of the currents

$$x_i^\pm(w)=\sum\nolimits_{r\in\mathbb{Z}}x_{i,r}^\pm w^r,$$

$$\phi_i^\pm(z)=\sum\nolimits_{m\geq0}\phi_{i,\pm m}^\pm z^{\pm m}=k_{\pm r_i\alpha_i^\vee}\exp(\pm(q-q^{-1})\sum\nolimits_{m'\geq1}h_{i,\pm m'}z^{\pm m'}).$$

We also set $\phi_{i,m}^+=0$ for $m<0$ and $\phi_{i,m}^-=0$ for $m>0$. (In this paper we omit the central elements because they act trivially on the considered representations).

Definition 2.2. *The quantum affinization* $\mathcal{U}_q(\hat{\mathfrak{g}})$ *is the* $\mathbb{C}$*-algebra with generators* $x_{i,r}^\pm$ *(*$i\in I,r\in\mathbb{Z}$*),* k_h *(*$h\in\mathfrak{h}$*),* $h_{i,m}$ *(*$i\in I,m\in\mathbb{Z}-\{0\}$*) and relations (*$h,h'\in\mathfrak{h}$*,* $i,j\in I$*) :*

$$k_hk_{h'}=k_{h+h'}\ ,\ k_0=1\ ,\ k_h\phi_i^\pm(z)=\phi_i^\pm(z)k_h,\tag{1}$$

$$k_hx_j^\pm(z)=q^{\pm\alpha_j(h)}x_j^\pm(z)k_h,\tag{2}$$

$$\phi_i^+(z)x_j^\pm(w)=\frac{q^{\pm B_{i,j}}w-z}{w-q^{\pm B_{i,j}}z}x_j^\pm(w)\phi_i^+(z),\tag{3}$$

$$\phi_i^-(z)x_j^\pm(w)=\frac{q^{\pm B_{i,j}}w-z}{w-q^{\pm B_{i,j}}z}x_j^\pm(w)\phi_i^-(z),\tag{4}$$

$$x_i^+(z)x_j^-(w) - x_j^-(w)x_i^+(z) = \frac{\delta_{i,j}}{q_i - q_i^{-1}}(\delta(\frac{w}{z})\phi_i^+(w) - \delta(\frac{z}{w})\phi_i^-(z)), \tag{5}$$

$$(w - q^{\pm B_{i,j}}z)x_i^{\pm}(z)x_j^{\pm}(w) = (q^{\pm B_{i,j}}w - z)x_j^{\pm}(w)x_i^{\pm}(z), \tag{6}$$

$$\sum\nolimits_{\pi\in\Sigma_s}\sum\nolimits_{k=0...s}(-1)^k \begin{bmatrix} s \\ k \end{bmatrix}_{q_i} x_i^{\pm}(w_{\pi(1)})...x_i^{\pm}(w_{\pi(k)})x_j^{\pm}(z)x_i^{\pm}(w_{\pi(k+1)})...x_i^{\pm}(w_{\pi(s)}) = 0, \tag{7}$$

where the last relation holds for all $i \neq j$, $s = 1 - C_{i,j}$, *all sequences of integers* $r_1, ..., r_s$. Σ_s *is the symmetric group on* s *letters. In the equation (3) (resp. equation (4)),* $\frac{q^{\pm B_{i,j}}w - z}{w - q^{\pm B_{i,j}}z}$ *is viewed as a formal power series in* z/w *expanded around* ∞ *(resp.* 0*).*

When C is of finite type, $\mathcal{U}_q(\hat{\mathfrak{g}})$ is called a quantum affine algebra, and when C is of affine type, $\mathcal{U}_q(\hat{\mathfrak{g}})$ is called a quantum toroidal algebra. There is a huge amount of very interesting papers on quantum affine algebras (see the introduction for references). They have the very particular property to be also quantum Kac-Moody algebras [Dr2, B]. The quantum toroidal algebras are also of particular importance and are closely related via a Frobenius-Schur-Weyl duality [VV1] to double affine Hecke algebras (Cherednik's algebras, see the introduction for references). In general a quantum affinization (for example a quantum toroidal algebra) is not isomorphic to a quantum Kac-Moody algebra.

Note that formulas (3), (4) are equivalent to (see for example [H3]) :

$$h_{i,m}x_{j,r}^{\pm} - x_{j,r}^{\pm}h_{i,m} = \pm\frac{1}{m}[mB_{i,j}]_q x_{j,m+r}^{\pm}. \tag{8}$$

Relations (7) are called affine quantum Serre relations.

