Hydrodynamic Limits

for the Boltzmann Equation

François Golse
Université Paris 7 & Laboratoire J.-L. Lions
golse@math.jussieu.fr

Academia Sinica, Taipei, December 2004

LECTURE 2

FORMAL INCOMPRESSIBLE HYDRODYNAMIC LIMITS

Dimensionless form of the Boltzmann equation

•Choose macroscopic scales of time T and length L, and a reference temperature Θ; this defines 2 velocity scales:

$$V = \frac{L}{T} \text{ (macroscopic velocity) }, \quad \text{and } c = \sqrt{\Theta} \text{ (thermal speed)}$$

Finally, set $\mathcal{N}$ to be the total number of particles.

•Define dimensionless time, position, and velocity variables by

$$\hat{t} = \frac{t}{T}, \quad \hat{x} = \frac{x}{L}, \quad \hat{v} = \frac{v}{c}$$

and a dimensionless number density

$$\hat{F}(\hat{t}, \hat{x}, \hat{v}) = \frac{L^3 c^3}{\mathcal{N}} F(t, x, v)$$

• One finds that

$$\frac{L}{cT}\partial_{\hat{t}}\hat{F} + \hat{v}\cdot\nabla_{\hat{x}}\hat{F} = \frac{\mathcal{N}r^2}{L^2}\iint(\hat{F}'\hat{F}'_* - \hat{F}\hat{F}_*)|(\hat{v}-\hat{v}_*)\cdot\omega|d\omega d\hat{v}_*$$

• The pre-factor multiplying the collision integral is

$$L\times\frac{\mathcal{N}r^2}{L^3} = \frac{L}{\pi\times\text{mean free path}} = \frac{1}{\pi\mathsf{Kn}}$$

• The pre-factor multiplying the time derivative is

$$\frac{\frac{1}{T}\times L}{c} = \mathsf{St}\,,\qquad \text{(kinetic Strouhal number)}$$

$$\mathsf{St}\partial_{\hat{t}}\hat{F} + \hat{v}\cdot\nabla_{\hat{x}}\hat{F} = \frac{1}{\pi\mathsf{Kn}}\iint_{\mathbf{R}^3\times\mathbf{S}^2}(\hat{F}'\hat{F}'_* - \hat{F}\hat{F}_*)|(\hat{v}-\hat{v}_*)\cdot\omega|d\omega d\hat{v}_*$$

Compressible Euler scaling

• This limit corresponds to $\boxed{\mathrm{St} = 1}$ and $\boxed{\pi\mathrm{Kn} =: \epsilon \ll 1}$, leading to the singular perturbation problem

$$\boxed{\partial_t F_\epsilon + v \cdot \nabla_x F_\epsilon = \frac{1}{\epsilon}\mathcal{B}(F_\epsilon, F_\epsilon) = \frac{1}{\epsilon}\iint (F'_\epsilon F'_{\epsilon*} - F_\epsilon F_{\epsilon*})|(v - v_*) \cdot \omega| d\omega dv_*}$$

• One expects that, as $\epsilon \to 0$, $F_\epsilon \to F$ and $\mathcal{B}(F_\epsilon, F_\epsilon) \to \mathcal{B}(F, F) = 0$; hence $F(t, x, \cdot)$ is a Maxwellian for all (t, x), i.e.

$$\boxed{F(t, x, v) = \mathcal{M}_{(\rho(t,x),u(t,x),\theta(t,x))}(v) = \frac{\rho(t, x)}{(2\pi\theta(t, x))^{3/2}} e^{-\frac{|v-u(t,x)|^2}{2\theta(t,x)}}}$$

In other words, F is a local Maxwellian equilibrium.

• <u>Problem:</u> to find the governing equations for $\rho(t, x)$, $u(t, x)$ and $\theta(t, x)$.

