Fluid Dynamics from Kinetic Equations

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LECTURE 6

FROM BOLTZMANN TO INCOMPRESSIBLE NAVIER STOKES

CONVERGENCE PROOF

The incompressible Navier-Stokes scaling

•Consider the dimensionless Boltzmann equation in the incompressible Navier-Stokes scaling, i.e. with $\boxed{\mathrm{St} = \pi \mathrm{Kn} = \epsilon \ll 1}$:

$$\epsilon \partial_t F_\epsilon + v \cdot \nabla_x F_\epsilon = \frac{1}{\epsilon} \mathcal{B}(F_\epsilon, F_\epsilon)$$

•Start with an initial data that is a perturbation of some uniform Maxwellian (say, the centered reduced Gaussian $M = M_{1,0,1}$) with Mach number $\boxed{\mathrm{Ma} = O(\epsilon)}$:

$$F_\epsilon^{in} = M_{1,0,1} + \epsilon f_\epsilon^{in}$$

•Example 1: pick $u^{in} \in L^2(\mathbf{R}^3)$ a divergence-free vector field; then the distribution function

$$F_\epsilon^{in}(x,v) = M_{1,\epsilon u^{in}(x),1}(v)$$

is of the type above.

•Example 2: If in addition $\theta^{in} \in L^2 \cap L^\infty(\mathbf{R}^3)$, the distribution function

$$F_\epsilon^{in}(x,v) = M_{1-\epsilon\theta^{in}(x), \frac{\epsilon u^{in}(x)}{1-\epsilon\theta^{in}(x)}, \frac{1}{1-\epsilon\theta^{in}(x)}}(v)$$

is also of the type above. (Pick $0 < \epsilon < \frac{1}{\|\theta^{in}\|_{L^\infty}}$, then $1 - \epsilon\theta^{in} > 0$ a.e.).

• Problem : to prove that

$$\frac{1}{\epsilon}\int_{\mathbf{R}^3} vF_\epsilon(t,x,v)dv \to u(t,x) \text{ as } \epsilon \to 0$$

where u solves the incompressible Navier-Stokes equations

$$\partial_t u + u \cdot \nabla_x u + \nabla_x p = \nu \Delta_x u\,, \quad \text{div}_x\, u = 0$$
$$u\Big|_{t=0} = u^{in}$$

The viscosity ν is given by the same formula as in the Chapman-Enskog expansion.

A priori estimates

•The only a priori estimate satisfied by renormalized solutions to the Boltzmann equation is the DiPerna-Lions entropy inequality:

$$H(F_\epsilon|M)(t) + \frac{1}{\epsilon^2}\int_0^t \int_{\mathbf{R}^3} \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} d(F_\epsilon)|(v-v_*)\cdot\omega| dv dv_* d\omega dx ds \leq H(F_\epsilon^{in}|M)$$

•Notation:

$$H(f|g) = \iint_{\mathbf{R}^3\times\mathbf{R}^3} \left(f \ln\left(\frac{f}{g}\right) - f + g\right) dx dv \quad \text{(relative entropy)}$$

$$d(f) = \tfrac{1}{4}(f'f'_* - ff_*) \ln\left(\frac{f'f'_*}{ff_*}\right) \quad \text{(dissipation integrand)}$$

•Introduce the relative number density, and the relative number density fluctuation:

$$G_\epsilon = \frac{F_\epsilon}{M}\,, \qquad g_\epsilon = \frac{F_\epsilon - M}{\epsilon M}$$

•Pointwise inequalities: one easily checks that

$$\begin{aligned}(\sqrt{G_\epsilon} - 1)^2 &\le C(G_\epsilon \ln G_\epsilon - G_\epsilon + 1)\\ \left(\sqrt{G'_\epsilon G'_{\epsilon *}} - \sqrt{G_\epsilon G_{\epsilon *}}\right)^2 &\le \tfrac{1}{4}(G'_\epsilon G'_{\epsilon *} - G_\epsilon G_{\epsilon *}) \ln\left(\frac{G'_\epsilon G'_{\epsilon *}}{G_\epsilon G_{\epsilon *}}\right)\\ &= d(G_\epsilon)\end{aligned}$$

•Notice that $Z \ln Z - Z + 1 \sim \frac{1}{2}(Z-1)^2$ near $Z = 1$.

