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Nonresonant velocity averaging and the Vlasov–Maxwell system

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3.1 Mean-field equations of Vlasov type

The Vlasov equation governs the number density in single-particle phase space of a large particle system (typically a rarefied ionized gas or plasma), subject to some external force field (for instance the Lorentz force acting on charged particles). Most importantly, collisions between particles are neglected in the Vlasov equation, unlike the case of the Boltzmann equation. Hence the only possible source of nonlinearity in the Vlasov equation for charged particles is the self-consistent electromagnetic field created by charges in motion: each particle is subject to the electromagnetic field created by all the particles other than itself.

The Vlasov equation reads

$$\partial_t \mathbf{f} + \mathrm{div}_x(\mathbf{v}(\xi)\mathbf{f}) + \mathrm{div}_\xi(\mathbf{F}(t,x)\mathbf{f}) = 0, \tag{3.1.1}$$

where $\mathbf{f} \equiv \mathbf{f}(t,x,\xi) \in M_N(\mathbb{R})$ is the diagonal matrix of number densities for the system of particles considered. Specifically, there are N different species of particles in the system, and

$$\mathbf{f}(t,x,\xi) = \begin{pmatrix} f_1(t,x,\xi) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & f_N(t,x,\xi) \end{pmatrix},$$

where $f_j(t,x,\xi)$ is the density of particles of the jth species located at the position $x \in \mathbb{R}^3$ with momentum $\xi \in \mathbb{R}^3$ at time t. Likewise

$$\mathbf{v}(\xi) = \begin{pmatrix} v_1(\xi) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & v_N(\xi) \end{pmatrix},$$

where $v_j(\xi)$ is the velocity of particles of the jth species with momentum ξ, while

$$\mathbf{F}(t,x) = \begin{pmatrix} F_1(t,x) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & F_N(t,x) \end{pmatrix},$$

where $F_j(t,x)$ is the force field at time t and position x acting on particles of the jth species. The divergence operators act entrywise on their arguments, meaning that

$$\begin{aligned} &\operatorname{div}_x(\mathbf{v}(\xi)\mathbf{f}(t,x,\xi)) \\ &\quad = \begin{pmatrix} \operatorname{div}_x(v_1(\xi)f_1(t,x,\xi)) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \operatorname{div}_x(v_N(\xi)f_N(t,x,\xi)) \end{pmatrix}, \end{aligned}$$

while

$$\begin{aligned} &\operatorname{div}_\xi(\mathbf{F}(t,x)\mathbf{f}(t,x,\xi)) \\ &\quad = \begin{pmatrix} \operatorname{div}_\xi(F_1(t,x)f_1(t,x,\xi)) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \operatorname{div}_\xi(F_N(t,x)f_N(t,x,\xi)) \end{pmatrix}. \end{aligned}$$

Henceforth

$$\mathbf{m} = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & m_N \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} q_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & q_N \end{pmatrix},$$

where m_j and q_j are respectively the mass and the charge of particles of the jth species.

The relativistic Vlasov–Maxwell model

This is the fundamental model for relativistic particles with strong electromagnetic coupling. Hence, denoting by c the speed of light in a vacuum,

$$\mathbf{v}(\xi) = \nabla_\xi \mathbf{e}(\xi), \quad \mathbf{F}(t,x) = E(t,x)\mathbf{q} - \tfrac{1}{c}B(t,x) \times \nabla_\xi \mathbf{e}(\xi)\mathbf{q}, \tag{3.1.2}$$

where

$$\mathbf{e}(\xi) = (\mathbf{m}^2 c^4 + c^2|\xi|^2\mathbf{I})^{1/2}, \tag{3.1.3}$$

while $E \equiv E(t,x) \in \mathbb{R}^3$ and $B \equiv B(t,x) \in \mathbb{R}^3$ are respectively the electric and the magnetic field at time t and position x. They are governed by the system of Maxwell's equations

$$\begin{aligned} &\operatorname{div}_x B = 0, \quad \partial_t B + c \operatorname{curl}_x E = 0, \\ &\operatorname{div}_x E = \rho, \quad \partial_t E - c \operatorname{curl}_x B = -j. \end{aligned} \tag{3.1.4}$$

The charge density ρ is defined as

$$\rho(t,x) = \int_{\mathbb{R}^3} \mathrm{trace}(\mathbf{q}\mathbf{f}(t,x,\xi))d\xi\,, \tag{3.1.5}$$

while the current density $j \equiv j(t,x) \in \mathbb{R}^3$ is given by

$$j(t,x) = \int_{\mathbb{R}^3} \mathrm{trace}(\mathbf{q}\mathbf{v}(\xi)\mathbf{f}(t,x,\xi))d\xi\,. \tag{3.1.6}$$

The main mathematical problem concerning the Vlasov–Maxwell system is the question of global existence and uniqueness of smooth solutions of the Cauchy problem, which remains open at the time of this writing.