Formal Euler limit by the moment method

•Let $F^{in} = \mathcal{M}_{(\rho^{in},u^{in},\theta^{in})}$ be such that $H(F^{in}|M) < +\infty$; let F_ϵ be a renormalized solution relative to $M = \mathcal{M}_{(1,0,1)}$ of the Boltzmann equation in the compressible Euler scaling

$$\mathrm{St} = 1\,, \text{ and } \pi\mathrm{Kn} = \epsilon$$

•Assume that F_ϵ satisfies the local conservation laws of momentum and energy and the local H Theorem, and that

$$F_\epsilon \to F \text{ a.e., in } L^1_{loc}(dtdx; L^1((1+|v|^3)dv)) \text{ as } \epsilon \to 0$$
$$\text{and in } L\ln L_{loc}(dtdx; L\ln L((1+|v|)dv))$$

Theorem. (C. Bardos-F.G. C.R. Acad. Sci. 1984) *Then* $F = \mathcal{M}_{(\rho,u,\theta)}$, *where* (ρ, u, θ) *is an entropic solution to the compressible Euler system for perfect gases with* $(\rho, u, \theta)\Big|_{t=0} = (\rho^{in}, u^{in}, \theta^{in})$

Proof:

•Step 1: The H Theorem implies that F is a local Maxwellian i.e. is of the form $F(t, x, v) = \mathcal{M}_{(\rho(t,x),u(t,x),\theta(t,x))}(v)$: indeed

$$\int_0^{+\infty} \iint \mathcal{B}(F_\epsilon, F_\epsilon) \ln F_\epsilon dv dx dt \le \epsilon H(F^{in}|M) \to 0$$

as $\epsilon \to 0$; hence, by Fatou's lemma

$$\int_0^{+\infty} \iint \mathcal{B}(F, F) \ln F dv dx dt = 0$$

•Step 2: Passing to the limit in the local conservation laws + the entropy differential inequality leads to the system of conservation laws for (ρ, u, θ) with entropy condition

$$\partial_t \int_{\mathbf{R}^3} \mathcal{M}_{(\rho,u,\theta)} dv + \mathrm{div}_x \int_{\mathbf{R}^3} v\mathcal{M}_{(\rho,u,\theta)} dv = 0$$

$$\partial_t \int_{\mathbf{R}^3} v\mathcal{M}_{(\rho,u,\theta)} dv + \mathrm{div}_x \int_{\mathbf{R}^3} v^{\otimes 2}\mathcal{M}_{(\rho,u,\theta)} dv = 0$$

$$\partial_t \int_{\mathbf{R}^3} \tfrac{1}{2}|v|^2\mathcal{M}_{(\rho,u,\theta)} dv + \mathrm{div}_x \int_{\mathbf{R}^3} v\tfrac{1}{2}|v|^2\mathcal{M}_{(\rho,u,\theta)} dv = 0$$

as well as the differential inequality

$$\partial_t \int_{\mathbf{R}^3} \mathcal{M}_{(\rho,u,\theta)} \ln \mathcal{M}_{(\rho,u,\theta)} dv + \mathrm{div}_x \int_{\mathbf{R}^3} v\mathcal{M}_{(\rho,u,\theta)} \ln \mathcal{M}_{(\rho,u,\theta)} dv \leq 0$$

•The following formulas for the moments of a Maxwellian

$$\int \mathcal{M}_{(\rho,u,\theta)} dv = \rho\,, \quad \int v\mathcal{M}_{(\rho,u,\theta)} dv = \rho u\,,$$
$$\int v^{\otimes 2}\mathcal{M}_{(\rho,u,\theta)} dv = \rho(u^{\otimes 2} + \theta I)\,, \quad \int \tfrac{1}{2}|v|^2\mathcal{M}_{(\rho,u,\theta)} dv = \tfrac{1}{2}\rho(|u|^2 + 3\theta)$$
$$\int v\tfrac{1}{2}|v|^2\mathcal{M}_{(\rho,u,\theta)} dv = \tfrac{1}{2}\rho u(|u|^2 + 5\theta)$$

and for its entropy and entropy flux

$$\int \mathcal{M}_{(\rho,u,\theta)} \ln \mathcal{M}_{(\rho,u,\theta)} dv = \rho \ln\left(\frac{\rho}{(2\pi\theta)^{3/2}}\right) - \tfrac{3}{2}\rho$$
$$\int v\mathcal{M}_{(\rho,u,\theta)} \ln \mathcal{M}_{(\rho,u,\theta)} dv = \rho u \ln\left(\frac{\rho}{(2\pi\theta)^{3/2}}\right) - \tfrac{3}{2}\rho u$$

show that (ρ, u, θ) is an admissible (entropic) solution of Euler's system.