•Express that the initial data is a perturbation of the uniform Maxwellian M with Mach number Ma $= O(\epsilon)$:

$$H(F_\epsilon^{in}) \le C^{in}\epsilon^2$$

•With the DiPerna-Lions entropy inequality, and the pointwise inequalities above, one gets the following uniform in ϵ bounds

$$\iint_{\mathbf{R}^3\times\mathbf{R}^3} (\sqrt{G_\epsilon}-1)^2 M\,dv\,dx \le C\epsilon^2$$

$$\int_0^{+\infty}\int_{\mathbf{R}^3}\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2}\left(\sqrt{G'_\epsilon G'_{\epsilon *}}-\sqrt{G_\epsilon G_{\epsilon *}}\right)^2 d\mu\,dx\,dt \le C\epsilon^4$$

where μ is the collision measure:

$$d\mu(v,v_*,\omega) = |(v-v_*)\cdot\omega|\,d\omega M_* dv_* M dv$$

References

•C. Bardos, F. G., D. Levermore: CPAM 1993 (Stokes limit+stationary incompressible Navier-Stokes, assuming local conservation of momentum + nonlinear compactness estimate)

•P.-L. Lions, N. Masmoudi: ARMA 2000 (evolution Navier-Stokes under the same assumptions)

•C.B.-F.G.-D.L.: ARMA 2000 + F.G.-D.L.: CPAM 2002 (local conservation of momentum and energy PROVED in the hydrodynamic limit, for the acoustic and Stokes limits)

•L. Saint-Raymond (CPDEs 2002 + Ann. Sci. ENS 2003): complete derivation of incompressible Navier-Stokes from BGK

•F.G.+L.S.-R.: (Invent. Math. 2004) complete derivation of incompressible Navier-Stokes from Boltzmann for cutoff Maxwell molecules

•L.S.-R. (Bull. Sci. Math. 2002 + ARMA 2003): complete derivation of dissipative solutions to incompressible Euler from BGK and Boltzmann equations

•N. M.+L.S.-R. (CPAM 2003) Stokes limit for the boundary value problem

The BGL Program (CPAM 1993)

•Let $F_\epsilon^{in} \geq 0$ be any sequence of measurable functions satisfying the entropy bound $H(F_\epsilon^{in}|M) \leq C^{in}\epsilon^2$, and let F_ϵ be a renormalized solution of the scaled Boltzmann equation

$$\epsilon\partial_t F_\epsilon + v \cdot \nabla_x F_\epsilon = \frac{1}{\epsilon}\mathcal{B}(F_\epsilon, F_\epsilon)\,, \quad F_\epsilon\Big|_{t=0} = F_\epsilon^{in}$$

•Let $g_\epsilon \equiv g_\epsilon(x,v)$ be such that $G_\epsilon := 1 + \epsilon g_\epsilon \geq 0$ a.e.. We say that $g_\epsilon \to g$ entropically at rate ϵ as $\epsilon \to 0$ iff

$$g_\epsilon \to g \text{ in } w - L^1_{loc}(Mdvdx)\,, \text{ and } \frac{1}{\epsilon^2}H(MG_\epsilon|M) \to \tfrac{1}{2}\iint g^2 Mdvdx$$

