THE t-ANALOGS OF q-CHARACTERS AT ROOTS OF UNITY FOR QUANTUM AFFINE ALGEBRAS AND BEYOND

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Abstract. The q-characters were introduced by Frenkel and Reshetikhin [FR2] to study finite dimensional representations of the untwisted quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$ for q generic. The ϵ-characters at roots of unity were constructed by Frenkel and Mukhin [FM2] to study finite dimensional representations of various specializations of $\mathcal{U}_q(\hat{\mathfrak{g}})$ at $q^s = 1$. In the finite simply laced case Nakajima [N2][N3] defined deformations of q-characters called q, t-characters for q generic and also at roots of unity. The definition is combinatorial but the proof of the existence uses the geometric theory of quiver varieties which holds only in the simply laced case. In [He2] we proposed an algebraic general (non necessarily simply laced) new approach to q, t-characters for q generic. In this paper we treat the root of unity case. Moreover we construct q-characters and q, t-characters for a large class of generalized Cartan matrices (including finite and affine cases except $A_1^{(1)}$, $A_2^{(2)}$) by extending the approach of [He2]. In particular we generalize the construction of analogs of Kazhdan-Lusztig polynomials at roots of unity of [N3] to those cases. We also study properties of various objects used in this article : deformed screening operators at roots of unity, t-deformed polynomial algebras, bicharacters arising from symmetrizable Cartan matrices, deformation of the Frenkel-Mukhin's algorithm.

Contents

1. Introduction

V.G. Drinfel'd [D1] and M. Jimbo [J] associated, independently, to any symmetrizable Kac-Moody algebra $\mathfrak{g}$ and any complex number $q \in \mathbb{C}^*$ a Hopf algebra $\mathcal{U}_q(\mathfrak{g})$ called quantum group or quantum Kac-Moody algebra.

First we suppose that $q \in \mathbb{C}^*$ is not a root of unity. In the case of a semi-simple Lie algebra $\mathfrak{g}$ of rank n, the structure of the Grothendieck ring $\mathrm{Rep}(\mathcal{U}_q(\mathfrak{g}))$ of finite dimensional representations of the quantum

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finite algebra $\mathcal{U}_q(\mathfrak{g})$ is well understood. It is analogous to the classical case $q=1$. In particular we have ring isomorphisms:

$$\mathrm{Rep}(\mathcal{U}_q(\mathfrak{g}))\simeq \mathrm{Rep}(\mathfrak{g})\simeq \mathbb{Z}[\Lambda]^W\simeq \mathbb{Z}[T_1,...,T_n]$$

deduced from the injective homomorphism of characters χ:

$$\chi(V)=\sum_{\lambda\in\Lambda}\dim(V_\lambda)\lambda$$

where V_λ are weight spaces of a representation V and Λ is the weight lattice.

For the general case of Kac-Moody algebras the picture is less clear. The representation theory of the quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$ is of particular interest (see [CP1], [CP2]). In this case there is a crucial property of $\mathcal{U}_q(\hat{\mathfrak{g}})$: it has two realizations, the usual Drinfel'd-Jimbo realization and a new realization (see [D2] and [Be]) as a quantum affinization of the quantum finite algebra $\mathcal{U}_q(\mathfrak{g})$.

To study the finite dimensional representations of $\mathcal{U}_q(\hat{\mathfrak{g}})$ Frenkel and Reshetikhin [FR2] introduced q-characters which encode the (pseudo)-eigenvalues of some commuting elements in the Cartan subalgebra $\mathcal{U}_q(\hat{\mathfrak{h}})\subset\mathcal{U}_q(\hat{\mathfrak{g}})$ (see also [Kn]). The morphism of q-characters is an injective ring homomorphism:

$$\chi_q:\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))\rightarrow\mathbb{Z}[Y_{i,a}^{\pm}]_{i\in I,a\in\mathbb{C}^*}$$

where $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ is the Grothendieck ring of finite dimensional (type 1)-representations of $\mathcal{U}_q(\hat{\mathfrak{g}})$ and $I=\{1,...,n\}$. In particular $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ is commutative and isomorphic to $\mathbb{Z}[X_{i,a}]_{i\in I,a\in\mathbb{C}^*}$.

The morphism of q-characters has a symmetry property analogous to the classical action of the Weyl group $\mathrm{Im}(\chi)=\mathbb{Z}[\Lambda]^W$: Frenkel and Reshetikhin [FR2] defined n screening operators S_i and showed that $\mathrm{Im}(\chi_q)=\bigcap\limits_{i\in I}\mathrm{Ker}(S_i)$ for $\mathfrak{g}=sl_2$. The result was proved by Frenkel and Mukhin for all finite $\mathfrak{g}$ in [FM1].

In the simply laced case Nakajima [N2][N3] introduced t-analogs of q-characters. The motivations are the study of filtrations induced on representations by (pseudo)-Jordan decompositions, the study of the decomposition in irreducible modules of tensorial products and the study of cohomologies of certain quiver varieties. The morphism of q,t-characters is a $\mathbb{Z}[t^{\pm}]$-linear map

$$\chi_{q,t}:\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))\rightarrow\mathbb{Z}[Y_{i,a}^{\pm},t^{\pm}]_{i\in I,a\in\mathbb{C}^*}$$

which is a deformation of χ_q and multiplicative in a certain sense. A combinatorial axiomatic definition of q,t-characters is given. But the existence is non-trivial and is proved with the geometric theory of quiver varieties which holds only in the simply laced case.

In [He2] we defined and constructed q,t-characters in the general (non necessarily simply laced) case with a new approach motivated by the non-commutative structure of $\mathcal{U}_q(\hat{\mathfrak{h}})\subset\mathcal{U}_q(\hat{\mathfrak{g}})$, the study of screening currents of [FR1] and of deformed screening operators $S_{i,t}$ of [He1]. In particular we have a symmetry property: the image of $\chi_{q,t}$ is a completion of $\bigcap\limits_{i\in I}\mathrm{Ker}(S_{i,t})$.

The representation theory of the quantum affine algebras $\mathcal{U}_q(\hat{\mathfrak{g}})$ depends crucially whether q is a root of unity or not (see [CP3]). Frenkel and Mukhin [FM1] generalized q-characters at roots of unity : if ϵ is a s^{th}-primitive root of unity the morphism of ϵ-characters is:

$$\chi_\epsilon:\mathrm{Rep}(\mathcal{U}_\epsilon^{\mathrm{res}}(\hat{\mathfrak{g}}))\rightarrow\mathbb{Z}[Y_{i,a}^{\pm}]_{i\in I,a\in\mathbb{C}^*}$$

where $\mathrm{Rep}(\mathcal{U}_\epsilon^{\mathrm{res}}(\hat{\mathfrak{g}}))$ is the Grothendieck ring of finite dimensional (type 1)-representations of the restricted specialization $\mathcal{U}_\epsilon^{\mathrm{res}}(\hat{\mathfrak{g}})$ of $\mathcal{U}_q(\hat{\mathfrak{g}})$ at $q=\epsilon$. In particular $\mathrm{Rep}(\mathcal{U}_\epsilon^{\mathrm{res}}(\hat{\mathfrak{g}}))$ is commutative and isomorphic to $\mathbb{Z}[X_{i,a}]_{i\in I,a\in\mathbb{C}^*}$.

Moreover χ_ϵ can be characterized by

$$\chi_\epsilon(\prod_{i\in I,l\in\mathbb{Z}/s\mathbb{Z}}X_{i,\epsilon^l}^{x_{i,l}})=\tau_s(\chi_q(\prod_{i\in I,0\leq l\leq s-1}X_{i,q^l}^{x_{i,[l]}}))$$

where $\tau_s:\mathbb{Z}[Y_{i,q^l}^{\pm}]_{i\in I,l\in\mathbb{Z}}\rightarrow\mathbb{Z}[Y_{i,\epsilon^l}^{\pm}]_{i\in I,l\in\mathbb{Z}/s\mathbb{Z}}$ is the ring homomorphism such that $\tau_s(Y_{i,q^l}^{\pm})=Y_{i,\epsilon^{[l]}}^{\pm}$ (for $l\in\mathbb{Z}$ we denote by $[l]$ its image in $\mathbb{Z}/s\mathbb{Z}$).

In the simply laced case Nakajima generalized the theory of q,t-characters at roots of unity with the help of quiver varieties [N3].

In this paper we construct q,t-characters at roots of unity in the general (non necessarily simply laced) case by extending the approach of [He2]. As an application we construct analogs of Kazhdan-Lusztig polynomials at roots of unity in the same spirit as Nakajima did for the simply laced case. We also study properties of various objects used in this paper: deformed screening operators at roots of unity, t-deformed polynomial algebras, bicharacters arising from general symmetrizable Cartan matrices, deformation of the Frenkel-Mukhin's algorithm.

The construction is also extended beyond the case of a quantum affine algebra, that is to say by replacing the finite Cartan matrix by a generalized symmetrizable Cartan matrix: the construction of q-characters as well as q,t-characters (generic and roots of unity cases) is explained in this paper for (non necessarily finite) Cartan matrices such that $i \neq j \Rightarrow C_{i,j}C_{j,i} \leq 3$ (it includes finite and affine types except $A_1^{(1)}$, $A_2^{(2)}$). The notion of a quantum affinization is more general than the construction of a quantum affine algebra from a quantum finite algebra: it can be extended to any general symmetrizable Cartan matrix (see [N1]). For example for an affine Cartan matrix one gets a quantum toroidal algebra (see [VV1]). In general a quantum affinization is not a quantum Kac-Moody algebra and few is known about the representation theory outside the quantum affine algebra case. However for an integrable representation one can define q-characters as Frenkel-Reshetikhin did for quantum affine algebras. So the q-characters constructed in this paper for some generalized symmetrizable Cartan matrix are to be linked with representation theory of the associated quantum affinization. We will address further developments on this point in a separate publication.

This paper is organized as follows: after some backgrounds in section 2, we generalize in section 3 the construction of t-deformed polynomial algebras of [He2] to the root of unity case. We give a "concrete" construction using Heisenberg algebras. We show that this twisted multiplication can also be "abstractly" defined with two bicharacters d_1, d_2 as Nakajima did for the simply laced case (for which there is only one bicharacter $d_1 = d_2$).

In section 4 we remind how q,t-characters are constructed for q generic and C finite in [He2]. We extend the construction of q-characters and of q,t-characters to symmetrizable (non necessarily finite) Cartan matrices such that $i \neq j \Rightarrow C_{i,j}C_{j,i} \leq 3$, in particular for affine Cartan matrices (except $A_1^{(1)}$ and $A_2^{(2)}$). The q,t-characters can be computed by the algorithm described in [He2] which is a deformation of the algorithm of Frenkel-Mukhin [FM1].

In section 5 we construct q,t-characters at roots of unity. Let us explain the crucial technical point of this section: we can not use directly a t-deformation of the definition of Frenkel-Mukhin because there is no analog of τ_s which is an algebra homomorphism for the t-deformed structures. But we can construct $\tau_{s,t}$ which is multiplicative for some ordered products (see section 5.2.1). In particular $\tau_{s,t}$ has nice properties and we can define $\chi_{\epsilon,t}$ such that "$\chi_{\epsilon,t} = \tau_{s,t} \circ \chi_{q,t}$". We give properties of $\chi_{\epsilon,t}$ analogous to the property of χ_ϵ (proposition 4.11, theorems 5.10 and 5.16). In particular in the ADE-case we get a formula which is Axiom 4 of [N3], and so the construction coincides with the construction of [N3] for the ADE-case.

In section 6 we give some applications about Kazhdan-Lusztig polynomials and quantization of the Grothendieck ring. If C is finite the technical point in the root of unity case is to show that the algorithm produces a finite number of dominant monomials. We give a conjecture about the multiplicity of an irreducible module in a standard module at roots of unity. For the ADE-case it is a result of Nakajima [N3]. An analogous conjecture was given in [He2] for q generic. We also study the non finite cases.

In section 7 we give some complements: first we discuss the finiteness of the algorithm; at $t = 1$ it stops if C is finite and it does not stop if C is affine. We relate the structure of the deformed ring in the affine $A_r^{(1)}$-case to the structure of quantum toroidal algebras. We study some combinatorial properties of the Cartan matrices which are related to the bicharacters d_1 and d_2 (propositions 7.9, 7.11, 7.12 and theorem 7.10).

For convenience of the reader we give at the end of this article an index of notations defined in the main body of the text.

In the course of writing this paper we were informed by H. Nakajima that the t-analogs of q-characters for some quantum toroidal algebras are also mentioned in the remark 6.9 of [N5]. This incited us to add the construction of analogs of Kazdhan-Lusztig polynomials at roots of unity also in the non finite cases (section 6.2.4).

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2. Background

2.1. **Cartan matrices.** A generalized Cartan matrix is $C = (C_{i,j})_{1\leq i,j\leq n}$ such that $C_{i,j} \in \mathbb{Z}$ and:

$$C_{i,i} = 2$$

$$i \neq j \Rightarrow C_{i,j} \leq 0$$

$$C_{i,j} = 0 \Leftrightarrow C_{j,i} = 0$$

Let $I = \{1, ..., n\}$.

C is said to be decomposable if it can be written in the form $C = P \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} P^{-1}$ where P is a permutation matrix, A and B are square matrices. Otherwise C is said to be indecomposable.

C is said to be symmetrizable if there is a matrix $D = \text{diag}(r_1, ..., r_n)$ ($r_i \in \mathbb{N}^*$) such that $B = DC$ is symmetric. In particular if C is symmetric then it is symmetrizable with $D = I_n$.

If C is indecomposable and symmetrizable then there is a unique choice of $r_1, ..., r_n > 0$ such that $r_1 \wedge ... \wedge r_n = 1$: indeed if $C_{j,i} \neq 0$ we have the relation $r_i = \frac{C_{j,i}}{C_{i,j}} r_j$.

In the following C is a symmetrizable and indecomposable generalized Cartan matrix. For example:

C is said to be of finite type if all its principal minors are positive (see [Bo] for a classification).

C is said to be of affine type if all its proper principal minor are positive and $\det(C) = 0$ (see [Ka] for a classification).

Let $r^\vee = \max\{r_i - 1 - C_{j,i})/i \neq j\} \cup \{1\}$. If C is finite we have $r^\vee = \max\{r_i / i \in I\} = \max\{-C_{i,j}/i \neq j\}$. In particular if C is of type ADE we have $r^\vee = 1$, if C is of type $B_l C_l F_4$ ($l \geq 2$) we have $r^\vee = 2$, if C of type G_2 we have $r^\vee = 3$.

Let z be an indeterminate and $z_i = z^{r_i}$. The matrix $C(z) = (C_{i,j}(z))_{1\leq i,j\leq n}$ with coefficients in $\mathbb{Z}[z^\pm]$ is defined by $C_{i,i}(z) = [2]_{z_i} = z_i + z_i^{-1}$ and $C_{i,j} = [C_{i,j}]_z$ for $i \neq j$ where for $l \in \mathbb{Z}$, we use the notation:

$$[l]_z = \frac{z^l - z^{-l}}{z - z^{-1}} \; (= z^{-l+1} + z^{-l+3} + ... + z^{l-1} \text{ for } l \geq 1)$$

Let $B(z) = D(z)C(z)$ where $D(z)$ is the diagonal matrix $D_{i,j}(z) = \delta_{i,j}[r_i]_z$, that is to say $B_{i,j}(z) = [r_i]_z C_{i,j}(z)$.

In particular, the coefficients of $C(z)$ and $B(z)$ are symmetric Laurent polynomials (invariant under $z \mapsto z^{-1}$).

In the following we suppose that $\det(C(z)) \neq 0$. It includes finite and affine Cartan matrices (if C is of type $A_1^{(1)}$ we set $r_1 = r_2 = 2$) and also the matrices such that $i \neq j \Rightarrow C_{i,j}C_{j,i} \leq 3$ which will appear later (see lemma 6.9 and section 7.3 for complements).

2.2. **Quantum affine algebras.** In the following q is a complex number $q \in \mathbb{C}^*$. If q is not a root of unity we set $s = 0$ and we say that q is generic. Otherwise $s \geq 1$ is set such that q is a s^{th} primitive root of unity.

We suppose in this section that C is finite. We refer to [FM2] for the definition of the untwisted quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$ associated to C (for q generic) and of the restricted specialization $\mathcal{U}_\epsilon^{\mathrm{res}}(\hat{\mathfrak{g}})$ of $\mathcal{U}_q(\hat{\mathfrak{g}})$ at $q = \epsilon$ (for ϵ root of unity).

We briefly describe the construction of $\mathcal{U}_\epsilon^{\mathrm{res}}(\hat{\mathfrak{g}})$ from $\mathcal{U}_q(\hat{\mathfrak{g}})$: we consider a $\mathbb{Z}[q, q^{-1}]$-subalgebra of $\mathcal{U}_q(\hat{\mathfrak{g}})$ containing the $(x_i^\pm)^{(r)} = \frac{(x_i^\pm)^s}{[r]_{q_i}!}$ (where $[r]_q! = [r]_q[r-1]_q...[1]_q$) for some generators $x_i^\pm$, and we set $q = \epsilon$.

One can define a Hopf algebra structure on $\mathcal{U}_q(\hat{\mathfrak{g}})$ and $\mathcal{U}_\epsilon^{\mathrm{res}}(\hat{\mathfrak{g}})$, and so we consider the Grothendieck ring of finite dimensional (type 1)-representations: $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ and $\mathrm{Rep}(\mathcal{U}_\epsilon^{\mathrm{res}}(\hat{\mathfrak{g}}))$.

The morphism of q-characters χ_q (Frenkel-Reshetikhin [FR2]) and the morphism of ϵ-character χ_ϵ (Frenkel-Mukhin [FM2]) are injective ring homomorphisms:

$$\chi_q : \mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}})) \to \mathbb{Z}[Y_{i,a}^\pm]_{i\in I, a\in\mathbb{C}^*} \ , \ \chi_\epsilon : \mathrm{Rep}(\mathcal{U}_\epsilon^{\mathrm{res}}(\hat{\mathfrak{g}})) \to \mathbb{Z}[Y_{i,a}^\pm]_{i\in I, a\in\mathbb{C}^*}$$

In particular $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ and $\mathrm{Rep}(\mathcal{U}_\epsilon^{\mathrm{res}}(\hat{\mathfrak{g}}))$ are commutative and isomorphic to $\mathbb{Z}[X_{i,a}]_{i\in I, a\in\mathbb{C}^*}$.

Frenkel and Mukhin [FM1][FM2] have proven that for $i \in I$, $a \in \mathbb{C}^*$:

$$\chi_q(X_{i,a}) \in \mathbb{Z}[Y_{i,aq^m}^\pm]_{i\in I, m\in\mathbb{Z}} \text{ and } \chi_\epsilon(X_{i,a}) \in \mathbb{Z}[Y_{i,a\epsilon^m}^\pm]_{i\in I, m\in\mathbb{Z}}$$

Indeed it suffices to study (see [He2] for details):

$$\chi_q : \mathrm{Rep} = \mathbb{Z}[X_{i,l}]_{i\in I, l\in\mathbb{Z}} \to \mathcal{Y} = \mathbb{Z}[Y_{i,l}^\pm]_{i\in I, l\in\mathbb{Z}}$$

(where $X_{i,l} = X_{i,q^l}$, $Y_{i,l}^\pm = Y_{i,q^l}^\pm$), and:

$$\chi_\epsilon^s : \mathrm{Rep}^s = \mathbb{Z}[X_{i,l}]_{i\in I, l\in\mathbb{Z}/s\mathbb{Z}} \to \mathcal{Y}^s = \mathbb{Z}[Y_{i,l}^\pm]_{i\in I, l\in\mathbb{Z}/s\mathbb{Z}}$$

(where $X_{i,l} = X_{i,\epsilon^l}$, $Y_{i,l}^\pm = Y_{i,\epsilon^l}^\pm$).

3. t-deformed polynomial algebras

3.1. **The t-deformed algebra $\hat{\mathcal{Y}}_t^s$.** In this section we generalize at roots of unity the construction of [He2] of t-deformed polynomial algebras.

3.1.1. *Construction.* In this section we suppose that $B(z)$ is symmetric.

Definition 3.1. *$\hat{\mathcal{H}}$ is the $\mathbb{C}$-algebra defined by generators $a_i[m], y_i[m]$ ($i \in I, m \in \mathbb{Z} - \{0\}$), central elements c_r ($r > 0$) and relations ($i, j \in I, m, r \in \mathbb{Z} - \{0\}$):*

$$[a_i[m], a_j[r]] = \delta_{m,-r}(q^m - q^{-m})B_{i,j}(q^m)c_{|m|} \tag{1}$$

$$[a_i[m], y_j[r]] = (q^{mr_i} - q^{-r_i m})\delta_{m,-r}\delta_{i,j}c_{|m|} \tag{2}$$

$$[y_i[m], y_j[r]] = 0 \tag{3}$$

Let $\hat{\mathcal{H}}_h = \hat{\mathcal{H}}[[h]]$. For $i \in I$, $l \in \mathbb{Z}/s\mathbb{Z}$ we define $\tilde{Y}_{i,l}^\pm, \tilde{A}_{i,l}^\pm, t_l^\pm \in \hat{\mathcal{H}}_h$ such that:

$$\tilde{Y}_{i,l} = \exp(\sum_{m>0} h^m q^{lm} y_i[m])\exp(\sum_{m>0} h^m q^{-lm} y_i[-m])$$

$$\tilde{A}_{i,l} = \exp(\sum_{m>0} h^m q^{lm} a_i[m])\exp(\sum_{m>0} h^m q^{-lm} a_i[-m])$$

$$t_l = \exp(\sum_{m>0} h^{2m} q^{lm} c_m)$$

and for $R=\sum\limits_{l\in\mathbb{Z}}R_lz^l\in\mathbb{Z}[z^{\pm}]$:

$$t_R=\prod_{l\in\mathbb{Z}}t_l^{R_l}=\exp(\sum_{m>0}h^{2m}R(q^m)c_m)\in\hat{\mathcal{H}}_h$$

Note that the root of unity condition, that is to say $s\geq 1$, is a periodic condition ($\tilde{Y}_{i,l+s}=\tilde{Y}_{i,l}$).

Lemma 3.2. *([He2]) We have the following relations in $\hat{\mathcal{H}}_h$:*

$$\tilde{A}_{i,l}\tilde{Y}_{j,k}\tilde{A}_{i,l}^{-1}\tilde{Y}_{j,k}^{-1}=t_{\delta_{i,j}(z^{-r_i}-z^{r_i})(-z^{(l-k)}+z^{(k-l)})}$$

$$\tilde{A}_{i,l}\tilde{A}_{j,k}\tilde{A}_{i,l}^{-1}\tilde{A}_{j,k}^{-1}=t_{B_{i,j}(z)(z^{-1}-z)(-z^{(l-k)}+z^{(k-l)})}$$

Definition 3.3. *$\hat{\mathcal{Y}}_u^s$ is the $\mathbb{Z}$-subalgebra of $\hat{\mathcal{H}}_h$ generated by the $\tilde{Y}_{i,l},\tilde{A}_{i,l}^{-1},t_l$ ($i\in I,l\in\mathbb{Z}/z\mathbb{Z}$).*

Note that if $s\geq 1$, the elements $\tilde{A}_{i,0}^{-1}\tilde{A}_{i,1}^{-1}...\tilde{A}_{i,s-1}^{-1}$ and $\tilde{Y}_{i,0}\tilde{Y}_{i,1}...\tilde{Y}_{i,s-1}$ are central in $\hat{\mathcal{Y}}_u$.

Definition 3.4. *$\hat{\mathcal{Y}}_t^s$ is the quotient-algebra of $\hat{\mathcal{Y}}_u^s$ by relations $t_l=1$ if $l\in\mathbb{Z}/s\mathbb{Z}-\{0\}$.*

We keep the notations $\tilde{Y}_{i,l},\tilde{A}_{i,l}^{-1}$ for their image in $\hat{\mathcal{Y}}_t^s$. We denote by t the image of $t_0=\exp(\sum\limits_{m>0}h^{2m}c_m)$ in $\hat{\mathcal{Y}}_t^s$. In particular the image of t_R is t^{R_0}. We denote by $\hat{\mathcal{Y}}_t=\hat{\mathcal{Y}}_t^0$ the algebra in the generic case.

3.1.2. *Structure.* For $a,b\in\mathbb{Z}/s\mathbb{Z}$, let $\delta_{a,b}=1$ if $a=b$ and $\delta_{a,b}=0$ if $a\neq b$.

The following theorem gives the structure of $\hat{\mathcal{Y}}_t^s$:

Theorem 3.5. *The algebra $\hat{\mathcal{Y}}_t^s$ is defined by generators $\tilde{Y}_{i,l},\tilde{A}_{i,l}^{-1},t^{\pm}$ ($i\in I,l\in\mathbb{Z}/s\mathbb{Z}$) and relations ($i,j\in I,k,l\in\mathbb{Z}/s\mathbb{Z}$):*

$$\tilde{Y}_{i,l}\tilde{Y}_{j,k}=\tilde{Y}_{j,k}\tilde{Y}_{i,l}$$

$$\tilde{A}_{i,l}^{-1}\tilde{A}_{j,k}^{-1}=t^{\alpha(i,l,j,k)}\tilde{A}_{j,k}^{-1}\tilde{A}_{i,l}^{-1}$$

$$\tilde{Y}_{j,k}\tilde{A}_{i,l}^{-1}=t^{\beta(i,l,j,k)}\tilde{A}_{i,l}^{-1}\tilde{Y}_{j,k}$$

where $\alpha,\beta:(I\times\mathbb{Z}/s\mathbb{Z})^2\rightarrow\mathbb{Z}$ are given by ($l,k\in\mathbb{Z}/s\mathbb{Z}$, $i,j\in I$):

$$\alpha(i,l,i,k)=2(\delta_{l-k,-2r_i}-\delta_{l-k,2r_i})$$

$$\alpha(i,l,j,k)=2\sum_{r=C_{i,j}+1,C_{i,j}+3,...,-C_{i,j}-1}(\delta_{l-k,r+r_i}-\delta_{l-k,r-r_i})\text{ (if } i\neq j)$$

$$\beta(i,l,j,k)=2\delta_{i,j}(-\delta_{l-k,r_i}+\delta_{l-k,-r_i})$$

Note that for $i,j\in I$ and $l,k\in\mathbb{Z}/s\mathbb{Z}$ we have $\alpha(i,l,j,k)=-\alpha(j,k,i,l)$ and $\beta(i,l,j,k)=-\beta(j,k,i,l)$.

