Nonresonant Velocity Averaging and the Vlasov-Maxwell System

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Non resonant wave + transport systems

•We consider systems of the form

$$\Box_{t,x} u = f \, , \qquad (\partial_t + v(\xi) \cdot \nabla_x) f = D_\xi^m g \, , \qquad x, \xi \in \mathbf{R}^N \, , \; t > 0$$

where $g \equiv g(t, x, \xi)$ is a data, while $(u, f) \equiv (u(t, x, \xi), f(t, x, \xi))$ is the unknown.

•We call the above system nonresonant iff $|v(\xi)| < 1$ for each $\xi \in \mathbf{R}^N$ — here 1 is the speed of propagation associated to the d'Alembert operator $\Box_{t,x} = \partial_{tt} - \Delta_x$.

•The above system is supplemented with the initial conditions

$$u\Big|_{t=0} = u_0 \, , \quad \partial_t u\Big|_{t=0} = u_0' \, , \qquad f\Big|_{t=0} = f_0 \, .$$

An example: the relativistic Vlasov-Maxwell system

• Consider the relativistic Vlasov equation

$$(\partial_t + v(\xi)\cdot\nabla_x)f) = -\,\mathrm{div}_\xi((E + v(\xi)\times B)f)\,, \quad x,\xi\in\mathbf{R}^3\,,\ t>0$$

coupled to the system of Maxwell equations

$$\partial_t E + \mathrm{curl}_x\, B = -j_f\,, \qquad \mathrm{div}_x\, E = \rho_f\,,$$
$$\partial_t B - \mathrm{curl}_x\, E = 0 \qquad \mathrm{div}_x\, B = 0\,,$$

where

$$v(\xi) = \frac{\xi}{\sqrt{1+|\xi|^2}}\,, \quad \rho_f(t,x) = \int_{\mathbf{R}^3} f d\xi\,, \quad j_f(t,x) = \int_{\mathbf{R}^3} v(\xi) f d\xi$$

•Equivalently, one can represent the electromagnetic field (E, B) in terms of a distribution of gauge potentials $u \equiv u(t, x, \xi)$ (distribution of Liénard-Wiechert potentials) that satisfies

$$\Box_{t,x} u(t, x, \xi) = f(t, x, \xi)\,, \quad x, \xi \in \mathbf{R}^3\,,\; t > 0$$

•Then (modulo fixing the initial conditions)

$$E = -\partial_t \int_{\mathbf{R}^3} v(\xi) u d\xi - \nabla_x \int_{\mathbf{R}^3} u d\xi\,, \qquad B = \mathrm{curl}_x \int_{\mathbf{R}^3} v(\xi) u d\xi$$

•The electromagnetic potential satisfies the Lorentz gauge

$$\partial_t \int_{\mathbf{R}^3} u d\xi + \mathrm{div}_x \int_{\mathbf{R}^3} v(\xi) u d\xi = 0\,,$$

as a consequence of the continuity equation

$$\partial_t \rho_f + \mathrm{div}_x\, j_f = 0\,.$$

• Then, the relativistic Vlasov-Maxwell system becomes

$$\Box_{t,x} u = f\,, \quad (\partial_t + v(\xi)\cdot\nabla_x) f = \mathrm{div}_\xi (F[u] f)\,, \quad x, \xi \in \mathbf{R}^3\,,\ t > 0$$

$$F[u] = \partial_t \left(A^0 + \int_{\mathbf{R}^3} v(\xi') u d\xi' \right) + \nabla_x \int_{\mathbf{R}^3} u d\xi' \\ - v(\xi) \times \mathrm{curl}_x \left(A^0 + \int_{\mathbf{R}^3} v(\xi') u d\xi' \right)$$

• The initial conditions are

$$u\Big|_{t=0} = \partial_t u\Big|_{t=0} = 0\,, \qquad f\Big|_{t=0} = f^{in}\,,$$

while

$$\Box_{t,x} A^0 = 0\,,\ A^0\Big|_{t=0} = A_I\,,\ \partial_t A^0\Big|_{t=0} = -E^{in}$$

with

$$\mathrm{curl}_x A_I = B^{in}\,, \quad \mathrm{div}_x A_I = 0$$

•The Vlasov-Maxwell system is nonresonant since

$$|v(\xi)| = \frac{|\xi|}{\sqrt{1+|\xi|^2}} < 1 \text{ for each } \xi \in \mathbf{R}^3$$

(the speed of massive particles is less than the speed of light).