Since the time-dependent vector field

$$\mathbb{R}^3 \times \mathbb{R}^3 \ni (x,\xi) \mapsto (v_j(\xi), F_j(t,x)) \in \mathbb{R}^3 \times \mathbb{R}^3$$

is divergence free for each $j = 1, \dots, N$, the quantity

$$\|f_j(t,\cdot,\cdot)\|_{L^p(\mathbb{R}^3\times\mathbb{R}^3)} = \mathrm{Const.}$$

is an invariant of the motion for each $j = 1, \dots, N$ and each $1 \le p \le \infty$.

The total energy of the particle system is also an invariant of the motion. It reads

$$\frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \mathrm{trace}(\mathbf{e}(\xi)\mathbf{f}(t,x,\xi))dxd\xi + \frac{1}{2}\int_{\mathbb{R}^3}(|E|^2+|B|^2)(t,x)dx = \ \mathrm{Const.}$$

Since all the matrices $\mathbf{m}$, $\mathbf{e}(\xi)$ and $\mathbf{f}(t,x,\xi)$ have nonnegative entries, there are no cancellations in the expression above, so that the energy conservation implies a priori estimates of the form

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} \sqrt{m_j^2c^4 + c^2|\xi|^2} f_j(t,x,\xi)dxd\xi \le \mathrm{Const.}$$

and

$$\int_{\mathbb{R}^3}(|E|^2+|B|^2)(t,x)dx \le \mathrm{Const.}$$

for the Vlasov–Maxwell system.

3.2 A kinetic formulation of the Maxwell equations

Henceforth, we consider a classical solution $(\mathbf{f}, E, B)$ of the relativistic Vlasov–Maxwell system (3.1.1), (3.1.2), (3.1.4), (3.1.5), (3.1.6), with initial data

$$\mathbf{f}\big|_{t=0} = \mathbf{f}^{in}\,, \quad E\big|_{t=0} = E^{in}\,, \quad B\big|_{t=0} = B^{in}\,, \tag{3.2.1}$$

where $\mathbf{f}^{in}$, E^{in} and B^{in} are at least of class C^∞ in all their variables, and satisfy the compatibility conditions

$$\operatorname{div}_x E^{in} = \int_{\mathbb{R}^3} \operatorname{trace}(\mathbf{q}\mathbf{f}^{in}) d\xi\,, \quad \operatorname{div}_x B^{in} = 0\,. \tag{3.2.2}$$

For simplicity, we shall assume moreover that f^{in}, E^{in} and B^{in} are compactly supported.

It will be especially convenient to represent the electromagnetic field in terms of the distribution of Liénard–Wiechert potentials created by each one of the moving charged particles in the system considered. For a classical presentation of Liénard–Wiechert potentials, see for instance §14.1 in [14], or §63 in [16]. Here, we propose a slightly different (and yet equivalent) formulation of that notion.

Define $\mathbf{u} \equiv \mathbf{u}(t,x,\xi) \in M_N(\mathbb{R})$ to be the solution of

$$\begin{aligned} &\Box_{t,x}\mathbf{u} = \mathbf{f}\,, \quad t > 0\,,\ x,\xi \in \mathbb{R}^3\,, \\ &\mathbf{u}\big|_{t=0} = \partial_t \mathbf{u}\big|_{t=0} = 0\,, \end{aligned} \tag{3.2.3}$$

where $\Box_{t,x} = \partial_t^2 - c^2 \Delta_x$ is the d'Alembert operator. The jth diagonal entry of $\mathbf{u}(t,x,\xi)$ is exactly the distribution of Liénard–Wiechert potentials created at time t by particles of the jth species distributed under $f_j^{in} \equiv f_j^{in}(x,\xi)$ initially.

Define then the electromagnetic potential

$$\begin{aligned} \boldsymbol{\Phi}(t,x) &= \int_{\mathbb{R}^3} \mathbf{q}\mathbf{u}(t,x,\xi) d\xi\,, \\ \mathbf{A}(t,x) &= \int_{\mathbb{R}^3} \mathbf{q}\mathbf{v}(\xi)\mathbf{u}(t,x,\xi) d\xi\,. \end{aligned} \tag{3.2.4}$$

