Hydrodynamic Limits for the Boltzmann Equation

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LECTURE 4

EXISTENCE THEORY FOR THE BOLTZMANN EQUATION

Notion of renormalized solution

•A nonnegative function $F \in C(\mathbf{R}_+; L^1(\mathbf{R}^3 \times \mathbf{R}^3))$ is a renormalized solution of the Boltzmann equation iff $\frac{\mathcal{B}(F,F)}{\sqrt{1+F}} \in L^1_{loc}(dtdxdv)$ and for each $\beta \in C^1(\mathbf{R}_+)$ s.t. $\beta'(Z) \leq \frac{C}{\sqrt{1+Z}}$ for all $Z \geq 0$, one has

$$(\partial_t + v \cdot \nabla_x)\beta(F) = \beta'(F)\mathcal{B}(F,F)$$

in the sense of distributions on $\mathbf{R}_+^* \times \mathbf{R}^3 \times \mathbf{R}^3$.

Theorem. (DiPerna-Lions, Ann. Math. 1990) *Let $F^{in} \geq 0$ a.e. satisfy*

$$\iint (1 + |x|^2 + |v|^2 + |\ln F^{in}|)F^{in} dxdv < +\infty$$

Then, there exists a renormalized solution of the Boltzmann equation such that $F\big|_{t=0} = F^{in}$.

Remark: For $F \in C(\mathbf{R}_+; L^1(\mathbf{R}^3 \times \mathbf{R}^3))$ such that $\frac{\mathcal{B}_\pm(F,F)}{1+F} \in L^1_{loc}(dtdxdv)$ the following conditions are equivalent

• F is a renormalized solution of the Boltzmann equation; and

• F is a mild solution of the Boltzmann equation, i.e. for a.e. $x, v \in \mathbf{R}^3$, $\mathcal{B}(F,F)^\sharp(t,x,v) \in L^1_{loc}(dt)$ and, denoting $f^\sharp(t,x,v) = f(t, x+tv, v)$

$$F^\sharp(t) = F^\sharp(0) + \int_0^t \mathcal{B}(F,F)^\sharp(s)ds \text{ for all } t > 0$$

• likewise, for a.e. $x, v \in \mathbf{R}^3$

$$F^\sharp(t) = e^{-A^\sharp(t)}F^\sharp(0) + \int_0^t e^{-(A^\sharp(t)-A^\sharp(s))}\mathcal{B}_+(F,F)^\sharp(s)ds \text{ for all } t > 0$$

where $A^\sharp(t,x,v) = \int_0^t \left(\frac{\mathcal{B}_-(F,F)}{F}\right)^\sharp (s,x,v)ds$

Properties of renormalized solutions

- Continuity equation + global conservation of momentum

$$\partial_t \int F dv + \mathrm{div}_x \int vF dv = 0\,, \quad \iint vF(t) dx dv = Cst$$

- Energy inequality

$$\iint \tfrac{1}{2}|v|^2 F(t,x,v) dx dv \leq \iint \tfrac{1}{2}|v|^2 F^{in} dx dv$$

- Entropy inequality

$$\iint F \ln F(t) dx dv + \tfrac{1}{4}\int_0^t ds \int dx \iiint (F'F'_* - FF_*) \ln\left(\frac{F'F'_*}{FF_*}\right) b\, d\omega\, dv\, dv_* \leq \iint F^{in} \ln F^{in} dx dv$$

The approximation scheme

- Bounded collision kernel: $0 \leq b \in L^\infty$, $b \geq 0$ a.e. and $\mathcal{B} = \mathcal{B}_+ - \mathcal{B}_-$

$$\text{where } \mathcal{B}_+(F,F) = \iint F'F'_* b(v - v_*, \omega)\, dv_* d\omega \,,$$
$$\mathcal{B}_-(F,F) = \iint F F_* b(v - v_*, \omega)\, dv_* d\omega$$

- Let F_n be the solution to the truncated Boltzmann equation on $\mathbf{R}^3 \times \mathbf{R}^3$:

$$\partial_t F_n + v \cdot \nabla_x F_n = \frac{\mathcal{B}(F_n, F_n)}{1 + \frac{1}{n}\int F_n dv} =: \mathcal{B}^n(F_n, F_n)\,; \qquad F_n\Big|_{t=0} = F^{in}$$

under the condition $\iint (1 + |x|^2 + |v|^2 + |\ln F^{in}|) F^{in} dx dv < +\infty$.

