On the statistics of free-path lengths for the periodic Lorentz gas

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The Lorentz kinetic model

Lorentz (1905) proposed to describe the motion of electrons in metals by a kinetic model

$$(\partial_t + v \cdot \nabla_x + \tfrac{1}{m} F(t,x) \cdot \nabla_v) f(t,x,v) = N_{at} r_{at}^2 |v| \mathcal{C}(f(t,x,\cdot))(v)$$

with collision integral

$$\mathcal{C}(\phi)(v) = \int_{\substack{|\omega|=1 \\ \omega \cdot v > 0}} \Big(\phi(\mathcal{R}_\omega v) - \phi(v) \Big) \cos(v,\omega) d\omega$$

where $\mathcal{R}_\omega$ is the specular reflection:

$$\mathcal{R}_\omega(v) = v - 2(v \cdot \omega)\omega$$

•Notation: $f \equiv f(t,x,v)$ is the unknown electronic phase-space density;

•$F \equiv F(t,x)$ is the electric force;

•m is the mass of the electron;

•N_{at} is the density of metallic atoms;

•r_{at} is the radius of metallic atoms.

•The Lorentz kinetic model with $F \equiv 0$ has been rigorously justified by Gallavotti (1972) as the Boltzmann-Grad limit of a system of point particles undergoing specular collisions with randomly distributed circular obstacles, in space dimension 2.

(Collisions involving two — or more than two — point particles are neglected, which explains why the limiting model is linear)

Periodic distribution of obstacles

•is the Lorentz kinetic model still valid in the case of a periodic configuration of circular obstacles?

•can the absorption coefficient appearing in the Lorentz kinetic model

$$N_{at} r_{at}^2 \int_{\substack{|\omega|=1 \\ \omega \cdot v > 0}} \cos(v, \omega) d\omega$$

be interpreted as the reciprocal mean-free path of the point particles (neglecting inter-particle collisions)?

The free-path length

Assume that $|v| = 1$ for $0 < r < \frac{1}{2}$, consider

$$Z_r = \{x \in \mathbf{R}^D \,|\, \mathrm{dist}(x, \mathbf{Z}^D) > r\}\,, \qquad Y_r = Z_r/\mathbf{Z}^D$$

and set

$$\Gamma_r^+ = \{(x, v) \in \partial Y_r \times \mathbf{S}^{D-1} \,|\, v \cdot n_y > 0\}$$

For $x \in Y_r$, define

$$\tau_r(x, v) = \inf\{t > 0 \,|\, x + tv \in \partial Y_r\}$$

this definition can be extended by continuity along trajectories to the case of $(x, v) \in \Gamma_r^+$.

•First notion of mean free-path: defined as the mean of $\tau_r(x, v)$ for (x, v) uniformly distributed on $Y_r \times \mathbf{S}^{D-1}$. Unfortunately

$$\frac{1}{|Y_r||\mathbf{S}^{D-1}|}\int_{Y_r \times \mathbf{S}^{D-1}} \tau_r(x, v)\,dx\,dv = +\infty$$

•Another notion of mean free-path: defined as the mean of $\tau_r(x, v)$ for (x, v) on Γ_r^+ with distribution proportional to

$$d\nu_r(x, v) = (v \cdot n_x) dS(x) dv \text{ on } \Gamma_r^+$$

One finds (explicit formula due to Santalò)

$$m.f.p. = \frac{1}{\nu_r(\Gamma_r^+)} \int_{\Gamma_r^+} \tau_r(x, v) d\nu_r(x, v)$$

$$= \frac{|Y_r||\mathbf{S}^{D-1}|}{\nu_r(\Gamma_r^+)} = \frac{r^{-(D-1)}}{|\mathbf{B}^{D-1}|} - \frac{|\mathbf{B}^D| r}{|\mathbf{B}^{D-1}|}$$

If $D = 3$ the reciprocal leading order term as $r \to 0$ coincides with the damping coefficient in the Lorentz kinetic model ($N_{at} = 1$, $r_{at} = r$):

$$N_{at} r_{at}^2 \int_{\substack{|\omega|=1 \\ \omega \cdot v > 0}} \cos(\omega, v) d\omega = \pi r^2$$

Distribution of exit times

If $m \in C(\mathbf{S}^{D-1})$ and $m > 0$, let $\mathbf{P}_m$ be the probability measure on $Y_r \times \mathbf{S}^{D-1}$ proportional to $m(v)dxdv$; define

$$\Phi_r^m(t) = \mathbf{P}_m(\{(x,v) \in Y_r \times \mathbf{S}^{D-1} \,|\, \tau_r(x,v) > t\})$$

Theorem. *For each positive $m \in C(\mathbf{S}^{D-1})$ there exist two positive constants C_m and C'_m such that, for all $r \in (0, \frac{1}{2})$ and $t > 1/r^{D-1}$*

$$\frac{C_m}{tr^{D-1}} \leq \Phi_r^m(t) \leq \frac{C'_m}{tr^{D-1}}$$

- Bourgain-Golse-Wennberg (CMP 1998):

  upper bound + lower bound for $D = 2$

- Golse-Wennberg (M2AN 2000):

  lower bound for all D + simulations

Asymptotic evaluation of Φ_r^m

In the case of space dimension $D=2$:

Theorem. *For each positive $m \in C(\mathbf{S}^1)$,*

$$\overline{\lim_{\epsilon\to 0^+}} \frac{1}{|\ln\epsilon|}\int_\epsilon^{1/4} \Phi_r^m\left(\frac{t}{r}\right)\frac{dr}{r} = \frac{1}{\pi^2 t} + O\left(\frac{1}{t^2}\right)$$

$$\underline{\lim_{\epsilon\to 0^+}} \frac{1}{|\ln\epsilon|}\int_\epsilon^{1/4} \Phi_r^m\left(\frac{t}{r}\right)\frac{dr}{r} = \frac{1}{\pi^2 t} + O\left(\frac{1}{t^2}\right)$$

as $t \to +\infty$.

•Proof by Caglioti-Golse (CMP 2003) using a new partition of $\mathbf{T}^2$ + ergodic properties of continued fractions

•later, Boca-Zaharescu announced a proof of explicit formulae conjectured by Dahlqvist (see Nonlinearity 1997) for

$$\lim_{r\to 0^+} \Phi_r^1(t/r)$$

using the same partition of $\mathbf{T}^2$ + Farey trees

Idea of the proof

•On $\mathbf{T}^2 \setminus (\{0\} \times [0, R])$ with $0 < R < 1$ consider the linear flow with irrational slope $0 < \alpha < 1$; each orbit has length $\in \{l_A < l_B < l_C\}$. (Proved by Blank-Krikorian, Int. J. Math. 1993)

•Let p_n/q_n be the sequence of convergents in

$$\alpha = \frac{1\,|}{|\,a_1} + \frac{1\,|}{|\,a_2} + \ldots + \frac{1\,|}{|\,a_n} + \ldots$$

and let $d_n = |q_n\alpha - p_n|$.

| If $d_n \leq R < d_{n-1}$ with $k = -[\frac{R-d_{n-1}}{d_n}]$ | then

$$l_A = q_n\,, \quad l_B = q_{n-1} + kq_n\,, \quad l_C = l_A + l_B$$

•The union of orbits of type A, resp. B, C, defines a 3-set partition of $\mathbf{T}^2 \setminus$ slit that gives an $O(r)$-approximation of

$$\text{Prob}\{x \in Y_r \,|\, \tau_r(x, v) > t\} \qquad \text{a.e. in } v$$

•Average in v and r using ergodicity of the Gauss map $x \mapsto 1/\alpha - [1/\alpha]$ on $(0, 1)$.

Applications to kinetic theory

•Set $\Omega_n = \{x \in \mathbf{R}^2 \,|\, \mathrm{dist}(x, \frac{1}{n}\mathbf{Z}^2) > \frac{1}{n^2}\}$, $n > 2$.

•For $\rho^{in} \equiv \rho^{in}(x) \geq 0$, let f_n be the solution to

$$\begin{aligned} \partial_t f_n(t,x,v) + v \cdot \nabla_x f_n(t,x,v) &= 0 \text{ on } \quad \Omega_n \times \mathbf{S}^1 \\ f(t,x,v) - f(t,x,\mathcal{R}_{n_x} v) &= 0 \text{ on } \partial\Omega_n \times \mathbf{S}^1 \\ f_n\Big|_{t=0} &= \rho^{in}\Big|_{\Omega_n} \end{aligned}$$

(with $\mathcal{R}_{n_x} v = v - 2(v \cdot n_x) n_x$).

Theorem. *For some ρ^{in} periodic in each variable x_1, x_2 with period 1, neither f_n nor any subsequence thereof converges in L^∞ weak-* to the solution of the Lorentz kinetic equation*

$(\partial_t + v \cdot \nabla_x) f = \mathcal{C}(f)$ *on* $\mathbf{R}^2 \times \mathbf{S}^1$, $\quad f\Big|_{t=0} = \rho^{in}$
with collision integral

$$\mathcal{C}(\phi)(v) = \int_{\substack{|\omega|=1 \\ \omega \cdot v > 0}} \Big(\phi(\mathcal{R}_\omega v) - \phi(v)\Big) \cos(v,\omega) d\omega$$

Method of proof

•Spectral arguments show that the solution of the Lorentz equation satisfies

$$\left\| f(t,\cdot,\cdot) - \int_{\mathbf{T}^2} \rho^{in}(x)dx \right\|_{L^2_{x,v}} = O(e^{-\gamma t})\|\rho^{in}\|_{L^2}$$

for some $\gamma > 0$ independent of ρ^{in};

•By the lower bound on the distribution of free-path lengths, if $f_n \to f$ in L^∞ weak-*,

$$\int_{\mathbf{T}^2\times\mathbf{S}^1} f(t,x,v)dxdv \geq \frac{C_1}{t}\int_{\mathbf{T}^2} \rho^{in}(x)dx$$

for some $C_1 > 0$ independent of ρ^{in};

•Both inequalities are incompatible: choose a family ρ^{in}_δ such that

$$\|\rho^{in}_\delta\|_{L^2} = 1\,, \qquad \lim_{\delta\to 0^+}\int_{\mathbf{T}^2} \rho^{in}_\delta(x)dx = 0$$