We also define (a diagonal matrix of) vector potentials $\mathbf{A}^0 \equiv \mathbf{A}^0(t,x)$ so that

$$\Box_{t,x}\mathbf{A}^0 = 0 \tag{3.2.5}$$

with the following compatibility conditions:

$$\operatorname{div}_x \mathbf{A}^0\big|_{t=0} = 0\,, \quad \operatorname{div}_x \partial_t \mathbf{A}^0\big|_{t=0} = -\int_{\mathbb{R}^3} \mathbf{q}\mathbf{f}^{in}(x,\xi) d\xi\,, \tag{3.2.6}$$

as well as

$$\operatorname{curl}_x \operatorname{trace}(\mathbf{A}^0)\big|_{t=0} = B^{in}\,, \quad \operatorname{trace}(\partial_t A^0)\big|_{t=0} = -E^{in}\,. \tag{3.2.7}$$

One easily checks that $(\boldsymbol{\Phi}, \mathbf{A}^0 + \mathbf{A})$ is the electromagnetic potential leading to the electromagnetic field (E,B) by the formulas

$$E = -\partial_t \operatorname{trace}(\mathbf{A}^0 + \mathbf{A}) - \nabla_x \boldsymbol{\Phi}\,, \quad B = \operatorname{curl}_x \operatorname{trace}(\mathbf{A}^0 + \mathbf{A})$$

and satisfying the Lorentz gauge entrywise:

$$\partial_t \boldsymbol{\Phi} + \operatorname{div}_x(\mathbf{A}^0 + \mathbf{A}) = 0\,.$$

Hence one can replace the Vlasov–Maxwell system (3.1.1), (3.1.2), (3.1.4), (3.1.5), (3.1.6) with the equivalent system (3.1.1), (3.2.3) with the following formula for the Lorentz force:

$$\begin{aligned}\mathbf{F}(t,x) = -\tfrac{1}{c}\operatorname{curl}_x \operatorname{trace}\left(\mathbf{A}^0 + \int_{\mathbb{R}^3} \mathbf{q}\mathbf{v}(\xi)\mathbf{u}(t,x,\xi)d\xi\right) \times \nabla_\xi \mathbf{e}(\xi)\mathbf{q} \\ - \operatorname{trace}\left(\partial_t \mathbf{A}^0 + \partial_t \int_{\mathbb{R}^3} \mathbf{q}\mathbf{v}(\xi)\mathbf{u}(t,x,\xi)d\xi + \nabla_x \int_{\mathbb{R}^3} \mathbf{q}\mathbf{u}(t,x,\xi)d\xi\right)\mathbf{q}\,.\end{aligned} \tag{3.2.8}$$

Observe that the (diagonal matrix of) vector potentials $\mathbf{A}^0$ can be chosen as smooth as the initial data (f^{in}, E^{in}, B^{in}), i.e., of class C^∞ in all its variables, since the wave equation (3.2.5) propagates the regularity of the initial data (3.2.6), (3.2.7) of $\mathbf{A}^0$. Hence the only possibility for a finite time blow-up of classical solutions of the relativistic Vlasov–Maxwell model would therefore come from the **f**-**u** coupling in the system (3.1.1), (3.2.3). In the next section, we shall analyze carefully some conditional smoothing mechanisms for such systems.

3.3 Nonresonant velocity averaging for transport+wave systems

Throughout this section, we set $c = 1$. We are concerned with coupled transport+wave systems of the form

$$\begin{aligned}\Box_{t,x} u(t,x,\xi) = f(t,x,\xi) \\ (\partial_t + v(\xi)\cdot\nabla_x) f(t,x,\xi) = P(D_\xi) g(t,x,\xi),\end{aligned} \tag{3.3.1}$$

where $P(D_\xi)$ is a differential operator of order $m \geq 0$. Specifically, we are interested in the local regularity in (t,x) of averages of u of the form

$$U(t,x) = \int_{\mathbb{R}^D} u(t,x,\xi)\phi(\xi)d\xi\,.$$

Observe that the expression of the Lorentz force in (3.2.8) involves precisely averages of $\mathbf{u}$ of this type, instead of $\mathbf{u}$ itself.

For simplicity, we consider first regularity estimates in L^2-based Sobolev spaces. Assume that $f, u, g \in L^2_{loc}(\mathbb{R}\times\mathbb{R}^D\times\mathbb{R}^D)$ while $v \in C^\infty(\mathbb{R}^D;\mathbb{R}^D)$ and $\phi \in C^\infty_c(\mathbb{R}^D)$.