- Exercise: Prove existence+uniqueness of the solution to the TBE

A priori bounds and weak L^1 compactness

•The truncation by the macroscopic density does not affect (i) the symmetries of the Boltzmann collision integral leading to the local conservation laws, and (ii) the H Theorem:

$$\iint (1+|x|^2+|v|^2+|\ln F_n(t)|)F_n(t)dxdv \le C(1+t^2)$$

where C is independent of n.

Proposition. *For each $\delta > 0$, the sequences*

$$\frac{\mathcal{B}_-^n(F_n,F_n)}{1+\delta F_n} \quad \textit{and} \quad \frac{\mathcal{B}_+^n(F_n,F_n)}{1+\delta F_n}$$

are both bounded in $L^1_{loc}(\mathbf{R}_+;L^1(\mathbf{R}^3\times\mathbf{R}^3))$ and relatively compact in $L^1_{loc}(\mathbf{R}_+\times\mathbf{R}^3\times\mathbf{R}^3)$ weak.

Proof: L^1 bound and uniform integrability obvious for $\mathcal{B}_-^n$, since

$$\frac{\mathcal{B}_-^n(F_n,F_n)}{1+\delta F_n} = L_n(F_n)\frac{F_n}{1+\delta F_n}$$

where

$$L_n(F) = \frac{\overline{b}\star_v F}{1+\frac{1}{n}\int F\,dv}, \qquad \overline{b}(z) = \int b(z,\omega)\,d\omega$$

— in other words

$$L_n(F) = \frac{\iint F_* b(v-v_*,\omega)\,d\omega\,dv_*}{1+\frac{1}{n}\int F\,dv} = \frac{\int F_*\overline{b}(v-v_*)\,dv_*}{1+\frac{1}{n}\int F\,dv}$$

so that

$$0 \leq L_n(F) \leq \|b\|_{L^\infty}\int F\,dv$$

As for $\mathcal{B}^n_+$, pick $R >> 1$ and write

$$\frac{\mathcal{B}^n(F_n,F_n)}{1+\delta F_n} = \frac{1}{1+\delta F_n}\iint \frac{F'_nF'_{n*}-F_nF_{n*}}{1+\frac{1}{n}\int F_n dv}\mathbf{1}_{F'_nF'_{n*}\le RF_nF_{n*}} b dv_* d\omega$$
$$+\frac{1}{1+\delta F_n}\iint \frac{F'_nF'_{n*}-F_nF_{n*}}{1+\frac{1}{n}\int F_n dv}\mathbf{1}_{F'_nF'_{n*}> RF_nF_{n*}} b dv_* d\omega$$

The first term is bounded pointwise by

$$(R-1)\frac{\mathcal{B}^n_-(F_n,F_n)}{1+\delta F_n}$$

while the $L^1([0,t]\times\mathbf{R}^3\times\mathbf{R}^3)$ norm of the second is bounded by the entropy production

$$\frac{1}{\ln R}\int_0^t\int dx\iiint \frac{F'_nF'_{n*}-F_nF_{n*}}{1+\frac{1}{n}\int F_n dv}\ln\left(\frac{F'_nF'_{n*}}{F_nF_{n*}}\right) b dv dv_* d\omega = O\left(\frac{1}{\ln R}\right)$$

CONCLUSION: hence, for each $R >> 1$,

$$\frac{\mathcal{B}^n(F_n, F_n)}{1+\delta F_n} \in B\left(0, \frac{1}{\ln R}\right)_{L^1} + K_R$$

where K_R is locally uniformly integrable on $\mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3$ for each R.