The incompressible Navier-Stokes limit

•This limit holds under the scaling assumption on the Boltzmann equation

$$\mathsf{St} = \pi\mathsf{Kn} = \epsilon \ll 1$$

•Moreover, the number density should correspond to a flow with small Mach number

$$\mathsf{Ma} = \epsilon$$

Example: $F_\epsilon(t,x,v) = \mathcal{M}_{(1,\epsilon u(t,x),1)}(v)$

•More generally, the number density should be a fluctuation of order ϵ about a uniform Maxwellian state

$$F_\epsilon(t,x,v) = \mathcal{M}_{(1,0,1)}(v) + \epsilon f(t,x,v)$$

Formal derivation of the incompressible Navier-Stokes equations

following a moment method due to C. Bardos-F.G.-D. Levermore (C.R. Acad. Sci. 1988)

- Introduce the relative number density fluctuation g_ϵ:

$$g_\epsilon(t,x,v) = \frac{F_\epsilon(t,x,v) - M(v)}{\epsilon M(v)}\,, \qquad \text{where } M(v) = \frac{1}{(2\pi)^{3/2}} e^{-\frac{|v|^2}{2}}$$

- In terms of g_ϵ, the Boltzmann equation becomes

$$\epsilon \partial_t g_\epsilon + v \cdot \nabla_x g_\epsilon + \frac{1}{\epsilon} \mathcal{L} g_\epsilon = \mathcal{Q}(g_\epsilon, g_\epsilon)$$

where the linearized collision operator $\mathcal{L}$ and $\mathcal{Q}$ are defined by

$$\mathcal{L} g = -2M^{-1}\mathcal{B}(M, Mg)\,, \qquad \mathcal{Q}(g,g) = M^{-1}\mathcal{B}(Mg, Mg)$$

Lemma. (Hilbert, Math. Ann. 1912) *The operator* $\mathcal{L}$ *is self-adjoint, Fredholm, unbounded on* $L^2(\mathbf{R}^3; M dv)$ *with domain* $L^2(\mathbf{R}^3; (1 + |v|) M dv)$ *and nullspace* $\ker \mathcal{L} = \operatorname{span}\{1, v_1, v_2, v_3, |v|^2\}$

1. Asymptotic fluctuations

- Multiplying the Boltzmann equation by ϵ and letting $\epsilon \to 0$ suggests that

$$g_\epsilon \to g \quad \text{with } \mathcal{L} g = 0$$

By Hilbert's lemma, g is an infinitesimal Maxwellian, i.e. is of the form

$$g(t, x, v) = \rho(t, x) + u(t, x) \cdot v + \tfrac{1}{2}\theta(t, x)(|v|^2 - 3)$$

Notice that g is parametrized by its own moments, since

$$\rho = \langle g \rangle\,, \quad u = \langle vg \rangle\,, \quad \text{and } \theta = \langle (\tfrac{1}{3}|v|^2 - 1)g \rangle$$

• NOTATION:

$$\langle \phi \rangle = \int_{\mathbf{R}^3} \phi(v) M(v) dv$$

2. Local conservation laws

• The continuity equation (local conservation of mass) reads

$$\epsilon \partial_t \langle g_\epsilon \rangle + \mathrm{div}_x \langle v g_\epsilon \rangle = 0\,, \quad \text{and thus } \mathrm{div}_x \langle vg \rangle = \mathrm{div}_x\, u = 0$$

which is the incompressibility condition in the Navier-Stokes equations.