Under the full-rank condition

$$\operatorname{rank}\nabla_\xi v(\xi) = D \quad \text{for each } \xi \in \operatorname{supp}\phi, \tag{3.3.2}$$

the classical velocity averaging lemma implies that

$$\int_{\mathbb{R}^D} f(t,x,\xi)\phi(\xi)d\xi \in H_{loc}^{1/2(m+1)}(\mathbb{R}\times\mathbb{R}^D)\,.$$

On the other hand

$$\Box_{t,x}U = \int_{\mathbb{R}^D} f\phi d\xi$$

and the usual energy estimate for the wave equation on u, obtained by multiplying each side of that equation by $\partial_t U$, leads to

$$\partial_t \frac{1}{2}\left(|\partial_t U|^2 + |\nabla_x U|^2\right) - \operatorname{div}_x(\partial_t U \nabla_x U) = \partial_t U \int_{\mathbb{R}^D} f d\xi\,.$$

After localizing in (t,x) and integrating in x, this clearly shows that U gains one derivative in L^2 in each variable t,x over the average of f:

$$\int_{\mathbb{R}^D} f(t,x,\xi)\phi(\xi)d\xi\,.$$

Hence

$$U \in H_{loc}^{1+\frac{1}{2(m+1)}}(\mathbb{R}\times\mathbb{R}^D)\,.$$

Going back to our formulation (3.1.1), (3.2.3), we see that, in order for the vector field

$$\mathbb{R}^3\times\mathbb{R}^3 \ni (v_j(\xi), F_j(t,x)) \in \mathbb{R}^3\times\mathbb{R}^3$$

to generate a unique characteristic flow, F_j should be locally Lipschitz continuous in x uniformly in t, for each $j = 1,\ldots,N$. In terms of the distribution of Liénard–Wiechert potentials $\mathbf{u}$, this amounts precisely to controlling second-order derivatives of averages of $\mathbf{u}$, as can be seen from (3.2.8). Unfortunately, the strategy based on the classical velocity averaging lemma as explained above (in the most favorable L^2 setting) fails to gain that much regularity—in fact, $m = 1$ in the Vlasov equation, so that the best one can hope for with this method is a gain of $1+\frac{1}{4}$ derivatives in (t,x), which is not enough to allow us to define characteristics for the Vlasov–Maxwell system.

However, this approach to the regularity question leaves aside an important feature of the Vlasov–Maxwell system. The electromagnetic field consists of waves that propagate at the speed of light, whereas the charged particles, all of which have positive mass, move at a lesser speed. Indeed, the speed of a particle of mass m and momentum ξ is

$$\frac{c^2|\xi|}{\sqrt{m^2c^4 + c^2|\xi|^2}} < c\,.$$

The fact that the speed of propagation in the wave equation (3.2.3) is larger than the speed of particles $v(\xi)$ in the Vlasov equation (3.1.1) leads to a new regularizing mechanism, which we now explain.

Consider the system (3.3.1). We shall call this system *nonresonant* if

$$|v(\xi)| < 1\,, \qquad \text{for each } \xi \in \mathbb{R}^D\,. \tag{3.3.3}$$

Theorem 3.3.1 (Bouchut-Golse-Pallard [2]) *Let* u, f, g *in* $L^2_{loc}(\mathbb{R}_t \times \mathbb{R}^D_x \times \mathbb{R}^D_\xi)$ *satisfy* (3.3.1) *with* $v \in C^\infty(\mathbb{R}^D; \mathbb{R}^D)$. *Assume that this system is nonresonant. Then, for each* $\phi \in C^\infty_c(\mathbb{R}^D_\xi)$, *one has*

$$U(t,x) = \int_{\mathbb{R}^D} u(t,x,\xi)\phi(\xi)d\xi \in H^2_{loc}(\mathbb{R} \times \mathbb{R}^D)\,.$$

This result generalizes the fact that the operator $\Box_{t,x}$ is microlocally elliptic on the null space of the transport operator whenever $|v(\xi)| < 1$.

There is also an interesting difference with the strategy based on the usual velocity averaging lemma described above. Indeed, this new method leads to a gain of 2 derivatives on momentum averages of u in the nonresonant case—without gaining more than $1 + \frac{1}{2(m+1)}$ derivatives in (t,x) on momentum averages of f itself.

To see the importance of the nonresonance condition (3.3.3), we briefly sketch the proof of Theorem 3.3.1.

Proof (Sketch of the proof). Set $T^\pm_\xi = \partial_t \pm v(\xi) \cdot \nabla_x$ and consider the second-order differential operator

$$Q_\xi = \Box_{t,x} - \lambda T^-_\xi T^+_\xi .$$

First, one checks that

$$\begin{aligned} Q_\xi u = f - \lambda T^-_\xi \Box^{-1}_{t,x} D^m_\xi g &= f - \lambda D^m_\xi \Box^{-1}_{t,x} T^-_\xi g - \lambda \Box^{-1}_{t,x}[T^-_\xi, D^m_\xi] g \\ &= f - \lambda D^m_\xi \Box^{-1}_{t,x} T^-_\xi g - \lambda \Box^{-1}_{t,x} D^m_\xi v(\xi) \cdot \nabla_x g \\ &= a + d^m_\xi b \in L^2_{loc}(dtdxd\xi) + D^m_\xi L^2_{loc}(dtdxd\xi). \end{aligned}$$