This theorem is a generalization of theorem 3.11 of [He2]. It is proved in the same way except for lemma 3.7 of [He2] whose proof is changed at roots of unity: for $N\geq 1$ we denote by $\mathbb{Z}_N[z]\subset\mathbb{Z}[z]$ the subset of polynomials of degree lower that N. The following lemma is a generalization of lemma 3.7 of [He2] at roots of unity :

Lemma 3.6. *We suppose that $s\geq 1$. Let $J=\{1,...,r\}$ be a finite set of cardinal r and Λ be the polynomial commutative algebra*
$\Lambda=\mathbb{C}[\lambda_{j,m}]_{j\in J,m\geq 0}$. For $R=(R_1,...,R_r)\in\mathbb{Z}_{s-1}[z]^r$, consider:

$$\Lambda_R=exp(\sum_{j\in I,m>0}h^mR_j(q^m)\lambda_{j,m})\in\Lambda[[h]]$$

Then the $(\Lambda_R)_{R\in\mathbb{Z}_{s-1}[z]^r}$ are $\mathbb{C}$-linearly independent. In particular the $\Lambda_{j,l}=\Lambda_{(0,...,0,z^l,0,...,0)}$ ($j\in I$, $0\leq l\leq s-1$) are $\mathbb{C}$-algebraically independent.

Proof: Suppose we have a linear combination ($\mu_R \in \mathbb{C}$, only a finite number of $\mu_R \neq 0$):

$$\sum_{R \in \mathbb{Z}_{s-1}[z]^r} \mu_R \Lambda_R = 0$$

In the proof of lemma 3.7 of [He2] we saw that for $N \geq 0$, $j_1, ..., j_N \in J$, $l_1, ..., l_N > 0$, $\alpha_1, ..., \alpha_N \in \mathbb{C}$ we have:

$$\sum_{R \in \mathbb{Z}_{s-1}[z]^r / R_{j_1}(q^{l_1}) = \alpha_1, ..., R_{j_N}(q^{l_R}) = \alpha_N} \mu_R = 0$$

We set $N = sr$ and

$$((j_1, l_1), ..., (j_N, l_N)) = ((1,1), (1,2), ..., (1,s), (2,1), ..., (2,s), (3,1), ..., (r,s))$$

We get for all $\alpha_{j,l} \in \mathbb{C}$ ($j \in J, 1 \leq l \leq L$):

$$\sum_{R \in \mathbb{Z}_{s-1}[z]^r / \forall j \in J, 1 \leq l \leq s, R_j(q^l) = \alpha_{j,l}} \mu_R = 0$$

It suffices to show that there is at most one term is this sum. But consider $P, Q \in \mathbb{Z}_{s-1}[z]$ such that for all $1 \leq l \leq s$, $P(q^l) = P'(q^l)$. As q is primitive the q^l are different and so $P - P' = 0$. □

3.2. **Bicharacters, monomials and involution.**

3.2.1. *Presentation with bicharacters.* The definition of the algebra $\hat{\mathcal{Y}}_t^s$ with the Heisenberg algebra $\hat{\mathcal{H}}$ is a "concrete" construction. It can also be defined "abstractly" with bicharacters in the same spirit as Nakajima [N3] did for the simply laced case :

We define π_+ as the algebra homomorphism:

$$\pi_+ : \hat{\mathcal{Y}}_t^s \to \hat{\mathcal{Y}}^s = \mathbb{Z}[Y_{i,l}, A_{i,l}^{-1}]_{i \in I, l \in \mathbb{Z}/s\mathbb{Z}}$$

such that $\pi_+(\tilde{Y}_{i,l}^{\pm}) = Y_{i,l}^{\pm}$, $\pi_+(\tilde{A}_{i,l}^{\pm}) = A_{i,l}^{\pm}$ and $\pi_+(t) = 1$ ($\hat{\mathcal{Y}}^s$ is commutative).

We say that $m \in \hat{\mathcal{Y}}_t^s$ is a $\hat{\mathcal{Y}}_t^s$-monomial if it is a product of the $\tilde{A}_{i,l}^{-1}, \tilde{Y}_{i,l}, t^{\pm}$. For m a $\hat{\mathcal{Y}}_t^s$-monomial, $i \in I$, $l \in \mathbb{Z}/s\mathbb{Z}$ we define $y_{i,l}(m), v_{i,l}(m) \geq 0$ such that $\pi_+(m) = \prod_{i \in I, l \in \mathbb{Z}/s\mathbb{Z}} Y_{i,l}^{y_{i,l}} A_{i,l}^{-v_{i,l}}$. In order to simplify the formulas for a Laurent polynomial let $P(z) = \sum_{k \in \mathbb{Z}} P_k z^k \in \mathbb{Z}[z^{\pm}]$ ($i \in I, l \in \mathbb{Z}/s\mathbb{Z}$):

$$(P(z))_{op} \mathcal{V}_{i,l}(m) = \sum_{k \in \mathbb{Z}} P_k v_{i,l+[k]}(m)$$

We define $u_{i,l}(m) \in \mathbb{Z}$ by :

$$u_{i,l}(m) = y_{i,l}(m) - \sum_{j \in I} (C_{i,j}(z))_{op} \mathcal{V}_{j,l}(m)$$

In particular if $C_{i,j} = 0$ we have $u_{i,l}(\tilde{A}_{j,k}^{-1}) = 0$ and if $C_{i,j} < 0$:

$$u_{i,l}(\tilde{A}_{j,k}^{-1}) = -([C_{i,j}]_z)_{op} \mathcal{V}_{j,l}(\tilde{A}_{j,k}^{-1}) = \sum_{r = C_{i,j}+1 ... -C_{i,j}-1} \delta_{l+r,k}$$

In the ADE-case the coefficients of C are $-1, 0$ or 2, and we have the expression:

$$u_{i,l}(m) = y_{i,l}(m) - [z + z^{-1}]_{op} \mathcal{V}_{i,l}(m) + \sum_{j \in I / C_{i,j} = -1} v_{j,l}(m)$$

$$= y_{i,l}(m) - v_{i,l+1}(m) - v_{i,l-1}(m) + \sum_{j \in I / C_{i,j} = -1} v_{j,l}(m)$$

which is the formula used in [N3].

Definition 3.7. *For m_1, m_2 $\hat{\mathcal{Y}}_t^s$-monomials we define:*

$$d_1(m_1,m_2)=\sum_{i\in I, l\in\mathbb{Z}/s\mathbb{Z}} v_{i,l+r_i}(m_1)u_{i,l}(m_2)+y_{i,l+r_i}(m_1)v_{i,l}(m_2)$$

$$d_2(m_1,m_2)=\sum_{i\in I, l\in\mathbb{Z}/s\mathbb{Z}} u_{i,l+r_i}(m_1)v_{i,l}(m_2)+v_{i,l+r_i}(m_1)y_{i,l}(m_2)$$

For m a $\hat{\mathcal{Y}}_t^s$-monomial we have always $d_1(m,m)=d_2(m,m)$ (see section 7.3). In the ADE-case we have $d_1=d_2$ and it is the bicharacter of Nakajima [N3].

Proposition 3.8. *For m_1, m_2 $\hat{\mathcal{Y}}_t^s$-monomials, we have in $\hat{\mathcal{Y}}_t^s$:*

$$m_1m_2=t^{2d_1(m_1,m_2)-2d_2(m_2,m_1)}m_2m_1=t^{2d_2(m_1,m_2)-2d_1(m_2,m_1)}m_2m_1$$

Proof: First we check that $m_1m_2=t^{2d_1(m_1,m_2)-2d_2(m_2,m_1)}m_2m_1$ on generators:

$$2d_1(\tilde{A}_{i,l}^{-1},\tilde{A}_{i,k}^{-1})-2d_2(\tilde{A}_{i,k}^{-1},\tilde{A}_{i,l}^{-1})=2u_{i,l-r_i}(\tilde{A}_{i,k}^{-1})-2u_{i,l+r_i}(\tilde{A}_{i,k}^{-1})=2(\delta_{l-k,-2r_i}-\delta_{l-k,2r_i})=\alpha(i,l,i,k)$$

$$2d_1(\tilde{A}_{i,l}^{-1},\tilde{A}_{j,k}^{-1})-2d_2(\tilde{A}_{j,k}^{-1},\tilde{A}_{i,l}^{-1})=2u_{i,l-r_i}(\tilde{A}_{j,k}^{-1})-2u_{i,l+r_i}(\tilde{A}_{j,k}^{-1})=\alpha(i,l,j,k)$$

$$2d_1(\tilde{A}_{i,l}^{-1},\tilde{Y}_{j,k})-2d_2(\tilde{Y}_{j,k},\tilde{A}_{i,l}^{-1})=2u_{i,l-r_i}(\tilde{Y}_{j,k})-2u_{i,l+r_i}(\tilde{Y}_{j,k})=-\beta(i,l,j,k)$$

$$2d_1(\tilde{Y}_{i,l},\tilde{A}_{j,k}^{-1})-2d_2(\tilde{A}_{j,k}^{-1},\tilde{Y}_{i,l})=2v_{i,l-r_i}(\tilde{A}_{j,k}^{-1})-2v_{i,l+r_i}(\tilde{A}_{j,k}^{-1})=-\beta(i,l,j,k)=\beta_{j,k,i,l}$$

The other equality $m_1m_2=t^{2d_2(m_1,m_2)-2d_1(m_2,m_1)}m_2m_1$ is checked in the same way. □

If $B(z)$ is not symmetric, the product is defined in section 7.3.4.

3.2.2. *Involution.* We consider the $\mathbb{Z}[t^\pm]$-antilinear antimultiplicative involution of $\hat{\mathcal{Y}}_t^s$ such that $\overline{\tilde{Y}_{i,l}}=\tilde{Y}_{i,l}$, $\overline{\tilde{A}_{i,l}^{-1}}=\tilde{A}_{i,l}^{-1}$, $\overline{t}=t^{-1}$.

In [He2] we gave a "concrete" construction of this involution for the generic case: in $\hat{\mathcal{Y}}_u$ the involution is defined by $c_m\to -c_m$.

Lemma 3.9. *There is a $\mathbb{Z}[t^\pm]$-basis $\overline{A}^s$ of $\hat{\mathcal{Y}}_t^s$ such that all $m\in\overline{A}^s$ is a $\hat{\mathcal{Y}}_t^s$-monomial and:*

$$\overline{m}=t^{2d_1(m,m)}m=t^{2d_2(m,m)}m$$

Moreover for $m_1,m_2\in\overline{A}^s$ we have $m_1m_2t^{-d_1(m_1,m_2)-d_2(m_1,m_2)}\in\overline{A}^s$.

Proof: For the first point it suffices to show that for m a $\hat{\mathcal{Y}}_t^s$-monomial there is a unique $\alpha\in\mathbb{Z}$ such that $\overline{t^\alpha m}=t^{2d_1(m,m)+\alpha}m$, that is to say for m a $\hat{\mathcal{Y}}_t^s$-monomial we have $m\overline{m}^{-1}\in t^{2\mathbb{Z}}$. This is proved as in lemma 6.12 of [He2].

For the second point we compute:

$\overline{t^{-d_1(m_1,m_2)-d_2(m_1,m_2)}m_1m_2}=t^{d_1(m_1,m_2)+d_2(m_1,m_2)}\overline{m_2m_1}$
$=t^{2d_1(m_2,m_2)+2d_1(m_1,m_1)+d_1(m_1,m_2)+d_2(m_1,m_2)}m_2m_1$
$=t^{2d_1(m_2,m_2)+2d_1(m_1,m_1)+2d_1(m_2,m_1)+d_1(m_1,m_2)-d_2(m_1,m_2)}m_1m_2$
$=t^{2d_1(m_1m_2,m_1m_2)}(t^{-d_1(m_1,m_2)-d_2(m_1,m_2)}m_1m_2)$ □

For example we have $\tilde{Y}_{i,l}\in\overline{A}^s$ (because $d_1(\tilde{Y}_{i,l},\tilde{Y}_{i,l})=0$) and if $s=0$ or $s>2r_i$ we have $t\tilde{A}_{i,l}^{-1}\in\overline{A}^s$ (because $d_1(\tilde{A}_{i,l}^{-1},\tilde{A}_{i,l}^{-1})=-1$).

For $m_1,m_2\in\overline{A}^s$ we set $m_1.m_2=m_1m_2t^{-d_1(m_1,m_2)-d_2(m_1,m_2)}\in\overline{A}^s$. We have $m_1.m_2=m_2.m_1$. The non commutative multiplication can be defined from . by setting ($m_1,m_2\in\overline{A}^s$):

$$m_1m_2=t^{d_1(m_1,m_2)+d_2(m_1,m_2)}m_1.m_2$$

In the ADE-case it is the point of view adopted in [N3]. In particular if $s=0$ or $s>2r_i$, $\tilde{Y}_{i,l}$ (resp. $\tilde{A}_{i,l}^{-1}$) is denoted by $W_{i,l}$ (resp. $t^{-1}V_{i,l}$) in [N3].

Let $\overline{A}=\overline{A}^0$ and for $s\geq 0$ there is a surjective map $p_s:\overline{A}\rightarrow\overline{A}^s$ such that for $m\in\overline{A}$, $p_s(m)$ is the unique element of $\overline{A}^s$ such that for $i\in I, l\in\mathbb{Z}/s\mathbb{Z}$:

$$y_{i,l}(p_s(m))=\sum_{l'\in\mathbb{Z}/[l']=l}y_{i,l}(m)\ ,\ v_{i,l}(p_s(m))=\sum_{l'\in\mathbb{Z}/[l']=l}v_{i,l}(m)$$

In particular it gives a $\mathbb{Z}[t^{\pm}]$-linear map $p_s:\hat{\mathcal{Y}}_t\rightarrow\hat{\mathcal{Y}}_t^s$.

3.2.3. *Notations and technical complements.* A $\hat{\mathcal{Y}}_t^s$-monomial is said to be i-dominant (resp. i-antidominant) if $\forall l\in\mathbb{Z}/s\mathbb{Z}$, $u_{i,l}(m)\geq 0$ (resp. $u_{i,l}(m)\leq 0$). We denote by $\overline{B}_i^s$ the set of i-dominant monomials m such that $m\in\overline{A}^s$.

A $\hat{\mathcal{Y}}_t^s$-monomial is said to be dominant (resp. antidominant) if $\forall l\in\mathbb{Z}/s\mathbb{Z},\forall i\in I$, $u_{i,l}(m)\geq 0$ (resp. $u_{i,l}(m)\leq 0$). We denote by $\overline{B}^s$ the set of dominant monomials m such that $m\in\overline{A}^s$. In the generic case let $\overline{A}=\overline{A}^0$, $\overline{B}_i=\overline{B}_i^0$, $\overline{B}=\overline{B}^0$.

We denote by $\hat{A}^s=\{m=\prod_{i\in I,l\in\mathbb{Z}/s\mathbb{Z}}Y_{i,l}^{y_{i,l}}A_{i,l}^{-v_{i,l}}/u_{i,l},v_{i,l}\geq 0\}\subset\hat{\mathcal{Y}}^s$ the set of $\hat{\mathcal{Y}}^s$-monomials. It is a $\mathbb{Z}$-basis of $\hat{\mathcal{Y}}^s$ and $\pi_+(\overline{A}^s)=\hat{A}^s$. Let $\hat{B}_i^s=\{m\in A^s/\forall l\in\mathbb{Z}/s\mathbb{Z},u_{i,l}(m)\geq 0\}=\pi_+(\overline{B}_i^s)$, $\hat{B}^s=\bigcap_{i\in I}\hat{B}_i^s=\pi_+(\overline{B}^s)$.

We define $\hat{\Pi}:\hat{\mathcal{Y}}_t^s\rightarrow\mathcal{Y}^s=\mathbb{Z}[Y_{i,l}^{\pm}]_{i\in I,l\in\mathbb{Z}/s\mathbb{Z}}$ as the ring morphism such that for m a $\hat{\mathcal{Y}}_t$-monomial $\hat{\Pi}(m)=\prod_{i\in I,l\in\mathbb{Z}/s\mathbb{Z}}Y_{i,l}^{u_{i,l}(m)}$ ($\mathcal{Y}^s$ is commutative).

In particular for $i\in I$, $l\in\mathbb{Z}/s\mathbb{Z}$, we have:

$$\hat{\Pi}(\tilde{A}_{i,l}^{-1})=Y_{i,l-r_i}^{-1}Y_{i,l+r_i}^{-1}\prod_{j/C_{j,i}<0}\prod_{k=C_{j,i}+1,C_{j,i}+3,...,-C_{j,i}-1}Y_{j,l+2k}$$

and we denote this term by $A_{i,l}^{-1}=\hat{\Pi}(\tilde{A}_{i,l}^{-1})$. Let $A^s=\{m=\prod_{i\in I,l\in\mathbb{Z}/s\mathbb{Z}}Y_{i,l}^{u_{i,l}(m)}/u_{i,l}(m)\in\mathbb{Z}\}=\hat{\Pi}(\overline{A}^s)$ the set of $\mathcal{Y}^s$-monomials, $B_i^s=\{m\in A^s/\forall l\in\mathbb{Z}/s\mathbb{Z},u_{i,l}(m)\geq 0\}=\hat{\Pi}(\overline{B}_i^s)$, $B^s=\bigcap_{i\in I}B_i^s=\hat{\Pi}(\overline{B}^s)$.

If q is generic then for $M\in\overline{A}$ and $m\in A$ there at most one $m'\in\overline{A}^s$ of the form $m'=t^\alpha M\tilde{A}_{i_1,l_1}^{-1}...\tilde{A}_{i_K,l_K}^{-1}$ such that $\hat{\Pi}(m')=m$ (the $A_{i,l}^{-1}$ are algebraically independant because we have supposed $\det(C(z))\neq 0$, see [He2]).

If q is a root of unity the situation can be different: for example we suppose that C is of type $A_2^{(1)}$ and $s=3$ (so $\det(C(q))=q^{-3}(q^3-1)^2=0$). Then for all $L\geq 0$, we have:

$$\hat{\Pi}(\tilde{Y}_{1,0}\tilde{A}_{1,1}^{-L}\tilde{A}_{2,2}^{-L}\tilde{A}_{3,3}^{-L})=Y_{1,0}$$

and $\hat{\Pi}^{-1}(Y_{1,0})$ is infinite.

If C is finite the situation is better. We have a generalization of lemma 3.14 of [He2] at roots of unity:

Lemma 3.10. *We suppose that C is finite and that $s\geq 1$. Let M be in $\overline{A}^s$. Then:*

i) There is at most a finite number of $m'\in\overline{A}^s$ of the form $m'=t^\alpha M\tilde{A}_{i_1,l_1}^{-1}...\tilde{A}_{i_K,l_K}^{-1}$ such that m' is dominant.

ii) For $m\in A^s$ there is at most a finite number of $m'\in\overline{A}^s$ of the form $m'=t^\alpha M\tilde{A}_{i_1,l_1}^{-1}...\tilde{A}_{i_K,l_K}^{-1}$ such that $\hat{\Pi}(m')=m$.

Proof: First let us show (i) : let m' be in $\overline{A}^s$ with $m'=t^\alpha M\prod_{i\in I,l\in\mathbb{Z}/s\mathbb{Z}}A_{i,l}^{-v_{i,l}}$ and the $v_{i,l}\geq 0$. It suffices to show that the condition m' dominant implies that the $v_i=\sum_{l'\in\mathbb{Z}/s\mathbb{Z}}v_{i,l}$ are bounded (because $\mathbb{Z}/s\mathbb{Z}$ is finite). This condition implies :

$$u_i(m') = -2v_i + \sum_{j\neq i}(-C_{i,j}v_j) + u_i(M) \geq 0$$

Let U be the column vector with coefficients $(u_1(M), ..., u_n(M))$ and V the column vector with coefficients $(v_1, ..., v_n)$. So we have $U - CV \geq 0$. As C is finite, the theorem 4.3 of [Ka] implies that $C^{-1}U - V \geq 0$ and so the v_i are bounded.

For the (ii) we use the same proof with the condition :

$$u_i(m') = -2v_i + \sum_{j\neq i}(-C_{i,j}v_j) + u_i(M) = u_i(m)$$

□

In some cases we have another result. For $i \in I$ let $L_i = (C_{i,1}, ..., C_{i,n})$.

Lemma 3.11. *We suppose that $s \geq 1$ and that there are $(\alpha_i)_{i\in I} \in \mathbb{Z}^I$ such that $\alpha_i > 0$ and:*

$$\sum_{j\in I}\alpha_j L_j = 0$$

Then for $M \in A^s$ there are at most a finite number of dominant monomials $m \in B^s$ of the form $m = MA^{-1}_{i_1,l_1}A^{-1}_{i_2,l_3}...A^{-1}_{i_k,l_k}$.

In particular an affine Cartan matrix verifies the property of the lemma (see [Ka] for the coefficients α_j).

Proof: Consider $m' = \prod_{i\in I, l\in\mathbb{Z}/s\mathbb{Z}} A_{i,l}^{-v_{i,l}}$ and $m = Mm'$. For $i \in I$ let $v_i = \sum_{l\in\mathbb{Z}/s\mathbb{Z}} v_{i,l} \geq 0$. We have:

$$\sum_{i\in I}\alpha_i u_i(m') = \sum_{i\in I}\alpha_i\sum_{j\in I}(-C_{i,j})v_j = -\sum_{j\in I}v_j(\sum_{i\in I}\alpha_i C_{i,j}) = 0$$

We suppose that m is dominant, in particular $u_{i,l}(m') \geq -u_{i,l}(M)$. So:

$$u_{i,l}(m') = u_i(m') - \sum_{l'\in\mathbb{Z}/s\mathbb{Z}, l'\neq l} u_{i,l'}(m') \leq u_i(m') + \sum_{l'\in\mathbb{Z}/s\mathbb{Z}, l'\neq l} u_{i,l'}(M)$$

$$\leq \frac{1}{\alpha_i}\sum_{j\neq i}\alpha_j(-u_j(m')) + \sum_{l'\in\mathbb{Z}/s\mathbb{Z}, l'\neq l} u_{i,l'}(M) \leq \frac{1}{\alpha_i}\sum_{j\neq i}\alpha_j u_j(M) + \sum_{l'\in\mathbb{Z}/s\mathbb{Z}, l'\neq l} u_{i,l'}(M)$$

So the $u_{i,l}(m')$ $(i \in I, l \in \mathbb{Z}/s\mathbb{Z})$ are bounded and there is at most a finite number of m' such that m is dominant. □

4. q, t-characters in the generic case

In [He2] we defined q, t-characters for all finite Cartan matrices in the generic case. In this section we define q and q, t-characters for all symmetrizable (non necessarily finite) Cartan matrix such that $i \neq j \Rightarrow C_{i,j}C_{j,i} \leq 3$, in particular for Cartan matrices of affine type (except $A_1^{(1)}$, $A_2^{(2)}$). We suppose $s = 0$, that is to say q is generic. The root of unity case will be studied in section 5.

4.1. **Deformed screening operators.** Classical screening operators were introduced in [FR2] and t-deformed screening operators were introduced in [He1] for C finite. We define and study deformed screening operators in the general case:

Definition 4.1. *$\hat{\mathcal{Y}}_{i,u}$ is the $\hat{\mathcal{Y}}_u$-bimodule defined by generators $\tilde{S}_{i,l}$ $(i \in I, l \in \mathbb{Z})$ and relations :*

$$\tilde{S}_{i,l}\tilde{A}^{-1}_{j,k} = t_{-C_{i,j}(z)(z^{(k-l)}+z^{(l-k)})}\tilde{A}^{-1}_{j,k}\tilde{S}_{i,l}$$

$$\tilde{S}_{i,l}\tilde{Y}_{j,k} = t_{\delta_{i,j}(z^{(k-l)}+z^{(l-k)})}\tilde{Y}_{j,k}\tilde{S}_{i,l} \ , \ \tilde{S}_{i,l}t = t\tilde{S}_{i,l}$$

$$\tilde{S}_{i,l-r_i} - t_{-q^{-2r_i}-1}\tilde{A}^{-1}_{i,l}\tilde{S}_{i,l+r_i} = 0$$

In [He2] we made a concrete construction of $\hat{\mathcal{Y}}_{i,u}$ by realizing it in $\hat{\mathcal{H}}_h$. Note that $\hat{\mathcal{Y}}_t$ is a $\hat{\mathcal{Y}}_u$-bimodule using the projection $\hat{\mathcal{Y}}_u \to \hat{\mathcal{Y}}_t$.

Definition 4.2. *$\hat{\mathcal{Y}}_{i,t}$ is the $\hat{\mathcal{Y}}_t$-bimodule $\hat{\mathcal{Y}}_t \otimes_{\hat{\mathcal{Y}}_u} \hat{\mathcal{Y}}_{i,u} \otimes_{\hat{\mathcal{Y}}_u} \hat{\mathcal{Y}}_t$.*

For $l \in \mathbb{Z}$ we denote by $\tilde{S}_{i,l}$ the image of $\tilde{S}_{i,l}$ in $\hat{\mathcal{Y}}_{i,t}$. The $\hat{\mathcal{Y}}_t$-module $\hat{\mathcal{Y}}_{i,t}$ is torsion free.

For m a $\hat{\mathcal{Y}}_t^s$-monomial only a finite number of $[\tilde{S}_{i,l}, m] = (t^2-1)t^{u_{i,l}(m)-1}[u_{i,l}(m)]_t \tilde{S}_{i,l} \in (t^2-1)\hat{\mathcal{Y}}_{i,t}$ are not equal to 0, so we can define:

Definition 4.3. *The i^{th}-deformed screening operator is the map $S_{i,t} : \hat{\mathcal{Y}}_t \to \hat{\mathcal{Y}}_{i,t}$ defined by ($\lambda \in \hat{\mathcal{Y}}_t$):*

$$S_{i,t}(\lambda) = \frac{1}{t^2-1}\sum_{l\in\mathbb{Z}}[\tilde{S}_{i,l}, \lambda] \in \hat{\mathcal{Y}}_{i,t}$$

Let $\hat{\mathfrak{K}}_{i,t} = \mathrm{Ker}(S_{i,t})$. As $S_{i,t}$ is a derivation, $\hat{\mathfrak{K}}_{i,t}$ is a subalgebra of $\hat{\mathcal{Y}}_t$.

At $t=1$ we define $S_i : \mathcal{Y} \to \mathcal{Y}_i = \bigoplus_{l\in\mathbb{Z}} \mathcal{Y}S_{i,l} / \sum_{l\in\mathbb{Z}} \mathcal{Y}.(S_{i,l-r_i} - A_{i,l}^{-1}S_{i,l+r_i})$ such for $m \in A$, $S_i(m) = m\sum_{l\in\mathbb{Z}} u_{i,l}(m)S_{i,l}$. It is the classical screening operator (see [FR2]). For $m \in \hat{\mathcal{Y}}_t$ we have $S_i(\hat{\Pi}(m)) = \hat{\Pi}(S_{i,t}(m))$ where $\hat{\Pi} : \hat{\mathcal{Y}}_{i,t} \to \mathcal{Y}_i$ is defined by $\hat{\Pi}(m\tilde{S}_{i,l}) = \hat{\Pi}(m)S_{i,l}$.

