Hydrodynamic Limits
for the Boltzmann Equation

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LECTURE 6

THE NAVIER-STOKES LIMIT: CONVERGENCE PROOF

Conservation defects $\to 0$

(as in FG+DL, CPAM 2002, but simpler)

Proposition. $\mathbf{D}_\epsilon(v) \to 0$ *in* $L^1_{loc}(\mathbf{R}_+ \times \mathbf{R}^3)$ *as* $\epsilon \to 0$.

•Split the conservation defect as $\mathbf{D}_\epsilon(v) = \mathbf{D}^1_\epsilon(v) + \mathbf{D}^2_\epsilon(v)$ with

$$\mathbf{D}^1_\epsilon(v) = \frac{1}{\epsilon^3}\left\langle\!\!\left\langle v_{K_\epsilon}\widehat{\gamma}_\epsilon \left(\sqrt{G'_\epsilon G'_{\epsilon*}} - \sqrt{G_\epsilon G_\epsilon}\right)^2 \right\rangle\!\!\right\rangle$$

$$\mathbf{D}^2_\epsilon(v) = \frac{2}{\epsilon^3}\left\langle\!\!\left\langle v_{K_\epsilon}\widehat{\gamma}_\epsilon \left(\sqrt{G'_\epsilon G'_{\epsilon*}} - \sqrt{G_\epsilon G_\epsilon}\right) \sqrt{G_\epsilon G_\epsilon}\right\rangle\!\!\right\rangle$$

That $\mathbf{D}^1_\epsilon(v) \to 0$ comes from the entropy production estimate.

•Setting $\Xi_\epsilon = \frac{1}{\epsilon^2}\left(\sqrt{G'_\epsilon G'_{\epsilon *}} - \sqrt{G_\epsilon G_\epsilon}\right)\sqrt{G_\epsilon G_\epsilon}$, we further split $\mathbf{D}^2_\epsilon(v)$ into

$$\mathbf{D}^2_\epsilon(v) = -\frac{2}{\epsilon}\left\langle\!\left\langle v\mathbf{1}_{|v|^2>K_\epsilon}\widehat{\gamma}_\epsilon\Xi_\epsilon\right\rangle\!\right\rangle + \frac{2}{\epsilon}\left\langle\!\left\langle v\widehat{\gamma}_\epsilon(1-\widehat{\gamma}_{\epsilon *}\widehat{\gamma}'_\epsilon\widehat{\gamma}_{\epsilon *})\Xi_\epsilon\right\rangle\!\right\rangle$$
$$+\frac{1}{\epsilon}\left\langle\!\left\langle (v+v_1)\widehat{\gamma}_\epsilon\widehat{\gamma}_{\epsilon *}\widehat{\gamma}'_\epsilon\widehat{\gamma}_{\epsilon *}\Xi_\epsilon\right\rangle\!\right\rangle$$

The first and third terms are easily mastered by the entropy production bound and classical estimates on the tail of Gaussian distributions.

•Sending the second term to 0 requires knowing that

$$(1+|v|)\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)^2 \text{ is uniformly integrable on } [0,T]\times K\times\mathbf{R}^3$$

for the measure $dtdxMdv$, for each $T>0$ and each compact $K\subset\mathbf{R}^3$.

Asymptotic behavior of the momentum flux

Proposition. *Denoting by* Π *the* $L^2(Mdv)$*-orthogonal projection on* $\ker \mathcal{L}$

$$\mathbf{F}_\epsilon(A) = 2\left\langle A\left(\Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)^2\right\rangle - 2\left\langle \hat{A}\frac{1}{\epsilon^2}\mathcal{Q}(\sqrt{G_\epsilon},\sqrt{G_\epsilon})\right\rangle + o(1)_{L^1_{loc}(dtdx)}$$

The proof is based upon splitting $\mathbf{F}_\epsilon(A)$ as

$$\mathbf{F}_\epsilon(A) = \left\langle A_{K_\epsilon}\gamma_\epsilon\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)^2\right\rangle + \frac{2}{\epsilon}\left\langle A_{K_\epsilon}\gamma_\epsilon\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right\rangle$$

using the uniform integrability of $(1+|v|)\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)^2$ and the following consequence thereof

$$\lim_{\epsilon\to 0}\left\|\frac{\sqrt{G_\epsilon}-1}{\epsilon} - \Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right\|_{L^2_{loc}(dtdx;L^2((1+|v|)Mdv))} = 0$$

•By the entropy production estimate, modulo extraction of a subsequence

$$\frac{1}{\epsilon^2}\left(\sqrt{G'_\epsilon G'_{\epsilon *}} - \sqrt{G_\epsilon G_\epsilon}\right) \rightharpoonup q \text{ in } L^2(dtdxd\mu)$$

and passing to the limit in the scaled, renormalized Boltzmann equation leads to

$$\iint_{\mathbf{R}^3\times\mathbf{S}^2} q|(v-v_*)\cdot\omega|M_* dv_* d\omega = \tfrac{1}{2}v\cdot\nabla_x g = \tfrac{1}{2}A : \nabla_x u + \text{ odd in } v$$