There is an injective algebra morphism $\mathcal{U}_q(\mathfrak{g}) \rightarrow \mathcal{U}_q(\hat{\mathfrak{g}})$ defined by $k_h \mapsto k_h$, $x_i^{\pm} \mapsto x_{i,0}^{\pm}$ for $h \in \mathfrak{h}, i \in I$. For $i \in I$, the subalgebra $\hat{\mathcal{U}_i} \subset \mathcal{U}_q(\hat{\mathfrak{g}})$ generated by the $x_{i,r}^{\pm}, h_{i,m}, k_{p\alpha_i^\vee}$ $(r \in \mathbb{Z}, m \neq 0, p \in \mathbb{Q})$ is isomorphic to $\mathcal{U}_{q_i}(\hat{sl_2})$.

Let $\mathcal{U}_q(\hat{\mathfrak{g}})^{\pm}$ (resp. $\mathcal{U}_q(\hat{\mathfrak{h}})$) be the subalgebra of $\mathcal{U}_q(\hat{\mathfrak{g}})$ generated by the $x_{i,r}^{\pm}$ (resp. by the k_h and the $h_{i,r}$). We have a triangular decomposition :

Theorem 2.3. [B, H3] *The multiplication* $x^- \otimes h \otimes x^+ \mapsto x^- h x^+$ *defines an isomorphism of* $\mathbb{C}$*-vector space* $\mathcal{U}_q(\hat{\mathfrak{g}})^- \otimes \mathcal{U}_q(\hat{\mathfrak{g}}) \otimes \mathcal{U}_q(\hat{\mathfrak{g}})^+ \simeq \mathcal{U}_q(\hat{\mathfrak{g}})$.

The result was proved in [B, Proposition 6.1] for quantum affine algebras (see also [CP3, Proposition 12.2.2]) and the general case is treated in [H3, Theorem 3.2] (the result follows from [B, Proposition 6.1] for Cartan matrices satisfying $(i \neq j \Rightarrow C_{i,j}C_{j,i} \leq 3)$; in general the technical point is to check the compatibility with affine quantum Serre relations (7). The proof [H3] implies some combinatorial relations discovered by Jing [Jin, Theorem 6.1]).

2.4. The category Mod($\mathcal{U}_q(\hat{\mathfrak{g}})$).

In this section we recall results on the category of integrable $\mathcal{U}_q(\hat{\mathfrak{g}})$-modules which are in the category $\mathcal{O}$ defined below.

For V a $\mathcal{U}_q(\mathfrak{h})$-module and $\omega \in \mathfrak{h}^*$ we denote by V_ω the weight space of weight ω

$$V_\omega = \{v \in V | \forall h \in \mathfrak{h}, k_h.v = q^{\omega(h)}v\}.$$

We say that V is $\mathcal{U}_q(\mathfrak{h})$-diagonalizable if $V = \bigoplus_{\omega\in\mathfrak{h}^*} V_\omega$. We have the following standard definitions (see [Kac] for the classical definition) :

Definition 2.4. *A $\mathcal{U}_q(\mathfrak{h})$-module V is said to be integrable if V is $\mathcal{U}_q(\mathfrak{h})$-diagonalizable and for all $\omega \in \mathfrak{h}^*$, V_ω is finite dimensional, and for $i \in I$ there is $R \geq 0$ such that $(r \geq R \Rightarrow V_{\omega \pm r\alpha_i} = \{0\})$. A $\mathcal{U}_q(\hat{\mathfrak{g}})$-module (resp. a $\mathcal{U}_q(\mathfrak{g})$-module) V is said to be integrable if V is integrable as a $\mathcal{U}_q(\mathfrak{h})$-module.*

An integrable representation is not necessarily finite dimensional.

We have the natural analog of the classical category $\mathcal{O}$ of Kac-Moody algebras [Kac] :

Definition 2.5. *A $\mathcal{U}_q(\mathfrak{h})$-module V is said to be in the category $\mathcal{O}(\mathcal{U}_q(\mathfrak{h}))$ if*

i) V is $\mathcal{U}_q(\mathfrak{h})$-diagonalizable,

ii) for all $\omega \in \mathfrak{h}^*$, $dim(V_\omega) < \infty$,

iii) there is a finite number of element $\lambda_1, ..., \lambda_s \in \mathfrak{h}^*$ such that the weights of V are in $\bigcup_{j=1,\cdots,s} \mathcal{S}(\lambda_j)$.