Therefore

$$\frac{\mathcal{B}^n(F_n, F_n)}{1+\delta F_n} \text{ is locally uniformly integrable on } \mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3$$

Finally

$$\frac{\mathcal{B}^n_+(F_n, F_n)}{1+\delta F_n} = \frac{\mathcal{B}^n_-(F_n, F_n)}{1+\delta F_n} + \frac{\mathcal{B}^n(F_n, F_n)}{1+\delta F_n}$$

is the sum of two locally uniformly integrable sequences.

Applying Velocity Averaging

•We know that

$$(\partial_t + v \cdot \nabla_x)\tfrac{1}{\delta}\ln(1+\delta F_n) = \frac{\mathcal{B}^n(F_n, F_n)}{1+\delta F_n} = O(1)_{L^1([0,T]\times\mathbf{R}^3\times\mathbf{R}^3)}$$

$$\iint (1+|x|^2+|v|^2+|\ln F_n(t)|)F_n(t)\,dxdv \le C(1+t^2)$$

The 2nd bound implies F_n is uniformly integrable on $[0,T]\times\mathbf{R}^3\times\mathbf{R}^3$ and tight; for each $\delta > 0$, this is also true of $\frac{1}{\delta}\ln(1+\delta F_n)$, since

$$0 \le \tfrac{1}{\delta}\ln(1+\delta F_n) \le F_n$$

hence, by Velocity Averaging in L^1

$$\int \tfrac{1}{\delta}\ln(1+\delta F_n)dv \text{ is strongly relatively compact in } L^1([0,T]\times\mathbf{R}^3)$$

•In fact

$$0 \leq F_n - \frac{1}{\delta}\ln(1+\delta F_n) \leq \delta F_n^2 \mathbf{1}_{F_n \leq R} + F_n \mathbf{1}_{F_n > R}$$

so that

$$\|F_n - \frac{1}{\delta}\ln(1+\delta F_n)\|_{L^1_{x,v}} \leq R\delta \|F_n\|_{L^1_{x,v}} + \frac{1}{\ln R}\iint F_n \ln F_n dx dv$$

Hence

$F_n - \frac{1}{\delta}\ln(1+\delta F_n) \to 0$ in $L^\infty([0,T]; L^1_{x,v})$ as $\delta \to 0$ uniformly in n

•Therefore, one can remove the nonlinear normalizing function, so that

$\int F_n dv$ is strongly relatively compact in $L^1([0,T] \times \mathbf{R}^3)$

•Therefore, modulo extracting subsequences, for each $T > 0$:

$$F_n \rightharpoonup F \text{ in } L^1([0,T] \times \mathbf{R}^3 \times \mathbf{R}^3)\,, \quad \text{while}$$

$$\int F_n \phi dv \to \int F \phi dv \text{ in } L^1_{loc}(\mathbf{R}_+ \times \mathbf{R}^3) \text{ and a.e.}$$

for each $\phi \in L^\infty(\mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3)$; likewise

$$L_n(F_n) \to L(F) = \bar{b} \star_v F = \iint F_* b(v - v_*, \omega) d\omega dv_*$$

in $L^1_{loc}(\mathbf{R}_+ \times \mathbf{R}^3)$ and a.e..