•However, the Vlasov-Maxwell system is not UNIFORMLY nonresonant as $|\xi| \to +\infty$.

•The only a priori bounds are

$$\|f(t)\|_{L^p_{x,\xi}} = Const.$$

$$\iint_{\mathbf{R}^3\times\mathbf{R}^3} \sqrt{1+|\xi|^2} f(t,x,\xi)\,dx\,d\xi + \int_{\mathbf{R}^3} (|E|^2 + |B|^2)(t,x)\,dx = Const.$$

(in particular, there is no a priori bound on the support of f.)

The Glassey-Strauss theorem (ARMA, 1986)

Theorem. *Let* $f \in C([0,T) \times \mathbf{R}^3 \times \mathbf{R}^3)$ *and* $E, B \in C^1([0,T) \times \mathbf{R}^3)$ *be a solution of the Vlasov-Maxwell system with* $f^{in} \in C^1_c(\mathbf{R}^3 \times \mathbf{R}^3)$ *and* $E^{in}, B^{in} \in C^2_c(\mathbf{R}^3)$ *s.t.*

$$\mathrm{div}_x\, E^{in} = \int_{\mathbf{R}^3} f^{in} d\xi\,, \quad \mathrm{div}_x\, B^{in} = 0$$

If

$$\overline{\lim_{t\to T^-}}\, \|f(t)\|_{Lip_{x,\xi}} + \|(E,B)(t)\|_{Lip_x} = +\infty$$

then

$$\overline{\lim_{t\to T^-}}\, R_f(t) = +\infty$$

where

$$R_f(t) = \inf\{r > 0 \,|\, f(t,x,\xi) = 0 \text{ for each } x \in \mathbf{R}^3 \text{ and } |\xi| > r\}\,.$$

•Need to gain 1 derivative on the fields, i.e. 2 derivatives on the gauge potential which is given in terms of momentum-averages of u.

•Assume that

$$\Box_{t,x} u = f\,, \quad (\partial_t + v(\xi)\cdot\nabla_x) f = \mathrm{div}_\xi\, g \quad \text{with } f, g \in L^2_{loc}(dtdxd\xi)$$

•Classical velocity averaging gives, for each $\phi \in C^1_c(\mathbf{R}^3)$:

$$\int_{\mathbf{R}^3} f(t,x,\xi)\phi(\xi)d\xi \in H^{1/4}_{loc}(\mathbf{R}_t \times \mathbf{R}^3_x)$$

•One gains one more derivative by the energy estimate for the wave equation, so that

$$\int_{\mathbf{R}^3} u(t,x,\xi)\phi(\xi)d\xi \in H^{1+1/4}_{loc}(\mathbf{R}_t \times \mathbf{R}^3_x)$$

One gains $1 + \frac{1}{4}$ derivatives on momentum-averages of u: NOT ENOUGH

Resonant Velocity Averaging

•In fact one can gain 2 derivatives on momentum-averages of u in the nonresonant case — without gaining more on momentum-averages of f.

Theorem. (Bouchut-G-Pallard, Revistà Mat. Iberoam. 2004) *Let* f, g *in* $L^2_{loc}(\mathbf{R}_t \times \mathbf{R}^N_x \times \mathbf{R}^N_\xi)$ *satisfy*

$$\Box_{t,x} u = f\,, \quad (\partial_t + v(\xi) \cdot \nabla_x) f = D^m_\xi g$$

where $v \in C^\infty(\mathbf{R}^N; \mathbf{R}^N)$ *is nonresonant, i.e. satisfies* $|v(\xi)| < 1$*. Then, for each* $m \in \mathbf{N}$ *and each* $\phi \in C^\infty_c(\mathbf{R}^N_\xi)$*, one has*

$$\int_{\mathbf{R}^3} u(t, x, \xi)\phi(\xi) d\xi \in H^2_{loc}(\mathbf{R} \times \mathbf{R}^N)\,.$$

•This generalizes the fact that $\Box_{t,x}$ acts as an elliptic operator on the nullspace of the transport operator whenever $|v(\xi)| < 1$.

