Hydrodynamic Limits for the Boltzmann Equation

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LECTURE 1

THE BOLTZMANN EQUATION: BASIC STRUCTURE

Orders of magnitude, perfect gas

• For a monatomic gas at room temperature and atmospheric pressure, about 10^{20} gas molecules with radius $\simeq 10^{-8}$cm are to be found in any volume of 1cm^3

• Excluded volume (i.e. the total volume occupied by the gas molecules if tightly packed): $10^{20} \times \frac{4\pi}{3} \times (10^{-8})^{-3} \simeq 5 \cdot 10^{-4}cm^3 \ll 1$cm^3

EXCLUDED VOLUME NEGLIGEABLE $\Rightarrow$ PERFECT GAS

• Equation of state for a perfect gas:

$p = k\rho\theta$, where k =Boltzmann's constant$= 1.38 \cdot 10^{-23}$J/K

Notion of mean-free path

•Roughly speaking, the average distance between two successive collisions for any given molecule in the gas

•There are more than one precise mathematical definitions of that notion (for instance, one can use the empirical measure to compute the mean)

•Intuitively, the higher the gas density, the smaller the mean-free path; likewise, the bigger the molecules, the smaller the mean-free path; this suggests

$$\text{mean-free path } \approx \frac{1}{\mathcal{N}\times\mathcal{A}}$$

where $\mathcal{N}=$number of gas molecules per unit volume and $\mathcal{A}=$area of the section of any individual molecule

•For the same monatomic gas as before (at room temperature and atmospheric pressure), $\mathcal{N} = 10^{20}$ molecules/cm^3, while $\mathcal{A} = \pi \times (10^{-8})^2 \simeq 3 \cdot 10^{-16}$cm^2; hence the mean-free path is $\approx \frac{1}{3} \cdot 10^{-4}$cm $\ll$ 1cm.

SMALL MEAN-FREE PATH REGIMES CAN OCCUR IN PERFECT GASES

•While keeping the same temperature, lower the pressure at 10^{-4}atm; then $\mathcal{N} = 10^{16}$ molecules/cm^3 and the mean-free path becomes $\approx \frac{1}{3}$cm which is comparable to the size of the 1cm^3 container

DEGREE OF RAREFACTION MEASURED BY KNUDSEN NUMBER

$$\mathrm{Kn} := \frac{\text{mean free path}}{\text{macroscopic length scale}}$$

Kinetic vs. fluid regimes

•Fluid regimes are characterized by $\boxed{\text{Kn} \ll 1}$; the gas is in local thermodynamic equilibrium: its state is adequately described by:

$$p \equiv p(t,x) \text{ pressure, } \theta \equiv \theta(t,x) \text{ temperature, } \vec{u} \equiv \vec{u}(t,x) \text{ velocity field}$$

•Kinetic regimes are characterized by $\boxed{\text{Kn} = O(1)}$; since the gas is more rarefied, there are not enough collisions per unit of time for a local thermodynamic equilibrium to be reached. However, also because of rarefaction, correlations are weak $\Rightarrow$ state of the gas is adequately described by

$$F \equiv F(t,x,v) \text{ single-particle phase-space density}$$

Macroscopic observables

•One calls F the "distribution function" or "number density"; $F(t,x,v)$ is the density (with respect to the Lebesgue measure $dxdv$) of particles which, at time t, are to be found at the position x with velocity v.

•Macroscopic quantities (observables) are computed by averaging the corresponding quantity for a single particle w.r.t. the measure $F(t,x,v)dxdv$:

momentum $= \iint mvF(t,x,v)dxdv$, energy $= \iint \frac{1}{2}m|v|^2F(t,x,v)dxdv$

•Likewise, one can also define macroscopic densities (w.r.t. the Lebesgue measure dx):

momentum density $= \int mvF(t,x,v)dv$

The Boltzmann equation

•The number density F is governed by the Boltzmann equation: in the absence of external force

$$\partial_t F + v \cdot \nabla_x F = \mathcal{B}(F, F)$$

where $\mathcal{B}(F, F)$ is the Boltzmann collision integral.

•Because of rarefaction, collisions other than BINARY are neglected.