Definition 2.6. *For $\mathfrak{m}$ respectively equal to $\hat{\mathfrak{h}}$, $\mathfrak{g}$, $\hat{\mathfrak{g}}$, we denote by $Mod(\mathcal{U}_q(\mathfrak{m}))$ the category of $\mathcal{U}_q(\mathfrak{m})$-modules which are integrable and in the category $\mathcal{O}(\mathcal{U}_q(\mathfrak{h}))$ as a $\mathcal{U}_q(\mathfrak{h})$-module.*

The category $\mathrm{Mod}(\mathcal{U}_q(\mathfrak{g}))$ (in a slightly more general form) is considered in [L2]. Definitions for $\mathcal{U}_q(\hat{\mathfrak{g}})$-modules are given in [N1, section 1.2].

For $\lambda \in P$, denote by $L(\lambda)$ the simple highest weight $\mathcal{U}_q(\mathfrak{g})$-module of highest weight λ. We have (see for example [R1, L2]) :

Theorem 2.7. *$L(\lambda) \in Mod(\mathcal{U}_q(\mathfrak{g}))$ if and only if $\lambda \in P^+$.*

The category $\mathrm{Mod}(\mathcal{U}_q(\mathfrak{g}))$ is semi-simple. But $\mathrm{Mod}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ is not semi-simple.

By generalizing the approach of finite dimensional representations of quantum affine algebras developed in [CP3], and definitions in [N1, Mi1], we defined in [H3] for general quantum affinization $\mathcal{U}_q(\hat{\mathfrak{g}})$:

Definition 2.8. *An l-weight is a couple (λ, Ψ) where $\lambda \in P$ and $\Psi = (\Psi^\pm_{i,\pm m})_{i\in I, m\geq 0}$, such that $\Psi^\pm_{i,\pm m} \in \mathbb{C}^*$ and $\Psi^\pm_{i,0} = q_i^{\pm\lambda(\alpha_i^\vee)}$ for $i \in I$ and $m \geq 0$.*

A $\mathcal{U}_q(\hat{\mathfrak{g}})$-module V is said to be of l-highest weight (λ, Ψ) if there is $v \in V_\lambda$ (l-highest weight vector) such that $V = \mathcal{U}_q(\hat{\mathfrak{g}}).v$ and $x^+_{i,r}.v = 0$, $\phi^\pm_{i,\pm m}.v = \Psi^\pm_{i,\pm m} v$ for $i \in I$, $r \in \mathbb{Z}$ and $m \geq 0$.

An l-weight (λ, Ψ) is said to be dominant if there is an n-tuplet of (Drinfeld) polynomials $(P_i(u))_{i\in I} \in (\mathbb{C}[u])^n$ satisfying for $i \in I$, $P_i(0) = 1$ and the relation in $\mathbb{C}[[z]]$ (resp. in $\mathbb{C}[[z^{-1}]]$):

$$\sum\nolimits_{m\geq 0} \Psi^\pm_{i,\pm m} z^{\pm m} = q_i^{deg(P_i)} \frac{P_i(zq_i^{-1})}{P_i(zq_i)}. \tag{9}$$

The element Ψ in an l-weight (λ, Ψ) is called a pseudo-weight.

Theorem 2.3 implies that for (λ, Ψ) an l-weight there is a unique simple module $L(\lambda, \Psi)$ of l-highest weight (λ, Ψ). Moreover :

Theorem 2.9. [CP3, Mi1, N1, H3] *We have $L(\lambda, \Psi) \in Mod(\mathcal{U}_q(\hat{\mathfrak{g}}))$ if and only if (λ, Ψ) is dominant.*

This result was proved in [CP3, Theorem 12.2.6] for quantum affine algebras (moreover in this case these integrable representations are finite dimensional), in [Mi1, Theorem 1] for quantum toroidal algebras of type A, in [N1, Proposition 1.2.15] for general simply-laced quantum affinizations and in [H3, Theorem 4.9] for general quantum affinizations.

Examples. For $i \in I, a \in \mathbb{C}^*, r \geq 1$, consider the l-weight $(\Lambda_i, \Psi_{i,a,r})$ where $\Psi_{i,a,r}$ is given by equation (9) with the n-tuplet $P_i(u) = (1 - ua)(1 - uaq_i^2)$