Theorem. *Assume that*

$$\frac{F_\epsilon^{in}(x,v) - M(v)}{\epsilon M(v)} \to u^{in}(x) \cdot v$$

entropically at rate ϵ*. Then the family of bulk velocity fluctuations*

$$\frac{1}{\epsilon} \int_{\mathbf{R}^3} v F_\epsilon dv$$

is relatively compact in $w - L^1_{loc}(dtdx)$ *and each of its limit points as* $\epsilon \to 0$ *is a Leray solution of*

$$\partial_t u + \mathrm{div}_x(u \otimes u) + \nabla_x p = \nu \Delta_x u\,, \quad \mathrm{div}_x\, u = 0\,, \quad u\Big|_{t=0} = u^{in}$$

with viscosity given by the formula

$$\nu = \tfrac{1}{10} \int A : \hat{A} M dv$$

where $\hat{A} = \mathcal{L}^{-1} A$

Method of proof

•Renormalization: pick $\gamma \in C^\infty(\mathbf{R}_+)$ a nonincreasing function such that

$$\gamma\Big|_{[0,3/2]} \equiv 1\,, \quad \gamma\Big|_{[2,+\infty)} \equiv 0\,; \qquad \text{set } \widehat{\gamma}(z) = \frac{d}{dz}((z-1)\gamma(z))$$

•The Boltzmann equation is renormalized (relatively to M) as follows:

$$\partial_t(g_\epsilon\gamma_\epsilon) + \frac{1}{\epsilon}v\cdot\nabla_x(g_\epsilon\gamma_\epsilon) = \frac{1}{\epsilon^3}\widehat{\gamma}_\epsilon\mathcal{Q}(G_\epsilon, G_\epsilon)$$

where $\gamma_\epsilon := \gamma(G_\epsilon)$, $\widehat{\gamma}_\epsilon = \widehat{\gamma}(G_\epsilon)$ and $\mathcal{Q}(G,G) = M^{-1}\mathcal{B}(MG, MG)$

• Continuity equation Renormalized solutions of the Boltzmann equation satisfy the local conservation of mass:

$$\epsilon\partial_t\langle g_\epsilon\rangle + \mathrm{div}_x\langle vg_\epsilon\rangle = 0$$

• The entropy bound implies that

$$(1+|v|^2)g_\epsilon \text{ is relatively compact in } w - L^1_{loc}(dtdx; L^1(Mdv))$$

Modulo extraction of a subsequence

$$g_\epsilon \to g \text{ in } w - L^1_{loc}(dtdx; L^1(Mdv))$$

and hence $\langle vg_\epsilon\rangle \to \langle vg\rangle =: u$ in $w - L^1_{loc}(dtdx)$; passing to the limit in the continuity equation leads to the incompressibility condition

$$\mathrm{div}_x\, u = 0$$

•High velocity truncation: pick $K > 6$ and set $K_\epsilon = K|\ln \epsilon|$; for each function $\xi \equiv \xi(v)$, define $\xi_{K_\epsilon}(v) = \xi(v)\mathbf{1}_{|v|^2 \leq K_\epsilon}$

•Multiply both sides of the scaled, renormalized Boltzmann equation by each component of v_{K_ϵ}:

$$\partial_t \langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle + \mathrm{div}_x \, \mathbf{F}_\epsilon(A) + \nabla_x \frac{1}{\epsilon} \langle \tfrac{1}{3} |v|^2_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle = \mathbf{D}_\epsilon(v)$$

where

$$\mathbf{F}_\epsilon(A) = \tfrac{1}{\epsilon} \langle A_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle \,, \quad \mathbf{D}_\epsilon(v) = \frac{1}{\epsilon^3} \langle\!\langle v_{K_\epsilon} \hat{\gamma}_\epsilon (G'_\epsilon G'_{\epsilon *} - G_\epsilon G_{\epsilon *}) \rangle\!\rangle$$

•Notation: with $d\mu = |(v - v_*) \cdot \omega| M dv M_* dv_* d\omega$ (collision measure)

$$\langle \phi \rangle = \int_{\mathbf{R}^3} \phi(v) M dv \,, \quad \langle\!\langle \psi \rangle\!\rangle = \iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} \psi(v, v_*, \omega) d\mu$$

•The plan is to prove that, modulo extraction of a subsequence

$\langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle \to \langle vg \rangle =: u$	in $w - L^1_{loc}(\mathbf{R}_+ \times \mathbf{R}^3)$
$\mathbf{D}_\epsilon(v) \to 0$	in $L^1_{loc}(\mathbf{R}_+ \times \mathbf{R}^3)$ and
$P\,(\mathrm{div}_x\, \mathbf{F}_\epsilon(A)) \to P\,\mathrm{div}_x(u^{\otimes 2}) - \nu \Delta_x u$	in $w{-}L^1_{loc}(dt, W^{-s,1}_{x,loc})$

for $s > 1$ as $\epsilon \to 0$, where P is the Leray projection.