Here, we have denoted by $\Box^{-1}_{t,x}$ the operator defined by $\Box^{-1}_{t,x}\psi = \Psi$ where Ψ is the solution of the Cauchy problem

$$\begin{aligned} \Box_{t,x}\Psi &= 0\,, \quad x \in \mathbb{R}^D\,,\ t > 0\,, \\ \Psi\big|_{t=0} &= 0\,, \\ \partial_t\Psi\big|_{t=0} &= \psi\,. \end{aligned}$$

Next, we observe that, for $\xi \in \operatorname{supp}\phi$ and λ such that

$$\sup_{\xi \in \operatorname{supp}\phi} |v(\xi)| < \lambda < 1,$$

the operator Q_ξ is elliptic for each $\xi \in \operatorname{supp}\phi$.

More precisely, denoting by $q_\xi(\omega, k)$ the symbol of Q_ξ, one has

$$\sup_{\xi \in \operatorname{supp}\phi} \left| D^m_\xi \left(\frac{1}{q_\xi(\omega,k)} \right) \right| \le \frac{C_m}{\omega^2 + |k|^2},$$

where C_m may depends on m but is *uniform in* ξ. Then

$$\int_{\mathbb{R}^D} \hat{u}\phi(\xi)d\xi = \int \frac{\hat{a}}{q_\xi(\omega,k)}\phi(\xi)d\xi + (-1)^m \int_{\mathbb{R}^D} D_\xi^m \left(\frac{\phi(\xi)}{q_\xi(\omega,k)}\right)\hat{b}d\xi$$

with $\hat{a}$ and $\hat{b} \in L^2_{\omega,k,\xi}$ have H^2-decay in ω, k.

Remarks

(a) First, one easily checks that none of the assumptions in Theorem 3.3.1 can be dispensed with.

(b) That one gains 2 derivatives is special to the L^2-case, since $\square_{t,x}^{-1}$ gains 1 derivative in (t,x) by the energy estimate for the wave equation.

In L^p with $1 < p < \infty$, $\square_{t,x}^{-1}$ gains $1-(D-1)|\frac{1}{2}-\frac{1}{p}|$ derivatives in (t,x)—see for instance [18], [19])—whenever $|\frac{1}{2}-\frac{1}{p}| \le \frac{1}{D-1}$.

Using this result and the Mihlin-Hörmander theorem on L^p multipliers—see for instance Theorem 3 on p. 96 of [20]—the same method as above shows that

$$\int_{\mathbb{R}^D} u(t,x,\xi)\phi(\xi)d\xi \in W_{loc}^{1+\gamma,p}(\mathbb{R}\times\mathbb{R}^D) \text{ with } \gamma = 1-(D-1)\left|\frac{1}{2}-\frac{1}{p}\right| .$$

This result, due to C. Pallard [17] suggests a gain of 1 derivative in L^1 or L^∞ in space dimension 3. The regularity statement above is still true in these limiting cases of nonresonant velocity averaging; however, the proof rests on the explicit formula for $\square_{t,x}^{-1}$—the forward fundamental solution of the d'Alembert operator—in physical (instead of Fourier) space. In the case of 3 space dimensions, this fundamental solutions turns out to be a measure, and thus behaves nicely with L^∞ data. The proof also uses a "division lemma" discussed in the next section. See [17] for a complete proof of the L^p variant of nonresonant velocity averaging, including the aforementioned limiting cases.

3.4 Applications to the Vlasov–Maxwell system

3.4.1 A conditional regularity result

R. DiPerna and P.-L. Lions [4] have proved that the Cauchy problem for the Vlasov–Maxwell system has globally defined renormalized solutions for initial data with finite energy. However, their method does not allow defining characteristic curves for the Vlasov equation; i.e., trajectories for the charged particles governed by the Vlasov–Maxwell system. Their analysis, written in the case of the classical Vlasov–Maxwell system, i.e., for

$$\mathbf{v}(\xi) = \mathbf{m}^{-1}\xi,$$

obviously applies to the relativistic Vlasov–Maxwell system considered here, which is somewhat more consistent on physical grounds, since the Maxwell equations are themselves a relativistic model.

By using both the standard velocity averaging argument for large momenta and the nonresonant velocity averaging method for momenta below some threshold R, one arrives at the following conditional result, upon optimizing in R.