•The local conservation of momentum takes the form

$$\epsilon\partial_t\langle vg_\epsilon\rangle + \mathrm{div}_x\langle v\otimes vg_\epsilon\rangle = 0$$

Recall the incompressible Navier-Stokes motion equation

$$\partial_t u + u\cdot\nabla_x u - \nu\Delta_x u = -\nabla_x p$$

that involves the term $\nabla_x p$ as the Lagrange multiplier associated to the constraint $\mathrm{div}_x\, u = 0$. Accordingly, split

$$v\otimes v = \left(v\otimes v - \tfrac{1}{3}|v|^2 I\right) + \tfrac{1}{3}|v|^2 I$$

so that the local conservation of momentum is recast as

$$\epsilon\partial_t\langle vg_\epsilon\rangle + \mathrm{div}_x\langle Ag_\epsilon\rangle + \nabla_x\langle\tfrac{1}{3}|v|^2 g_\epsilon\rangle = 0$$

where

$$A(v) = v\otimes v - \tfrac{1}{3}|v|^2 I\ ;\ \text{notice that } A\perp \ker\mathcal{L}$$

•Passing to the limit in the local conservation of momentum above:

$$\mathrm{div}_x\langle Ag\rangle + \nabla_x\langle \tfrac{1}{3}|v|^2 g\rangle = 0$$

where g is a local Maxwellian:

$$g(t,x,v) = \rho(t,x) + u(t,x)\cdot v + \theta(t,x)\tfrac{1}{2}(|v|^2-3)$$

In other words, $g(t,x,\cdot)\in \ker\mathcal{L}$ so that

$$\langle Ag\rangle = 0\,, \quad \text{and thus } \nabla_x\langle \tfrac{1}{3}|v|^2 g\rangle = \nabla_x(\rho+\theta) = 0$$

If $g\in L^\infty(\mathbf{R}_+;L^2(\mathbf{R}^3;Mdvdx))$, this entails the Boussinesq relation

$$\rho+\theta = 0\,, \quad \text{so that } g(t,x,v) = u(t,x)\cdot v + \theta(t,x)\tfrac{1}{2}(|v|^2-5)$$

•It remains to derive the Navier-Stokes motion equation. Start from the local conservation of momentum in the form

$$\partial_t \langle v g_\epsilon \rangle + \mathrm{div}_x \frac{1}{\epsilon} \langle A g_\epsilon \rangle + \nabla_x \frac{1}{\epsilon} \langle \tfrac{1}{3} |v|^2 g_\epsilon \rangle = 0$$

•Fredholm's alternative $\Rightarrow$ $A = \mathcal{L}\hat{A}$ for some $\hat{A} \perp \ker \mathcal{L}$; thus

$$\frac{1}{\epsilon} \langle A g_\epsilon \rangle = \left\langle \hat{A} \frac{1}{\epsilon} \mathcal{L} g_\epsilon \right\rangle = \langle \hat{A} \mathcal{Q}(g_\epsilon, g_\epsilon) \rangle - \langle \hat{A} (\epsilon \partial_t + v \cdot \nabla_x) g_\epsilon \rangle$$
$$\to \langle \hat{A} \mathcal{Q}(g, g) \rangle - \langle \hat{A} v \cdot \nabla_x g \rangle$$

•Since g is an infinitesimal Maxwellian, using the incompressibility condition and Boussinesq's relation shows that

$$\begin{aligned} \langle \hat{A} v \cdot \nabla_x g \rangle &= \tfrac{1}{2} \langle \hat{A} \otimes A \rangle : D(u) + \langle \hat{A} \otimes \tfrac{1}{2}(|v|^2 - 3) v \rangle \cdot \nabla_x \theta \\ &= \tfrac{1}{2} \langle \hat{A} \otimes A \rangle : D(u) \text{ since } \hat{A} \text{ is even} \end{aligned}$$

where $D(u) = \nabla_x u + (\nabla_x u)^T$ is the deformation tensor of u.

Lemma. *For each $g \in \ker \mathcal{L}$, one has $\mathcal{Q}(g,g) = \frac{1}{2}\mathcal{L}(g^2)$*

Proof: Differentiate twice the relation $\mathcal{B}(\mathcal{M}_{(\rho,u,\theta)}, \mathcal{M}_{(\rho,u,\theta)}) = 0$.