Proposition. *For each* $\phi \in L^\infty(\mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3)$

$$\int \frac{\mathcal{B}^n_\pm(F_n, F_n)}{1 + \int F_n dv} \phi dv \to \int \frac{\mathcal{B}_\pm(F, F)}{1 + \int F dv} \phi dv \text{ in } L^1_{loc}(\mathbf{R}_+ \times \mathbf{R}^3)$$

Product Limit Theorems

We recall Egorov's Theorem: Assume that $v_n \to v$ a.e. on $K \subset\subset \mathbf{R}^N$. Then, for each $\epsilon > 0$, there exists a measurable $E \subset K$ such that

$$|K \setminus E| < \epsilon\,, \quad \text{and } v_n \to v \text{ UNIFORMLY on } E$$

Lemma. *Assume that $u_n \rightharpoonup u$ in L^1, that $\sup \|v_n\|_{L^\infty} < +\infty$, and that $v_n \to v$ a.e.. Then $u_n v_n \rightharpoonup uv$ in L^1. (If $v = 0$, $u_n v_n \to 0$ in L^1).*

Lemma. *Assume that, for each $\phi \in L^\infty(\mathbf{R}^N \times \mathbf{R}^N)$*

$$u_n \rightharpoonup u \text{ in } L^1(\mathbf{R}^N \times \mathbf{R}^N)\,, \quad \int u_n \phi dv \to \int u \phi dv \text{ in } L^1_{loc}(\mathbf{R}^N \times \mathbf{R}^N)$$

that $\sup \|v_n\|_{L^\infty(\mathbf{R}^N \times \mathbf{R}^N)} < +\infty$, and that $v_n \to v$ a.e.. Then

$$\int u_n v_n \phi dv \to \int uv\phi dv \text{ in } L^1_{loc}(\mathbf{R}^N) \text{ for each } \phi \in L^\infty(\mathbf{R}^N \times \mathbf{R}^N)$$

•Proof of the first lemma: Write $u_n v_n - uv = u_n(v_n - v) + v(u_n - u)$; since $v \in L^\infty$ and $u_n \rightharpoonup u$ in L^1, the second term $\rightharpoonup 0$ in L^1.

WLOG, one can assume that $\mathrm{supp}(u_n) \subset K$ compact; indeed, since $u_n \rightharpoonup u$ in L^1, the sequence u_n is tight. By Egorov's Theorem

$$u_n(v_n - v) = u_n \mathbf{1}_{K \setminus E}(v_n - v) + u_n \mathbf{1}_E (v_n - v)$$

the second term $\to 0$ in L^1, while the first term can be made arbitrarily small with ϵ, since u_n is uniformly integrable.

Proof of the second lemma: left as an exercise, following the same pattern.

Proof: One has

$$F_n \rightharpoonup F \text{ in } L^1_{loc}(dt; L^1_{x,v})\,, \quad \int F_n \phi dv \to \int F \phi dv \text{ in } L^1_{loc}(dtdx) \text{ and a.e.}$$

on the other hand

$$\frac{L_n(F_n)}{1 + \int F_n dv} \to \frac{L(F)}{1 + \int F dv} \text{ a.e.}$$

where

$$L(F) = \iint F_* b(v - v_*, \omega) d\omega dv_*\,, \quad \text{and } L_n(F) = \frac{L(F)}{1 + \frac{1}{n}\int F dv}$$

while

$$\left\| \frac{L_n(F_n)}{1 + \int F_n dv} \right\|_{L^\infty} \le \|b\|_{L^\infty}$$

•Applying the second lemma above shows that

$$\int \frac{F_n L_n(F_n)}{1+\int F_n dv}\phi dv \to \int \frac{FL(F)}{1+\int F dv}\phi dv \text{ in } L^1_{loc}(\mathbf{R}_+ \times \mathbf{R}^3)$$

•In other words

$$\boxed{\int \frac{\mathcal{B}^n_-(F_n,F_n)}{1+\int F_n dv}\phi dv \to \int \frac{\mathcal{B}_-(F,F)}{1+\int F dv}\phi dv \text{ in } L^1_{loc}(\mathbf{R}_+ \times \mathbf{R}^3)}$$

•The case of $\mathcal{B}^n_+(F_n,F_n)$ is easily reduced to the case of $\mathcal{B}^n_-(F_n,F_n)$ by exchanging (v,v_*) and (v',v'_*).