Conservation defects $\to 0$

(as in FG+DL, CPAM 2002, but simpler)

Proposition. *$D_\epsilon(v) \to 0$ in $L^1_{loc}(\mathbf{R}_+ \times \mathbf{R}^3)$ as $\epsilon \to 0$.*

•Split the conservation defect as

$$\mathbf{D}^1_\epsilon(v) = \frac{1}{\epsilon^3}\left\langle\!\!\left\langle v_{K_\epsilon}\widehat{\gamma}_\epsilon\left(\sqrt{G'_\epsilon G'_{\epsilon *}} - \sqrt{G_\epsilon G_\epsilon}\right)^2\right\rangle\!\!\right\rangle$$

$$\mathbf{D}^2_\epsilon(v) = \frac{1}{\epsilon^3}\left\langle\!\!\left\langle v_{K_\epsilon}\widehat{\gamma}_\epsilon\left(\sqrt{G'_\epsilon G'_{\epsilon *}} - \sqrt{G_\epsilon G_\epsilon}\right)\sqrt{G_\epsilon G_\epsilon}\right\rangle\!\!\right\rangle$$

That $\mathbf{D}^1_\epsilon(v) \to 0$ comes from the entropy production estimate.

•Setting $\Xi_\epsilon = \frac{1}{\epsilon^2}\left(\sqrt{G'_\epsilon G'_{\epsilon *}} - \sqrt{G_\epsilon G_\epsilon}\right)\sqrt{G_\epsilon G_\epsilon}$, we further split $\mathbf{D}^2_\epsilon(v)$ into

$$\mathbf{D}^2_\epsilon(v) = -\frac{2}{\epsilon}\left\langle\!\left\langle v\mathbf{1}_{|v|^2>K_\epsilon}\widehat{\gamma}_\epsilon \Xi_\epsilon\right\rangle\!\right\rangle + \frac{2}{\epsilon}\left\langle\!\left\langle v\widehat{\gamma}_\epsilon(1-\widehat{\gamma}_{\epsilon *}\widehat{\gamma}'_\epsilon\widehat{\gamma}_{\epsilon *})\Xi_\epsilon\right\rangle\!\right\rangle$$
$$+\frac{1}{\epsilon}\left\langle\!\left\langle (v+v_1)\widehat{\gamma}_\epsilon\widehat{\gamma}_{\epsilon *}\widehat{\gamma}'_\epsilon\widehat{\gamma}_{\epsilon *}\Xi_\epsilon\right\rangle\!\right\rangle$$

The first and third terms are easily mastered by the entropy production bound and classical estimates on the tail of Gaussian distributions.

•Sending the second term to 0 requires knowing that

$(1+|v|)\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)^2$ is uniformly integrable on $[0,T]\times K\times\mathbf{R}^3$

for the measure $dtdxMdv$, for each $T>0$ and each compact $K\subset\mathbf{R}^3$.

Asymptotic behavior of the momentum flux

Proposition. *Denoting by* Π *the* $L^2(Mdv)$*-orthogonal projection on* $\ker \mathcal{L}$

$$\mathbf{F}_\epsilon(A) = 2\left\langle A\left(\Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)^2\right\rangle - 2\left\langle \hat{A}\frac{1}{\epsilon^2}\mathcal{Q}(\sqrt{G_\epsilon},\sqrt{G_\epsilon})\right\rangle + o(1)_{L^1_{loc}(dtdx)}$$