•At the kinetic level of description, the size of particles is neglected everywhere but in the expression of the mean-free path: collisions are LOCAL and INSTANTANEOUS

$\Rightarrow \quad \mathcal{B}(F, F)$ operates only on the v-variable in F

The collision integral (hard sphere gas)

•For a gas of hard spheres with radius r, Boltzmann's collision integral is

$$\mathcal{B}(F,F)(v)=\iint_{\mathbf{R}^3\times\mathbf{S}^2}\Big(F(v')F(v'_*)-F(v)F(v_*)\Big)\,r^2|(v-v_*)\cdot\omega|d\omega dv_*$$

where the velocities v' and v'_* are defined in terms of v, v_* and ω by

$$\begin{aligned} v' &\equiv v'(v,v_*,\omega) = v-(v-v_*)\cdot\omega\omega \\ v'_* &\equiv v'_*(v,v_*,\omega) = v_*+(v-v_*)\cdot\omega\omega \end{aligned}$$

•Usual notation: F_*, F' and F'_* designate resp. $F(v_*)$, $F(v')$ and $F(v'_*)$

Pre- to post-collision relations

•Given any velocity pair $(v, v_*) \in \mathbf{R}^6$, the pair $(v'(v, v_*, \omega), v'_*(v, v_*, \omega))$ runs through the set of solutions to the system of 4 equations

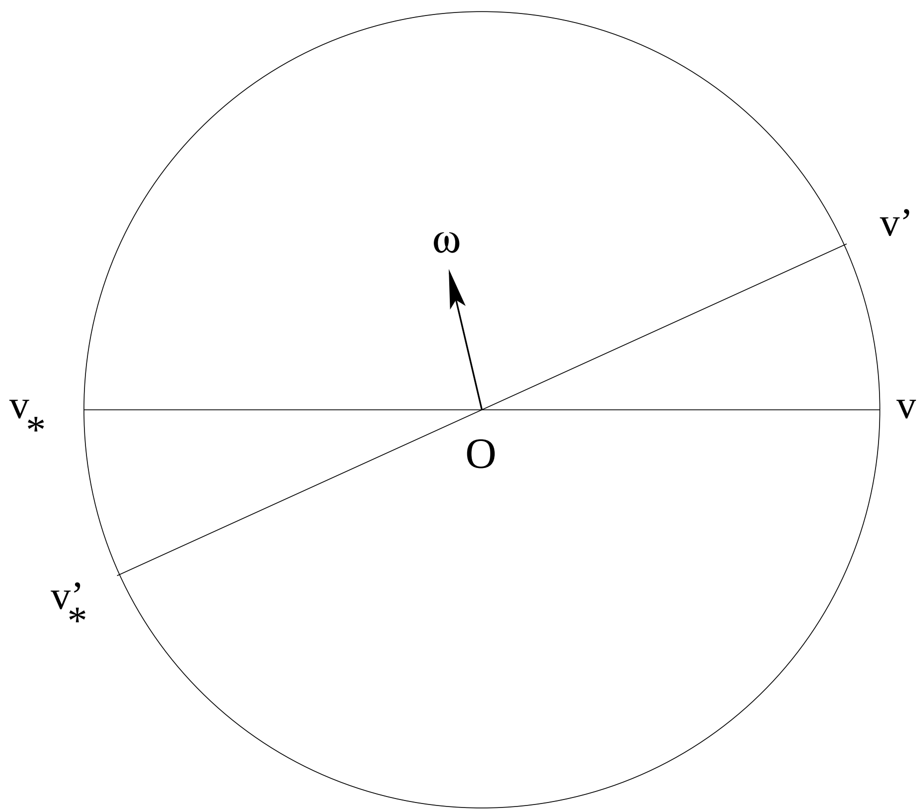
$$v' + v'_* = v + v_* \qquad \text{conservation of momentum}$$
$$|v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2 \qquad \text{conservation of kinetic energy}$$

as ω runs through $\mathbf{S}^2$.

•The geometric interpretation of these formulas is as follows: in the reference frame of the center of mass of the particle pair, the velocity pair before and after collisions is made of two opposite vectors, $\pm\frac{1}{2}(v' - v'_*)$ and $\pm\frac{1}{2}(v - v_*)$. Conservation of energy implies that $|v - v_*| = |v' - v'_*|$.