We set $\mathfrak{K}_i = \mathrm{Ker}(S_i)$ and $\mathfrak{K} = \bigcap_{i\in I}\mathfrak{K}_i$.

In the following a product $\overrightarrow{\prod_{l\in\mathbb{Z}}} M_l$ (resp. $\overleftarrow{\prod_{l\in\mathbb{Z}}} M_l$) is the ordered product $...M_{-2}M_{-1}M_0M_1...$ (resp. $...M_2M_1M_0M_{-1}...$).

Definition 4.4. *For $M \in \hat{\mathcal{Y}}_t$ a i-dominant monomial we define:*

$$\overleftarrow{E}_{i,t}(M) = M(\prod_{l\in\mathbb{Z}}\tilde{Y}_{i,l}^{u_{i,l}(M)})^{-1}\overleftarrow{\prod_{l\in\mathbb{Z}}}(\tilde{Y}_{i,l}(1+t\tilde{A}_{i,l+r_i}^{-1}))^{u_{i,l}(M)} \in \hat{\mathcal{Y}}_t$$

For example we have $\overleftarrow{E}_{i,t}(\tilde{Y}_{i,l}) = \tilde{Y}_{i,l}(1+t\tilde{A}_{i,l+r_i}^{-1})$, $\overleftarrow{E}_{i,t}(\tilde{A}_{i,l}^{-1}\tilde{Y}_{i,l+r_i}\tilde{Y}_{i,l-r_i}) = \tilde{A}_{i,l}^{-1}\tilde{Y}_{i,l+r_i}\tilde{Y}_{i,l-r_i}$ and for $j \neq i$: $\overleftarrow{E}_{i,t}(\tilde{Y}_{j,l}) = \tilde{Y}_{j,l}$.

Theorem 4.5. *([He1]) For all Cartan matrix C, the kernel $\hat{\mathfrak{K}}_{i,t}$ of $S_{i,t}$ is the $\mathbb{Z}[t^{\pm}]$-subalgebra of $\hat{\mathcal{Y}}_t$ generated by the ($l \in \mathbb{Z}, j \neq i$):*

$$\tilde{Y}_{i,l}(1+t\tilde{A}_{i,l+r_i}^{-1}) \ , \ \tilde{A}_{i,l}^{-1}\tilde{Y}_{i,l+r_i}\tilde{Y}_{i,l-r_i} \ , \ \tilde{Y}_{j,l} \ , \ \overleftarrow{E}_{i,t}(\tilde{A}_{j,l}^{-1})$$

For M a i-dominant monomial we have $\overleftarrow{E}_{i,t}(M) \in \hat{\mathfrak{K}}_{i,t}$, and:

$$\hat{\mathfrak{K}}_{i,t} = \bigoplus_{M\in\overline{B}_i}\mathbb{Z}[t^{\pm}]\overleftarrow{E}_{i,t}(M)$$

Note that the proof of [He1] works also if C is not finite : the point of this proof is that an element $\chi \in \hat{\mathfrak{K}}_{i,t} - \{0\}$ has at least one i-dominant monomial, which is shown as in the sl_2-case.

At $t=1$ it is a classical result of [FR2].

Note that in the ADE-case the identification (see section 3.2.2) between the $t\tilde{A}_{i,l}^{-1}$ and the $V_{i,l}$ shows that the notation $\hat{\mathfrak{K}}_{i,t}$ coincides with the notation of [N3].

4.2. **Reminder on the algorithm of Frenkel-Mukhin and on the deformed algorithm.**

4.2.1. *Completed algebras.* Let $\hat{\mathfrak{K}}_t=\bigcap_{i\in I}\mathrm{Ker}(S_{i,t})\subset\hat{\mathcal{Y}}_t$. It is a subalgebra of $\hat{\mathcal{Y}}_t$.

We recall that a partial ordering is defined on the $\hat{\mathcal{Y}}_t$-monomials by $m\in t^{\mathbb{Z}}\tilde{A}_{i,l}^{-1}m'\Leftrightarrow m<m'$.

We define a $\mathbb{N}$-graduation of $\hat{\mathcal{Y}}_t$ by putting $\deg(\tilde{A}_{i,l}^{-1})=1$, $\deg(\tilde{Y}_{i,l})=0$. Note that $m<m'\Rightarrow\deg(m)>\deg(m')$.

We define the algebra $\hat{\mathcal{Y}}_t^\infty\supset\hat{\mathcal{Y}}_t$ as the completion for this gradation. In particular the elements of $\hat{\mathcal{Y}}_t^\infty$ are (infinite) sums $\sum_{k\geq 0}\lambda_k$ such that λ_k is homogeneous of degree k.

In the same way we define $\hat{\mathfrak{K}}_{i,t}^\infty$ such that $\hat{\mathcal{Y}}_t^\infty\supset\hat{\mathfrak{K}}_{i,t}^\infty\supset\hat{\mathfrak{K}}_{i,t}$, that is to say $\chi\in\hat{\mathcal{Y}}_t^\infty$ is in $\hat{\mathfrak{K}}_{i,t}^\infty$ if and only if it is of the form $\chi=\sum_{k\geq 0}\chi_k$ where:

$$\chi_k\in\bigoplus_{M\in\overline{B}_i/\deg(M)=k}\mathbb{Z}[t^\pm]\overleftarrow{E}_{i,t}(M)$$

Let $\hat{\mathfrak{K}}_t^\infty=\bigcap_{i\in I}\hat{\mathfrak{K}}_{i,t}^\infty$.

In the same way for $t=1$ we define $\hat{\mathcal{Y}}^\infty\supset\hat{\mathcal{Y}}$, $\mathcal{Y}^\infty\supset\mathcal{Y}$, $\hat{\mathfrak{K}}^\infty\supset\hat{\mathfrak{K}}$, $\mathfrak{K}^\infty\supset\mathfrak{K}$. They are well defined because in $\hat{\mathcal{Y}}$ and in $\mathcal{Y}$ the $A_{i,l}^{-1}$ are algebraically independent (see section 3.2.3) and π_+ preserves the degree. In particular the maps π_+ and $\hat{\Pi}$ can be extended to maps $\pi_+:\hat{\mathcal{Y}}_t^\infty\to\hat{\mathcal{Y}}^\infty$ and $\hat{\Pi}:\hat{\mathcal{Y}}_t^\infty\to\mathcal{Y}^\infty$.

For m a $\hat{\mathcal{Y}}_t$-monomial let $u(m)=\max\{l\in\mathbb{Z}/\forall k<l,\forall i\in I,u_{i,k}(m)=0\}$. We define the subset $C(m)\subset\overline{A}$

$$C(m)=\{t^{\mathbb{Z}}m\tilde{A}_{i_1,l_1}^{-1}...\tilde{A}_{i_N,l_N}^{-1}/N\geq 0,l_1,...,l_N\geq u(m)\}\cap\overline{A}$$

We define the $\mathbb{Z}[t^\pm]$-submodule of $\hat{\mathcal{Y}}_t^\infty$:

$$\tilde{C}(m)=\{\chi\in\hat{\mathcal{Y}}_t^\infty/\chi=\sum_{m'\in C(m)}\lambda_{m'}(t)m'\}$$

Lemma 4.6. *An element of $\hat{\mathfrak{K}}_t^\infty-\{0\}$ has at least one dominant monomial. An element of $\hat{\mathfrak{K}}_t-\{0\}$ has at least one dominant monomial and one antidominant monomial.*

Proof: For $\chi\in\hat{\mathfrak{K}}_t^\infty$ let M be a maximal monomial of χ. Then in the decomposition $\chi=\sum_{k\geq 0}\chi_k^{(i)}$ where $\chi_k^{(i)}\in\bigoplus_{M\in\overline{B}_i/\deg(M)=k}\mathbb{Z}[t^\pm]\overleftarrow{E}_{i,t}(M)$ we see that M is i-dominant.

For $\chi\in\hat{\mathfrak{K}}_t$, we can consider a maximal and a minimal monomial, and so we have a dominant monomial and an antidominant monomial in χ. □

4.2.2. *Algorithms.* In [He2] we defined a deformed algorithm to compute q,t-characters for C finite. We had to show that this algorithm is well defined, that is to say that at each step the different ways to compute each term give the same result.

The formulas of [He2] gives also a (non necessarily well defined) deformed algorithm for all Cartan matrices, that is to say:

Let $m\in\overline{B}$. If the deformed algorithm beginning with m is well defined, it gives an element $\hat{F}_t(m)\in\hat{\mathfrak{K}}_t^\infty$ such that m is the unique dominant monomial of $\hat{F}_t(m)$.

An algorithm was also used by Nakajima in the ADE-case in [N2]. If we set $t=1$ and apply $\hat{\Pi}$ (where $\hat{\Pi}$ is defined in section 3.2.3) we get a classical algorithm (it is analogous to the algorithm constructed by Frenkel and Mukhin in [FM1]). So:

Let $m\in B$. If the classical algorithm beginning with m is well defined, it gives an element $F(m)\in\mathfrak{K}^\infty$ such that m is the unique dominant monomial of $F(m)$.

We say that the classical algorithm (resp. the deformed algorithm) is well defined if for all $m \in B$ (resp. all $m \in \overline{B}$) the classical algorithm (resp. deformed algorithm) beginning with m is well defined.

Lemma 4.7. *If the deformed algorithm is well defined then the classical algorithm is well defined.*

Proof: If the deformed algorithm beginning with m is well defined then the classical algorithm beginning with $\hat{\Pi}(m)$ is well defined and $F(\hat{\Pi}(m)) = \hat{\Pi}(\hat{F}_t(m))$. □

The following results are known:

If C is finite then the classical algorithm is well defined ([FM1]).

If C is finite and symmetric then the deformed algorithm is well defined ([N3]).

If C is finite then the deformed algorithm is well defined ([He2]).

In this section (theorem 4.9) we show that the classical and the deformed algorithms are well defined for a (non necessarily finite) Cartan matrix such that $i \neq j \Rightarrow C_{i,j}C_{j,i} \leq 3$.

4.3. **Morphism of q, t-characters.** The construction of [He2] is based on the fact that we can compute explicitly q, t-characters for the submatrices of format 2 of the Cartan matrix. So:

4.3.1. *The case $n = 2$.*

Proposition 4.8. *We suppose that C is a Cartan matrix of rank 2. The following properties are equivalent:*

i) For all $m \in B$, $F(m) \in \mathfrak{K}$

ii) C is finite

iii) $C_{1,2}C_{2,1} \leq 3$

iv) For $i = 1$ or 2, $\hat{\mathfrak{K}}_t \cap \tilde{C}(\tilde{Y}_{i,0}) \neq \{0\}$

v) For $i = 1$ or 2, $C(Y_{i,0})$ has an antidominant monomial

Proof: The Cartan matrices of rank 2 such that $C_{1,2}C_{2,1} \leq 3$ are matrices of type $A_1 \times A_1$, A_2, B_2, C_2, G_2 or G_2^t. Those are finite Cartan matrices of rank 2, so $(ii) \Leftrightarrow (iii)$. Moreover if C is finite, the classical theory of q-characters shows $(ii) \Rightarrow (i)$.

We have seen in [He2] that $(ii) \Rightarrow (iv)$. It follows from lemma 4.6 that $(iv) \Rightarrow (v)$ and $(i) \Rightarrow (v)$.

So it suffices to show that $(v) \Rightarrow (iii)$. We suppose there is an antidominant monomial $m \in C(Y_{1,0})$. We can suppose $C_{1,2} < 0$ and $C_{2,1} < 0$. m verifies $\hat{\Pi}(m) = Y_{1,0}A_{1,l_1}^{-1}...A_{1,l_L}^{-1}A_{2,l_1}^{-1}...A_{2,l_M}^{-1}$ where $L, M \geq 0$. In particular we have:

$$u_1(m) = 1 - 2L - MC_{1,2} \text{ and } u_2(m) = -2M - LC_{2,1}$$

As m is antidominant, we have $u_1(m), u_2(m) \leq 0$.

if $M = 0$, we have $u_2(m) = -LC_{2,1} \leq 0 \Rightarrow L = 0$ and $u_1(m) = 1 > 0$, impossible.

if $M > 0$, we have:

$$\frac{L}{M} > \frac{-C_{1,2}}{2} \text{ and } \frac{L}{M} \leq \frac{2}{-C_{2,1}}$$

$$\frac{-C_{1,2}}{2} < \frac{2}{-C_{2,1}} \Rightarrow C_{1,2}C_{2,1} \leq 3$$

□

4.3.2. *General case.*

Theorem 4.9. *If $i \neq j \Rightarrow C_{i,j}C_{j,i} \leq 3$, then the classical and the deformed algorithms are well defined.*

Proof: It suffices to show that the deformed algorithm is well defined (lemma 4.7).We follow the proof of theorem 5.13 of [He2]: it suffices to construct $\hat{F}_t(m)$ for $m = \tilde{Y}_{i,0}$ $(i \in I)$ and it suffices to see the property for the matrices $\begin{pmatrix} 2 & C_{i,j} \\ C_{j,i} & 2 \end{pmatrix}$. If $r_i \wedge r_j = 1$ this follows from proposition 4.8. If $r_i \wedge r_j > 1$ it suffices to replace r_i, r_j with $\frac{r_i}{r_i \wedge r_j}, \frac{r_j}{r_i \wedge r_j}$ (in fact it means that we replace q by $q^{r_i \wedge r_j}$). $\square$

In the following we suppose that $i \neq j \Rightarrow C_{i,j}C_{j,i} \leq 3$. For example C could be of finite or affine type (except $A_1^{(1)}$, $A_2^{(2)}$).

We conjecture that for C of type $A_1^{(1)}$ (with $r_1 = r_2 = 2$) and of type $A_2^{(2)}$ the algorithms are well defined. This conjecture is motivated by the remarks of the introduction about representation theory of quantum affinization algebras (note that for C of type $A_1^{(1)}$ and $r_1 = r_2 = 1$ the classical algorithm is not well defined).

4.3.3. *Definition of $\chi_{q,t}$.* We verify as in [He2] that $\hat{F}_t(\tilde{Y}_{i,l})\hat{F}_t(\tilde{Y}_{j,l}) = \hat{F}_t(\tilde{Y}_{j,l})\hat{F}_t(\tilde{Y}_{i,l})$. Let Rep $= \mathbb{Z}[X_{i,l}]_{i \in I, l \in \mathbb{Z}}$ as in section 2.2 and a Rep-monomials is a product of the $X_{i,l}$.

Definition 4.10. *The morphism of q,t-characters $\chi_{q,t} : Rep \rightarrow \hat{\mathfrak{K}}_t^\infty$ is the $\mathbb{Z}$-linear map such that:*

$$\chi_{q,t}(\prod_{i\in I, l\in\mathbb{Z}} X_{i,l}^{x_{i,l}}) = \overrightarrow{\prod_{l\in\mathbb{Z}}}\prod_{i\in I}\hat{F}_t(\tilde{Y}_{i,l})^{x_{i,l}}$$

The morphism of q-characters $\chi_q : Rep \rightarrow \mathfrak{K}^\infty$ is defined by $\chi_q = \hat{\Pi} \circ \hat{\chi}_{q,t}$.

Theorem 4.11. *(*[He2]*) The $\mathbb{Z}$-linear map $\chi_{q,t} : Rep \rightarrow \hat{\mathcal{Y}}_t^\infty$ is injective and is characterized by the three following properties:*

1) For M a Rep-monomial define $m = \prod_{i\in I, l\in\mathbb{Z}} \tilde{Y}_{i,l}^{x_{i,l}(M)} \in \overline{B}$. Then we have :

$$\chi_{q,t}(M) = m + \sum_{m'<m} a_{m'}(t)m' \text{ (where } a_{m'}(t) \in \mathbb{Z}[t^\pm])$$

2) The image $Im(\chi_{q,t})$ is contained in $\hat{\mathfrak{K}}_t^\infty$.

3) Let M_1, M_2 be Rep-monomials such that $max\{l/\sum_{i\in I} x_{i,l}(M_1) > 0\} \leq min\{l/\sum_{i\in I} x_{i,l}(M_2) > 0\}$. We have :

$$\chi_{q,t}(M_1M_2) = \chi_{q,t}(M_1)\chi_{q,t}(M_2)$$

Those properties are generalizations of Nakajima's axioms [N3] for q generic, so:

Corollary 4.12. *If C is finite then we have $\pi_+(Im(\chi_{q,t})) \subset \hat{\mathcal{Y}}$ and $\chi_q : Rep \rightarrow \mathcal{Y}$ is the classical morphism of q-characters and $\chi_{q,t}$ is the morphism of* [He2]*. In particular if C is of type ADE then $\chi_{q,t}$ is the morphism of q,t-characters of* [N3].

5. ϵ, t-characters in the root of unity case

In this section we define and study ϵ, t-characters at roots of unity: let $\epsilon \in \mathbb{C}^*$ be a s^{th}-primitive root of unity. We suppose that $s > 2r^\vee$.

The case $t = 1$ was study in [FM2] (but classical screening operators in the root of unity case were not defined). The t-deformations were studied in the ADE-case by Nakajima in [N3] using quiver varieties. In this section we suppose that $i \neq j \Rightarrow C_{i,j}C_{j,i} \leq 3$ and $B(z)$ is symmetric. In particular C can be of finite type or of affine type (except $A_1^{(1)}$, $A_{2l}^{(2)}$, $l \geq 1$, see section 7.3.3). The deformed algorithm is well defined and $\chi_{q,t}$ exists (theorem 4.9).

5.1. **Reminder: classical ϵ-characters at roots of unity.** We define $\tau_s : \mathcal{Y} \to \mathcal{Y}_s$ as the ring homomorphism such that $\tau_s(Y_{i,l}) = Y_{i,[l]}$ where for $l \in \mathbb{Z}$ we denote by $[l]$ its image in $\mathbb{Z}/s\mathbb{Z}$.

If C is finite the morphism of ϵ-characters $\chi_\epsilon : \mathrm{Rep}^s \to \mathcal{Y}^s$ is defined by Frenkel and Mukhin (see section 2.2). We have the following characterization:

Theorem 5.1. *([FM2]) If C is finite, the morphism of ϵ-characters $\chi_\epsilon : Rep^s \to \mathcal{Y}^s$ verifies ($l_0 \in \mathbb{Z}$):*

$$\chi_\epsilon(\prod_{i\in I, l\in\mathbb{Z}/s\mathbb{Z}} X_{i,l}^{x_{i,l}}) = \tau_s(\chi_q(\prod_{i\in I, l_0\leq l\leq l_0+s-1} X_{i,l}^{x_{i,[l]}}))$$

Note that this formula suffices to characterize the $\mathbb{Z}$-linear map χ_ϵ.

If C is not finite, we can consider $\hat{\mathcal{Y}}^s = \mathbb{Z}[Y_{i,l}, A_{i,l}^{-1}]_{i\in I, l\in\mathbb{Z}/s\mathbb{Z}}$ and the completion $\hat{\mathcal{Y}}_t^{s,\infty}$ as in the generic case. We define $\hat{\chi}_\epsilon : \mathrm{Rep}^s \to \hat{\mathcal{Y}}^{s,\infty}$ with the formula of the theorem 5.1. The map $\hat{\chi}_\epsilon$ is also an injective ring homomorphism.

In the following we give an analogous construction in the deformed case $t \neq 1$.

5.2. **Construction of $\chi_{\epsilon,t}$.** The point for the t-deformation is that we can not define a natural t-analog of τ_s which is a ring homomorphism. In this section we construct an analog $\tau_{s,t}$ of τ_s which is not a ring homomorphism but has nice properties.

5.2.1. *Definition of $\tau_{s,t}$.* First let us briefly explain how $\tau_{s,t}$ is constructed. The main property is a compatibility with some ordered products: suppose that $l_1 > l_2$ ($l_1, l_2 \in \mathbb{Z}$), that $m_1 \in \hat{\mathcal{Y}}_t$ involves only the $\tilde{Y}_{i,l_1}, \tilde{A}_{i,l_1}^{-1}$ and that m_2 involves only the $\tilde{Y}_{i,l_2}, \tilde{A}_{i,l_2}^{-1}$. Then $\tau_{s,t}$ is defined such that $\tau_{s,t}(m_1 m_2) = \tau_{s,t}(m_1)\tau_{s,t}(m_2)$. Let us now write it in a formal way:

For m a $\hat{\mathcal{Y}}_t$-monomial and $l \in \mathbb{Z}$, let :

$$\pi_l(m) = (\prod_{i\in I} \tilde{Y}_{i,l}^{y_{i,l}(m)})(\prod_{i\in I} \tilde{A}_{i,l}^{-v_{i,l}(m)})$$

It is well defined because for $i, j \in I$ and $l \in \mathbb{Z}$ we have $\tilde{Y}_{i,l}\tilde{Y}_{j,l} = \tilde{Y}_{j,l}\tilde{Y}_{i,l}$, $\tilde{A}_{i,l}^{-1}\tilde{A}_{j,l}^{-1} = \tilde{A}_{j,l}^{-1}\tilde{A}_{i,l}^{-1}$ and for $i \neq j$, $\tilde{A}_{i,l}^{-1}\tilde{Y}_{j,l} = \tilde{Y}_{j,l}\tilde{A}_{i,l}^{-1}$ (theorem 3.5).

Let $\overrightarrow{m} = \overrightarrow{\prod_{l\in\mathbb{Z}}} \pi_l(m)$, $\overleftarrow{m} = \overleftarrow{\prod_{l\in\mathbb{Z}}} \pi_l(m)$, and :

$$\overrightarrow{A} = \{\overrightarrow{m} / \ m \ \hat{\mathcal{Y}}_t\text{-monomial}\} \text{ and } \overleftarrow{A} = \{\overleftarrow{m} / \ m \ \hat{\mathcal{Y}}_t\text{-monomial}\}$$

It follows from theorem 3.5 that $\overrightarrow{A}$ and $\overleftarrow{A}$ are $\mathbb{Z}[t^{\pm}]$-basis of $\hat{\mathcal{Y}}_t$.

Definition 5.2. *We define $\tau_{s,t} : \hat{\mathcal{Y}}_t \to \hat{\mathcal{Y}}_t^s$ as the $\mathbb{Z}[t^{\pm}]$-linear map such that for $m \in \overleftarrow{A}$:*

$$\tau_{s,t}(m) = \overleftarrow{\prod_{l\in\mathbb{Z}}}(\prod_{j\in I} \tilde{A}_{j,[l]}^{-v_{j,l}(m)})(\prod_{j\in I} \tilde{Y}_{j,[l]}^{y_{j,l}(m)})$$

Note that $\tau_{s,t}$ is not a ring homomorphism and is not injective.

5.2.2. *Definition of $\chi_{\epsilon,t}$.* We define a $\mathbb{N}$-gradation of $\hat{\mathcal{Y}}_t^s$, the completed algebra $\hat{\mathcal{Y}}_t^{s,\infty}$ in the same way as we did for the generic case (section 4.2.1). In particular $\tau_{s,t}$ is compatible with the gradations of $\hat{\mathcal{Y}}_t$ and $\hat{\mathcal{Y}}_t^s$ and is extended to a map $\tau_{s,t} : \hat{\mathcal{Y}}_t^\infty \to \hat{\mathcal{Y}}_t^{s,\infty}$.

Definition 5.3. *The morphism of q,t-characters at the s^{th}-primitive roots of unity $\chi_{\epsilon,t} : Rep^s \to \hat{\mathcal{Y}}_t^{s,\infty}$ is the $\mathbb{Z}$-linear map such that:*

$$\chi_{\epsilon,t}(\prod_{i\in I, l\in\mathbb{Z}/s\mathbb{Z}} X_{i,l}^{x_{i,l}}) = \tau_{s,t}(\chi_{q,t}(\prod_{i\in I, 0\leq l\leq s-1} X_{i,l}^{x_{i,[l]}}))$$

Proposition 5.4. *The morphism $\chi_{\epsilon,t}$ verifies the following properties:*

1) The following diagram is commutative:

$$\begin{array}{ccc} Rep^s & \xrightarrow{\chi_{\epsilon,t}} & Im(\chi_{\epsilon,t}) \\ id\downarrow & & \downarrow \quad \pi_+ \\ Rep^s & \xrightarrow{\hat{\chi}_\epsilon} & \hat{\mathcal{Y}}^{s,\infty} \end{array}$$

2) If C is finite we have $\pi_+(Im(\chi_{\epsilon,t}))\subset\hat{\mathcal{Y}}^s$ and $\hat{\Pi}\circ\chi_{\epsilon,t}=\chi_\epsilon$.

3) The map $\chi_{\epsilon,t}$ is injective.

4) For a Rep-monomial M define $m=\prod_{i\in I,l\in\mathbb{Z}/s\mathbb{Z}}\tilde{Y}_{i,l}^{x_{i,l}(M)}\in\overline{B}^s$. Then we have :

$$\chi_{q,t}(M)=m+\sum_{m'<m}a_{m'}(t)m' \text{ (where } a_{m'}(t)\in\mathbb{Z}[t^\pm]\text{)}$$

Proof:

1) Consequence of the definition and of $(\tau_{s,t})_{t=1}=\tau_s$.

2) Consequence of (1) and of theorem 5.1.

3) Consequence of (1) and of the injectivity of $\hat{\chi}_\epsilon$ (see section 2.2).

4) Consequence of the analogous property of $\chi_{q,t}$ (1. of theorem 4.11). □

Note that 2) means that in the finite case we get at $t=1$ the map of [FM1].

In the following we show other fundamental properties of $\chi_{\epsilon,t}$ (theorem 5.10 and theorem 5.16).

5.3. **Classical and deformed screening operators at roots of unity.** We define classical and deformed screening operators at roots of unity in order to have an analog of the property 2 of theorem 4.11 at roots of unity.