Theorem 3.4.1 (Bouchut-Golse-Pallard [2]) *Consider the relativistic Vlasov–Maxwell system* (3.1.1), (3.1.2), (3.1.4), (3.1.5), (3.1.6) *with initial condition (3.2.1). Assume that the initial data satisfy*

$$0 \leq \mathbf{f}^{in} \in L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)\,, \;\; E^{in} \;\textit{and}\; B^{in} \in H^1_{loc}(\mathbb{R}^3)$$

with the compatibility condition

$$\operatorname{div} B^{in} = 0\,, \quad \operatorname{div} E^{in} = \int_{\mathbb{R}^3} \operatorname{trace}(\mathbf{q}\mathbf{f}(t,x,\xi))d\xi$$

and the finite energy condition

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} \operatorname{trace}(\mathbf{e}(\xi)\mathbf{f}^{in}(x,\xi))d\xi dx + \frac{1}{2}\int_{\mathbb{R}^3}(|E^{in}|^2 + |B^{in}|^2)(x)dx < \infty\,.$$

Let (f, E, B) *be a renormalized solution of the Vlasov–Maxwell system with those initial data. If the macroscopic energy density satisfies*

$$\int_{\mathbb{R}}^{3} \operatorname{trace}(\mathbf{e}(\xi)\mathbf{f}^{in}(t,x,\xi))d\xi \in L^p_{loc}(\mathbb{R}_+ \times \mathbb{R}^3) \;\textit{with}\; \frac{3}{2} < p \leq 2,$$

then the electromagnetic field has Sobolev regularity

$$(E,B) \in H^s_{loc}(\mathbb{R}^*_+ \times \mathbb{R}^3) \;\textit{with}\; s < \frac{4p-6}{4p+3}\,.$$

See [2] for a proof of this result. The theorem above falls short of providing the amount of regularity on the electromagnetic field that one would need in order to define a characteristic flow, even in a generalized sense—see [1], [5]. Perhaps, its main interest is to indicate the relevance of the idea of nonresonant velocity averaging in the context of the Vlasov–Maxwell system. Most likely, further ideas are needed in order to apply the method of nonresonant velocity averaging to the Vlasov–Maxwell system with more convincing output.

3.4.2 A new proof of the Glassey–Strauss theorem

Consider the Cauchy problem for the Vlasov–Maxwell system (3.1.1), (3.1.2), (3.1.4), (3.1.5), (3.1.6) with initial condition (3.2.1). As mentioned above, R. DiPerna and P.-L. Lions [4] have proved that this Cauchy problem has globally defined renormalized solutions for initial data with finite energy.

However, such solutions are not known to be uniquely defined by their initial data. Besides, one would expect that the regularity of initial data propagates, so that it seems reasonable to seek classical solutions, with (f, E, B) at least of class C^1. The benefit of dealing with classical solutions is twofold: first, such solutions are uniquely defined by their initial data. Furthermore, one can define characteristic curves of the Vlasov equation (3.1.1) for C^1 solutions by a simple application of the Cauchy–Lipschitz theorem.

Unfortunately, global existence of classical solutions of the Vlasov–Maxwell system for any C^1 initial data of arbitrary size with good enough decay property at infinity remains an open problem. The best result in that direction is the following theorem.

Theorem 3.4.2 (R. Glassey-W. Strauss [13]) *Let* $\mathbf{f} \in C^1([0,T) \times \mathbb{R}^3 \times \mathbb{R}^3)$ *and* $E, B \in C^1([0,T) \times \mathbb{R}^3)$ *be a solution of the Vlasov–Maxwell system* (3.1.1)*,* (3.1.2)*,* (3.1.4)*,* (3.1.5)*,* (3.1.6) *with initial condition (3.2.1). Assume that* $\mathbf{f}^{in} \in C^1_c(\mathbb{R}^3 \times \mathbb{R}^3)$ *and that* $E^{in}, B^{in} \in C^2_c(\mathbb{R}^3)$ *with*

$$\operatorname{div}_x E^{in} = \int_{\mathbb{R}^3} \operatorname{trace}(\mathbf{q}\mathbf{f}^{in}) d\xi \,, \quad \operatorname{div}_x B^{in} = 0.$$

If

$$\overline{\lim}_{t \to T^-} \|\mathbf{f}(t)\|_{Lip_{x,\xi}} + \|(E,B)(t)\|_{Lip_x} = +\infty,$$

then

$$\overline{\lim}_{t \to T^-} R_{\mathbf{f}}(t) = +\infty,$$

where

$$R_{\mathbf{f}}(t) = \inf\{r > 0 \,|\, \mathbf{f}(t,x,\xi) = 0 \text{ for each } x \in \mathbb{R}^3 \text{ and } |\xi| > r\}\,.$$

The original proof of this result is fairly hard to read in detail, although the general strategy is very clearly explained in [13]. For this reason, together with the considerable interest in the result itself, there have been some attempts at finding alternatives to the original proof. One is due to S. Klainerman and G. Staffilani [15]; although perhaps not very much simpler than the original proof, it is based on a completely new and different idea which may be of great interest in further understanding the Vlasov–Maxwell system.