Geometric interpretation of collision relations

•Hence $v - v_*$ and $v' - v'_*$ are exchanged by some orthogonal symmetry, whose invariant plane is orthogonal to $\pm\omega$.



Symmetries of the collision integral

•The collision integrand is invariant if one exchanges v and v_*:

$$\int_{\mathbf{R}^3} \mathcal{B}(F,F)\phi dv = \iiint (F'F'_* - FF_*)\phi|(v-v_*)\cdot\omega| d\omega dv_* dv$$
$$= \iiint (F'F'_* - FF_*)\frac{\phi+\phi_*}{2}|(v-v_*)\cdot\omega| d\omega dv_* dv$$

•The collision integrand is changed into its opposite if, given $\omega \in \mathbf{S}^2$, one exchanges (v, v_*) and (v', v'_*) (in the center of mass reference frame, this is a symmetry, and thus an involution).

•Further, $(v, v_*) \mapsto (v', v'_*)$ is an isometry of $\mathbf{R}^6$ (conservation of kinetic energy), so that $\boxed{dv dv_* = dv' dv'_*}$. Finally $\boxed{(v - v_*)\cdot\omega = -(v' - v'_*)\cdot\omega}$.

Symmetries of the collision integral 2

Theorem. *Assume that $F \in L^1(\mathbf{R}^3)$ is rapidly decaying at infinity, i.e.*

$$F(v) = O(|v|^{-n}) \text{ as } |v| \to +\infty \text{ for all } n \geq 0$$

while $\phi \in C(\mathbf{R}^3)$ has at most polynomial growth at infinity, i.e.

$$\phi(v) = O(1 + |v|^m) \text{ as } |v| \to +\infty \text{ for some } m \geq 0$$

Then, one has:

$$\int_{\mathbf{R}^3} \mathcal{B}(F,F)\phi dv = \iiint FF_* \frac{\phi' + \phi'_* - \phi - \phi_*}{2} |(v - v_*) \cdot \omega| d\omega dv_* dv$$
$$= \iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} (F'F'_* - FF_*) \frac{\phi + \phi_* - \phi' - \phi'_*}{4} |(v - v_*) \cdot \omega| d\omega dv_* dv$$

Collision invariants

•These are the functions $\phi \equiv \phi(v) \in C(\mathbf{R}^3)$ such that

$$\phi(v) + \phi(v_*) - \phi(v') - \phi(v'_*) = 0 \text{ for all } (v, v_*) \in \mathbf{R}^3 \text{ and } \omega \in \mathbf{S}^2$$

Theorem. *Any collision invariant is of the form*

$$\phi(v) = a + b_1v_1 + b_2v_2 + b_3v_3 + c|v|^2, \quad a, b_1, b_2, b_3, c \in \mathbf{R}$$

•If ϕ is any collision invariant and $F \in L^1(\mathbf{R}^3)$ is rapidly decaying, then

$$\int_{\mathbf{R}^3} \mathcal{B}(F, F)\phi dv = 0$$

Local conservation laws

• In particular, if $F \equiv F(t,x,v)$ is a solution to the Boltzmann equation that is rapidly decaying in the v-variable

$$\int_{\mathbf{R}^3} \mathcal{B}(F,F)dv = \int_{\mathbf{R}^3} v_k \mathcal{B}(F,F)dv = \int_{\mathbf{R}^3} \tfrac{1}{2}|v|^2 \mathcal{B}(F,F)dv = 0$$

for $k = 1,2,3$.

• Therefore, one has the local conservation laws:

$$\partial_t \int_{\mathbf{R}^3} F dv + \operatorname{div}_x \int_{\mathbf{R}^3} vF dv = 0\,, \quad \text{(mass)}$$
$$\partial_t \int_{\mathbf{R}^3} vF dv + \operatorname{div}_x \int_{\mathbf{R}^3} v^{\otimes 2} F dv = 0\,, \quad \text{(momentum)}$$
$$\partial_t \int_{\mathbf{R}^3} \tfrac{1}{2}|v|^2 F dv + \operatorname{div}_x \int_{\mathbf{R}^3} v \tfrac{1}{2}|v|^2 F dv = 0\,, \quad \text{(energy)}$$