5.3.1. *Deformed bimodules.*

Definition 5.5. *$\hat{\mathcal{Y}}_{i,u}^s$ is the $\hat{\mathcal{Y}}_u^s$-bimodule defined by generators $\tilde{S}_{i,l}$ ($i\in I,l\in\mathbb{Z}/s\mathbb{Z}$) and relations :*

$$\tilde{S}_{i,l}\tilde{A}_{j,k}^{-1}=t_{-C_{i,j}(z)(z^{(k-l)}+z^{(l-k)})}\tilde{A}_{j,k}^{-1}\tilde{S}_{i,l}\ ,\ \tilde{S}_{i,l}\tilde{Y}_{j,k}=t_{\delta_{i,j}(z^{(k-l)}+z^{(l-k)})}\tilde{Y}_{j,k}\tilde{S}_{i,l}$$

$$\tilde{S}_{i,l}t=t\tilde{S}_{i,l}\ ,\ \tilde{S}_{i,l-r_i}-t_{-q^{-2r_i}-1}\tilde{A}_{i,l}^{-1}\tilde{S}_{i,l+r_i}\ ,\ \tilde{S}_{i,l+s}-\tilde{S}_{i,l}$$

Note that this structure is well-defined: if $s\geq 1$, for example we have $t_{-C_{i,j}(z)(z^{(k+s-l)}+z^{(l-k-s)})}=t_{-C_{i,j}(z)(z^{(k-l)}+z^{(l-k)})}$.

Note that $\hat{\mathcal{Y}}_t^s$ is a $\hat{\mathcal{Y}}_u^s$-bimodule using the projection $\hat{\mathcal{Y}}_u^s\to\hat{\mathcal{Y}}_t^s$.

Definition 5.6. *$\hat{\mathcal{Y}}_{i,t}^s$ is the $\hat{\mathcal{Y}}_t^s$-bimodule $\hat{\mathcal{Y}}_t^s\otimes_{\hat{\mathcal{Y}}_u^s}\hat{\mathcal{Y}}_{i,u}^s\otimes_{\hat{\mathcal{Y}}_u^s}\hat{\mathcal{Y}}_t^s$.*

For $l\in\mathbb{Z}/s\mathbb{Z}$ we denote by $\tilde{S}_{i,l}$ the image of $\tilde{S}_{i,l}$ in $\hat{\mathcal{Y}}_{i,t}^s$. If $s\geq 1$, the $\hat{\mathcal{Y}}_t^s$-module $\hat{\mathcal{Y}}_{i,t}^s$ has torsion:

$$\tilde{S}_0=t^\alpha\tilde{A}_{r_i}^{-1}\tilde{A}_{3r_i}^{-1}...\tilde{A}_{(2s-1)r_i}^{-1}\tilde{S}_{2r_is}=t^\alpha\tilde{A}_{r_i}^{-1}\tilde{A}_{3r_i}^{-1}...\tilde{A}_{(2s-1)r_i}^{-1}\tilde{S}_0$$

where $\alpha=-2s$ if $s|2r_i$ and $\alpha=-s$ otherwise.

5.3.2. *Deformed screening operators.* As in the generic case, we can define:

Definition 5.7. *The i^{th}-deformed screening operator is the map $S^s_{i,t} : \hat{\mathcal{Y}}^s_t \to \hat{\mathcal{Y}}^s_{i,t}$ defined by ($\lambda \in \hat{\mathcal{Y}}^s_t$):*

$$S^s_{i,t}(\lambda) = \frac{1}{t^2-1}\sum_{l\in\mathbb{Z}/s\mathbb{Z}}[\tilde{S}_{i,l},\lambda] \in \hat{\mathcal{Y}}^s_{i,t}$$

We define $\hat{\mathfrak{K}}^s_{i,t} = \mathrm{Ker}(S^s_{i,t})$ and we complete this algebra $\hat{\mathfrak{K}}^{s,\infty}_{i,t} \supset \hat{\mathfrak{K}}^s_{i,t}$.

5.3.3. *Classical screening operators at roots of unity.* We suppose in this section that $t = 1$.

The classical screening operators at roots of unity are

$$S^s_i : \mathcal{Y}^s \to \mathcal{Y}^s_i = \bigoplus_{l\in\mathbb{Z}/s\mathbb{Z}} \mathcal{Y}^s S_{i,l} / \sum_{l\in\mathbb{Z}/s\mathbb{Z}} \mathcal{Y}^s.(S_{i,l-r_i} - A^{-1}_{i,l}S_{i,l+r_i})$$

such that for $m \in A^s$, $S^s_i(m) = m\sum_{l\in\mathbb{Z}/s\mathbb{Z}} u_{i,l}(m)S_{i,l}$.

For $\lambda \in \hat{\mathcal{Y}}^s_t$ we have $S_i(\hat{\Pi}(\lambda)) = \hat{\Pi}(S_{i,t}(\lambda))$ where $\hat{\Pi} : \hat{\mathcal{Y}}^s_{i,t} \to \mathcal{Y}^s_i$ is defined by $\hat{\Pi}(m\tilde{S}_{i,l}) = \hat{\Pi}(m)S_{i,l}$.

The map $\tau_s : \mathcal{Y} \to \mathcal{Y}^s$ is a ring homomorphism. In particular we can define a $\mathbb{Z}$-linear map $\tau_s : \mathcal{Y}_i \to \mathcal{Y}^s_i$ such that:

$$\tau_s(mS_{i,l}) = \Pi_s(m)S_{i,[l]}$$

Indeed it suffices to see it agrees with the defining relations of $\mathcal{Y}_i$:

$$\tau_s(mA^{-1}_{i,l+r_i}S_{i,l+2r_i}) = \tau_s(mA^{-1}_{i,l+r_i})S_{i,[l+2r_i]} = \tau_s(m)A^{-1}_{i,[l+r_i]}S_{i,[l+2r_i]} = \tau_s(m)S_{i,[l]} = \tau_s(mS_{i,l})$$

Note that the crucial point is that τ_s is a ring homomorphism.

Lemma 5.8. *We have $\tau_s \circ S_i = S^s_i \circ \tau_s$.*

Proof: It suffices to see for m a $\mathcal{Y}$-monomial:

$$\tau_s(S_i(m)) = \sum_{l\in\mathbb{Z}} u_{i,l}(m)\tau_s(mS_{i,l}) = \tau_s(m)\sum_{l\in\mathbb{Z}} u_{i,l}(m)S_{i,[l]} = \tau_s(m)\sum_{0\le l\le s-1}\Big(\sum_{r\in\mathbb{Z}} u_{i,l+rs}(m)\Big)S_{i,[l]}$$

$$= \tau_s(m)\sum_{0\le l\le s-1} u_{i,[l]}(\tau_s(m))S_{i,[l]} = S^s_i(\tau_s(m))$$

□

For $m \in B^s_i$, we set $E_i(m) = m\prod_{l\in\mathbb{Z}/s\mathbb{Z}}(1 + A^{-1}_{i,l+r_i})^{u_{i,l}(m)}$. Let $\mathfrak{K}^s_i = \mathrm{Ker}(S^s_i)$.

Proposition 5.9. *$\tau_s(\mathfrak{K}_i)$ is a subalgebra of $\mathfrak{K}^s_i$. Moreover:*

$$\tau_s(\mathfrak{K}_i) = \bigoplus_{m\in B_s} E_i(m)$$

In particular if $\chi \in \tau_s(\mathfrak{K}_i)$ has no i-dominant monomial then $\chi = 0$.

Proof: The lemma 5.8 gives $\tau_s(\mathfrak{K}_i) \subset \mathfrak{K}^s_i$ and τ_s is an algebra homomorphism.

For $m \in \overline{B}$ we have $\tau_s(E_i(m)) = E_i(\tau_s(m))$ and so it follows from theorem 4.5 that $\tau_s(\mathfrak{K}_i) = \bigoplus_{m\in B_s} E_i(m)$.

□

5.4. **The image of $\chi_{\epsilon,t}$.** In this section we show an analog of the property 2 of theorem 4.11 at roots of unity.

Theorem 5.10. *The image of $\chi_{\epsilon,t}$ is contained in $\hat{\mathfrak{K}}^{s,\infty}_t$.*

With the help of theorem 4.11 it suffices to show that $\tau_{s,t}(\hat{\mathfrak{K}}_{i,t}) \subset \hat{\mathfrak{K}}^s_{i,t}$ which will be done in proposition 5.15.

5.4.1. *The bicharacters D_1, D_2.* For m a $\hat{\mathcal{Y}}_t$-monomial and $k\in\mathbb{Z}$ let:

$$m[k]=m(\overleftarrow{m})^{-1}\overleftarrow{\prod_{l\in\mathbb{Z}}}(\prod_{j\in I}\tilde{Y}_{j,l}^{y_{j,l+k}(m)})(\prod_{j\in I}\tilde{A}_{j,l}^{-v_{j,l+k}(m)})$$

Note that $\tau_{s,t}(m[ks])=\tau_{s,t}(m)$ and for $m\in\overleftarrow{A}$, $k\in\mathbb{Z}$ we have $m[k]\in\overleftarrow{A}$.

For m_1,m_2 $\hat{\mathcal{Y}}_t$-monomials, and $k\in\mathbb{Z}$ we have :

$$d_1(m_1,m_2[k])=d_1(m_1[-k],m_2)\text{ and }d_2(m_1,m_2[k])=d_2(m_1[-k],m_2)$$

Moreover there is only a finite number of $k\in\mathbb{Z}$ such that $d_1(m_1,m_2[k])\neq 0$ or $d_2(m_1,m_2[k])\neq 0$. So we can define:

Definition 5.11. *For m_1,m_2 $\hat{\mathcal{Y}}_t$-monomials we define:*

$$D_1(m_1,m_2)=\sum_{r\in\mathbb{Z}}d_1(m_1,m_2[rs])=\sum_{r\in\mathbb{Z}}d_1(m_1[rs],m_2)$$

$$D_2(m_1,m_2)=\sum_{r\in\mathbb{Z}}d_2(m_1,m_2[rs])=\sum_{r\in\mathbb{Z}}d_2(m_1[rs],m_2)$$

Lemma 5.12. *For m_1,m_2 $\hat{\mathcal{Y}}_t$-monomials we have:*

$$D_1(m_1,m_2)=d_1(\tau_{s,t}(m_1),\tau_{s,t}(m_2))\ ,\ D_2(m_1,m_2)=d_2(\tau_{s,t}(m_1),\tau_{s,t}(m_2))$$

In particular we have in $\hat{\mathcal{Y}}_t^s$:

$$\tau_{s,t}(m_1)\tau_{s,t}(m_2)=t^{D_1(m_1,m_2)-D_2(m_2,m_1)}\tau_{s,t}(m_2)\tau_{s,t}(m_1)$$

Proof: For example for d_1 we compute:

$$\begin{aligned}&d_1(\tau_{s,t}(m_1),\tau_{s,t}(m_2))\\&=\sum_{i\in I,l\in\mathbb{Z}/s\mathbb{Z}}v_{i,l+r_i}(\tau_{s,t}(m_1))u_{i,l}(\tau_{s,t}(m_2))+w_{i,l+r_i}(\tau_{s,t}(m_1))v_{i,l}(\tau_{s,t}(m_2))\\&=\sum_{i\in I,0\leq l\leq s-1,r\in\mathbb{Z},r'\in\mathbb{Z}}v_{i,l+r_i+rs}(m_1)u_{i,l+r's}(m_2)+w_{i,l+r_i+rs}(m_1)v_{i,l+r's}(m_2)\\&=\sum_{i\in I,l\in\mathbb{Z},r\in\mathbb{Z}}v_{i,l+r_i}(m_1)u_{i,l+rs}(m_2)+w_{i,l+r_i}(m_1)v_{i,l+rs}(m_2)\\&=\sum_{r\in\mathbb{Z}}d_1(m_1,m_2[rs])\end{aligned}$$

□

5.4.2. *Technical lemmas.*

Lemma 5.13. *Let m be a $\hat{\mathcal{Y}}_t$-monomial of the form $m=Z_1Z_2...Z_K$ where $Z_k=\tilde{Y}_{i_k,l_k}$ or $Z_k=\tilde{A}_{i_k,l_k}^{-1}$. We suppose that $k>k'$ implies $l_k\leq l_{k'}+r^\vee$ and $(Z_k,Z_{k'})\notin\{(\tilde{A}_{i,l}^{-1},\tilde{A}_{i,l'}^{-1})/i\in I,l'<l\}$. Then we have:*

$$\tau_{s,t}(m)=\tau_{s,t}(Z_1)\tau_{s,t}(Z_2)...\tau_{s,t}(Z_K)$$

Proof: First we order the factors of m:

$$m=t^{2\sum_{k<k'/l_k<l_{k'}}d_1(Z_k,Z_{k'})-d_2(Z_{k'},Z_k)}\overleftarrow{m}$$

So we can apply $\tau_{s,t}$:

$$\tau_{s,t}(m)=t^{2\sum_{k<k'/l_k<l_{k'}}d_1(Z_k,Z_{k'})-d_2(Z_{k'},Z_k)}\tau_{s,t}(\overleftarrow{m})$$

where:

$$\tau_{s,t}(\overleftarrow{m})=\overleftarrow{\prod_{l\in\mathbb{Z}}}(\prod_{j\in I}\tilde{Y}_{j,[l]}^{y_{j,l}(m)})(\prod_{j\in I}\tilde{A}_{j,[l]}^{-v_{j,l}(m)})$$

If we order the factors of $\tau_{s,t}(Z_1)\tau_{s,t}(Z_2)...\tau_{s,t}(Z_k)$, we get:

$$\tau_{s,t}(Z_1)\tau_{s,t}(Z_2)...\tau_{s,t}(Z_k)=t^{2\sum_{k<k'/l_k<l_{k'}}(D_1(Z_k,Z_{k'})-D_2(Z_{k'},Z_k))}\tau_{s,t}(\overleftarrow{m})$$

So it suffices to show that $k < k'$ and $l_k < l_{k'}$ implies $d_1(Z_k, Z_{k'}) - d_2(Z_{k'}, Z_k) = D_1(Z_k, Z_{k'}) - D_2(Z_{k'}, Z_k)$. But we have $0 < l_{k'} - l_k \leq r^\vee$ and $s > 2r^\vee$. So for $p \in \mathbb{Z}$ such that $p \neq 0$ we have $|l_k - l_{k'} + ps| > r^\vee$. But in general for k_1, k_2, we have:

$$[Z_{k_1}, Z_{k_2}] \neq 0 \Rightarrow (Z_{k_1}, Z_{k_2}) = (\tilde{A}^{-1}_{i_k, l_k}, \tilde{A}^{-1}_{i_k, l_k \pm 2r_{i_k}}) \text{ or } |l_{k_1} - l_{k_2}| \leq r^\vee$$

So in our situation we have $d_1(Z_k, Z_{k'}[ps]) = d_2(Z_{k'}, Z_k[ps]) = 0$. In particular:

$$D_1(Z_k, Z_{k'}) - D_2(Z_{k'}, Z_k)$$
$$= d_1(Z_k, Z_{k'}) - d_2(Z_{k'}, Z_k) + \sum_{p\neq 0}(d_1(Z_k, Z_{k'}[ps]) - d_2(Z_{k'}, Z_k[ps])) = d_1(Z_k, Z_{k'}) - d_2(Z_{k'}, Z_k)$$

□

Lemma 5.14. *Let m be a $\hat{\mathcal{Y}}^s_t$-monomial and $l, l' \in \mathbb{Z}$.*

$$l' \geq l + s - r_i \Rightarrow u_{i,l'}(\pi_l(m)) = 0$$
$$l' \leq l + r_i - s + 1 \Rightarrow u_{i,l'}(\pi_l(m)\pi_{l-1}(m)...) = u_{i,l'}(m)$$

Proof: First notice that for $l, l' \in \mathbb{Z}$, we have:

$$u_{i,l'}(\tilde{Y}_{i,l}) \neq 0 \Rightarrow l' = l \ , \ u_{i,l'}(\tilde{A}_{i,l}) \neq 0 \Rightarrow l' = l \pm r_i$$
$$i \neq j \ , \ u_{i,l'}(\tilde{A}_{j,l}) \neq 0 \Rightarrow |l' - l| \leq -C_{j,i} - 1 \leq r^\vee - 1$$

As $r_i \leq r^\vee$ we have: $u_{i,l'}(\pi_l(m)) \neq 0 \Rightarrow l - r^\vee \leq l' \leq l + r^\vee$.

If we suppose $l' \geq l + s - r_i \geq l + 2r^\vee + 1 - r_i \geq l + r^\vee + 1$ we have $u_{i,l'}(\pi_l(m)) = 0$ and this gives the first point.

We suppose that $l' \leq l + r_i - s + 1$. If $k \geq l + 1 \geq l' + s - r_i \geq l' + r^\vee + 1$ we have $u_{i,l'}(\pi_k(m)) = 0$. So: $u_{i,l'}(\pi_l(m)\pi_{l-1}(m)...) = u_{i,l'}(m) - \sum_{k>l} u_{i,l'}(\pi_k(m)) = u_{i,l'}(m)$. □

5.4.3. *Elements of $\hat{\mathfrak{K}}^s_{i,t}$.*

Proposition 5.15. *We have $\tau_{s,t}(\hat{\mathfrak{K}}_{i,t}) \subset \hat{\mathfrak{K}}^s_{i,t}$. Moreover for m a i-dominant monomial:*

$$\tau_{s,t}(\overleftarrow{E}_{i,t}(m)) = \tau_{s,t}(m)\tau_{s,t}(\hat{m}^i)^{-1}\overleftarrow{\prod_{l\in\mathbb{Z}}}(\tilde{Y}_{i,[l]}(1 + t\tilde{A}^{-1}_{i,[l+r_i]}))^{u_{i,l}(m)}$$

where $\hat{m}^i = \prod_{l\in\mathbb{Z}} \tilde{Y}^{u_{i,l}(m)}_{i,l} \in \overline{B}_i$.

Proof: We have to show that for m a i-dominant monomial, $\tau_{s,t}(\overleftarrow{E}_{i,t}(m)) \in \hat{\mathfrak{K}}^s_{i,t}$. The proof has three steps:

1) First we suppose that $m = \tilde{Y}_{i,l}$ where $l \in \mathbb{Z}$. We have $\overleftarrow{E}_{i,t}(\tilde{Y}_{i,l}) = \tilde{Y}_{i,l}(1 + t\tilde{A}^{-1}_{i,l+r_i})$, and:

$$\tau_{s,t}(\overleftarrow{E}_{i,t}(\tilde{Y}_{i,l})) = \tau_{s,t}((1 + t^{-1}\tilde{A}^{-1}_{i,l+r_i})\tilde{Y}_{i,l}) = 1 + t^{-1}\tilde{A}^{-1}_{i,[l+r_i]})\tilde{Y}_{i,[l]} = \tilde{Y}_{i,[l]}(1 + t\tilde{A}^{-1}_{i,[l+r_i]})$$

and so:

$$S^s_{i,t}(\tau_{s,t}(\overleftarrow{E}_{i,t}(\tilde{Y}_{i,l})) = \tilde{Y}_{i,[l]}S_{i,[l]} - t^{-2}t\tilde{Y}_{i,[l]}\tilde{A}^{-1}_{i,[l+r_i]}S_{i,[l+2r_i]} = \tilde{Y}_{i,[l]}(S_{i,[l]} - t^{-1}\tilde{A}^{-1}_{i,[l+r_i]}S_{i,[l+2r_i]}) = 0$$

2) Next we suppose that $m = \prod_{l\in\mathbb{Z}} \tilde{Y}^{u_{i,l}}_{i,l}$. We have $\overleftarrow{E}_{i,t}(m) = \overleftarrow{\prod_{l\in\mathbb{Z}}}(\overleftarrow{E}_{i,t}(\tilde{Y}_{i,l}))^{u_{i,l}}$. But $r_i \leq r^\vee$, and in $(\overleftarrow{E}_{i,t}(\tilde{Y}_{i,l}))^{u_{i,l}}$ there are only $\tilde{Y}_{i,l}$ and $\tilde{A}^{-1}_{i,l+r_i}$. So we are in the situation of the lemma 5.13, and:

$$\tau_{s,t}(\overleftarrow{E}_{i,t}(m)) = \overleftarrow{\prod_{l\in\mathbb{Z}}}(\tau_{s,t}(\overleftarrow{E}_{i,t}(\tilde{Y}_{i,l})))^{u_{i,l}}$$

As $\hat{\mathfrak{K}}^s_{i,t}$ is a subalgebra of $\hat{\mathcal{Y}}^s_t$, it follows from the first step that $\tau_{s,t}(\overleftarrow{E}_{i,t}(m)) \in \hat{\mathfrak{K}}^s_{i,t}$.

3) Finally let $m \in \overline{B}_i$ be an i-dominant monomial. As for all $l \in \mathbb{Z}$, $u_{i,l}(m) = u_{i,l}(\hat{m}^i)$, we have:

$$(\tau_{s,t}(m))^{-1}S^s_{i,t}(\tau_{s,t}(m)) = (\tau_{s,t}(\hat{m}^i))^{-1}S^s_{i,t}(\tau_{s,t}(\hat{m}^i))$$

It follows from the second point that $\tau_{s,t}(\overleftarrow{E}_{i,t}(\hat{m}^i)) \in \hat{\mathfrak{K}}^s_{i,t}$. Let $\hat{\chi}$ be in $\hat{\mathcal{Y}}^s_t$ defined by:

$$\hat{\chi} = \tau_{s,t}(\hat{m}^i)^{-1}\tau_{s,t}(\overleftarrow{E}_{i,t}(\hat{m}^i))$$

We have $\tau_{s,t}(m)\hat{\chi} \in \hat{\mathfrak{K}}^s_{i,t}$, because:

$S^s_{i,t}(\tau_{s,t}(m)\hat{\chi}) = S^s_{i,t}(\tau_{s,t}(m))\hat{\chi} + \tau_{s,t}(m)S^s_{i,t}(\hat{\chi})$
$= \tau_{s,t}(m)(\tau_{s,t}(\hat{m}^i))^{-1}(S^s_{i,t}(\tau_{s,t}(\hat{m}^i))\hat{\chi} + \tau_{s,t}(\hat{m}^i)S^s_{i,t}(\hat{\chi}))$
$= \tau_{s,t}(m)(\tau_{s,t}(\hat{m}^i))^{-1}S^s_{i,t}(\tau_{s,t}(\hat{m}^i)\hat{\chi})$
$= S^s_{i,t}(\tau_{s,t}(\overleftarrow{E}_{i,t}(m))) = 0$

So it suffices to show that $\tau_{s,t}(\overleftarrow{E}_{i,t}(m)) = \tau_{s,t}(m)\hat{\chi}$.

Let χ be in $\hat{\mathcal{Y}}_t$ defined by:

$$\chi = (\hat{m}^i)^{-1}\overleftarrow{E}_{i,t}(\hat{m}^i)$$

By definition of $\overleftarrow{E}_{i,t}(m)$, we have in $\hat{\mathcal{Y}}_t$:

$$\overleftarrow{E}_{i,t}(m) = m\chi$$

In particular we want to show that $\tau_{s,t}(m\chi) = \tau_{s,t}(m)\tau_{s,t}(\hat{m}^i)^{-1}\tau_{s,t}(\hat{m}^i\chi)$. Let $\lambda_{m'}(t)$ be in $\mathbb{Z}[t^{\pm}]$ such that:

$$\chi = \sum_{m' \in \overline{A}} \lambda_{m'}(t)m'$$

If $\lambda_{m'}(t) \neq 0$ then m' is of the form $m' = \tilde{A}^{-1}_{i,l_1}...\tilde{A}^{-1}_{i,l_k}$. As $\tau_{s,t}$ is $\mathbb{Z}[t^{\pm}]$-linear, it suffices to show that for all m' of this form, we have:

$$\tau_{s,t}(m)\tau_{s,t}(\hat{m}^i)^{-1}\tau_{s,t}(\hat{m}^i m') = \tau_{s,t}(mm')$$

That is to say $\alpha = \beta$ where $\alpha, \beta \in \mathbb{Z}$ are defined by:

$$\tau_{s,t}(mm') = t^{\alpha}\tau_{s,t}(m)\tau_{s,t}(m') \text{ and } \tau_{s,t}(\hat{m}^i m') = t^{\beta}\tau_{s,t}(\hat{m}^i)\tau_{s,t}(m')$$

We can suppose without loss of generality that $m \in \overleftarrow{A}$ and $m' \in \overleftarrow{A}$ (because $\tau_{s,t}$ is $\mathbb{Z}[t^{\pm}]$-linear). Let us compute α. First we have in $\hat{\mathcal{Y}}_t$:

$$mm' = t^{2\sum\limits_{l'>l} d_2(\pi_l(m),\pi_{l'}(m')) - d_1(\pi_{l'}(m'),\pi_l(m))} \overleftarrow{\prod_{l\in\mathbb{Z}}} \pi_l(m)\pi_l(m')$$

We are in the situation of lemma 5.13, so:

$$\tau_{s,t}(mm') = t^{2\sum\limits_{l'>l} d_2(\pi_l(m),\pi_{l'}(m')) - d_1(\pi_{l'}(m'),\pi_l(m))} \overleftarrow{\prod_{l\in\mathbb{Z}}} \tau_{s,t}(\pi_l(m))\tau_{s,t}(\pi_l(m'))$$

But we have in $\hat{\mathcal{Y}}^s_t$ (lemma 5.12):

$$\tau_{s,t}(m)\tau_{s,t}(m') = t^{2\sum\limits_{l'>l} D_2(\pi_l(m),\pi_{l'}(m')) - D_1(\pi_{l'}(m'),\pi_l(m))} \overleftarrow{\prod_{l\in\mathbb{Z}}} \tau_{s,t}(\pi_l(m))\tau_{s,t}(\pi_l(m'))$$

And we get:

$$\alpha = 2\sum_{l'>l} d_2(\pi_l(m),\pi_{l'}(m')) - d_1(\pi_{l'}(m'),\pi_l(m)) - 2\sum_{l'>l} D_2(\pi_l(m),\pi_{l'}(m')) - D_1(\pi_{l'}(m'),\pi_l(m))$$

And so we have from lemma 5.12:

$$\alpha = 2\sum_{l'>l}(d_2(\pi_l(m),\pi_{l'}(m'))-d_1(\pi_{l'}(m'),\pi_l(m)))-2\sum_{l'>l,r\in\mathbb{Z}}(d_2(\pi_l(m)[rs],\pi_{l'}(m'))-d_1(\pi_{l'}(m'),\pi_l(m)[rs]))$$
$$= -2\sum_{l'>l,r\neq 0}(d_2(\pi_l(m)[rs],\pi_{l'}(m'))-d_1(\pi_{l'}(m'),\pi_l(m)[rs]))$$

But we have $\pi_{l'}(m')$ of the form $\tilde{A}_{i,l'}^{-v_{i,l'}}$, and so:

$$\alpha = -2\sum_{l'>l,r\neq 0} v_{i,l'}(m')(u_{i,l'+r_i}(\pi_l(m)[rs]) - u_{i,l'-r_i}(\pi_l(m)[rs]))$$
$$= -2\sum_{l'\in\mathbb{Z}} v_{i,l'}(m')\sum_{l<l',r\neq 0}(u_{i,l'+r_i-rs}-u_{i,l'-r_i-rs})(\pi_l(m))$$
$$= -2\sum_{l'\in\mathbb{Z}} v_{i,l'}(m')\sum_{r\neq 0}(u_{i,l'+r_i-rs}-u_{i,l'-r_i-rs})(\pi_{l'-1}(m)\pi_{l'-2}(m)...)$$