The proof is based upon splitting $\mathbf{F}_\epsilon(A)$ as

$$\mathbf{F}_\epsilon(A) = \left\langle A_{K_\epsilon}\gamma_\epsilon\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)^2\right\rangle + \frac{2}{\epsilon}\left\langle A_{K_\epsilon}\gamma_\epsilon\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right\rangle$$

using the uniform integrability of $(1+|v|)\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)^2$ and the following consequence thereof

$$\lim_{\epsilon\to 0}\left\|\frac{\sqrt{G_\epsilon}-1}{\epsilon} - \Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right\|_{L^2_{loc}(dtdx;L^2((1+|v|)Mdv))} = 0$$

•By the entropy production estimate, modulo extraction of a subsequence

$$\frac{1}{\epsilon^2}\left(\sqrt{G'_\epsilon G'_{\epsilon *}} - \sqrt{G_\epsilon G_\epsilon}\right) \to q$$

and passing to the limit in the scaled, renormalized Boltzmann equation leads to

$$\iint_{\mathbf{R}^3\times\mathbf{S}^2} q|(v-v_*)\cdot\omega| M_* dv_* d\omega = v\cdot\nabla_x g = \tfrac{1}{2}A : \nabla_x u + \text{ odd in } v$$

•Since $\frac{\sqrt{G_\epsilon}-1}{\epsilon} \simeq \frac{1}{2}g_\epsilon\gamma_\epsilon$, one gets

$$\mathbf{F}_\epsilon(A) = A(\langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon\rangle) - \nu(\nabla_x u + (\nabla_x u)^T) + o(1)_{w-L^1_{loc}(dtdx)}$$

(remember that $A(u) = u\otimes u - \frac{1}{3}|u|^2 I$), while

$$\langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon\rangle \to u \text{ in } w - L^1_{loc}(\mathbf{R}_+ \times \mathbf{R}^3)$$

Strong compactness

•In order to pass to the limit in the quadratic term $A(\langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle)$, one needs strong-$L^2$ compactness of $\langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle$.

•Velocity averaging provides strong compactness in the x-variable:

$$\left(\frac{\sqrt{\epsilon^\alpha + G_\epsilon} - 1}{\epsilon}\right)^2 \text{ is locally uniformly integrable on } \mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3$$

$$(\epsilon \partial_t + v \cdot \nabla_x) \frac{\sqrt{\epsilon^\alpha + G_\epsilon} - 1}{\epsilon} \text{ is bounded in } L^1_{loc}(\mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3)$$

This implies that, for each $T > 0$ and each compact $K \subset \mathbf{R}^3$,

$$\int_0^T \int_K |\langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle(t, x + y) - \langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle(t, x)|^2 dx dt \to 0$$

$$\text{as } |y| \to 0\,, \text{ uniformly in } \epsilon > 0$$

•It remains to get compactness in the time variable. Observe that

$\partial_t P\langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon\rangle = P(\mathbf{D}_\epsilon(v) - \mathrm{div}_x\, \mathbf{F}_\epsilon(A))$ is bounded in $L^1_{loc}(dt, W^{-s,1}_{x,loc})$

(Recall that $\mathbf{D}_\epsilon(v) \to 0$ while $\mathbf{F}_\epsilon(A)$ is bounded in $L^1_{loc}(dtdx)$).

•Together with the compactness in the x-variable that follows from velocity averaging, this implies that

$$P\langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon\rangle \to u \text{ in } L^2_{loc}(dtdx)$$

•Recall that $\langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon\rangle \to u$ in $w - L^2_{loc}(dtdx)$; we do not seek to prove that

$$\langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon\rangle \to u \text{ strongly in } L^2_{loc}(dtdx)$$

Filtering acoustic waves (PLL+NM, ARMA 2002)

•Instead, we prove that

$$P\,\mathrm{div}_x\left(\langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon\rangle^{\otimes 2}\right) \to P\,\mathrm{div}_x\left(u^{\otimes 2}\right) \text{ in } \mathcal{D}'(\mathbf{R}_+^* \times \mathbf{R}^3) \text{ as } \epsilon \to 0$$