The proof of the Glassey–Strauss theorem sketched below originates from [3]. The underlying strategy is essentially the same as in the original proof; however, it is much simpler in two very different respects. A first, considerable simplification over [13] comes from the representation of the electromagnetic field (E, B) in terms of the number densities $\mathbf{f}$: whereas the original argument led the reader through complicated manipulations of integrals involving vector analysis, using the kinetic formulation of the Maxwell equations in terms of the distribution of Liénard–Wiechert potentials as in Section 3.2 reduces that burden to performing similar manipulations on *scalar* solutions of the wave equation.

But the most important part of the Glassey–Strauss analysis was a subtle decomposition of the vector fields corresponding to space- and time-derivatives into their projection on the wave cone and the free streaming operator. This decomposition was then used in the representation of the electromagnetic field with several integrations by parts to smooth out the singularities of the integral kernels involved.

In our analysis, this last step is replaced by a "division lemma" bearing on the fundamental solution of the d'Alembert operator, which is vaguely reminiscent of the classical "Preparation Theorem." The main advantage of this argument is that it does not depend at all on the explicit form of the fundamental solution, and remains the same for other space dimensions—whereas the 2-dimensional analogue of the Glassey–Strauss analysis required a different argument, since the 2-dimensional fundamental solution of the d'Alembert operator is not concentrated on the wave cone in even space dimensions.

The division lemma

At variance with the Glassey–Strauss analysis, our argument uses only the following symmetries of the d'Alembert operator. Denote the Lorentz boosts on $\mathbb{R}_t \times \mathbb{R}^D_x$ by

$$L_j = x_j \partial_t + t \partial_{x_j}\,, \qquad j = 1, \dots, D\,.$$

We recall that these Lorentz boosts commute with the d'Alembert operator $\Box_{t,x}$ on $\mathbb{R}_t \times \mathbb{R}^D_x$:

$$[\Box_{t,x}, L_j] = 0\,, \quad j = 1, \dots, D\,.$$

Let Y be the forward fundamental solution of $\Box_{t,x}$, i.e.,

$$\Box_{t,x} Y = \delta_{(0,0)}\,, \quad \operatorname{supp} Y \subset \mathbb{R}_+ \times \mathbb{R}^D$$

—for instance, in space dimension $D = 3$, one has

$$Y(t,x) = \mathbf{1}_{t \geq 0} \frac{\delta(t - |x|)}{4\pi t}\,.$$

Then, since L_j commutes with $\Box_{t,x}$, one finds that

$$\Box_{t,x} L_j Y = L_j \delta_{(0,0)} = 0\,, \quad \operatorname{supp} L_j Y \subset \mathbb{R}_+ \times \mathbb{R}^D,$$

whence, by the uniqueness of the solution to the Cauchy problem for the wave equation,

$$L_j Y = 0\,, \quad j = 1, \dots, D\,.$$

Lemma 3.4.3 *Let $D \geq 2$. For each $\xi \in \mathbb{R}^D$, there exists $b^k_{ij} \equiv b^k_{ij}(t,x,\xi)$ in C^∞ on $\mathbb{R}^{D+1} \setminus 0$ and homogeneous of degree $-k$ in (t,x) such that*

(i) *the homogeneous distribution $b^2_{ij} Y$ of degree $-D-1$ on $\mathbb{R}^{D+1} \setminus 0$ has null residue at the origin, and*

(ii) *there exists an extension of* $b^2_{ij}Y$ *as a homogeneous distribution of degree* $-D-1$ *on* $\mathbb{R}^{D+1}\setminus 0$*, still denoted* $b^2_{ij}Y$*, that satisfies*

$$\partial_{ij}Y = T^2(b^0_{ij}Y) + T(b^1_{ij}Y) + b^2_{ij}Y\,, \quad i,j=0,\dots,D\,.$$

Here T denotes the advection operator $T=\partial_t + v(\xi)\cdot\nabla_x$.

Remark. The null residue condition reads

$$\int_{\mathbb{S}^2} b^2_{ij}(1,y)d\sigma(y) = 0 \quad \text{if } D=3\,,$$

$$\int_{|y|\le 1} b^2_{ij}(1,y)\frac{dy}{\sqrt{1-|y|^2}} = 0 \quad \text{if } D=2\,.$$

In the first formula, $d\sigma$ designates the surface element on the unit sphere.