We use lemma 5.14:

$$\alpha = -2\sum_{l'\in\mathbb{Z}} v_{i,l'}(m')\sum_{r>0}(u_{i,l'+r_i-rs}-u_{i,l'-r_i-rs})(\pi_{l'-1}(m)\pi_{l'-2}(m)...)$$
$$= -2\sum_{l'\in\mathbb{Z}} v_{i,l'}(m')\sum_{r>0}(u_{i,l'+r_i-rs}-u_{i,l'-r_i-rs})(m)$$

It depends only of the $u_{i,l}(m)$, so with the same computation we get:

$$\beta = -2\sum_{l'\in\mathbb{Z}} v_{i,l'}(m')\sum_{r>0}(u_{i,l'+r_i-rs}-u_{i,l'-r_i-rs})(\hat{m}^i)$$

and we can conclude $\alpha=\beta$ because for all $l\in\mathbb{Z}$, $u_{i,l}(m) = u_{i,l}(\hat{m}^i)$. □

Note that there is another more direct proof if C is symmetric (in particular if C is of type ADE):

Proof: Let m be an i-dominant monomial.

$$\overleftarrow{m} = \overleftarrow{\prod_{l\in\mathbb{Z}}}\prod_{j\in I}\tilde{A}_{j,l+1}^{-v_{j,l+1}}\prod_{j\in I}\tilde{Y}_{j,l}^{y_{j,l}} = \overleftarrow{\prod_{l\in\mathbb{Z}}}\tilde{A}_{i,l+1}^{-v_{i,l+1}}\tilde{Y}_{i,l}^{y_{i,l}}(\prod_{j\neq i}\tilde{Y}_{j,l}^{y_{j,l}}\prod_{j\neq i}\tilde{A}_{j,l}^{-v_{j,l}})$$

For $l\in\mathbb{Z}$, let $M_l = \prod_{j\neq i}\tilde{Y}_{j,l}^{y_{j,l}}\prod_{j/C_{i,j}=0}\tilde{A}_{j,l}^{-v_{j,l}}$. We have $\overleftarrow{E}_{i,t}(M_l) = M_l$. The $\tilde{Y}_{i,l}$ and the $\tilde{A}_{j,l}^{-1}$ with $C_{i,j}=-1$ have the same relations with the $\tilde{A}_{i,l}^{-1}$, so we use indifferently the notation $Z_{i,l}$ for $\tilde{Y}_{i,l}$ or $\tilde{A}_{j,l}^{-1}$. The power of $Z_{i,l}$ is:

$$z_{i,l} = y_{i,l} + \sum_{j/C_{j,i}=-1} v_{j,l+1}+v_{j,l-1} = u_{i,l}+v_{i,l-1}+v_{i,l+1}$$

In particular we have:

$$\overleftarrow{E}_{i,t}(m) = \overleftarrow{\prod_{l\in\mathbb{Z}}}(Z_{i,l}\tilde{A}_{i,l+1}^{-1})^{v_{i,l+1}}\overleftarrow{E}_{i,t}(Z_{i,l}^{u_{i,l}})M_l Z_{i,l}^{v_{i,l-1}}$$

and it follows from lemma 5.13 that:

$$\tau_{s,t}(\overleftarrow{E}_{i,t}(m)) = \overleftarrow{\prod_{l\in\mathbb{Z}}}(\tau_{s,t}(Z_{i,l}\tilde{A}_{i,l+1}^{-1}))^{v_{i,l+1}}\tau_{s,t}(\overleftarrow{E}_{i,t}(Z_{i,l}))^{u_{i,l}}\tau_{s,t}(M_l)\tau_{s,t}(Z_{i,l}^{v_{i,l-1}}) \in \hat{\mathfrak{K}}_{i,t}^s$$

□

5.5. **Description of $\chi_{\epsilon,t}$.** In this section we prove the following theorem (the map p_s is defined in section 3.2.2):

Theorem 5.16. *If* $\chi_{q,t}(\prod_{i\in I,0\leq l\leq s-1}X_{i,l}^{x_{i,[l]}}) = \sum_{m\in\overline{A}}\lambda_m(t)m$, *then:*

$$\chi_{\epsilon,t}(\prod_{i\in I,0\leq l\leq s-1}X_{i,[l]}^{x_{i,l}}) = \sum_{m\in\overline{A}}\lambda_m(t)t^{D_1^-(m)+D_2^-(m)}p_s(m)$$

where for m a $\hat{\mathcal{Y}}_t$-monomial:

$$D_1^-(m) = \sum_{k<0}d_1(m,m[ks])\ ,\ D_2^-(m) = \sum_{k<0}d_2(m,m[ks])$$

Note that this result is a generalization of the axiom 4 of [N3] to the non necessarily finite simply laced case. In particular our construction fits with [N3] in the ADE-case.

5.5.1. *Description of the basis* $\overline{A}$.

Lemma 5.17. *For m a $\hat{\mathcal{Y}}_t$-monomial we have $t^\gamma\overrightarrow{m} \in \overline{A}$ and $t^{-\gamma-2d_1(m,m)}\overleftarrow{m} \in \overline{A}$ where:*

$$\gamma = \sum_{l\in\mathbb{Z}}(\sum_{i\in I}v_{i,l}^2(m) - \sum_{i,j/C_{i,j}+r_i=-1}v_{i,l}(m)v_{j,l}(m) - \sum_{i,j/C_{i,j}=-3 \text{ and } r_i=1}v_{i,l}(m)(v_{j,l+1}(m)+v_{j,l-1}(m)))$$

Proof: We have $\overline{\overrightarrow{m}} = \overleftarrow{m} = t^{2\beta}\overrightarrow{m}$ where:

$$\beta = \sum_{l>l'}d_1(\pi_l(m),\pi_{l'}(m)) - d_2(\pi_{l'}(m),\pi_l(m))$$
$$= d_1(m,m) - \sum_{l\in\mathbb{Z}}d_1(\pi_l(m),\pi_l(m)) - \sum_{l<l'}d_1(\pi_l(m),\pi_{l'}(m)) + d_2(\pi_l(m),\pi_{l'}(m))$$

So $\overline{t^\gamma\overrightarrow{m}} = t^{2d_1(m,m)}t^\gamma\overrightarrow{m}$ where :

$$\gamma = -\sum_{l\in\mathbb{Z}}d_1(\pi_l(m),\pi_l(m)) - \sum_{l<l'}d_1(\pi_l(m),\pi_{l'}(m)) + d_2(\pi_l(m),\pi_{l'}(m))$$

But for $l\in\mathbb{Z}$ we have

$$d_1(\pi_l(m),\pi_l(m)) = -\sum_{i\in I}v_{i,l}^2(m) + \sum_{i,j/C_{i,j}=-2 \text{ and } r_i=-1}v_{i,l}(m)v_{j,l}(m) + \sum_{i,j/C_{i,j}=-3 \text{ and } r_i=2}v_{i,l}(m)v_{j,l}(m)$$
$$= -\sum_{i\in I}v_{i,l}^2(m) + \sum_{i,j/C_{i,j}+r_i=-1}v_{i,l}(m)v_{j,l}(m)$$

For $l<l'$ we have:

$$d_1(\pi_l(m),\pi_{l'}(m)) = \delta_{l'=l+1}\sum_{i,j/C_{i,j}=-3 \text{ and } r_i=1}v_{i,l}(m)v_{j,l+1}(m)$$
$$d_2(\pi_l(m),\pi_{l'}(m)) = \delta_{l'=l+1}\sum_{i,j/C_{i,j}=-3 \text{ and } r_i=1}v_{i,l+1}(m)v_{j,l}(m)$$

and we get for γ the annonced value.

For the second point we show that $t^{-\gamma-2d_1(m,m)}\overleftarrow{m} \in \overline{A}$:

$$\overline{t^{-\gamma-2d_1(m,m)}\overleftarrow{m}} = t^{\gamma+2d_1(m,m)}\overline{\overleftarrow{m}} = t^{\gamma+2d_1(m,m)-2\beta}\overleftarrow{m} = t^{-\gamma}\overleftarrow{m} = t^{2d_1(m,m)}(t^{-\gamma-2d_1(m,m)}\overleftarrow{m})$$

□

5.5.2. *Description of* $\tau_{s,t}$.

Proposition 5.18. *For $m\in\overline{A}$ we have:*

$$\tau_{s,t}(m) = t^{D_1^-(m)+D_2^-(m)}p_s(m)$$

Proof: Using lemma 5.17 we can write $m = t^{-\gamma-2d_1(m,m)}\overleftarrow{m}$. So we have:

$$\tau_{s,t}(m) = t^{-\gamma-2d_1(m,m)}\overleftarrow{\prod_{l\in\mathbb{Z}}}\tau_{s,t}(\pi_l)$$

where $\pi_l = \pi_l(m)$. So we have $\overline{\tau_{s,t}(m)} = t^{2\alpha}\tau_{s,t}(m)$ where:

$$\alpha = \gamma + 2d_1(m,m) + \sum_{l<l'}d_1(\tau_{s,t}(\pi_l),\tau_{s,t}(\pi_{l'})) - d_2(\tau_{s,t}(\pi_{l'}),\tau_{s,t}(\pi_l))$$
$$= \gamma + 2d_1(m,m) + \sum_{l<l'}D_1(\pi_l,\pi_{l'}) - D_2(\pi_{l'},\pi_l)$$

So it suffices to show that $\alpha = -D_1^-(m) - D_2^-(m) + d_1(p_s(m), p_s(m))$. But we have:

$$d_1(p_s(m), p_s(m)) = D_1(m,m) = \sum_{l<l'} D_1(\pi_l, \pi_{l'}) + \sum_{l\geq l'} D_1(\pi_l, \pi_{l'})$$

So we want to show:

$$-D_2(\pi_{l'}, \pi_l) = \sum_{l\geq l'} D_1(\pi_l, \pi_{l'}) - (d_1(m,m) + D_1^-(m)) - (d_2(m,m) + D_2^-(m)) - \gamma$$

The second term is:

$$\sum_{l\geq l', r\in\mathbb{Z}} d_1(\pi_l, \pi_{l'}[rs]) - \sum_{l,l'\in\mathbb{Z}, r\leq 0} (d_1(\pi_l, \pi_{l'}[rs]) + d_2(\pi_l, \pi_{l'}[rs])) + \sum_{l\in\mathbb{Z}} d_2(\pi_l, \pi_l) + \sum_{l<l'} (d_1(\pi_l, \pi_{l'}) + d_2(\pi_l, \pi_{l'}))$$

But for $l < l'$ and $r < 0$ (resp. $l \geq l'$ and $r > 0$) we have $d_1(\pi_l, \pi_{l'}[rs]) = d_2(\pi_l, \pi_{l'}[rs]) = 0$. So this term is:

$$\sum_{l\geq l', r\leq 0} d_1(\pi_l, \pi_{l'}[rs]) - \sum_{l\geq l', r\leq 0} (d_1(\pi_l, \pi_{l'}[rs]) + d_2(\pi_l, \pi_{l'}[rs])) + \sum_{l\in\mathbb{Z}} d_2(\pi_l, \pi_l)$$
$$= -\sum_{l> l', r\leq 0} d_2(\pi_l, \pi_{l'}[rs]) = -\sum_{l> l', r\in\mathbb{Z}} d_2(\pi_l, \pi_{l'}[rs]) = -\sum_{l>l'} D_2(\pi_l, \pi_{l'})$$

□

6. Applications

In this section we see how we can generalize at roots of unity results of [He2] about Kazhdan-Lusztig polynomials and quantization of the Grothendieck ring. We suppose that $i \neq j \Rightarrow C_{i,j}C_{j,i} \leq 3$.

Such constructions were made by Nakajima [N3] in the simply laced case.

6.1. **Reminder: Kazhdan-Lusztig polynomials in the generic case** [N3][He2]**.** In this section we suppose that $s = 0$. The involution of $\hat{\mathcal{Y}}_t$ is naturally extended to an involution of $\hat{\mathcal{Y}}_t^\infty$.

For m a dominant $\hat{\mathcal{Y}}_t$-monomial we set:

$$\overrightarrow{E}_t(m) = m\Big(\overrightarrow{\prod_{l\in\mathbb{Z}}}\prod_{i\in I} \tilde{Y}_{i,l}^{u_{i,l}(m)}\Big)^{-1} \overrightarrow{\prod_{l\in\mathbb{Z}}}\prod_{i\in I} \hat{F}_t(\tilde{Y}_{i,l})^{u_{i,l}(m)}$$

We denote by $\hat{\mathfrak{K}}_t^{f,\infty} \subset \hat{\mathfrak{K}}_t^\infty$ the subset of elements with only a finite number of dominant monomials.

We show as in [He2] that for $m \in \overline{B}$, $C(m) \cap \overline{B}$ is finite, $\overrightarrow{E}_t(m) \in \hat{\mathfrak{K}}_t^{f,\infty}$, and:

Proposition 6.1. *(*[He2]*)* $\hat{\mathfrak{K}}_t^{f,\infty}$ *is a subalgebra of* $\hat{\mathfrak{K}}_t^\infty$*, and:*

$$\hat{\mathfrak{K}}_t^{f,\infty} = \bigoplus_{m\in\overline{B}} \mathbb{Z}[t^\pm]\hat{F}_t(m) = \bigoplus_{m\in\overline{B}} \mathbb{Z}[t^\pm]\overrightarrow{E}_t(m)$$

Moreover $\hat{\mathfrak{K}}_t^{f,\infty}$ *is stable by the involution.*

For m a $\hat{\mathcal{Y}}_t^s$-monomial there is a unique $\alpha(m) \in \mathbb{Z}$ such that $\overline{t^{\alpha(m)}m} = t^{\alpha(m)}m$ (see the proof of lemma 6.12 of [He2]).

Let $\hat{A}^{\mathrm{inv}} = \{t^{\alpha(m)}m / m \in \overline{A}\}$ and $\hat{B}^{\mathrm{inv}} = \{t^{\alpha(m)}m / m \in \overline{B}\}$.

The following theorem was given in [N3] for the ADE-case and in [He2] for the general finite case:

Theorem 6.2. *For* $m \in \hat{B}^{inv}$ *there is a unique* $\hat{L}_t(m) \in \hat{\mathfrak{K}}_t^{f,\infty}$ *such that:*

$$\overline{\hat{L}_t(m)} = \hat{L}_t(m)$$

$$\overrightarrow{E}_t(m) = \hat{L}_t(m) + \sum_{m'<m, m'\in\hat{B}^{inv}} P_{m',m}(t)\hat{L}_t(m')$$

where $P_{m',m}(t)\in t^{-1}\mathbb{Z}[t^{-1}]$.

6.2. **Kazhdan-Lusztig polynomials at roots of unity.** In this section we suppose that $s>2r^\vee$. The involution of $\hat{\mathcal{Y}}_t^s$ is extended to an involution of $\hat{\mathcal{Y}}_t^{s,\infty}$.

6.2.1. *Construction of stable subalgebras.* For $m\in\overline{B}_i^s$ a i-dominant $\hat{\mathcal{Y}}_t^s$-monomial we set:

$$\overleftarrow{E}_{i,t}(m)=m(\prod_{i\in I,l=0..s-1}\tilde{Y}_{i,[l]}^{u_{i,[l]}(m)})^{-1}\overleftarrow{\prod_{i\in I,l=0..s-1}}(\tilde{Y}_{i,[l]}(1+t\tilde{A}_{i,[l+r_i]}^{-1}))^{u_{i,[l]}(m)}$$

In particular the formula of proposition 5.15 implies:

$$\overleftarrow{E}_{i,t}(m)=m(\tau_{s,t}(M))^{-1}\tau_{s,t}(\overleftarrow{E}_{i,t}(M))\text{ where }M=\prod_{l=0...s-1}\tilde{Y}_{i,l}^{u_{i,[l]}(m)}$$

We define:

$$\tilde{\mathfrak{K}}_{i,t}^s=\bigoplus_{m\in\overline{B}_i^s}\mathbb{Z}[t^\pm]\overleftarrow{E}_{i,t}(m)$$

In particular if $\chi\in\tilde{\mathfrak{K}}_{i,t}^s$ has no i-dominant monomial then $\chi=0$.

Lemma 6.3. *We have* $\tau_{s,t}(\hat{\mathfrak{K}}_{i,t})\subset\tilde{\mathfrak{K}}_{i,t}^s\subset\hat{\mathfrak{K}}_{i,t}^s$. *Moreover* $\tilde{\mathfrak{K}}_{i,t}^s$ *is a subalgebra of* $\hat{\mathfrak{K}}_{i,t}^s$ *and is stable by the involution.*

Proof: As $\hat{\mathfrak{K}}_{i,t}^s$ is a subalgebra of $\hat{\mathcal{Y}}_t^s$ and $\tilde{Y}_{i,[l]}(1+t\tilde{A}_{i,[l+r_i]}^{-1})\in\hat{\mathfrak{K}}_{i,t}^s$, $m(\prod_{i\in I,l=0..s-1}\tilde{Y}_{i,[l]}^{u_{i,[l]}(m)})^{-1}\in\hat{\mathfrak{K}}_{i,t}^s$ we have $\tilde{\mathfrak{K}}_{i,t}^s\subset\hat{\mathfrak{K}}_{i,t}^s$.

Let us show that $\bigoplus_{m\in\overline{B}_i^s}\mathbb{Z}[t^\pm]\overleftarrow{E}_{i,t}(m)$ is a subalgebra of $\tilde{\mathfrak{K}}_{i,t}^s$ (note that in the generic case $s=0$ this point needs no proof because $\tilde{\mathfrak{K}}_{i,t}=\hat{\mathfrak{K}}_{i,t}$). For this point our proof is analogous to theorem 3.8 of [N3]. It suffices to show that for $0\leq k\leq s-1$, $M=\prod_{l\in\mathbb{Z}/s\mathbb{Z}}\tilde{Y}_{i,l}^{u_{i,l}}$ we have $\overleftarrow{E}_{i,t}(M)\overleftarrow{E}_{i,t}(\tilde{Y}_{i,k})\in\bigoplus_{m\in\overline{B}_i^s}\mathbb{Z}[t^\pm]\overleftarrow{E}_{i,t}(m)$.

We can suppose without loss of generality that we are in the sl_2-case and that $r_i=r_1=1$. The $\overleftarrow{E}_t(\tilde{Y}_k)$ do not commute with $\overleftarrow{E}_t(\tilde{Y}_{k-2}^{u_{k-2}})$ and $\overleftarrow{E}_t(\tilde{Y}_{k+2}^{u_{k+2}})$. So if $k\geq 2$ that fact that $s\neq 0$ do not change anything and the result follows from the generic case. If $k=0$, we have:

$$\overleftarrow{E}_t(m)\overleftarrow{E}_t(\tilde{Y}_{i,0})=\overleftarrow{E}_t(m\tilde{Y}_{i,0})+\overleftarrow{E}_t(\tilde{Y}_{i,0}^{u_{i,0}}\tilde{Y}_{i,1}^{u_{i,1}})[\overleftarrow{E}_t(\tilde{Y}_{i,2}^{u_{i,2}}),\overleftarrow{E}_t(\tilde{Y}_{i,0})]\overleftarrow{E}_t(\tilde{Y}_{i,3}^{u_{i,3}}...\tilde{Y}_{i,s-1}^{u_{i,s-1}})$$

$$+\overleftarrow{E}_t(\tilde{Y}_{i,0}^{u_{i,0}}...\tilde{Y}_{i,s-3}^{u_{i,s-3}})[\overleftarrow{E}_t(\tilde{Y}_{i,s-2}^{u_{i,s-2}}),\overleftarrow{E}_t(\tilde{Y}_{i,0})]\overleftarrow{E}_t(\tilde{Y}_{i,s-1}^{u_{i,s-1}})$$

It follows from the study of the generic case that:

$$[\overleftarrow{E}_t(\tilde{Y}_{i,2}^{u_{i,2}}),\overleftarrow{E}_t(\tilde{Y}_{i,0})]\in\bigoplus_{0\leq r<u_{i,2}}\mathbb{Z}[t^\pm]\overleftarrow{E}_t(\tilde{Y}_{i,2})^r$$

$$[\overleftarrow{E}_t(\tilde{Y}_{i,s-2}^{u_{i,s-2}}),\overleftarrow{E}_t(\tilde{Y}_{i,0})]\in\bigoplus_{0\leq r<u_{i,s-2}}\mathbb{Z}[t^\pm]\overleftarrow{E}_t(\tilde{Y}_{i,s-2})^r$$

and we can conclude by induction. The case $k=1$ is studied in the same way.

Let us study the stability by the involution: we see that $\overline{\overleftarrow{E}_{i,t}(\tilde{Y}_{i,l})}=\overleftarrow{E}_{i,t}(\tilde{Y}_{i,l})$, and:

$$\overline{\overleftarrow{E}_{i,t}(m)}=\overrightarrow{\prod_{i\in I,l=s-1,s-2,...0}}\overline{\overleftarrow{E}_{i,t}(\tilde{Y}_{i,[l]})}^{u_{i,[l]}(m)}\overleftarrow{E}_{i,t}(m(\overrightarrow{\prod_{i\in I,l=0..s-1}}\tilde{Y}_{i,[l]}^{u_{i,[l]}(m)})^{-1})\in\tilde{\mathfrak{K}}_{i,t}$$

Let us show that $\tau_{s,t}(\hat{\mathfrak{K}}_{i,t}) \subset \tilde{\mathfrak{K}}^s_{i,t}$: the formula of proposition 5.15 implies that for $m \in \overline{B}_i$:

$$\tau_{s,t}(\overleftarrow{E}_{i,t}(m)) = \tau_{s,t}(m)\tau_{s,t}(\hat{m}^i)^{-1}\overleftarrow{\prod_{l\in\mathbb{Z}}}(\tilde{Y}_{i,[l]}(1+t\tilde{A}^{-1}_{i,[l+r_i]}))^{u_{i,l}(m)}$$

$$= \overleftarrow{E}_{i,t}(\tau_{s,t}(m)\tau_{s,t}(\hat{m}^i)^{-1})\overleftarrow{\prod_{l\in\mathbb{Z}}}\overleftarrow{E}_{i,t}(\tilde{Y}^{u_{i,l}(m)}_{i,[l]})$$

and we can conclude because $\tilde{\mathfrak{K}}^s_{i,t}$ is an algebra. □

We define the completion $\tilde{\mathfrak{K}}^{s,\infty}_{i,t} \subset \hat{\mathfrak{K}}^{s,\infty}_{i,t}$ (as in section 4.2.1) and:

$$\tilde{\mathfrak{K}}^{s,\infty}_t = \bigcap_{i\in I}\tilde{\mathfrak{K}}^{s,\infty}_{i,t}$$

For $m \in \overline{B}^s$ we define $\overrightarrow{E}_t(m) = m(\tau_{s,t}(M))^{-1}\tau_{s,t}(\overrightarrow{E}_t(M))$ where $M = \prod_{i\in I, l=0\ldots s}\tilde{Y}^{u_{i,[l]}(m)}_{i,l}$.

6.2.2. *Polynomials at roots of unity (finite case).* In this section we suppose that C is finite. Note that it follows from the lemma 3.10 that for $m \in \overline{B}^s$, the set $C(m) \cap \overline{B}^s$ is finite.

We denote by $\tilde{\mathfrak{K}}^{s,f,\infty}_t$ the set of elements of $\tilde{\mathfrak{K}}^{s,\infty}_t$ with only a finite number of dominant monomials.

Lemma 6.4. *$\tilde{\mathfrak{K}}^{s,f,\infty}_t$ is a subalgebra of $\hat{\mathcal{Y}}^{s,\infty}_t$, is stable by the involution, and:*

$$\tilde{\mathfrak{K}}^{s,f,\infty}_t = \bigoplus_{m\in\overline{B}^s}\mathbb{Z}[t^\pm]\overrightarrow{E}_t(m)$$

Proof: It follows from lemma 6.3 that $\tilde{\mathfrak{K}}^{s,\infty}_t$ is a subalgebra of $\hat{\mathcal{Y}}^{s,\infty}_t$. Let m be in $\overline{B}^s$. For all $i \in I$ we have $m(\tau_{s,t}(M))^{-1} = \overleftarrow{E}_{i,t}(m(\tau_{s,t}(M))^{-1})$ and so $m(\tau_{s,t}(M))^{-1} \in \tilde{\mathfrak{K}}^{s,\infty}_t$. But $\tau_{s,t}(\overrightarrow{E}_t(M)) \in \tilde{\mathfrak{K}}^{s,\infty}_{i,t}$ for all $i \in I$. So $\overrightarrow{E}_t(m) \in \tilde{\mathfrak{K}}^{s,\infty}_t$. Moreover lemma 3.10 shows that $\overrightarrow{E}_t(m)$ has only a finite number of dominant monomials, so $\overrightarrow{E}_t(m) \in \tilde{\mathfrak{K}}^{s,f,\infty}_t$. It follows from lemma 6.3 that a maximal monomial of an element of $\tilde{\mathfrak{K}}^{s,f,\infty}_t$ is dominant, and so we have the other inclusion $\tilde{\mathfrak{K}}^{s,f,\infty}_t \subset \bigoplus_{m\in\overline{B}^s}\mathbb{Z}[t^\pm]\overrightarrow{E}_t(m)$.

It follows from lemma 6.3 that $\tilde{\mathfrak{K}}^{s,\infty}_t$ is stable by the involution. But for m a dominant monomial, $\overline{m}$ is a dominant monomial and so $\tilde{\mathfrak{K}}^{s,f,\infty}_t$ is stable by the involution.

As $\tilde{\mathfrak{K}}^{s,\infty}_t$ is an algebra, $\tilde{\mathfrak{K}}^{s,f,\infty}_t$ is an algebra if for $m, m' \in \overline{B}^s$, $\overrightarrow{E}_t(m)\overrightarrow{E}_t(m')$ has only a finite number of dominant monomials. But the monomials of $\overrightarrow{E}_t(m)\overrightarrow{E}_t(m')$ are in $C(mm')$ and we can conclude with the help of lemma 3.10. □

Let $\hat{A}^{s,\mathrm{inv}} = \{t^{\alpha(m)}m / m \in \overline{A}^s\}$ and $\hat{B}^{s,\mathrm{inv}} = \{t^{\alpha(m)}m / m \in \overline{B}^s\}$ where $\alpha(m)$ is defined by $\overline{t^\alpha m} = t^{\alpha(m)}m$ (see the proof of lemma 6.12 of [He2]).