•Sketch of proof: Set $T_\xi^\pm = \partial_t \pm v(\xi)\cdot\nabla_x$ and consider the 2nd order differential operator

$$Q_\xi = \Box_{t,x} - \lambda T_\xi^- T_\xi^+$$

•First, one checks that

$$\begin{aligned} Q_\xi u = f - \lambda T_\xi^- \Box_{t,x}^{-1} D_\xi^m g &= f - \lambda D_\xi^m \Box_{t,x}^{-1} T_\xi^- g - \lambda \Box_{t,x}^{-1} [T_\xi^-, D_\xi^m] g \\ &= f - \lambda D_\xi^m \Box_{t,x}^{-1} T_\xi^- g - \lambda \Box_{t,x}^{-1} D_\xi^m v(\xi)\cdot\nabla_x g \\ &= a + d_\xi^m b \in L^2_{loc}(dtdxd\xi) + D_\xi^m L^2_{loc}(dtdxd\xi) \end{aligned}$$

•Then we observe that, for $\xi \in \operatorname{supp}\phi$ and λ such that

$$\sup_{\xi\in\operatorname{supp}\phi} |v(\xi)| < \lambda < 1$$

the operator Q_ξ is elliptic for each $\xi \in \operatorname{supp}\phi$.

•More precisely, denoting by $q_\xi(\omega,k)$ the symbol of Q_ξ, one has

$$\sup_{\xi\in\operatorname{supp}\phi}\left|D_\xi^m\left(\frac{1}{q_\xi(\omega,k)}\right)\right|\le\frac{C_m}{\omega^2+|k|^2}$$

where C_m may depends on m but is uniform in ξ.

•Then

$$\int\hat{u}\phi(\xi)d\xi=\int\frac{\hat{a}}{q_\xi(\omega,k)}\phi(\xi)d\xi+(-1)^m\int D_\xi^m\left(\frac{\phi(\xi)}{q_\xi(\omega,k)}\right)\hat{b}d\xi$$

with $\hat{a}$ and $\hat{b}\in L^2_{\omega,k,\xi}$ has H^2-decay in ω,k. QED

Remarks:

•First, one easily checks that all the assumptions in the Theorem above cannot be dispensed with.

•That one gains 2 derivatives is special to L^2, because $\square_{t,x}^{-1}$ gains 1 derivative in (t,x) by the energy estimate for the wave equation.

In L^p with $1 < p < \infty$, $\square_{t,x}^{-1}$ gains $1-(N-1)|\frac{1}{2}-\frac{1}{p}|$ derivatives in (t,x) (Peral, JFA 1980) whenever $|\frac{1}{2}-\frac{1}{p}| \leq \frac{1}{N-1}$. Using this result and the Mihlin-Hörmander theorem on L^p multipliers, the same proof as above shows that

$\int_{\mathbf{R}^N} u(t,x,\xi)\phi(\xi)d\xi \in W_{loc}^{1+\gamma,p}(\mathbf{R}\times\mathbf{R}^N)$ with $\gamma = 1-(N-1)|\frac{1}{2}-\frac{1}{p}|$

This result suggests a gain of 1 derivative in L^∞ — and in any case does not apply in L^∞.