•Observe that

$$\epsilon\partial_t\langle v_{K_\epsilon} g_\epsilon\gamma_\epsilon\rangle + \nabla_x\langle \tfrac{1}{3}|v|^2_{K_\epsilon} g_\epsilon\gamma_\epsilon\rangle \to 0 \text{ in } L^1_{loc}(\mathbf{R}_+; W^{-1,1}_{loc}(\mathbf{R}^3))$$

$$\epsilon\partial_t\langle \tfrac{1}{3}|v|^2_{K_\epsilon} g_\epsilon\gamma_\epsilon\rangle + \mathrm{div}_x\langle \tfrac{5}{3} v_{K_\epsilon} g_\epsilon\gamma_\epsilon\rangle \to 0 \text{ in } L^1_{loc}(\mathbf{R}_+; W^{-1,1}_{loc}(\mathbf{R}^3))$$

as $\epsilon \to 0$.

•Setting $\nabla_x \pi_\epsilon = (I-P)\langle v_{K_\epsilon} g_\epsilon\gamma_\epsilon\rangle$, the system above becomes

$$\epsilon\partial_t\nabla_x\pi_\epsilon + \nabla_x\langle \tfrac{1}{3}|v|^2_{K_\epsilon} g_\epsilon\gamma_\epsilon\rangle \to 0 \text{ in } L^1_{loc}(\mathbf{R}_+; W^{-1,1}_{loc}(\mathbf{R}^3))$$

$$\epsilon\partial_t\langle \tfrac{1}{3}|v|^2_{K_\epsilon} g_\epsilon\gamma_\epsilon\rangle + \tfrac{5}{3}\Delta_x\pi_\epsilon \to 0 \text{ in } L^1_{loc}(\mathbf{R}_+; W^{-1,1}_{loc}(\mathbf{R}^3))$$

•Straightforward computation shows that

$$\operatorname{div}_x\left((\nabla_x\pi_\epsilon)^{\otimes 2}\right)=\tfrac{1}{2}\nabla_x\left(|\nabla_x\pi_\epsilon|^2-\tfrac{3}{5}\langle\tfrac{1}{3}|v|^2_{K_\epsilon}g_\epsilon\gamma_\epsilon\rangle^2\right)+o(1)_{L^1_{loc}(dtdx)}$$

•On the other hand, because the limiting velocity field is divergence-free, one has

$$\nabla_x\pi_\epsilon\to 0\text{ in }w-L^2_{loc}(dtdx)\text{ as }\epsilon\to 0$$

•Splitting

$$P\operatorname{div}_x\left(\langle v_{K_\epsilon}g_\epsilon\gamma_\epsilon\rangle^{\otimes 2}\right)=P\operatorname{div}_x\left((P\langle v_{K_\epsilon}g_\epsilon\gamma_\epsilon\rangle)^{\otimes 2}\right)+P\operatorname{div}_x\left((\nabla_x\pi_\epsilon)^{\otimes 2}\right)$$
$$+2P\operatorname{div}_x\left(P\langle v_{K_\epsilon}g_\epsilon\gamma_\epsilon\rangle\vee\nabla_x\pi_\epsilon\right)$$

The last two terms vanish with ϵ while the first converges to $P\operatorname{div}_x(u^{\otimes 2})$ since $P\langle v_{K_\epsilon}g_\epsilon\gamma_\epsilon\rangle\to u$ strongly in $L^2_{loc}(dtdx)$.

The key estimates (as in FG+LSR, Invent. Math. 2004)

Proposition. *For each* $T > 0$ *and each compact* $K \subset \mathbf{R}^3$*, the family* $\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)^2 (1+|v|)$ *is uniformly integrable on* $[0,T] \times K \times \mathbf{R}^3$ *for the measure* $dtdxMdv$*.*

Idea no. 1 We first prove that $\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right)^2 (1+|v|)$ is uniformly integrable on $[0,T] \times K \times \mathbf{R}^3$ for the measure $dtdxMdv$ in the v-variable .