Theorem 6.5. *For $m \in \hat{B}^{s,inv}$ there is a unique $\hat{L}^s_t(m) \in \hat{\mathfrak{K}}^{s,f,\infty}_t$ such that:*

$$\overline{\hat{L}^s_t(m)} = \hat{L}^s_t(m)$$

$$\overrightarrow{E}_t(m) = \hat{L}^s_t(m) + \sum_{m'<m, m'\in\hat{B}^{s,inv}} P^s_{m',m}(t)\hat{L}^s_t(m')$$

where $P^s_{m',m}(t) \in t^{-1}\mathbb{Z}[t^{-1}]$.

The proof is analogous to the proof of theorem 6.2 with the help of lemma 6.4. The result was first given by Nakajima [N3] for the ADE-case.

6.2.3. *Example and conjecture (finite case).* In the following example we suppose that we are in the sl_2-case and we study the decomposition with $m = \tilde{Y}_0\tilde{Y}_1\tilde{Y}_2$.

If $s = 0$, we have:

$$\begin{aligned}\overrightarrow{E}_t(\tilde{Y}_0\tilde{Y}_1\tilde{Y}_2) &= \tilde{Y}_0(1 + t\tilde{A}_1^{-1})\tilde{Y}_1(1 + t\tilde{A}_2^{-1})\tilde{Y}_2(1 + t\tilde{A}_3^{-1})\\ &= \hat{L}_t(\tilde{Y}_0\tilde{Y}_1\tilde{Y}_2) + t^{-1}\hat{L}_t(t^2\tilde{Y}_0\tilde{A}_1^{-1}\tilde{Y}_1\tilde{Y}_2)\end{aligned}$$

where:

$$\begin{aligned}\hat{L}_t(\tilde{Y}_0\tilde{Y}_1\tilde{Y}_2) &= \tilde{Y}_0\tilde{Y}_1\tilde{Y}_2(1 + t\tilde{A}_3^{-1}(1 + t\tilde{A}_1^{-1}))(1 + t\tilde{A}_2^{-1})\\ \hat{L}_t(t^2\tilde{Y}_0\tilde{A}_1^{-1}\tilde{Y}_1\tilde{Y}_2) &= t^2\tilde{Y}_0\tilde{A}_1^{-1}\tilde{Y}_1\tilde{Y}_2(1 + t\tilde{A}_2^{-1})\end{aligned}$$

If $s = 3$, we have:

$$\begin{aligned}&\overrightarrow{E}_t(\tilde{Y}_0\tilde{Y}_1\tilde{Y}_2) = \tau_{s,t}(\tilde{Y}_0(1 + t\tilde{A}_1^{-1})\tilde{Y}_1(1 + t\tilde{A}_2^{-1})\tilde{Y}_2(1 + t\tilde{A}_3^{-1}))\\ &= \tilde{Y}_0\tilde{Y}_1\tilde{Y}_2 + t\tilde{Y}_0\tilde{A}_1^{-1}\tilde{Y}_1\tilde{Y}_2 + t^{-1}\tilde{Y}_0\tilde{Y}_1\tilde{A}_2^{-1}\tilde{Y}_2 + t^{-1}\tilde{Y}_0\tilde{Y}_1\tilde{A}_2^{-1}\tilde{Y}_2\\ &+ t^2\tilde{Y}_0\tilde{A}_1^{-1}\tilde{Y}_1\tilde{Y}_2\tilde{A}_2^{-1} + \tilde{Y}_0\tilde{Y}_1\tilde{A}_2^{-1}\tilde{Y}_2\tilde{A}_3^{-1} + \tilde{Y}_0\tilde{Y}_1\tilde{A}_1^{-1}\tilde{A}_2^{-1}\tilde{Y}_2 + t^{-3}\tilde{Y}_0\tilde{A}_1^{-1}\tilde{Y}_1\tilde{A}_2^{-1}\tilde{Y}_2\tilde{A}_3^{-1}\end{aligned}$$

and so:

$$\overrightarrow{E}_t(\tilde{Y}_0\tilde{Y}_1\tilde{Y}_2) = \hat{L}_t^s(\tilde{Y}_0\tilde{Y}_1\tilde{Y}_2) + t^{-1}\hat{L}_t^s(t^2\tilde{Y}_0\tilde{A}_1^{-1}\tilde{Y}_1\tilde{Y}_2) + t^{-1}\hat{L}_t^s(\tilde{Y}_0\tilde{Y}_1\tilde{A}_2^{-1}\tilde{Y}_2) + t^{-1}\hat{L}_t^s(\tilde{Y}_0\tilde{Y}_1\tilde{Y}_2\tilde{A}_3^{-1})$$

where:

$$\begin{aligned}\hat{L}_t^s(\tilde{Y}_0\tilde{Y}_1\tilde{Y}_2) &= \tilde{Y}_0\tilde{Y}_1\tilde{Y}_2 + t^{-3}\tilde{Y}_0\tilde{A}_1^{-1}\tilde{Y}_2\tilde{A}_3^{-1}\tilde{Y}_4\tilde{A}_5^{-1}\\ \hat{L}_t^s(t^2\tilde{Y}_0\tilde{A}_1^{-1}\tilde{Y}_1\tilde{Y}_2) &= t^2\tilde{Y}_0\tilde{A}_1^{-1}\tilde{Y}_1\tilde{Y}_2(1 + \tilde{A}_3^{-1})\\ \hat{L}_t^s(\tilde{Y}_0\tilde{Y}_1\tilde{A}_2^{-1}\tilde{Y}_2) &= \tilde{Y}_0\tilde{Y}_1\tilde{A}_2^{-1}\tilde{Y}_2(1 + t\tilde{A}_2^{-1})\\ \hat{L}_t^s(\tilde{Y}_0\tilde{Y}_1\tilde{Y}_2\tilde{A}_3^{-1}) &= \tilde{Y}_0\tilde{Y}_1\tilde{Y}_2\tilde{A}_3^{-1}(1 + t\tilde{A}_2^{-1})\end{aligned}$$

In particular we see in this example that the decomposition of $\overrightarrow{E}_t(m)$ in general is not necessarily the same if $s = 0$ or $s \neq 0$.

We recall that irreducible representations of $\mathcal{U}_q(\hat{\mathfrak{g}})$ (resp. $\mathcal{U}_\epsilon^{\text{res}}(\hat{\mathfrak{g}})$) are classified by dominant monomials of $\mathcal{Y}$ (resp. $\mathcal{Y}^s$) or by Drinfel'd polynomials (see [CP1], [CP3], [FR2], [FM2]).

For $m \in B$ (resp. $m \in B^s$) we denote by $V_m^0 = V_m \in \text{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ (resp. $V_m^s \in \text{Rep}(\mathcal{U}_\epsilon^{\text{res}}(\hat{\mathfrak{g}}))$) the irreducible module of highest weight m. In particular for $i \in I, l \in \mathbb{Z}/s\mathbb{Z}$ let $V_{i,l}^s = V_{Y_{i,l}}$. The simple modules $V_{i,l}^s$ are called fundamental representations. In the ring Rep^s it is denoted by $X_{i,l}$.

For $m \in B$ (resp. $m \in B^s$) we denote by $M_m^s \in \text{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ (resp. $M_m^s \in \text{Rep}(\mathcal{U}_\epsilon^{\text{res}}(\hat{\mathfrak{g}}))$) the module $M_m^s = \bigotimes_{i\in I, l\in\mathbb{Z}/s\mathbb{Z}} V_{i,l}^{s,\otimes u_{i,l}(m)}$. It is called a standard module and in Rep^s it is denoted by $\prod_{i\in I,l\in\mathbb{Z}/s\mathbb{Z}} X_{i,l}^{u_{i,l}(m)}$.

The irreducible $\mathcal{U}_q(\hat{sl}_2)$-representation with highest weight m is $V_m = V_{Y_0Y_2} \otimes V_{Y_1}$ (see [CP1] or [FR2]). In particular $\dim(V_m) = 6$, that is to say the number of monomials of $\hat{L}_t(m)$.

For ϵ such that $s = 3$, the irreducible $\mathcal{U}_\epsilon^{\text{res}}(\hat{\mathfrak{g}})$-representation with highest weight m is V_m^s the pull back by the Frobenius morphism of the $\overline{\mathcal{U}}(\hat{sl}_2)$-module $\overline{V}$ of Drinfel'd polynomial $(1-u)$ (see [CP3] or [FM2]). In particular $\dim(V_m^s) = 2$, that is to say the number of monomials of $\hat{L}_t^s(m)$.

Those observations would be explained by the following conjecture which is a generalization of the conjecture 7.3 of [He2] to the root of unity case. We know from [N3] that the result is true in the simply laced case (in particular in the last example).

For $m = \prod_{i\in I,l\in\mathbb{Z}/s\mathbb{Z}} Y_{i,l}^{u_{i,l}}$ a dominant $\mathcal{Y}^s$-monomial let $M = \prod_{i\in I,l\in\mathbb{Z}/s\mathbb{Z}} \tilde{Y}_{i,l}^{u_{i,l}} \in \hat{\mathcal{Y}}_t^s$. We suppose that C is finite.

Conjecture 6.6. *For m a dominant $\mathcal{Y}_s$-monomial, $\hat{\Pi}(\hat{L}_t^s(M))$ is the ϵ-character of the irreducible $\mathcal{U}_\epsilon^{res}(\hat{\mathfrak{g}})$-representation V_m^s associated to m. In particular for m' another dominant $\mathcal{Y}_s$-monomials the multiplicity of $V_{m'}^s$ in the standard module M_m^s associated to m is:*

$$\sum_{m''\in\hat{B}^{s,inv}/\hat{\Pi}(m'')=m'} P^s_{m'',M}(1)$$

Let us look at an application of the conjecture in the non-simply laced case: we suppose that $C = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$ and $m = \tilde{Y}_{1,0}\tilde{Y}_{1,1}$. We have for $l \in \mathbb{Z}/s\mathbb{Z}$:

$$\tilde{A}_{1,l}^{-1} =: \tilde{Y}_{1,l-1}\tilde{Y}_{1,l+1}\tilde{Y}_{2,l} : \ , \ \tilde{A}_{2,l}^{-1} =: \tilde{Y}_{2,l-2}^{-1}\tilde{Y}_{2,l+2}^{-1}\tilde{Y}_{1,l-1}\tilde{Y}_{1,l+1} :$$

First we suppose that $s = 0$. The formulas for $\hat{F}_t(\tilde{Y}_{1,0})$ and $\hat{F}_t(\tilde{Y}_{1,1})$ are given in [He2]:

$$\hat{F}_t(\tilde{Y}_{1,0}) = \tilde{Y}_{1,0}(1 + t\tilde{A}_{1,1}^{-1}(1 + t\tilde{A}_{2,3}^{-1}(1 + t\tilde{A}_{1,5}^{-1})))$$

$$\hat{F}_t(\tilde{Y}_{1,1}) = \tilde{Y}_{1,1}(1 + t\tilde{A}_{1,2}^{-1}(1 + t\tilde{A}_{2,4}^{-1}(1 + t\tilde{A}_{1,6}^{-1})))$$

The product $\hat{F}_t(\tilde{Y}_{1,0})\hat{F}_t(\tilde{Y}_{1,1})$ has a unique dominant monomial $\tilde{Y}_{1,0}\tilde{Y}_{2,0}$, so:

$$\overrightarrow{E}_t(\tilde{Y}_{1,0}\tilde{Y}_{2,0}) = \hat{F}_t(\tilde{Y}_{1,0}\tilde{Y}_{2,0}) = \hat{L}_t(\tilde{Y}_{1,0}\tilde{Y}_{2,0}) = \hat{F}_t(\tilde{Y}_{1,0})\hat{F}_t(\tilde{Y}_{1,1})$$

In particular the $V_{1,0} \otimes V_{1,1}$ is irreducible. Note that it is not a consequence of the conjecture but of classical theory of q-characters.

We suppose now that $s = 5 > 4 = 2r^\vee$. There are two dominant monomials in $\tau_{s,t}(\overrightarrow{E}_t(\tilde{Y}_{1,0}\tilde{Y}_{1,1}))$:

$$\tau_{s,t}(\tilde{Y}_{1,0}\tilde{Y}_{1,1}) = \tilde{Y}_{1,0}\tilde{Y}_{1,1} \text{ and } \tau_{s,t}(t^3\tilde{Y}_{1,0}\tilde{A}_{1,1}^{-1}\tilde{A}_{2,3}^{-1}\tilde{A}_{1,5}^{-1}\tilde{Y}_{1,1}) = t^{-1}$$

And so we have:

$$\tau_{s,t}(\overrightarrow{E}_t(\tilde{Y}_{1,0}\tilde{Y}_{1,1})) = \hat{L}_t^s(\tilde{Y}_{1,0}\tilde{Y}_{1,1}) + t^{-1}\hat{L}_t(1)$$

where $\hat{L}_t(1) = 1$. So if the conjecture is true, at $s = 5$ the $V_{1,0}^s \otimes V_{1,1}^s$ is not irreducible and contains the trivial representation with multiplicity one.

6.2.4. *Non finite cases.* In this section we suppose that $B(z)$ is symmetric and $s > 2r^\vee$. An important difference with the finite case is that an infinite number of dominant monomials can appear in the q,t-character : let us briefly explain it for the example of section 3.2.3. We consider the case C of type $A_2^{(1)}$ and $s = 3$. We have the following subgraph in the q-character given by the classical algorithm:

$$Y_{1,0} \to Y_{1,2}^{-1}Y_{2,1}Y_{3,1} \to Y_{3,2}Y_{3,1}Y_{2,3}^{-1} \to Y_{3,4}^{-1}Y_{3,1}Y_{1,0}$$

But at $s = 3$ we have $Y_{3,4}^{-1}Y_{3,1}Y_{1,0} \simeq Y_{1,0}$. So we have a periodic chain and an infinity of dominant monomials in $\tau_{s,t}(\hat{F}_t(\tilde{Y}_{1,0}))$.

However we propose a construction of analogs of Kazdhan-Lusztig polynomials. As there is an infinity of dominant monomials, we have to begin the induction from the highest weight monomial. Let us describe it in a more formal way:

For $m \in \hat{B}^{s,\mathrm{inv}}$ and $k \geq 0$ we denote by $\hat{B}_k^s(m) \subset \hat{B}^{s,\mathrm{inv}}$ the set of dominant monomials of the form $m' = t^\alpha m\tilde{A}_{i_1,l_1}^{-1}...\tilde{A}_{i_k,l_k}^{-1}$. We set also $B^s(m) = \bigcup_{k\geq 0} B_k^s(m)$.

For $m \in \hat{B}^{s,\mathrm{inv}}$, $\overrightarrow{E}_t(m) \in \hat{\mathfrak{K}}_t^{s,\infty}$ is defined as in section 6.2.1. It will be useful to construct the element $\hat{F}_t^s(m) \in \hat{\mathfrak{K}}_t^{s,\infty}$ with a unique dominant monomial m: we denote by $m_0 = m > m_1 > m_2 > ...$ the dominant monomials appearing in $\overrightarrow{E}_t(m)$ with a total ordering compatible with the partial ordering and the degree (the set is countable because there is a finite number of monomials of degree k). We define $\lambda_k(t) \in \mathbb{Z}[t^{\pm}]$ inductively as the coefficient of m_k in $\overrightarrow{E}_t(m) - \sum_{1\leq l\leq k-1} \lambda_l(t)\overrightarrow{E}_t(m_l)$. We define :

$$\hat{F}_t^s(m) = \overrightarrow{E}_t(m) - \sum_{l\geq 1}\lambda_l(t)\overrightarrow{E}_t(m_l) \in \tilde{\mathfrak{K}}_t^{s,\infty}$$

(this infinite sum is allowed in $\tilde{\mathfrak{K}}_t^{s,\infty}$). The unique dominant monomial of $\hat{F}_t^s(m)$ is m. In particular $\overline{\hat{F}_t^s(m)} = \hat{F}_t^s(m)$ (see section 6.2.1). In the following theorem the infinite sums are well-defined in $\tilde{\mathfrak{K}}_t^{s,\infty}$:

Theorem 6.7. *For $m \in \hat{B}^{s,inv}$ there is a unique $\hat{L}_t^s(m) \in \tilde{\mathfrak{K}}_t^{s,\infty}$ of the form $\hat{L}_t^s(m) = m + \sum_{m'<m} \mu_{m',m}(t)m'$ such that:*

$$\overline{\hat{L}_t^s(m)} = \hat{L}_t^s(m)$$
$$\overrightarrow{E}_t(m) = \hat{L}_t^s(m) + \sum_{m'\in\hat{B}_k^s(m), k\geq 1} P_{m',m}^s(t)\hat{L}_t^s(m')$$

where $P_{m',m}^s(t) \in t^{-1}\mathbb{Z}[t^{-1}]$. Moreover we have:

$$\hat{\Pi}(m) = \hat{\Pi}(m') \Rightarrow m^{-1}\hat{L}_t^s(m) = m'^{-1}\hat{L}_t^s(m')$$

Proof: We aim at defining the $\mu_{m'm}(t) \in \mathbb{Z}[t^{\pm}]$ such that:

$$\hat{L}_t^s(m) = \sum_{m'\in\hat{B}^s(m)} \mu_{m',m}(t)\hat{F}_t(m')$$

The condition $\overline{\hat{L}_t^s(m)} = \hat{L}_t^s(m)$ means that $\mu_{m',m}(t^{-1}) = \mu_{m',m}(t)$.

We define by induction on $k \geq 0$, for $m' \in \hat{B}_k^s(m)$ the $P_{m',m}^s(t)$ and the $\mu_{m',m}(t)$ such that:

$$\hat{E}_t(m) - \sum_{k\geq l\geq 0, m'\in\hat{B}_l^s(m)} P_{m',m}^s(t) \sum_{k\geq r\geq 0, m''\in\hat{B}_r^s(m')} \mu_{m'',m'}(t)\hat{F}_t(m'')$$
$$\in \sum_{m'\in\hat{B}_{k+1}^s(m)} (\mu_{m',m}(t) + P_{m',m}^s(t))\hat{F}_t(m') + \sum_{l>k+1, m'\in\hat{B}_l^s(m')} \mathbb{Z}[t^{\pm}]\hat{F}_t(m')$$

For $k = 0$ we have $P_{m,m}^s(t) = \mu_{m,m}(t) = 1$. And the the equation determines uniquely $P_{m',m}^s(t) \in t^{-1}\mathbb{Z}[t^{-1}]$ and $\mu_{m',m}(t) \in \mathbb{Z}[t^{\pm}]$ such that $\mu_{m',m}(t) = \mu_{m',m}(t^{-1})$.

For the last point we see also by induction on k that for $m_1, m_2 \in \hat{B}^{s,\text{inv}}$ such that $\hat{\Pi}(m_1) = \hat{\Pi}(m_2)$ and $m_1' \in \hat{B}^s(m_1)$, $m_2' \in \hat{B}^s(m_2)$ such that $m_1^{-1}m_1' \in t^{\mathbb{Z}}m_2^{-1}m_2'$ we have:

$$\mu_{m_1',m_1}(t) = \mu_{m_2',m_2}(t) \text{ , } P_{m_1',m_1}^s(t) = P_{m_2',m_2}^s(t)$$

□

Let us look at an example: we suppose that C is of type $A_2^{(1)}$. In the generic case, the classical algorithm gives the q-characters beginning with $Y_{1,0}$, and the first terms are:

$$\begin{tikzcd}
 & Y_{1,0} \arrow[d, "1,1"] & \\
 & Y_{1,2}^{-1}Y_{3,1}Y_{2,1} \arrow[dl, "3,2"'] \arrow[dr, "2,2"] & \\
Y_{2,0}^{-1}Y_{3,1}Y_{3,2} & & Y_{3,3}^{-1}Y_{2,1}Y_{2,2}
\end{tikzcd}$$

The deformed algorithm gives:

$$\overrightarrow{E}_t(\tilde{Y}_{1,0}) = \tilde{Y}_{1,0}(1 + t\tilde{A}_{1,1}^{-1}(1 + t\tilde{A}_{2,2}^{-1} + t\tilde{A}_{3,2})) + \text{terms of higher degree}$$

We suppose now that $s=3$. First let $m=\tilde{Y}_{1,0}\tilde{Y}_{1,2}$, $m'=t^2\tilde{Y}_{1,0}\tilde{A}_{1,1}^{-1}\tilde{Y}_{1,2}$. We have:

$$\overrightarrow{E}_t(m)=\hat{F}_t(m)+t^{-1}\hat{F}_t(m')+...$$

In particular $P^s_{m',m}(t)=t^{-1}$.

Let $m=\tilde{Y}_{2,1}\tilde{Y}_{3,1}$, $m'=t\tilde{Y}_{2,1}\tilde{Y}_{3,1}\tilde{A}_{3,2}^{-1}\tilde{A}_{2,3}^{-1}$, $m''=t\tilde{Y}_{2,1}\tilde{Y}_{3,1}\tilde{A}_{2,2}^{-1}\tilde{A}_{3,3}^{-1}$. We have:

$$\overrightarrow{E}_t(m)=\hat{F}_t(m)+t^{-1}\hat{F}_t(m')+t^{-1}\hat{F}_t(m'')+...$$

In particular $P^s_{m',m}(t)=t^{-1}$ and $P^s_{m'',m}(t)=t^{-1}$.

Let us go back to general case and we want to define $P^s_{m',m}(t)$ for $m,m'\in B^s$. We can not set as in the finite case $P^s_{m',m}(t)=\sum_{M'\in\hat{B}^s(M)/\hat{\Pi}(M')=m'}P_{M',M}(t)$ (where $M\in\hat{B}^{s,\text{inv}}$ verifies $\hat{\Pi}(M)=m$) because this sum is not finite in general. However we propose the following construction. For $m,m'\in B^s$, we define $k(m,m')\geq 0$ such that for $M\in\hat{\Pi}^{-1}(m)$ we have $k(m,m')=\min\{k\geq 0/\exists M'\in\hat{B}^s_k(M),\hat{\Pi}(M')=m'\}$.

Definition 6.8. *For $m,m'\in B^s$ we define $P^s_{m',m}(t)\in\mathbb{Z}[t^{\pm}]$ by:*

$$P^s_{m',m}(t)=\sum_{M'\in\hat{B}^s(M)/\hat{\Pi}(M')=m' \text{ and } deg(M')=deg(M)+k(m,m')}P_{M',M}(t)$$

where M an element of $\hat{B}^{s,inv}\cap\hat{\Pi}^{-1}(m)$.

Note that if C affine it follows from lemma 3.11 that for each $m\in B^s$, there is a finite number of $m'\in B^s$ such that $P^s_{m',m}(t)\neq 0$. In particular in this situation the proof of the theorem gives an algorithm to compute the polynomials with a finite number of steps (although there could be an infinite number of monomials in the ϵ,t-character).

For example if C is of type $A_2^{(1)}$ and $s=3$ we have:

$$P_{Y_{3,1}Y_{2,1},Y_{1,0}Y_{1,2}}(t)=t^{-1}\ ,\ P_{Y_{1,0},Y_{1,2},Y_{3,1}Y_{2,1}}=2t^{-1}$$

6.3. **Quantization of the Grothendieck ring.**

6.3.1. *General quantization.* We set $\text{Rep}^s_t=\text{Rep}^s\otimes\mathbb{Z}[t^{\pm}]=\mathbb{Z}[X_{i,l},t^{\pm}]_{i\in I,l\in\mathbb{Z}/s\mathbb{Z}}$ and we extend $\chi_{\epsilon,t}$ to a $\mathbb{Z}[t^{\pm}]$-linear injective map $\chi_{\epsilon,t}:\text{Rep}^s_t\to\tilde{\mathfrak{K}}^{\infty,s}_t$. We set $B^s=\{m=\prod_{i\in I,l\in\mathbb{Z}/s\mathbb{Z}}\tilde{Y}_{i,l}^{u_{i,l}(m)}\}\subset\overline{B}^s$. We have a map $\pi:\overline{B}^s\to B^s$ defined by $\pi(m)=\prod_{i\in I,l\in\mathbb{Z}/s\mathbb{Z}}\tilde{Y}_{i,l}^{u_{i,l}(m)}$.

We have:

$$\text{Im}(\chi_{\epsilon,t})=\bigoplus_{m\in B^s}\mathbb{Z}[t^{\pm}]\overrightarrow{E}_t(m)\subset\tilde{\mathfrak{K}}^{s,\infty}_t$$

But in general $\text{Im}(\chi_{\epsilon,t})$ is not a subalgebra of $\tilde{\mathfrak{K}}^{s,\infty}_t$.

If $s=0$ or C is finite we have $\text{Im}(\chi_{\epsilon,t})\subset\tilde{\mathfrak{K}}^{s,f,\infty}_t=\bigoplus_{m\in\overline{B}^s}\mathbb{Z}[t^{\pm}]\overrightarrow{E}_t(m)$ and we have a $\mathbb{Z}[t^{\pm}]$-linear map $\pi:\tilde{\mathfrak{K}}^{s,f,\infty}_t\to\text{Im}(\chi_{q,t})$ such that for $m\in\overline{B}^s$:

$$\pi(\overrightarrow{E}_t(m))=\overrightarrow{E}_t(\pi(m))$$

If $s>2r^\vee$ and C verifies the property of lemma 3.11 (for example C is affine) then there is a $\mathbb{Z}[t^{\pm}]$-linear map $\pi:\tilde{\mathfrak{K}}^{s,\infty}_t\to\text{Im}(\chi_{q,t})$ such that for $m\in\overline{B}^s$ of the form $m=Mm'$ where $M\in B^s$ and $m'=t^\alpha\tilde{A}_{i_1,l_1}^{-1}...\tilde{A}_{i_k,l_k}^{-1}$ (see the definition of $k(m_1,m_2)\in\mathbb{Z}$ in section 6.2.4):

$$\pi(\overrightarrow{E}_t(m))=\overrightarrow{E}_t(\pi(m))\text{ if }k=k(\hat{\Pi}(m),\hat{\Pi}(M))$$

$$\pi(\overrightarrow{E}_t(m))=0\text{ if }k>k(\hat{\Pi}(m),\hat{\Pi}(M))$$

In both cases, as $\chi_{\epsilon,t}$ is injective, we can define a $\mathbb{Z}[t^{\pm}]$-bilinear map $*$ such that for $\alpha,\beta\in\mathrm{Rep}_t^s$:

$$\alpha*\beta=\chi_{q,t}^{-1}(\pi(\chi_{q,t}(\alpha)\chi_{q,t}(\beta)))$$

This is a deformed multiplication on Rep_t^s. But in general this multiplication is not associative.