•Since $\frac{\sqrt{G_\epsilon}-1}{\epsilon} \simeq \frac{1}{2}g_\epsilon\gamma_\epsilon$, one gets

$$\mathbf{F}_\epsilon(A) = A(\langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon\rangle) - \nu(\nabla_x u + (\nabla_x u)^T) + o(1)_{w-L^1_{loc}(dtdx)}$$

(remember that $A(u) = u\otimes u - \frac{1}{3}|u|^2 I$), while

$$\langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon\rangle \rightharpoonup u \text{ in } L^1_{loc}(\mathbf{R}_+\times\mathbf{R}^3)$$

Strong compactness

•In order to pass to the limit in the quadratic term $A(\langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle)$, one needs strong-$L^2$ compactness of $\langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle$.

•Velocity averaging provides strong compactness in the x-variable:

$$\left(\frac{\sqrt{\epsilon^\alpha + G_\epsilon} - 1}{\epsilon}\right)^2 \text{ is locally uniformly integrable on } \mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3$$

$$(\epsilon \partial_t + v \cdot \nabla_x) \frac{\sqrt{\epsilon^\alpha + G_\epsilon} - 1}{\epsilon} \text{ is bounded in } L^1_{loc}(\mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3)$$

This implies that, for each $T > 0$ and each compact $C \subset \mathbf{R}^3$,

$$\int_0^T \int_C |\langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle(t, x+y) - \langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle(t,x)|^2 dx dt \to 0$$

$$\text{as } |y| \to 0\,, \text{ uniformly in } \epsilon > 0$$

•It remains to get compactness in the time variable. Observe that

$\partial_t P\langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle = P(\mathbf{D}_\epsilon(v) - \mathrm{div}_x \mathbf{F}_\epsilon(A))$ is bounded in $L^1_{loc}(dt, W^{-s,1}_{x,loc})$

(Recall that $\mathbf{D}_\epsilon(v) \to 0$ while $\mathbf{F}_\epsilon(A)$ is bounded in $L^1_{loc}(dtdx)$).

•Together with the compactness in the x-variable that follows from velocity averaging, this implies that

$$P\langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle \to u \text{ in } L^2_{loc}(dtdx)$$

•Recall that $\langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle \rightharpoonup u$ in $L^2_{loc}(dtdx)$; we DO NOT seek to prove that

$$\langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle \to u \text{ strongly in } L^2_{loc}(dtdx)$$

Filtering acoustic waves (PLL+NM, ARMA 2002)

• Instead, we prove that

$$P\,\mathrm{div}_x\left(\langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon\rangle^{\otimes 2}\right) \to P\,\mathrm{div}_x\left(u^{\otimes 2}\right) \text{ in } \mathcal{D}'(\mathbf{R}_+^* \times \mathbf{R}^3) \text{ as } \epsilon \to 0$$

• Observe that

$$\epsilon \partial_t \langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle + \nabla_x \langle \tfrac{1}{3}|v|^2_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle \to 0 \text{ in } L^1_{loc}(\mathbf{R}_+; W^{-1,1}_{loc}(\mathbf{R}^3))$$
$$\epsilon \partial_t \langle \tfrac{1}{3}|v|^2_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle + \mathrm{div}_x \langle \tfrac{5}{3} v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle \to 0 \text{ in } L^1_{loc}(\mathbf{R}_+; W^{-1,1}_{loc}(\mathbf{R}^3))$$

as $\epsilon \to 0$.

• Setting $\nabla_x \pi_\epsilon = (I-P)\langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle$, the system above becomes

$$\epsilon \partial_t \nabla_x \pi_\epsilon + \nabla_x \langle \tfrac{1}{3}|v|^2_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle \to 0 \text{ in } L^1_{loc}(\mathbf{R}_+; W^{-s,1}_{loc}(\mathbf{R}^3)), \quad s > 1$$
$$\epsilon \partial_t \langle \tfrac{1}{3}|v|^2_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle + \tfrac{5}{3}\Delta_x \pi_\epsilon \to 0 \text{ in } L^1_{loc}(\mathbf{R}_+; W^{-1,1}_{loc}(\mathbf{R}^3))$$

•Straightforward computation shows that

$$\mathrm{div}_x \left((\nabla_x \pi_\epsilon)^{\otimes 2} \right) = \tfrac{1}{2} \nabla_x \left(|\nabla_x \pi_\epsilon|^2 - \tfrac{3}{5} \langle \tfrac{1}{3} |v|^2_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle^2 \right) + o(1)_{L^1_{loc}(dtdx)}$$

•On the other hand, because the limiting velocity field is divergence-free, one has

$$\nabla_x \pi_\epsilon \rightharpoonup 0 \text{ in } L^2_{loc}(dtdx) \text{ as } \epsilon \to 0$$

•Splitting

$$P\,\mathrm{div}_x \left(\langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle^{\otimes 2} \right) = P\,\mathrm{div}_x \left((P \langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle)^{\otimes 2} \right) + P\,\mathrm{div}_x \left((\nabla_x \pi_\epsilon)^{\otimes 2} \right)$$
$$+ 2P\,\mathrm{div}_x \left(P \langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle \vee \nabla_x \pi_\epsilon \right)$$

The last two terms vanish with ϵ while the first converges to $P\,\mathrm{div}_x(u^{\otimes 2})$ since $P\langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle \to u$ strongly in $L^2_{loc}(dtdx)$.