Boltzmann's H Theorem

•Assume that $0 < F \in L^1(\mathbf{R}^3)$ is rapidly decaying and such that $\ln F$ has polynomial growth at infinity. Then

$$\int_{\mathbf{R}^3} \mathcal{B}(F,F) \ln F dv =$$
$$-\frac{1}{4}\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} (F'F'_* - FF_*) \ln\left(\frac{F'F'_*}{FF_*}\right) |(v-v_*)\cdot\omega| d\omega dv dv_* \le 0$$

•The following conditions are equivalent:

$$\int_{\mathbf{R}^3} \mathcal{B}(F,F) \ln F dv = 0 \;\Leftrightarrow\; \mathcal{B}(F,F) = 0 \text{ a.e. } \;\Leftrightarrow\; F \text{ is a Maxwellian}$$

i.e. $F(v)$ is of the form

$$F(v) = \mathcal{M}_{\rho,u,\theta}(v) := \frac{\rho}{(2\pi\theta)^{3/2}} e^{-\frac{|v-u|^2}{2\theta}} \text{ for some } \rho,\theta > 0 \text{ and } u \in \mathbf{R}^3$$

$\boxed{\mathcal{B}(F,F) = 0 \text{ implies } F \text{ is a Maxwellian}}$

Lemma. (Perthame) *Let $F > 0$ a.e. be a measurable function s.t.*

$$\int (1 + |v|^2) F(v) dv < +\infty$$

If $F(v)F(v_*) = F(v')F(v'_*)$ a.e. in $(v, v_*, \omega) \in \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2$, then F is a Maxwellian.

•WLOG, by translation and scaling, one can assume

$$\int_{\mathbf{R}^3} F(v) dv = 1\,, \qquad \int_{\mathbf{R}^3} vF(v) dv = 0$$

•Apply the Fourier transform in (v, v_*) to the equality satisfied by F:

$$\widehat{F}(\xi)\widehat{F}(\xi_*) = \iint e^{-i\xi\cdot v - i\xi_*\cdot v_*} F(v')F(v'_*)dvdv_*$$
$$= \iint e^{-i\xi\cdot v' - i\xi_*\cdot v'_*} F(v)F(v_*)dvdv_* \,.$$

•In other words

$$\widehat{F}(\xi)\widehat{F}(\xi_*) = \iint e^{-i\xi\cdot v - i\xi_*\cdot v_*} e^{i(\xi-\xi_*)\cdot\omega(v-v_*)\cdot\omega} F(v)F(v_*)dvdv_*$$

Differentiate this in ω while keeping ξ and ξ_* fixed; at any $\omega_0 \perp \xi - \xi_*$

$$0 = \iint e^{-i\xi\cdot v - i\xi_*\cdot v_*}(v - v_*)\cdot\omega_0 F(v)F(v_*)dvdv_*$$

•Hence, for each $\xi, \xi_* \in \mathbf{R}^3$

$$(\nabla_\xi - \nabla_{\xi_*})\widehat{F}(\xi)\widehat{F}(\xi_*) \text{ is colinear to } \xi - \xi_*$$

•In particular, for $\xi \neq 0$ and $\xi_* = 0$, one has

$$\nabla_\xi \widehat{F}(\xi) \text{ is colinear to } \xi \,;$$

(notice that the assumptions on F imply that $\widehat{F} \in C^2(\mathbf{R}^3)$).

•In other words, the foliations of $\mathbf{R}^3$ by level surfaces of $\widehat{F}$ and of $\frac{1}{2}|\xi|^2$ coincide. Hence there exists a function $f \in C^1(\mathbf{R}_+)$ such that

$$\widehat{F}(\xi) \text{ is of the form } \widehat{F}(\xi) = f(|\xi|^2)$$

•Going back to $(\nabla_\xi - \nabla_{\xi_*})\widehat{F}(\xi)\widehat{F}(\xi_*)$, one sees that

$$f'(|\xi|^2)f(|\xi_*|^2)\xi - f(|\xi|^2)f'(|\xi_*|^2)\xi_* \text{ is colinear to } \xi - \xi_*$$

Whenever ξ and ξ_* are not colinear, i.e. for a dense subset of $\mathbf{R}^3_\xi \times \mathbf{R}^3_{\xi_*}$ this last relation implies that

$$\boxed{f'(|\xi|^2)f(|\xi_*|^2) = f(|\xi|^2)f'(|\xi_*|^2)}$$

•Since f is continuous, this relation holds everywhere on $\mathbf{R}^3_\xi \times \mathbf{R}^3_{\xi_*}$; clearly, if f' vanishes somewhere, the above relation shows that f must be a constant, hence $f = 0$ since $\widehat{f}(|\xi|^2)$ is the Fourier transform of a L^1 function.