6.3.2. *Associative quantization.* In some cases it is possible to define an associative quantization (see [VV2], [N3], [He2]). The point is to use a t-deformed algebra $\mathcal{Y}_t=\mathbb{Z}[\tilde{Y}_{i,l}^{\pm},t^{\pm}]_{i\in I,l\in\mathbb{Z}}$ instead of $\hat{\mathcal{Y}}_t$: in this case $\mathrm{Im}(\chi_{q,t})$ is an algebra and we have an associative quantization of the Grothendieck ring (see [He2] for details). In this section we see how this construction can be generalized to other Cartan matrices. We suppose that $s=0$ and that q is transcendental.

Lemma 6.9. *Let C be a Cartan matrix such that:*

$$C_{i,j}<-1\Rightarrow -C_{j,i}\leq r_i$$

Then:

$$det(C(z))=z^{-R}+\alpha_{-R+1}z^{-R+1}+...+\alpha_{R-1}z^{R-1}+z^R$$

where $R=\sum\limits_{i=1...n}r_i$ and $\alpha(-l)=\alpha(l)\in\mathbb{Z}$.

In particular finite and affine Cartan matrices ($A_1^{(1)}$ with $r_1=r_2=2$) verify the property of lemma 6.9. Note that the condition $C_{i,j}<0\Rightarrow C_{i,j}=-1$ or $C_{j,i}=-1$ is sufficient; in particular Cartan matrices such that $i\neq j\Rightarrow C_{i,j}C_{j,i}\leq 3$ verify the property.

Proof: For $\sigma\in S_n$ let us look at the term $\det_\sigma=\prod\limits_{i\in I}C_{i,\sigma(i)}(z)$ of $\det(C(z))$. If $\sigma=\mathrm{Id}$ then the degree $\deg(\det_{\mathrm{Id}})$ is $\sum\limits_{i\in I}r_i$. So it suffices to show that for $\sigma\neq\mathrm{Id}$ we have $\deg(\det_\sigma)<\sum\limits_{i\in I}r_i$. If $i\neq\sigma(i)$, we have the following cases:

if $C_{i,\sigma(i)}=0$ or -1, $\deg([C_{i,\sigma(i)}]_z)\leq 0<r_{\sigma(i)}$

if $C_{i,\sigma(i)}<-1$, we have $C_{\sigma(i),i}=-1$ and so $r_iC_{i,\sigma(i)}=-r_{\sigma(i)}$ and so

$$\deg([C_{i,\sigma(i)}]_z)=-C_{i,\sigma(i)}-1=-\frac{r_{\sigma(i)}C_{\sigma(i),i}}{r_i}-1\leq r_{\sigma(i)}-1<r_{\sigma(i)}$$

So if $\sigma\neq\mathrm{Id}$ we have:

$$\deg(\det_\sigma)=\sum_{i\in I/i=\sigma(i)}r_i+\sum_{i\in I/i\neq\sigma(i)}\deg([C_{i,\sigma(i)}]_{z_i})<\sum_{i\in I/i=\sigma(i)}r_i+\sum_{i\in I/i\neq\sigma(i)}r_{\sigma(i)}=\sum_{i\in I}r_i$$

For the last point $\det(C(z))$ is symmetric polynomial because the coefficients of $C(z)$ are symmetric. $\square$

We suppose in this section that C verifies the property of lemma 6.9.

In particular $\det(C(z))\neq 0$ and $C(z)$ has an inverse $\tilde{C}(z)$ with coefficients of the form $\frac{P(z)}{Q(z^{-1})}$ where $P(z)\in\mathbb{Z}[z^{\pm}]$, $Q(z)\in\mathbb{Z}[z]$, $Q(0)=\pm1$ and the dominant coefficient of Q is ±1. We denote by $\mathfrak{V}\subset\mathbb{Z}((z^{-1}))$ the set of rational fractions of this form. Note that $\mathfrak{V}$ is a subring of $\mathbb{Q}(z)$, and for $R(z)\in\mathfrak{V},m\in\mathbb{Z}$ we have $R(z^m)\in\mathfrak{V}$. In particular for $m\in\mathbb{Z}-\{0\}$, $\tilde{C}(q^m)$ makes sense.

We denote by $\mathbb{Z}((z^{-1}))$ the ring of series of the form $P=\sum\limits_{r\leq R_P}P_rz^r$ where $R_P\in\mathbb{Z}$ and the coefficients $P_r\in\mathbb{Z}$. We have an embedding $\mathfrak{V}\subset\mathbb{Z}((z^{-1}))$ by expanding $\frac{1}{Q(z^{-1})}$ in $\mathbb{Z}[[z^{-1}]]$ for $Q(z)\in\mathbb{Z}[z]$ such that $Q(0)=1$. So we can introduce maps (π_r, $r\in\mathbb{Z}$):

$$\pi_r:\mathfrak{V}\to\mathbb{Z}\ ,\ P=\sum_{r\leq R_P}P_rz^r\mapsto P_r$$

We denote by $\mathcal{H}$ the algebra with generators $a_i[m],y_i[m],c_r$, relations 1, 2 (of definition 3.1) and ($j\in I,m\neq 0$):

$$y_j[m]=\sum_{i\in I}\tilde{C}_{i,j}(q^m)a_i[m]\qquad(4)$$

Note that the relations 4 are compatible with the relations 2.

We define $\mathcal{Y}_u$ as the subalgebra of $\mathcal{H}[[h]]$ generated by the $\tilde{Y}_{i,l}^{\pm}$, $\tilde{A}_{i,l}^{\pm}$ $(i \in I, l \in \mathbb{Z})$, t_R $(R \in \mathfrak{V})$.

Let the algebra $\mathcal{Y}_t$ be the quotient of $\mathcal{Y}_u$ by relations

$$t_R = t_{R'} \text{ if } \pi_0(R) = \pi_0(R')$$

We keep the notations $\tilde{Y}_{i,l}^{\pm}, \tilde{A}_{i,l}^{\pm}$ for their image in $\mathcal{Y}_t$. We denote by t the image of $t_1 = \exp(\sum_{m>0} h^{2m} c_m)$ in $\mathcal{Y}_t$.

The following theorem is a generalization of theorem 3.11 of [He2]:

Theorem 6.10. *([He2]) The algebra $\mathcal{Y}_t$ is defined by generators $\tilde{Y}_{i,l}^{\pm}$, $(i \in I, l \in \mathbb{Z})$ central elements $t^{\pm}$ and relations $(i,j \in I, k,l \in \mathbb{Z})$:*

$$\tilde{Y}_{i,l}\tilde{Y}_{j,k} = t^{\gamma(i,l,j,k)}\tilde{Y}_{j,k}\tilde{Y}_{i,l}$$

where $\gamma : (I \times \mathbb{Z})^2 \to \mathbb{Z}$ is given by:

$$\gamma(i,l,j,k) = \sum_{r\in\mathbb{Z}} \pi_r(\tilde{C}_{j,i}(z))(-\delta_{l-k,-r_j-r} - \delta_{l-k,r-r_j} + \delta_{l-k,r_j-r} + \delta_{l-k,r_j+r})$$

7. Complements

7.1. **Finiteness of algorithms.** In the construction of q,t and ϵ,t-character we deal with completed algebras $\hat{\mathcal{Y}}_t^{s,\infty}$, so the algorithms can produce an infinite number of monomials. In some cases we can say when this number is finite:

7.1.1. *Finiteness of the classical and deformed algorithms.*

Definition 7.1. *We say that the classical algorithm stops if the classical algorithm is well defined and for all $m \in B$, $F(m) \in \mathfrak{K}$.*

It follows from the classical theory of q-characters that if C is finite then the classical algorithm stops.

For $i \in I$ let $L_i = (C_{i,1}, ..., C_{i,n})$.

Proposition 7.2. *We suppose that there are $(\alpha_i)_{i\in I} \in \mathbb{Z}^I$ such that $\alpha_i > 0$ and:*

$$\sum_{j\in I} \alpha_j L_j = 0$$

Then the classical algorithm does not stop.

In particular if C is an affine Cartan matrix then the classical algorithm does not stop.

Proof: It follows from lemma 4.6 at $t = 1$ that it suffices to show that there is no antidominant monomial in $C(Y_{1,0})$. So let $m = Y_{1,0} \prod_{i\in I, l\in\mathbb{Z}} A_{i,l}^{-v_{i,l}}$ be in $C(Y_{1,0})$. We see as in lemma 3.11 $u_i(Y_{1,0}^{-1}m) = 0$. In particular $u_1(m) = 1$ and m is not antidominant. □

Note that in the $A_r^{(1)}$-case $(r \geq 2)$ we have a more "intuitive" proof : for all $l \in \mathbb{Z}$, $i \in I$ we have $A_{i,l}^{-1} = Y_{i,l+1}^{-1} Y_{i,l-1}^{-1} Y_{i+1,l} Y_{i-1,l}$, and:

$$u(A_{i,l}^{-1}) = \sum_{j\in I, k\in\mathbb{Z}} u_{j,k}(A_{i,l}^{-1}) = (-u_{i,l+1} - u_{i,l-1} + u_{i+1,l} + u_{i-1,l})(A_{i,l}^{-1}) = 0$$

where we set in I: $(1) - 1 = r + 1$ and $(r + 1) + 1 = 1$. So for all $m \in C(Y_{1,0})$ we have $u(m) = 1$ and m is not antidominant.

7.1.2. *Finiteness of the deformed algorithm.*

Proposition 7.3. *The following properties are equivalent:*

i) For all $i \in I$, $\hat{F}_t(\tilde{Y}_{i,0}) \in \hat{\mathfrak{K}}_t$.

ii) For all $m \in \overline{B}$, $\hat{F}_t(m) \in \hat{\mathfrak{K}}_t$.

iii) $Im(\chi_{q,t}) \subset \hat{\mathfrak{K}}_t$.

Definition 7.4. *If the properties of the proposition 7.3 are verified we say that the deformed algorithm stops.*

Let us give some examples:

-If C is of type ADE then the deformed algorithm stops: [N3] (geometric proof) and [N4] (algebraic proof in AD cases)

-If C is of rank 2 ($A_1 \times A_1$, A_2, B_2, C_2, G_2) then the deformed algorithm stops: [He2] (algebraic proof)

-In [He2] we give an alternative algebraic proof for Cartan matrices of type A_n ($n \geq 1$) and we conjecture that for all finite Cartan matrices the deformed algorithm stops. The cases F_4, B_n, C_n ($n \leq 10$) have been checked on a computer (with the help of T. Schedler).

Lemma 7.5. *If the deformed algorithm stops then the classical algorithm stops.*

Proof: This is a consequence of the formula $F(\hat{\Pi}(m)) = \hat{\Pi}(\hat{F}_t(m))$ (see section 4.2.2). □

In particular if C is affine then the deformed algorithm does not stop.

Let C be a Cartan matrix such that $i \neq j \Rightarrow C_{i,j}C_{j,i} \leq 3$. We conjecture that the deformed algorithm stops if and only if the classical algorithm stops.

7.2. q,t**-characters of affine type and quantum toroidal algebras.** We have seen in [He2] that if C is finite then the defining relations of $\hat{\mathcal{H}}$:

$$[a_i[m], a_j[r]] = \delta_{m,-r}(q^m - q^{-m})B_{i,j}(q^m)c_{|m|}$$

appear in the $\mathbb{C}$-subalgebra $\mathcal{U}_q(\hat{\mathfrak{h}})$ of $\mathcal{U}_q(\hat{\mathfrak{g}})$ generated by the $h_{i,m}, c^{\pm}$ ($i \in I, m \in \mathbb{Z} - \{0\}$): it suffices to send $a_i[m]$ to $(q - q^{-1})h_{i,m}$ and c_r to $\frac{c^r - c^{-r}}{r}$.

In this section we see that in the affine case $A_n^{(1)}$ ($n \geq 2$) the relations of $\hat{\mathcal{H}}$ appear in the structure of the quantum toroidal algebra. In particular we hope that q,t-characters will play a role in representation theory of quantum toroidal algebras (see the introduction).

7.2.1. *Reminder on quantum toroidal algebras* [VV1]*.* Let be $d \in \mathbb{C}^*$ and $n \geq 3$. In the quantum toroidal algebra of type sl_n there is a subalgebra $\mathcal{Z}$ generated by the $k_i^{\pm}, h_{i,l}$ ($i \in \{1, ..., n\}$, $l \in \mathbb{Z} - \{0\}$) with relations :

$$k_i k_i^{-1} = cc^{-1} = 1 \text{ , } [k_{\pm,i}(z), k_{\pm,j}(w)] = 0$$

$$\theta_{-a_{i,j}}(c^2 d^{-m_{i,j}} wz^{-1})k_{+,i}(z)k_{-,j}(w) = \theta_{-a_{i,j}}(c^{-2}d^{-m_{i,j}} wz^{-1})k_{-,j}k_{+,i}(z) \tag{5}$$

where $k_{\pm,i}(z) \in \mathcal{Z}[[z]]$ is defined by:

$$k_{\pm,i}(z) = k_i^{\pm}\exp(\pm(q - q^{-1})\sum_{k>0} h_{i,\pm k} z^{\mp k})$$

$\theta_m(z) \in \mathbb{C}[[z]]$ is the expansion of $\frac{q^m z-1}{z-q^m}$, $A = (a_{i,j})_{0\leq i,j\leq n}$ is the affine Cartan matrix of type $A^{(1)}_{n-1}$:

$$A = \begin{pmatrix} 2 & -1 & ... & 0 & -1 \\ -1 & 2 & ... & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & ... & 2 & -1 \\ -1 & 0 & ... & -1 & 2 \end{pmatrix}$$

and $M = (m_{i,j})_{1\leq i,j\leq n}$ is given by:

$$M = \begin{pmatrix} 0 & -1 & ... & 0 & 1 \\ 1 & 0 & ... & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & ... & 0 & -1 \\ -1 & 0 & ... & 1 & 0 \end{pmatrix}$$

7.2.2. *Relations of the Heisenberg algebra.*

Lemma 7.6. *The relation (5) are consequences of:*

$$[h_{i,l}, h_{j,m}] = \delta_{l,-m}\frac{q^{la_{i,j}} - q^{-la_{i,j}}}{(q-q^{-1})^2} d^{-|l|m_{i,j}}\frac{c^{2l}-c^{-2l}}{l}$$

Proof: First for $m \in \mathbb{Z}$, we have in $\mathbb{C}[[z]]$:

$$\theta_m(z) = \frac{q^m z - 1}{z - q^m} = q^{-m}\exp(\ln(1-q^m z) - \ln(1-q^{-m}z)) = q^{-m}\exp(\sum_{r\geq 1}(-\frac{(q^m z)^r}{r} + \frac{(q^{-m}z)^r}{r}))$$

and so $k_{+,i}(z)k_{-,j}(z)k_{+,i}(z)^{-1}k_{-,j}(w)^{-1} = \theta_{-a_{i,j}}(c^{-2}d^{-m_{i,j}}wz^{-1})\theta_{-a_{i,j}}(c^2 d^{-m_{i,j}}wz^{-1})^{-1}$ is given by:

$$\theta_{-a_{i,j}}(c^{-2}d^{-m_{i,j}}wz^{-1})\theta_{-a_{i,j}}(c^2 d^{-m_{i,j}}wz^{-1})^{-1} = \exp(\sum_{r\geq 1}(q^{-ra_{i,j}} - q^{ra_{i,j}})d^{-rm_{i,j}}(wz^{-1})^r\frac{c^{2r}-c^{-2r}}{r}))$$

But following the proof of lemma 3.2 we see that the relation of lemma 7.6 give:

$$[-(q-q^{-1})\sum_{r\geq 1}h_{j,-r}w^r, -(q-q^{-1})\sum_{l\geq 1}h_{i,l}z^{-r}] = \sum_{r\geq 1}(q-q^{-1})^2\frac{q^{-ra_{i,j}} - q^{ra_{i,j}}}{(q-q^{-1})^2}d^{-rm_{i,j}}(wz^{-1})^r\frac{c^{2r}-c^{-2r}}{r}$$

□

In particular for $d=1$, $a_i[m] = \frac{h_{i,m}}{q-q^{-1}}$ and $c_m = \frac{c^{2m}-c^{-2m}}{m}$, we get the defining relation of the Heisenberg algebra $\hat{\mathcal{H}}$ of section 3.1.1 in the affine case $A^{(1)}_{n-1}$:

$$[a_i[m], a_j[r]] = \delta_{m,-r}(q^m - q^{-m})[B_{i,j}]_{q^m}c_{|m|}$$

In the case $d \neq 1$ we have to extend the former construction:

7.2.3. *Twisted multiplication with two variables.* Let us study the case $d \neq 1$: in this section we suppose that q, d are indeterminate and we construct a t-deformation of $\mathbb{Z}[\tilde{A}^{\pm}_{i,l,k}]_{i\in I, l,k\in\mathbb{Z}}$.

We define the $\mathbb{C}[q^{\pm}, d^{\pm}]$-algebra $\hat{\mathcal{H}}_d$ by generators $a_i[m]$ ($i \in I = \{1, ..., n\}, m \in \mathbb{Z}$) and relations:

$$[a_i[m], a_j[r]] = \delta_{m,-r}(q^m - q^{-m})[A_{i,j}]_{q^m}d^{-|m|m_{i,j}}c_{|m|}$$

For $i \in I, l, k \in \mathbb{Z}$ we define $\tilde{A}_{i,l,k} \in \hat{\mathcal{H}}_d[[h]]$ by:

$$\tilde{A}_{i,l,k} = \exp(\sum_{m>0}h^m q^{lm}d^{km}a_i[m])\exp(\sum_{m>0}h^m q^{-lm}d^{-km}a_i[-m])$$

and for $R(q,d) \in \mathbb{Z}[q^{\pm}, d^{\pm}]$, $t_R \in \hat{\mathcal{H}}_d[[h]]$ by:

$$t_{R(q,d)} = \exp(\sum_{m>0}h^m R(q^m, d^m)c_m)$$

A computation analogous to the proof of lemma 3.2 gives:

$$\tilde{A}_{i,l,p}\tilde{A}_{j,k,r}\tilde{A}_{i,l,p}^{-1}\tilde{A}_{j,k,r}^{-1}=t_{(q-q^{-1})[A_{i,j}]_q(-q^{l-k}d^{p-r}+q^{k-l}d^{r-p})d^{-m_{i,j}}}$$

In particular, in the quotient of $\hat{\mathcal{H}}_d[[h]]$ by relations $t_R=1$ if $R\neq 0$, we have:

$$\tilde{A}_{i,l,p}^{-1}\tilde{A}_{j,k,r}^{-1}=t^{\alpha(i,j,k,l,p,r)}\tilde{A}_{j,k,r}^{-1}\tilde{A}_{i,l,p}^{-1}$$

where $\alpha:(I\times\mathbb{Z}\times\mathbb{Z})^2\rightarrow\mathbb{Z}$ is given by ($l,k\in\mathbb{Z}$, $i,j\in I$):

$$\alpha(i,i,l,k,p,r)=2(\delta_{l-k,2r_i}-\delta_{l-k,-2r_i})\delta_{r,p}$$

$$\alpha(i,j,l,k,p,r)=\sum_{r=r_iC_{i,j}+1,r_iC_{i,j}+3,...,-r_iC_{i,j}-1}(-\delta_{l-k,r+r_i}\delta_{p-r,m_{i,j}}+\delta_{l-k,r-r_i}\delta_{r-p,m_{i,j}})\text{ (if } i\neq j)$$

In particular this would lead to the construction of q,t-characters with variables $\tilde{Y}_{i,l,p},\tilde{A}_{i,l,p}^{-1}$ associated to quantum toroidal algebras. But we shall leave further discussion of this point to another place.

7.3. **Combinatorics of bicharacters and Cartan matrices.** In this section $C=(C_{i,j})_{1\leq i,j\leq n}$ is an indecomposable generalized (non necessarily symmetrizable) Cartan matrix and $(r_1,...,r_n)$ are positive integers. Let $D=\mathrm{diag}(r_1,...,r_n)$ and $B=DC$ (which is non necessarily symmetric).

We show that the quantization of $\hat{\mathcal{Y}}^s\otimes\mathbb{Z}[t^{\pm}]=\mathbb{Z}[Y_{i,l},V_{i,l},t^{\pm}]_{i\in I,l\in\mathbb{Z}/s\mathbb{Z}}$ is linked to fundamental combinatorial properties of C and $(r_1,...,r_n)$ (propositions 7.9, 7.11, 7.12 and theorem 7.10). Let us begin with some general background about twisted multiplication defined by bicharacters.

7.3.1. *Bicharacters and twisted multiplication.* Let Λ be a set, Y be the commutative polynomial ring:

$$Y=\mathbb{Z}[X_\alpha,t^{\pm}]_{\alpha\in\Lambda}$$

and A the set of monomials of the form $m=\prod_{\alpha\in\Lambda}X_\alpha^{x_\alpha(m)}\in Y$. The usual commutative multiplication of Y is denoted by . in the following.

Definition 7.7. *A bicharacter on A is a map $d:A\times A\rightarrow\mathbb{Z}$ such that ($m_1,m_2,m_3\in A$):*

$$d(m_1.m_2,m_3)=d(m_1,m_3)+d(m_2,m_3)\ ,\ d(m_1,m_2.m_3)=d(m_1,m_2)+d(m_1,m_3)$$

The symmetric bicharacter $\mathfrak{S}d$ and the antisymmetric bicharacter $\mathfrak{A}d$ of d are defined by:

$$\mathfrak{S}d(m_1,m_2)=\frac{1}{2}(d(m_1,m_2)+d(m_2,m_1))\ ,\ \mathfrak{A}d(m_1,m_2)=\frac{1}{2}(d(m_1,m_2)-d(m_2,m_1))$$

and we have $d=\mathfrak{A}d+\mathfrak{S}d$.

Let be d be a bicharacter on A. One can define a $\mathbb{Z}[t^{\pm}]$-bilinear map $*:Y\times Y\rightarrow Y$ such that:

$$m_1*m_2=t^{d(m_1,m_2)}m_1.m_2$$

This map is associative[1] and we get a $\mathbb{Z}[t^{\pm}]$-algebra structure on Y. We say that the new multiplication is the twisted multiplication associated to the bicharacter d, and it is given by formulas:

$$m_1*m_2=t^{d(m_1,m_2)-d(m_2,m_1)}m_2*m_1=t^{2\mathfrak{A}d(m_1,m_2)}m_2*m_1$$

Lemma 7.8. *Let d_1,d_2 be two bicharacters. One can define a multiplication on Y such that ($m_1,m_2\in A$):*

$$m_1*m_2=t^{2d_1(m_1,m_2)-2d_2(m_2,m_1)}m_2*m_1$$

if and only if $\mathfrak{S}d_1=\mathfrak{S}d_2$.

In this case, the multiplication is the twisted multiplication associated to the bicharacter $d=d_1+d_2$:

$$m_1*m_2=t^{d_1(m_1,m_2)+d_2(m_1,m_2)}m_1.m_2$$

[1]In fact it suffices that $-d(m_2,m_3)+d(m_1m_2,m_3)-d(m_1,m_2m_3)+d(m_1,m_2)=0$.

Proof: It follows immediately from the definition of $*$:

$$m_1 * m_2 = t^{2d_1(m_1,m_2)-2d_2(m_2,m_1)} m_2 * m_1 = t^{4(\mathfrak{S}d_1-\mathfrak{S}d_2)(m_1,m_2)} m_1 * m_2$$

If $\mathfrak{S}d_1 = \mathfrak{S}d_2$, let $*$ be the twisted multiplication associated with the bicharacter $d = d_1 + d_2$. We have:

$$m_1 * m_2 = t^{d_1(m_1,m_2)+d_2(m_1,m_2)-d_1(m_2,m_1)-d_2(m_2,m_1)} m_2 * m_1 = t^{2d_1(m_1,m_2)-2d_2(m_2,m_1)} m_2 * m_1$$

□

7.3.2. *Definition of d_1 and d_2.* For $s \geq 0$ let $\Lambda_s = I \times (\mathbb{Z}/s\mathbb{Z})$ and $\overline{A}^s$ be the set of monomials of $\hat{\mathcal{Y}}^s$, that is to say elements of the form $m = \prod_{(i,l)\in\Lambda_s} Y_{i,l}^{y_{i,l}(m)} V_{i,l}^{v_{i,l}(m)}$. Let $D(z) = \text{diag}([r_1]_z, ..., [r_n]_z)$.

For $\alpha \in \Lambda_s$, we define a character [2] u_α on $\overline{A}^s$ as in section 3.2.1. In particular $u_\alpha(Y_\beta) = \delta_{\alpha,\beta}$.

We define d_1, d_2 the bicharacters on $\overline{A}^s$ as in section 3.2.1, that is to say ($m_1, m_2 \in \overline{A}^s$):

$$d_1(m_1, m_2) = \sum_{\alpha\in\Lambda_s} v_{b(\alpha)}(m_1)u_\alpha(m_2) + y_{b(\alpha)}(m_1)v_\alpha(m_2)$$

$$d_2(m_1, m_2) = \sum_{\alpha\in\Lambda_s} u_{b(\alpha)}(m_1)v_\alpha(m_2) + v_{b(\alpha)}(m_1)y_\alpha(m_2)$$

where $b : \Lambda_s \to \Lambda_s$ is the bijection defined by $b(i, l) = (i, l + r_i)$.

Proposition 7.9. *The following properties are equivalent:*

i) For $s \geq 0$, $d_1 = d_2$

ii) For $s \geq 0$, $\forall \alpha, \beta \in \Lambda_s, u_\alpha(V_\beta) = u_{b(\beta)}(V_{b(\alpha)})$

iii) C is symmetric and $\forall i, j \in I$, $r_i = r_j$.

Proof: We have always:

$$d_1(Y_\alpha, Y_\beta) = d_2(Y_\alpha, Y_\beta) = 0$$

For $\alpha, \beta \in \Lambda_s$, we have $u_\alpha(Y_\beta) = \delta_{\alpha,\beta}$. In particular:

$$d_1(Y_\alpha, V_\beta) = \delta_{b(\beta),\alpha} = u_{b(\beta)}(Y_\alpha) = d_2(Y_\alpha, V_\beta)$$

$$d_1(V_\beta, Y_\alpha) = u_{b^{-1}(\beta)}(Y_\alpha) = \delta_{b(\alpha),\beta} = d_2(V_\beta, Y_\alpha)$$

So the condition $d_1 = d_2$ means $\forall \alpha, \beta \in \Lambda_s$, $d_1(V_\alpha, V_\beta) = d_2(V_\alpha, V_\beta)$. But the equation (ii) means:

$$d_1(V_\alpha, V_\beta) = u_{b^{-1}(\alpha)}(V_\beta) = u_{b(\beta)}(V_\alpha) = d_2(V_\alpha, V_\beta)$$

In particular we have $(i) \Leftrightarrow (ii)$.