Supersolution

•Write the truncated Boltzmann equation along characteristics:

$$\frac{d}{dt}F_n^\sharp + \left(\frac{L(F_n)}{1+\frac{1}{n}\int F_n dv}\right)^\sharp F_n^\sharp = \left(\frac{\mathcal{B}_+(F_n,F_n)}{1+\frac{1}{n}\int F_n dv}\right)^\sharp$$

with the notation $f^\sharp(t,x,v) = f(t,x+tv,v)$. Setting

$$A_n^\sharp(t,x,v) = \int_0^t \left(\frac{L(F_n)}{1+\frac{1}{n}\int F_n dv}\right)^\sharp (s,x,v)ds$$

we see that (modulo extraction of a subsequence)

$$A_n^\sharp \to A^\sharp \equiv \int_0^t L(F)^\sharp(s,x,v)ds \text{ in } C([0,T],L^1_{loc}(dxdv)) \text{ and a.e.}$$

Pick β_R to be a mollified version of $z \mapsto \sup(z, R)$; then

$$F_n^\sharp(t) \geq F^\sharp(0)e^{-A_n^\sharp(t)} + \int_0^t e^{-(A_n^\sharp(t)-A_n^\sharp(s))} \left(\frac{\mathcal{B}_+(\beta_R(F_n), \beta_R(F_n))}{1+\frac{1}{n}\int F_n dv}\right)^\sharp (s)ds$$

By Velocity Averaging applied to $\beta_R(F_n)$, one sees that

$$\beta_R(F_n) \rightharpoonup F^R, \quad \mathcal{B}_+(\beta_R(F_n), \beta_R(F_n)) \rightharpoonup \mathcal{B}_+(F^R, F^R) \text{ in } L^1_{loc}(dtdx; L^1_v)$$

Passing to the limit in the inequality above leads to

$$F^\sharp(t) \geq F^\sharp(0)e^{-A^\sharp(t)} + \int_0^t e^{-(A^\sharp(t)-A^\sharp(s))} \mathcal{B}_+(F^R, F^R)^\sharp(s)ds$$

It follows from the entropy bound that

$$F^R \uparrow F \text{ in } L^1_{loc}(dt; L^1_{x,v}) \text{ as } R \to +\infty$$

•Therefore, by monotone convergence, one eventually finds that

$$F^\sharp(t) \geq F^\sharp(0)e^{-A^\sharp(t)} + \int_0^t e^{-(A^\sharp(t)-A^\sharp(s))}\mathcal{B}_+(F,F)^\sharp(s)ds$$

•Remark: This implies in particular that, for each $t > 0$, the function

$$(s,x,v) \mapsto e^{-(A^\sharp(t,x,v)-A^\sharp(s,x,v))}\mathcal{B}_+(F,F)^\sharp(s,x,v)$$

belongs to $L^1([0,t] \times \mathbf{R}^3 \times \mathbf{R}^3)$. Because of the inequality

$$\mathcal{B}_-(F,F) \leq R\mathcal{B}_+(F,F) + \frac{1}{\ln R}\iint (F'F'_* - FF_*) \ln\left(\frac{F'F'_*}{FF_*}\right) bd\omega dv_*$$

the function

$$(s,x,v) \mapsto e^{-(A^\sharp(t,x,v)-A^\sharp(s,x,v))}\mathcal{B}_-(F,F)^\sharp(s,x,v)$$

also belongs to $L^1([0,t] \times \mathbf{R}^3 \times \mathbf{R}^3)$.