Proof (Sketch of the proof). Observe that

$$\sum_{j=1}^{D} v_j(\xi)L_j = v(\xi)\cdot x\partial_t + tv(\xi)\cdot\nabla_x = (v(\xi)\cdot x - t)\partial_t + tT\,.$$

Since $L_jY=0$ for $j=1,\dots,D$, one has

$$(t-v(\xi)\cdot x)\partial_t Y = tTY\,.$$

Furthermore, since $\operatorname{supp}\partial_t Y\cap\{t-v(\xi)\cdot x=0\}=\{(0,0)\}$,

$$\partial_t Y - a^0_0 TY = 0\,.$$

Indeed, Y is a homogeneous distribution of degree $1-D$ on $\mathbb{R}^{D+1}$, so that $\partial_t Y - a^0_0 TY$ is a homogeneous distribution of degree $-D$ on $\mathbb{R}^{D+1}\setminus 0$. It has therefore a unique extension to $\mathbb{R}^{D+1}$ as a distribution of degree $-D$; since this distribution is supported at the origin, it is a linear combination of $\delta_{(0,0)}$ and its derivatives. Because $\delta_{(0,0)}$ is homogeneous of degree $-D-1$ on $\mathbb{R}^{D+1}$, this linear combination must be 0. Hence

$$\partial_t Y = T(a^0_0 Y) - (Ta^0_0)Y\,.$$

One finds analogous formulas for $\partial_{x_j}Y$ with $j=1,\dots,D$ by combining the formula above with the fact that $L_jY=0$ for $j=1,\dots,D$.

Statement (ii) is obtained by iterating the argument above once in each variable.

As for statement (i), observe that $b^k_{ij}Y$ is a homogeneous distribution of degree $1-k-D$ on $\mathbb{R}^{D+1}\setminus 0$. Hence, whenever $k=0,1$, $b^k_{ij}Y$ has a unique extension as a homogeneous distribution of degree $1-k-D$ on $\mathbb{R}^{D+1}$. Since

$$\beta^2_{ij}Y = \partial_{ij}Y - T^2(b^0_{ij}Y) - T(b^1_{ij}Y)$$

and the right-hand side is a homogeneous dsitribution on $\mathbb{R}^{D+1}$, the left-hand side is a homogeneous distribution of degree $-1-D$ on $\mathbb{R}^{D+1}\setminus 0$ that has a homogeneous extension to $\mathbb{R}^{D+1}$. Hence, it has null residue at $(0,0)$: see for instance §3 in chapter 3 of [6].

Application to the Glassey–Strauss theorem

We use the division lemma above to estimate the first-order derivatives of the electromagnetic field. This amounts to estimating the second-order derivatives of the momenum averages of the distribution of Liénard–Wiechert potentials:

$$\partial_{ij} \int m(\xi)\mathbf{u}(t,x,\xi)d\xi = \sum_{k=0}^{2} \int m(\xi) \left(b_{ij}^{k-l} Y \star T^l(\mathbf{1}_{t\geq 0}\mathbf{f})(t,x,\xi)\right) d\xi.$$

Here, m denotes any C^∞ function with compact support that coincides with either 1 or each component of $v(\xi)$ on the ξ-support of f.

The idea is to use the Vlasov equation to compute $T^l(\mathbf{1}_{t\geq 0}\mathbf{f})$ and integrate by parts to bring the ξ-derivatives to bear on b_{ij}^{k-l} and m.

In fact, the worst term is for $l = 0$:

$$\int m(\xi) \left(b_{ij}^2 Y \star (\mathbf{1}_{t\geq 0}\mathbf{f})(t,x,\xi)\right) d\xi\,.$$

By using the null residue condition, we write this term in the form

$$\int m(\xi) \int_\epsilon^t \int_{\mathbb{S}^2} b_{ij}^2(1,\omega,\xi)\mathbf{f}(t-s,x-s\omega,\xi)\frac{d\sigma(\omega)}{4\pi s} ds d\xi$$
$$+ \int m(\xi) \int_0^\epsilon \int_{\mathbb{S}^2} b_{ij}^2(1,\omega,\xi)\frac{\mathbf{f}(t-s,x-s\omega,\xi)-\mathbf{f}(t,x,\omega)}{4\pi s} d\sigma(\omega) ds d\xi.$$

If the size $R_\mathbf{f}(t)$ of the ξ-support of $\mathbf{f}$ is bounded on $[0,T)$, i.e., if

$$\overline{\lim}_{t\to T^-} R_\mathbf{f}(t) < +\infty,$$

this term is bounded by

$$C(1 + \ln_+(t\|\nabla_x \mathbf{f}\|_{L^\infty})).$$

Hence, the Lipschitz semi-norm $N(t) = \|\nabla_{x,\xi}\mathbf{f}(t,\cdot,\cdot)\|_{L^\infty}$ satisfies a logarithmic Gronwall inequality of the form

$$N(t) \leq N(0) + \int_0^t (1 + \ln_+ N(s))N(s)ds\,, \quad t \in [0,T)\,.$$

Therefore, N is uniformly bounded on $[0,T]$, which implies in turn that the fields $(E,B) \in L^\infty([0,T];W^{1,\infty}(\mathbb{R}^3))$.

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