For $i, j \in I$ and $l, k \in \mathbb{Z}/s\mathbb{Z}$ we have:

$$u_{i,l}(V_{j,k}) = \sum_{r=C_{i,j}+1...-C_{i,j}-1} \delta_{l+r,k} = \sum_{r=C_{i,j}+1...-C_{i,j}-1} \delta_{l-k,r}$$

$$u_{j,k+r_j}(V_{i,l+r_i}) = \sum_{r=C_{j,i}+1...-C_{j,i}-1} \delta_{k+r+r_j,l+r_i} = \sum_{r=C_{j,i}+1...-C_{j,i}-1} \delta_{l-k,r_j-r_i+r}$$

If $s = 0$, those terms are equal for all $l, k \in \mathbb{Z}$ if and only if $C_{i,j} \neq 0$ implies $C_{i,j} = C_{j,i}$ and $r_i = r_j$. So as C is indecomposable we have $(ii) \Leftrightarrow (iii)$.

If $s \geq 0$ and (iii) is verified we see in the same way that those terms are equal, so $(iii) \Rightarrow (ii)$. □

In particular if C is of type ADE, we get the bicharacter of [N3] and $d_1 = d_2$ is the equation ([N3], 2.1).

[2] ie. $u_\alpha(m_1.m_2) = u_\alpha(m_1) + u_\alpha(m_2)$

7.3.3. *Bicharacters and symmetrizable Cartan matrices.* We have seen in lemma 7.8 that we can define a twisted multiplication if and only if $\mathfrak{S}d_1 = \mathfrak{S}d_2$, so we investigate those cases:

Theorem 7.10. *The following properties are equivalent:*

i) For $s \geq 0$, we have $\mathfrak{S}d_1 = \mathfrak{S}d_2$

ii) For $s \geq 0$, $\forall \alpha, \beta \in \Lambda_s, u_\alpha(V_{b(\beta)}) - u_{b^2(\alpha)}(V_{b(\beta)}) = u_{b^2(\beta)}(V_{b(\alpha)}) - u_\beta(V_{b(\alpha)})$

iii) For $s \geq 0$ and $m \in \overline{A}^s$, $d_1(m,m) = d_2(m,m)$

iv) $B(z)$ is symmetric

v) B is symmetric and $C_{i,j} \neq C_{j,i} \Longrightarrow (r_i = -C_{j,i}$ and $r_j = -C_{i,j})$

Proof:

First we show that $(i) \Leftrightarrow (ii)$. We have always:

$$\mathfrak{S}d_1(Y_\alpha, Y_\beta) = \mathfrak{S}d_2(Y_\alpha, Y_\beta) = 0$$

and:

$$2\mathfrak{S}d_1(Y_\alpha, V_\beta) = u_{b(\beta)}(Y_\alpha) - u_{b^{-1}(\beta)}(Y_\alpha) = \delta_{b(\beta),\alpha} - \delta_{b(\alpha),\beta} = 2\mathfrak{S}d_2(Y_\alpha, V_\beta)$$

But the equation (ii) means:

$$d_1(V_\alpha, V_\beta) - d_2(V_\beta, V_\alpha) = d_2(V_\alpha, V_\beta) - d_1(V_\beta, V_\alpha)$$

that is to say:

$$2\mathfrak{S}d_1(V_\alpha, V_\beta) = 2\mathfrak{S}d_2(V_\alpha, V_\beta)$$

and we can conclude because d_1, d_2 are bicharacters.

Let us show that $(iv) \Leftrightarrow (v)$: the matrix $B(z)$ is symmetric if and only if for all $i \neq j$ we have:

$$(z^{r_i} - z^{-r_i})(z^{C_{i,j}} - z^{-C_{i,j}}) = (z^{r_j} - z^{-r_j})(z^{C_{j,i}} - z^{-C_{j,i}})$$

If $C_{i,j} = C_{j,i} = 0$ it is obvious. If $C_{i,j} = C_{j,i} \neq 0$, the equation means $r_i = r_j$. If $C_{i,j} \neq C_{j,i}$, the equality means $(r_i = -C_{j,i}$ and $r_j = -C_{i,j})$.

The equation (ii) means:

$$\sum_{r=C_{i,j}+1\dots -C_{i,j}-1} \delta_{l-k,r_j-r} - \delta_{l-k,r_j-2r_i-r} = \sum_{r=C_{j,i}+1\dots -C_{j,i}-1} \delta_{l-k,2r_j+r-r_i} - \delta_{l-k,r-r_i}$$

At $s=0$, the formula holds for all $l,k \in \mathbb{Z}$, if and only the coefficients of Kronecker's functions are equal, that is to say in $\mathbb{Z}[X^\pm]$:

$$\sum_{r=C_{i,j}+1\dots -C_{i,j}-1} X^{r_j-r} - X^{r_j-2r_i-r} = \sum_{r=C_{j,i}+1\dots -C_{j,i}-1} X^{2r_j+r-r_i} - X^{r-r_i}$$

$$(X^{r_j} - X^{r_j-2r_i})X^{C_{i,j}+1}\frac{1-X^{-2C_{i,j}}}{1-X^2} = (X^{2r_j-r_i} - X^{-r_i})X^{C_{j,i}+1}\frac{1-X^{-2C_{j,i}}}{1-X^2}$$

$$X^{r_j-2r_i+C_{i,j}}(1-X^{-2C_{i,j}})(1-X^{2r_i}) = X^{-r_i+C_{j,i}}(1-X^{-2C_{j,i}})(1-X^{2r_j})$$

$$(C_{i,j} = C_{j,i} = 0) \text{ or } (r_i = r_j \text{ and } C_{i,j} = C_{j,i} \neq 0) \text{ or } (r_j = -C_{i,j} \text{ and } r_i = -C_{j,i})$$

and so $(ii) \Rightarrow (v)$. If we suppose that (iv) is true, then the above equation is also verified in $\mathbb{Z}[X^\pm]/(X^s = 1)$ and (ii) is true.

To conclude it suffices to show that $(iii) \Leftrightarrow (i)$. If (iii) is verified we have for $m, m' \in \overline{A}^s$:

$$d_1(m,m') + d_1(m',m) = d_1(mm',mm') - d_1(m,m) - d_1(m',m') = d_2(m,m') + d_2(m',m)$$

and (i) is verified. If (i) is verified we have for $m \in \overline{A}^s$: $2d_1(m,m) = 2d_2(m,m)$. $\square$

7.3.4. *Bicharacters and q-symmetrizable Cartan matrices.* There is a way to define a deformation multiplication if $B(z)$ is non necessarily symmetric. First we define the matrix $C'_{i,j}(z) = [C_{i,j}]_{z_i}$ and the characters :

$$u'_{i,l}(m) = y_{i,l}(m) - \sum_{j\in I}(C'_{i,j}(z))_{op}\mathcal{V}_{j,l}(m)$$

We define the bicharacters d'_1 and d'_2 from $\tilde{u}_{i,l}$ in the same way d_1 and d_2 were defined from $u_{i,l}$ (section 7.3.2).

We also define $B'_{i,j}(z) = [B_{i,j}]_z$. Note that we have always $B'(z) = D(z)C'(z)$. Indeed:

$$B'_{i,j}(z) = \frac{z^{r_iC_{i,j}} - z^{-r_iC_{i,j}}}{z - z^{-1}} = \frac{z_i - z_i^{-1}}{z - z^{-1}}\frac{z_i^{C_{i,j}} - z_i^{-C_{i,j}}}{z_i - z_i^{-1}} = D_{i,i}(z)C'_{i,j}(z)$$

Proposition 7.11. *The following properties are equivalent:*

i) For $s \geq 0$, $\mathfrak{S}d'_1 = \mathfrak{S}d'_2$

ii) For $s \geq 0$, $\forall\alpha,\beta\in\Lambda_s, u'_\alpha(V_{b(\beta)}) - u'_{b^2(\alpha)}(V_{b(\beta)}) = u'_{b^2(\beta)}(V_{b(\alpha)}) - u'_\beta(V_{b(\alpha)})$

iii) B is symmetric

iv) $B'(z)$ is symmetric

In particular if C is symmetrizable we can define the deformed structure for all $s \geq 0$.

Proof: First we have $(iii) \Leftrightarrow (iv)$ because $B'_{i,j}(z) = [B_{i,j}]_z$.

We show as in theorem 7.10 that $(ii) \Leftrightarrow (i)$.

Let us write the equation (ii):

$$u'_{i,l}(V_{j,k+r_j}) - u'_{i,l+2r_i}(V_{j,k+r_j}) = u'_{j,k+2r_j}(V_{i,l+r_i}) - u'_{j,k}(V_{i,l+r_i})$$

If $i = j$, we are in the symmetric case, and it follows from proposition 7.9 that this equation is verified. In the case $i \neq j$, if $C_{i,j} = 0$ then all is equal to 0. In the cases $C_{i,j} < 0$ the equation reads:

$$\sum_{r=C_{i,j}+1\dots -C_{i,j}-1}\delta_{l+r_ir,k+r_j} - \delta_{l+2r_i+rr_i,k+r_j} = \sum_{l=C_{j,i}+1\dots -C_{j,i}-1}\delta_{k+2r_j+lr_j,l+r_i} - \delta_{k+rr_j,l+r_i}$$

$$\sum_{r=C_{i,j}+1\dots -C_{i,j}-1}\delta_{l-k,r_j-rr_i} - \delta_{l-k,r_j-2r_i-r_ir} = \sum_{r=C_{j,i}+1\dots -C_{j,i}-1}\delta_{l-k,2r_j+rr_j-r_i} - \delta_{l-k,rr_j-r_i}$$

$$\delta_{l-k,r_j-r_i-r_iC_{i,j}} - \delta_{l-k,r_j-r_i+r_iC_{i,j}} = \delta_{l-k,r_j-r_i-r_jC_{j,i}} - \delta_{l-k,r_j-r_i+r_jC_{j,i}}$$

That is to say:

$$(2r_iC_{i,j} \in s\mathbb{Z} \text{ and } 2r_jC_{j,i} \in s\mathbb{Z}) \text{ or } r_iC_{i,j} - r_jC_{j,i} \in s\mathbb{Z}$$

If $s = 0$, the equation means $r_iC_{i,j} = r_jC_{j,i}$ that is to say $B = DC$ symmetric. So $(ii) \Leftrightarrow (iii)$.

If $s \geq 0$ and B symmetric we have $r_iC_{i,j} - r_jC_{j,i} \in s\mathbb{Z}$. So $(iii) \Rightarrow (ii)$. □

In some situations the two constructions are the same:

Proposition 7.12. *The following properties are equivalent:*

i) For $s \geq 0$, $u' = u$

ii) For $s \geq 0$, $d'_1 = d_1$

iii) For $s \geq 0$, $d'_2 = d_2$

iv) $C'(z) = C(z)$

v) $B'(z) = B(z)$

vi) $i \neq j \Rightarrow (r_i = 1 \text{ or } C_{i,j} = -1 \text{ or } C_{i,j} = 0)$

Proof: We have $(iv) \Leftrightarrow (v)$ because $B(z) = D(z)C(z)$, $B'(z) = D(z)C'(z)$ and $D(z)$ is invertible.

The $(i) \Rightarrow (ii)$ (resp. $(i) \Rightarrow (iii)$) is clear and we get $(ii) \Rightarrow (i)$ (resp. $(iii) \Rightarrow (i)$) by looking at $d_1(V_{i,l}, V_{j,k}) = d'_1(V_{i,l}, V_{j,k})$ (resp. $d_2(V_{i,l}, V_{j,k}) = d'_2(V_{i,l}, V_{j,k})$).

The $(iv) \Rightarrow (i)$ is clear. If (i) is true we have for $i \neq j$ and all $l, k \in \mathbb{Z}$:

$$u_{i,l}(V_{j,k}) = \sum_{r=C_{i,j}+1, C_{i,j}+3, \dots, -1C_{i,j}-1} \delta_{l-k,r} = \sum_{r=C_{i,j}+1, C_{i,j}+3, \dots, -1C_{i,j}-1} \delta_{l-k,rr_i} = u'_{i,l}(V_{j,k})$$

and so $\frac{z^{C_{i,j}} - z^{-C_{i,j}}}{z - z^{-1}} = \frac{z_i^{C_{i,j}} - z_i^{-C_{i,j}}}{z_i - z_i^{-1}}$ that is to say (iv).

So it suffices to show that $(v) \Leftrightarrow (vi)$. We have always:

$$B_{i,i}(z) = \frac{z^{r_i} - z^{-r_i}}{z - z^{-1}}(z^{r_i} + z^{-r_i}) = \frac{z^{2r_i} - z^{-2r_i}}{z - z^{-1}} = [2r_i]_z = [B_{i,i}]_z$$

If $i \neq j$, the equality $B_{i,j}(z) = B'_{i,j}(z)$ means:

$$z^{r_i + C_{i,j}} + z^{-r_i - C_{i,j}} - z^{C_{i,j} - r_i} - z^{r_i - C_{i,j}} = z^{r_i C_{i,j} + 1} + z^{-1 - r_i C_{i,j}} - z^{r_i C_{i,j} - 1} - z^{1 - r_i C_{i,j}}$$

If $r_i = 1$ or $C_{i,j} = -1$ or $C_{i,j} = 0$ the equality is clear and so $(vi) \Rightarrow (v)$. Suppose that (v) is true and let be $i \neq j$. We have to study different cases:

$r_i + C_{i,j} = C_{i,j} - r_i \Rightarrow r_i = 0$ (impossible)

$r_i + C_{i,j} = r_i - C_{i,j} \Rightarrow C_{i,j} = 0$

$r_i + C_{i,j} = r_i C_{i,j} + 1$ and $r_i C_{i,j} - 1 = C_{i,j} - r_i \Rightarrow r_i = 1$

$r_i + C_{i,j} = r_i C_{i,j} + 1$ and $-r_i C_{i,j} + 1 = C_{i,j} - r_i \Rightarrow C_{i,j} = 1$ (impossible)

$r_i + C_{i,j} = -r_i C_{i,j} - 1$ and $r_i C_{i,j} - 1 = C_{i,j} - r_i \Rightarrow C_{i,j} = -1$

$r_i + C_{i,j} = -r_i C_{i,j} - 1$ and $-r_i C_{i,j} + 1 = C_{i,j} - r_i \Rightarrow r_i = -1$ (impossible)

and so we get (vi). □

Lemma 7.13. *If the properties of the proposition 7.12 are verified and $B = DC$ is symmetric then the properties of the proposition 7.11 are verified.*

Proof: We verify the property (iv) of proposition 7.11: we suppose that $C_{i,j} \neq C_{j,i}$. So $C_{i,j} \neq 0$, $C_{j,i} \neq 0$ and we do not have $C_{i,j} = C_{j,i} = -1$. As $r_i C_{i,j} = r_j C_{j,i}$, we do not have $r_i = r_j = 1$. So we have (property (vi) of proposition 7.12) $r_i = -C_{j,i} = 1$ or $r_j = -C_{i,j} = 1$. For example in the first case, $r_i C_{i,j} = r_j C_{j,i}$ leads to $C_{i,j} = -r_j$. □

Definition 7.14. *We say that C is q-symmetrizable if $B = DC$ is symmetric and:*

$$i \neq j \Rightarrow (r_i = 1 \text{ or } C_{i,j} = -1 \text{ or } C_{i,j} = 0)$$

In particular C q-symmetrizable verifies the properties of proposition 7.11, 7.12 and of theorem 7.10.

7.3.5. *Examples.* If C is symmetric then for all $i \in I$ we have $r_i = 1$ and so C is q-symmetrizable.

Lemma 7.15. *The Cartan matrices of finite or affine type (except $A_1^{(1)}$, $A_{2l}^{(2)}$ case, $l \geq 2$) are q-symmetrizable. The affine Cartan matrices $A_1^{(1)}$, $A_{2l}^{(2)}$ with $l \geq 2$ are not q-symmetrizable.*

In particular if C is finite then $u = \tilde{u}$ and the presentation adopted in this paper fits with former articles, in particular in the non symmetric cases ([FR2], [FM1], [FM2], [He2]).

Proof: As those matrices are symmetrizable, it suffices to check the property (vi) of proposition 7.12:

the finite Cartan matrices A_l $(l \geq 1)$, D_l $(l \geq 4)$, E_6, E_7, E_8 and the affine Cartan matrices $A_l^{(1)}$ $(l \geq 1)$, $D_l^{(1)}$ $(l \geq 4)$, $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$ are symmetric and so q-symmetrizable.

the finite Cartan matrices B_l $(l \geq 2)$, G_2 and the affine Cartan matrices $B_l^{(1)}$ $(l \geq 3)$, $G_2^{(1)}$ verify $r_n = 1$ and for $i \neq j$: $i \leq n-1 \Rightarrow C_{i,j} = -1$ or 0.

the finite Cartan matrices C_l $(l \geq 2)$, the affine Cartan matrices $A_{2l-1}^{(2)}$ $(l \geq 3)$, $D_4^{(3)}$ verify $r_1 = ... = r_{n-1} = 1$, $C_{n,1} = ... = C_{n,n-2} = 0$ and $C_{n,n-1} = -1$.

the affine Cartan matrices $C_l^{(1)}$ $(l \geq 2)$ verify $r_2 = ... = r_{n-1} = 1$ and $C_{1,3} = ... = C_{1,n} = 0$, $C_{1,2} = -1$, $C_{n,1} = ... = C_{n,n-2} = 0$, $C_{n,n-1} = -1$.

the affine Cartan matrices $D_{l+1}^{(2)}$ $(l \geq 2)$ verify $r_1 = r_n = 1$ and for $i \neq j$: $2 \leq i \leq n-1 \Rightarrow C_{i,j} = -1$ or 0.

The other particular cases are studied one after one:

for the finite Cartan matrix $F_4 = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$ we have $(2,2,1,1)$

for the affine Cartan matrix $F_4^{(1)} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -2 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$ we have $(2,2,2,1,1)$

for the affine Cartan matrix $A_2^{(2)} = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$ we have $(1,4)$

for the affine Cartan matrix $E_6^{(2)} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -2 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$ we have $(1,1,1,2,2)$.

Finally the affine Cartan matrices $A_1^{(1)}$ and $A_{2l}^{(2)}$ $(l \geq 2)$ are not q-symmetrizable because $C_{n-1,n} = -2$ and $r_{n-1} = 2$. □

One can understand "intuitively" the fact that $A_{2l}^{(2)}$ $(l \geq 2)$ is not q-symmetrizable: in the Dynkin diagram there is an oriented path without loop with two arrows in the same direction.

There are q-symmetrizable Cartan matrices which are not finite and not affine: here is an example such that for all $i,j \in I$, $C_{i,j} \geq -2$:

$$C = \begin{pmatrix} 2 & -2 & -2 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -2 & -2 & 2 \end{pmatrix}$$
$$(r_1, r_2, r_3, r_4) = (1,2,2,1)$$

Notations

A^s	set of $\mathcal{Y}^s$-monomials	p 9
$\overline{A}, \overline{A}^s$	sets of $\hat{\mathcal{Y}}_t, \hat{\mathcal{Y}}_t^s$-monomials	p 8
$\hat{A}^{\mathrm{inv}}, \hat{B}^{\mathrm{inv}}$	sets of $\hat{\mathcal{Y}}_t$-monomials	p 23
$\hat{A}^{s,\mathrm{inv}}, \hat{B}^{s,\mathrm{inv}}$	sets of $\hat{\mathcal{Y}}_t^s$-monomials	p 25
$\overrightarrow{A}, \overleftarrow{A}$	sets of $\hat{\mathcal{Y}}_t$-monomials	p 15
α	map $(I\times\mathbb{Z}/s\mathbb{Z})^2\rightarrow\mathbb{Z}$	p 6
$\alpha(m)$	character	p 23
$a_i[m]$	element of $\hat{\mathcal{H}}$	p 5
$\tilde{A}_{i,l}, \tilde{A}_{i,l}^{-1}$	elements of $\hat{\mathcal{Y}}_u$ or $\hat{\mathcal{Y}}_t$	p 5
$A_{i,l}, A_{i,l}^{-1}$	elements of $\hat{\mathcal{Y}}$	p 9
b	bijection of Λ_s	p 35
$\overline{B}_i, \overline{B}$	sets of $\hat{\mathcal{Y}}_t$-monomials	p 9
$\overline{B}_i^s, \overline{B}^s$	sets of $\hat{\mathcal{Y}}_t^s$-monomials	p 9
B^s, B_i^s	sets of $\mathcal{Y}^s$-monomials	p 9
$B=(B_{i,j})$	symmetrized Cartan matrix	p 4
$B(z)$	deformation of B	p 4
$B'(z)$	deformation of B	p 37
β	map $(I\times\mathbb{Z}/s\mathbb{Z})^2\rightarrow\mathbb{Z}$	p 6
$C=(C_{i,j})$	Cartan matrix	p 4
$C(z)$	deformation of B	p 4
$C'(z)$	deformation of B	p 37
$C(m)$	set of monomials	p 12
$(\tilde{C}_{i,j})$	inverse of C	p 30
c_r	central element of $\hat{\mathcal{H}}$	p 5
d_1, d_2	bicharacters	p 8
d'_1, d'_2	bicharacters	p 37
D_1, D_2	bicharacters	p 18
ϵ	root of unity	p 2
$E_i(m)$	element of $\mathfrak{K}_i, \mathfrak{K}_i^s$	p 17
$\overleftarrow{E}_{i,t}(m)$	element of $\hat{\mathfrak{K}}_{i,t}, \hat{\mathfrak{K}}_{i,t}^s$	p 11
$\overrightarrow{E}_t(m)$	element of $\hat{\mathfrak{K}}_t^\infty$	p 23
$F(m)$	element of $\hat{\mathfrak{K}}$	p 12
$\hat{F}_t(m)$	element of $\hat{\mathfrak{K}}_t^\infty$	p 12
γ	map $(I\times\mathbb{Z}/s\mathbb{Z})^2\rightarrow\mathbb{Z}$	p 31
$\hat{\mathcal{H}}$	Heisenberg algebra	p 5
$\hat{\mathcal{H}}_h$	formal series in $\hat{\mathcal{H}}$	p 5
$\mathfrak{K}_i, \mathfrak{K}$	subrings of $\mathcal{Y}$	p 11
$\hat{\mathfrak{K}}_{i,t}, \hat{\mathfrak{K}}_t$	subrings of $\hat{\mathcal{Y}}_t$	p 11
$\hat{\mathfrak{K}}_{i,t}^\infty, \hat{\mathfrak{K}}_t^\infty$	subrings of $\hat{\mathcal{Y}}_t^\infty$	p 12
$\mathfrak{K}_i^s, \mathfrak{K}^s$	subrings of $\mathcal{Y}^s$	p 17
$\hat{\mathfrak{K}}_{i,t}^s, \hat{\mathfrak{K}}_t^s$	subrings of $\hat{\mathcal{Y}}_t^s$	p 17
$\hat{\mathfrak{K}}_{i,t}^{s,\infty}, \hat{\mathfrak{K}}_t^\infty$	subrings of $\hat{\mathcal{Y}}_t^{s,\infty}$	p 17
$\tilde{\mathfrak{K}}_{i,t}^s$	subring of $\hat{\mathcal{Y}}_t^s$	p 24
$\tilde{\mathfrak{K}}_{i,t}^{s,\infty}, \tilde{\mathfrak{K}}_t^{s,\infty,f}$	subring of $\hat{\mathcal{Y}}_t^{s,\infty}$	p 25
χ_ϵ	morphism of ϵ-characters	p 5
$\chi_{\epsilon,t}$	morphism of ϵ,t-characters	p 15
χ_q	morphism of q-characters	p 5
$\chi_{q,t}$	morphism of q,t-characters	p 14
$[l]$	element of $\mathbb{Z}/s\mathbb{Z}$	p 2
$\hat{L}_t(m)$	element of $\hat{\mathfrak{K}}_t^\infty$	p 23
$\hat{L}_t^s(m)$	element of $\hat{\mathfrak{K}}_t^{s,\infty}$	p 25
Λ_s	set	p 35
$\overrightarrow{m}, \overleftarrow{m}$	$\hat{\mathcal{Y}}_t^s$-monomial	p 15
op	operator	p 7
p_s	morphism	p 9
π_r	map to $\mathbb{Z}$	p 30
π_+	ring homomorphism of	p 7
$\hat{\Pi}$	morphism	p 9
$P_{m',m}(t)$	polynomial	p 24
$P^s_{m',m}(t)$	polynomial	p 25
q	complex number	p 5
$r^\vee$	integer	p 4
r_i	integer	p 4
Rep	Grothendieck ring	p 5
Rep^s	Grothendieck ring	p 5
Rep_t^s	deformed Grothendieck ring	p 29
s	integer	p 5
S_i	screening operator	p 11
S_i^s	screening operator	p 17
$\tilde{S}_{i,l}$	screening current	p 10
$S_{i,t}$	t-screening operator	p 11
$S_{i,t}^s$	t-screening operator	p 17
t	central element of $\mathcal{Y}_t$	p 6
t_R	central element of $\mathcal{Y}_u$	p 6
τ_s	morphism	p 15
$\tau_{s,t}$	morphism	p 15
$u_{i,l}$	character	p 7
$u'_{i,l}$	character	p 37
$X_{i,l}$	element of $\mathrm{Rep}, \mathrm{Rep}^s$	p 5
$y_i[m]$	element of $\hat{\mathcal{H}}$	p 5
$Y_{i,l}, Y_{i,l}^{-1}$	elements of $\mathcal{Y}$	p 7
$\tilde{Y}_{i,l}, \tilde{Y}_{i,l}^{-1}$	elements of $\hat{\mathcal{Y}}_u$ or $\hat{\mathcal{Y}}_t$	p 5
$\mathcal{Y}$	commutative algebra	p 2
$\hat{\mathcal{Y}}_t^s, \hat{\mathcal{Y}}_t$	quotient of $\hat{\mathcal{Y}}_u^s, \hat{\mathcal{Y}}_u$	p 6
$\hat{\mathcal{Y}}_u^s, \hat{\mathcal{Y}}_u$	subalgebra of $\hat{\mathcal{H}}_h$	p 6
$\hat{\mathcal{Y}}_{i,t}$	$\hat{\mathcal{Y}}_t$-module	p 11
$\hat{\mathcal{Y}}_{i,t}^s$	$\hat{\mathcal{Y}}_t^s$-module	p 16
$\hat{\mathcal{Y}}_t^\infty$	completion of $\hat{\mathcal{Y}}_t$	p 12
$\hat{\mathcal{Y}}_t^{s,\infty}$	completion of $\hat{\mathcal{Y}}_t^s$	p 15
z	indeterminate	p 5
$*$	t-product	p 30

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