•We say that $\phi_\epsilon \equiv \phi_\epsilon(x,y) \in L^1_{x,y}(d\mu(x)d\nu(y))$ is uniformly integrable in the y-variable for the measure $d\mu(x)d\nu(y)$ iff

$$\int \sup_{\nu(A)<\alpha} \int_A |\phi_\epsilon(x,y)| d\nu(y) d\mu(x) \to 0 \text{ as } \alpha \to 0 \text{ uniformly in } \epsilon$$

•Start from the formula

$$\mathcal{L}\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right) = \epsilon\mathcal{Q}\left(\frac{\sqrt{G_\epsilon}-1}{\epsilon}, \frac{\sqrt{G_\epsilon}-1}{\epsilon}\right) - \frac{1}{\epsilon}\mathcal{Q}\left(\sqrt{G_\epsilon}, \sqrt{G_\epsilon}\right)$$

and use the following estimate (G.-Perthame-Sulem, ARMA 1988)

$$\|\mathcal{Q}(f,f)\|_{L^2((1+|v|)^{-1}Mdv)} \leq C\|f\|_{L^2(Mdv)}\|f\|_{L^2((1+|v|)Mdv)}$$

to arrive at

$$\left(1 - O(\epsilon)\left\|\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right\|_{L^2(Mdv)}\right)\left\|\frac{\sqrt{G_\epsilon}-1}{\epsilon} - \Pi\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right\|_{L^2((1+|v|)Mdv)}$$
$$\leq O(\epsilon)_{L^2_{t,x}} + O(\epsilon)\left\|\frac{\sqrt{G_\epsilon}-1}{\epsilon}\right\|^2_{L^2(Mdv)}$$

•This estimates tells us that $\frac{\sqrt{G_\epsilon}-1}{\epsilon}$ stays close to its associated infinitesimal Maxwellian $\Rightarrow$ regularity+decay in v.

Idea no. 2 Use a L^1-variant of velocity averaging (FG+LSR, CRAS 2002).

Lemma. *Let $f_n \equiv f_n(x,v)$ be a bounded sequence in $L^1_{loc}(dxdv)$ such that $v \cdot \nabla_x f_n$ is also bounded in $L^1_{loc}(dxdv)$. Assume that f_n is locally uniformly integrable in v. Then*

- *f_n is locally uniformly integrable (in x, v)*

- *for each test function $\phi \in L^\infty_{comp}(\mathbf{R}^D_v)$, the sequence of averages*

$$\rho_n^\phi(x) = \int f_n(x,v)\phi(v)dv$$

is relatively compact in $L^1_{loc}(dx)$.

•Let's prove that the sequence of averages ρ_n^ϕ is locally uniformly integrable (LSR, CPDEs 2002). WLOG, assume that f_n and $\phi \geq 0$.

•Let $\chi \equiv \chi(t, x, v)$ be the solution to

$$\partial_t \chi + v \cdot \nabla_x \chi = 0\,, \qquad \chi(0, x, v) = \mathbf{1}_A(x)$$

Clearly, $\chi(t, x, v) = \mathbf{1}_{A_x(t)}(v)$ (χ takes the values 0 and 1 only). On the other hand,

$$|A_x(t)| = \int \chi(t, x, v)dv = \int \mathbf{1}_A(x - tv)dv = \frac{|A|}{t^D}$$

•Remark: this is the basic dispersion estimate for the free transport equation.

•Set $g_n(x,v) = f_n(x,v)\phi(v)$, and $v\cdot\nabla_x g_n(x,v) = \phi(v)(v\cdot\nabla_x f_n(x,v))$: $h_n(x,v) =$: g_n and h_n are bounded in $L^1_{x,v}$ and g_n is uniformly integrable in v.

•Observe that (hint: integrate by parts the 2nd integral in the r.h.s.)

$$\int_A \int g_n dv dx = \int \int_{A_x(t)} g_n dv dx - \int_0^t \iint h_n(x,v)\chi(s,x,v) dx dv ds$$

The second integral on the r.h.s. is $O(t) \sup \|h_n\|_{L^1_{x,v}} < \epsilon$ by choosing $t > 0$ small enough. For that value of t, $|A_x(t)| \to 0$ as $|A| \to 0$, hence the first integral on the r.h.s. vanishes by uniform integrability in v.