Subsolution

•Write the truncated Boltzmann equation for $\beta_\delta(F_n) = \frac{1}{\delta}\ln(1+\delta F_n)$:

$$(\partial_t + v\cdot\nabla_x)\beta_\delta(F_n) + \frac{L(F_n)}{1+\frac{1}{n}\int F_n dv}\beta_\delta(F_n) = \frac{\mathcal{B}_+(F_n,F_n)}{(1+\delta F_n)(1+\frac{1}{n}\int F_n dv)}$$
$$+\frac{L(F_n)}{1+\frac{1}{n}\int F_n dv}\left(\beta_\delta(F_n) - \frac{F_n}{1+\delta F_n}\right)$$

and integrate along characteristics:

$$\beta_\delta(F_n)^\sharp(t) = e^{-A_n^\sharp(t)}\beta_\delta(F(0))$$
$$+\int_0^t e^{-(A_n^\sharp(t)-A_n^\sharp(s))}\left(\frac{\mathcal{B}_+(F_n,F_n)}{(1+\delta F_n)(1+\frac{1}{n}\int F_n dv)}\right)^\sharp(s)ds$$
$$+\int_0^t e^{-(A_n^\sharp(t)-A_n^\sharp(s))}\left(\frac{L(F_n)}{1+\frac{1}{n}\int F_n dv}\right)^\sharp(s)\left(\beta_\delta(F_n)^\sharp(s) - \frac{F_n^\sharp(s)}{1+\delta F_n^\sharp(s)}\right)ds$$

Next, we let $n \to +\infty$, keeping $\delta > 0$ fixed, and recall that

$$\beta_\delta(F_n) \rightharpoonup F_\delta \text{ in } L^1_{loc}(dt; L^1_{x,v}), \quad A_n^\sharp \to A^\sharp \text{ in } C([0,T]; L^1_{loc}(dxdv))$$

while

$$\int F_n dv \to \int F dv \text{ a.e. and } \frac{\mathcal{B}_+(F_n, F_n)}{1+\delta F_n} \rightharpoonup \mathcal{B}_\delta^+ \text{ in } L^1_{loc}(dtdxdv)$$

Hence, by the product limit theorem, the second integral above satisfies

$$\int_0^t e^{-(A_n^\sharp(t)-A_n^\sharp(s))} \left(\frac{\mathcal{B}_+(F_n, F_n)}{(1+\delta F_n)(1+\frac{1}{n}\int F_n dv)} \right)^\sharp (s) ds$$
$$\rightharpoonup \int_0^t e^{-(A^\sharp(t)-A^\sharp(s))} \mathcal{B}_\delta^{+\sharp}(s) ds$$

Notice the inequality

$$\frac{\mathcal{B}_+(F_n,F_n)}{(1+\delta F_n)(1+\int F_n dv)} \le \frac{\mathcal{B}_+(F_n,F_n)}{1+\int F_n dv}$$

passing to the limit as $n \to +\infty$ in weak L^1_{loc} leads to

$$\frac{\mathcal{B}_\delta^+}{1+\int F dv} \le \frac{\mathcal{B}_+(F,F)}{1+\int F dv}, \quad \text{and hence } \mathcal{B}_\delta^+ \le \mathcal{B}_+(F,F)$$

Hence

$$F_\delta^\sharp(t) \le e^{-A^\sharp(t)}\beta_\delta(F(0)) + \int_0^t e^{-(A^\sharp(t)-A^\sharp(s))}\mathcal{B}^+(F,F)^\sharp(s)ds$$
$$+\text{REMAINDER}$$

In the limit as $\delta \to 0$, $F_\delta \uparrow F$ in $L^1_{loc}(dt, L^1_{x,v})$, while the remainder term is disposed of by a combination of arguments that involve the entropy bound, monotone convergence, and the fact that

$$\int_0^t e^{-(A^\sharp(t)-A^\sharp(s))} L(F)^\sharp(s) F^\sharp(s) ds < +\infty$$

Finally, we arrive at the inequality

$$\boxed{F^\sharp(t) \le F^\sharp(0) e^{-A^\sharp(t)} + \int_0^t e^{-(A^\sharp(t)-A^\sharp(s))} \mathcal{B}_+(F,F)^\sharp(s) ds}$$