•Hence one has

$$\langle \widehat{A}\mathcal{Q}(g,g)\rangle = \frac{1}{2}\langle \widehat{A}\mathcal{L}(g^2)\rangle = \frac{1}{2}\langle A g^2\rangle = \frac{1}{2}\langle A \otimes A\rangle : \left(u \otimes u - \frac{1}{3}|u|^2 I\right)$$

•Straightforward computations on Gaussian integrals give

$$\langle A_{ij}A_{kl}\rangle = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl}$$
$$\langle \widehat{A}_{ij}A_{kl}\rangle = \nu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl}\right)$$

so that

$$\frac{1}{\epsilon}\langle A g_\epsilon\rangle \to \left(u \otimes u - \frac{1}{3}|u|^2 I\right) - \nu D(u)$$

•Substituting this relation in the local conservation of momentum

$$\partial_t \langle v g_\epsilon \rangle + \text{div}_x \frac{1}{\epsilon} \langle A g_\epsilon \rangle + \nabla_x \frac{1}{\epsilon} \langle \tfrac{1}{3} |v|^2 g_\epsilon \rangle = 0$$

and passing to the limit shows that

$$\partial_t u + \text{div}_x (u \otimes u) - \nu \Delta_x u = 0 \text{ modulo gradients}$$

which is precisely the Navier-Stokes motion equation (since $\text{div}_x\, u = 0$, one has $\text{div}_x (u \otimes u) = u \cdot \nabla_x u$ and $\text{div}_x\, D(u) = \Delta_x u$).

•The relation above for $\langle \hat{A}_{ij} A_{kl} \rangle$ shows that

$$\nu = \tfrac{1}{10} \langle \hat{A} : A \rangle = \tfrac{1}{10} \langle \hat{A} : \mathcal{L} \hat{A} \rangle > 0$$

since $\mathcal{L} \geq 0$ and $\hat{A} \perp \ker \mathcal{L}$.

Other limits

•From Boltzmann to incompressible Euler: the scaling is

$$\mathrm{St} = \mathrm{Ma} = \epsilon \ll 1\,, \quad \pi \mathrm{Kn} = \epsilon^a \text{ with } a > 1$$

i.e. one seeks solutions of the scaled Boltzmann equation

$$\epsilon \partial_t F_\epsilon + v \cdot \nabla_x F_\epsilon = \frac{1}{\epsilon^a} \mathcal{B}(F_\epsilon, F_\epsilon)$$

in the form

$$F_\epsilon = \mathcal{M}_{(1,0,1)} + \epsilon f_\epsilon$$

•Formal argument by C. Bardos-F.G.-D. Levermore (J. Stat. Phys. 1991)

•Proof with dissipative solutions of Euler by L. Saint-Raymond (Arch. Rat. Mech. Anal. 2002)

•From Boltzmann to Stokes: the scaling is

$$\mathrm{St} = \pi \mathrm{Kn} = \epsilon \ll 1\,, \quad \mathrm{Ma} = \epsilon^a \text{ with } a > 1$$

i.e. one seeks solutions of the scaled Boltzmann equation

$$\epsilon \partial_t F_\epsilon + v \cdot \nabla_x F_\epsilon = \frac{1}{\epsilon} \mathcal{B}(F_\epsilon, F_\epsilon)$$

in the form

$$F_\epsilon = \mathcal{M}_{(1,0,1)} + \epsilon^a f_\epsilon$$

•Formal argument by C. Bardos-F.G.-D. Levermore (J. Stat. Phys. 1991)

•Proof by F.G.-D. Levermore (Comm. Pure Appl. Math. 2002)

•From Boltzmann to the acoustic system: the scaling is

$$\mathrm{St} = 1\,, \quad \pi\mathrm{Kn} = \epsilon$$

i.e. one seeks solutions of the scaled Boltzmann equation

$$\partial_t F_\epsilon + v \cdot \nabla_x F_\epsilon = \frac{1}{\epsilon}\mathcal{B}(F_\epsilon, F_\epsilon)$$

in the form

$$F_\epsilon = \mathcal{M}_{(1,0,1)} + \epsilon^c f_\epsilon\,, \text{ with } c > 0$$

•Proof by F.G.-D. Levermore (Comm. Pure Appl. Math. 2002) for $c > \frac{1}{2}$