•Hence one can assume that $f' \neq 0$ everywhere on $\mathbf{R}_+$; the relation above becomes

$$\frac{f(|\xi|^2)}{f'(|\xi|^2)} = \frac{f(|\xi_*|^2)}{f'(|\xi_*|^2)} \quad \Rightarrow \quad \frac{f(|\xi|^2)}{f'(|\xi|^2)} = Const.$$

so that

$$f(r) = e^{-\frac{1}{2}\alpha r}$$

which in turn implies that

$$\widehat{F}(\xi) = e^{-\frac{1}{2}\alpha|\xi|^2}$$

Hence F is a Gaussian in v, as announced.

Implications of conservation laws + H Theorem

•If $F \equiv F(t,x,v) > 0$ is a solution to the Boltzmann equation that is rapidly decaying and such that $\ln F$ has polynomial growth in the v-variable, then

$$\partial_t \int_{\mathbf{R}^3} F dv + \mathrm{div}_x \int_{\mathbf{R}^3} vF dv = 0\,, \quad \text{(mass)}$$

$$\partial_t \int_{\mathbf{R}^3} vF dv + \mathrm{div}_x \int_{\mathbf{R}^3} v^{\otimes 2} F dv = 0\,, \quad \text{(momentum)}$$

$$\partial_t \int_{\mathbf{R}^3} \tfrac{1}{2}|v|^2 F dv + \mathrm{div}_x \int_{\mathbf{R}^3} v\tfrac{1}{2}|v|^2 F dv = 0\,, \quad \text{(energy)}$$

$$\partial_t \int_{\mathbf{R}^3} F \ln F dv + \mathrm{div}_x \int_{\mathbf{R}^3} vF \ln F dv \le 0\,, \quad \text{(entropy)}$$

The last differential inequality bearing on the entropy density is reminiscent of the Lax-Friedrichs entropy condition that selects admissible solutions of hyperbolic systems of conservation laws.

Fluctuation setup

Hydrodynamic limits of kinetic theory leading to incompressible flows consider solutions to the Boltzmann equation that are fluctuations of some uniform Maxwellian state.

•WLOG, we henceforth set this uniform equilibrium state to be

$$M = \mathcal{M}_{(1,0,1)} \qquad \text{(the centered, reduced Gaussian distribution)}$$

•The size of the number density fluctuations around the equilibrium state M will be measured in terms of the relative entropy defined as

$$H(F|M) = \iint_{\mathbf{R}^3 \times \mathbf{R}^3} \left[F \ln\left(\frac{F}{M}\right) - F + M \right] dx dv \quad (\geq 0)$$

for each measurable $F \geq 0$ a.e. on $\mathbf{R}^3 \times \mathbf{R}^3$

•A formal computation shows that

$$\frac{d}{dt}H(F|M) = \iint_{\mathbf{R}^3\times\mathbf{R}^3} \partial_t \left[F \ln\left(\frac{F}{M}\right) - F + M\right] dxdv$$
$$= \iint_{\mathbf{R}^3\times\mathbf{R}^3} (\partial_t + v\cdot\nabla_x) \left[F \ln\left(\frac{F}{M}\right) - F + M\right] dxdv$$
$$= \iint_{\mathbf{R}^3\times\mathbf{R}^3} \ln\left(\frac{F}{M}\right)(\partial_t + v\cdot\nabla_x)F dxdv$$
$$= \iint_{\mathbf{R}^3\times\mathbf{R}^3} \mathcal{B}(F,F)\ln F dxdv \le 0$$

since $\ln M$ is a collision invariant. This suggests that

$H(F(t)|M)$ is a nonincreasing function of t, and, for each $t > 0$

$$H(F(t)|M) - \int_0^t \iint_{\mathbf{R}^3\times\mathbf{R}^3} \mathcal{B}(F,F)\ln F dxdv = H(F(0)|M)$$