The key estimates (as in FG+LSR, Invent. Math. 2004)

Proposition. *For each $T > 0$ and each compact $K \subset \mathbf{R}^3$, the family $\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)^2 (1+|v|)$ is uniformly integrable on $[0,T] \times K \times \mathbf{R}^3$ for the measure $dtdxMdv$.*

Idea no. 1 We first prove that $\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)^2 (1+|v|)$ is uniformly integrable on $[0,T] \times K \times \mathbf{R}^3$ for the measure $dtdxMdv$ in the v-variable .

•We say that $\phi_\epsilon \equiv \phi_\epsilon(x,y) \in L^1_{x,y}(d\mu(x)d\nu(y))$ is uniformly integrable in the y-variable for the measure $d\mu(x)d\nu(y)$ iff

$$\int \sup_{\nu(A)<\alpha} \int_A |\phi_\epsilon(x,y)| d\nu(y) d\mu(x) \to 0 \text{ as } \alpha \to 0 \text{ uniformly in } \epsilon$$

•Start from the formula

$$\mathcal{L}\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)=\epsilon\mathcal{Q}\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon},\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)-\frac{1}{\epsilon}\mathcal{Q}\left(\sqrt{G_\epsilon},\sqrt{G_\epsilon}\right)$$

and use the following estimate (G.-Perthame-Sulem, ARMA 1988)

$$\|\mathcal{Q}(f,f)\|_{L^2((1+|v|)^{-1}Mdv)}\le C\|f\|_{L^2(Mdv)}\|f\|_{L^2((1+|v|)Mdv)}$$

to arrive at

$$\left(1-O(\epsilon)\left\|\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right\|_{L^2(Mdv)}\right)\left\|\frac{\sqrt{G_\epsilon}-1}{\epsilon}-\Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right\|_{L^2((1+|v|)Mdv)}$$
$$\le O(\epsilon)_{L^2_{t,x}}+O(\epsilon)\left\|\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right\|^2_{L^2(Mdv)}$$

•This estimates tells us that $\frac{\sqrt{G_\epsilon}-1}{\epsilon}$ stays close to its associated infinitesimal Maxwellian $\Rightarrow$ regularity+decay in v.

Idea no. 2 Use a L^1-variant of velocity averaging (FG+LSR, CRAS 2002).

Lemma. *Let $f_n \equiv f_n(x,v)$ be a bounded sequence in $L^1_{loc}(dxdv)$ such that $v \cdot \nabla_x f_n$ is also bounded in $L^1_{loc}(dxdv)$. Assume that f_n is locally uniformly integrable in v. Then*

- *f_n is locally uniformly integrable (in x, v), and*

- *for each test function $\phi \in L^\infty_{comp}(\mathbf{R}^D_v)$, the sequence of averages*

$$\rho_n^\phi(x) = \int f_n(x,v)\phi(v)dv$$

  is relatively compact in $L^1_{loc}(dx)$.

•Let's prove that the sequence of averages ρ_n^ϕ is locally uniformly integrable (LSR, CPDEs 2002). WLOG, assume that f_n and $\phi \geq 0$.

•Let $\chi \equiv \chi(t,x,v)$ be the solution to

$$\partial_t \chi + v \cdot \nabla_x \chi = 0\,, \qquad \chi(0,x,v) = \mathbf{1}_A(x)$$

Clearly, $\chi(t,x,v) = \mathbf{1}_{A_x(t)}(v)$ (χ takes the values 0 and 1 only). On the other hand,

$$|A_x(t)| = \int \chi(t,x,v)dv = \int \mathbf{1}_A(x - tv)dv = \frac{|A|}{t^D}$$

•<u>Remark:</u> this is the basic dispersion estimate for the free transport equation.

•Set

$$g_n(x,v) := f_n(x,v)\phi(v)\,,\ \text{and}$$
$$h_n(x,v) := v\cdot\nabla_x g_n(x,v) = \phi(v)(v\cdot\nabla_x f_n(x,v))$$

Both g_n and h_n are bounded in $L^1_{x,v}$, while g_n is uniformly integrable in v.

•Observe that (hint: integrate by parts the 2nd integral in the r.h.s.)

$$\int_A\int g_n dvdx = \int\int_{A_x(t)} g_n dvdx - \int_0^t \iint h_n(x,v)\chi(s,x,v)dxdvds$$

The second integral on the r.h.s. is $O(t)\sup\|h_n\|_{L^1_{x,v}} < \epsilon$ by choosing $t>0$ small enough. For that value of t, $|A_x(t)|\to 0$ as $|A|\to 0$, hence the first integral on the r.h.s. vanishes by uniform integrability in v.