A division lemma

•Let Y be the forward fundamental solution of $\Box_{t,x}$, i.e.

$$\Box_{t,x} Y = \delta_{(0,0)}\,, \quad \operatorname{supp} Y \subset \mathbf{R}_+ \times \mathbf{R}^N$$

— for instance, in space dimension $N = 3$, one has

$$Y(t,x) = \mathbf{1}_{t\geq 0} \tfrac{\delta(t-|x|)}{4\pi t}$$

•Recall that the Lorentz boosts

$$L_j = x_j \partial_t + t\partial_{x_j}\,, \qquad j = 1, \ldots, N$$

commute with the d'Alembertian

$$[\Box_{t,x}, L_j] = 0\,, \text{ so that } L_j Y = 0\,, \quad j = 1, \ldots, N\,.$$

Lemma. *Let $N \geq 2$. For each $\xi \in \mathbf{R}^N$, there exists $b_{ij}^k \equiv b_{ij}^k(t,x,\xi)$ is C^∞ on $\mathbf{R}^{1+N} \setminus 0$ and homogeneous of degree $-k$ in (t,x) such that*

(i) the homogeneous distribution $b_{ij}^2 Y$ of degree $-1-N$ on $\mathbf{R}^{1+N} \setminus 0$ has null residue at the origin, and

(ii) there exists an extension of $b_{ij}^2 Y$ as a homogeneous distribution of degree $-1-N$ on $\mathbf{R}^{1+N} \setminus 0$, still denoted $b_{ij}^2 Y$, that satisfies

$$\partial_{ij} Y = T^2(b_{ij}^0 Y) + T(b_{ij}^1 Y) + b_{ij}^2 Y\,, \quad i,j = 0,\ldots,N\,.$$

Here T is the advection operator $T = \partial_t + v(\xi) \cdot \nabla_x$.

•The null residue condition reads

$$\int_{\mathbf{S}^2} b_{ij}^2(1,y)d\sigma(y) = 0 \quad \text{if } N = 3\,,$$
$$\int_{|y|\le 1} b_{ij}^2(1,y)\frac{dy}{\sqrt{1-|y|^2}} = 0 \quad \text{if } N = 2\,.$$

•Next we use the above lemma to estimate the derivatives of the fields

$$\partial_{ij}\int m(\xi)u(t,x,\xi)d\xi = \sum_{k=0}^{2}\int m(\xi)\left(b_{ij}^{k-l}Y \star T^l(\mathbf{1}_{t\ge 0}f)(t,x,\xi)\right)d\xi$$

Here, m denotes any C^∞ function with compact support that coincides with either 1 or each component of $v(\xi)$ on the ξ-support of f.

The idea is to use the Vlasov equation to compute $T^l(\mathbf{1}_{t\ge 0}f)$ and integrate by parts to bring the ξ-derivatives to bear on b_{ij}^{k-l} and m.

•Worst term is for $l = 0$:

$$\int m(\xi)\left(b_{ij}^2 Y \star (\mathbf{1}_{t\geq 0} f)(t,x,\xi)\right) d\xi\,.$$

By using the null residue condition, write this term as

$$\int m(\xi)\int_\epsilon^t \int_{\mathbf{S}^2} b_{ij}^2(1,\omega,\xi) f(t-s,x-s\omega,\xi)\frac{d\sigma(\omega)}{4\pi s} ds d\xi$$
$$+\int m(\xi)\int_0^\epsilon \int_{\mathbf{S}^2} b_{ij}^2(1,\omega,\xi)\frac{f(t-s,x-s\omega,\xi)-f(t,x,\omega)}{4\pi s} d\sigma(\omega) ds d\xi$$

•If the size $R_f(t)$ of the ξ-support of f is bounded on $[0,T)$, i.e. if $\overline{\lim}_{t\to T^-} R_f(t) < +\infty$, this term is bounded by

$$C(1+\ln_+(t\|\nabla_x f\|_{L^\infty}))$$

•Hence, the Lipschitz semi-norm $N(t) = \|\nabla_{x,\xi} f(t,\cdot,\cdot)\|_{L^\infty}$ satisfies a logarithmic Gronwall inequality

$$N(t) \leq N(0) + \int_0^t (1 + \ln_+ N(s))N(s)ds\,, \quad t \in [0,T)\,.$$

Therefore N is uniformly bounded on $[0,T]$, which implies in turn that the fields $(E,B) \in L^\infty(0,T;W^{1,\infty}(\mathbf{R}^3))$.