Fluid Dynamics from Kinetic Equations

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Fields Institute, 24–26 March 2004

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LECTURE 5:

GLOBAL SOLUTIONS AND GLOBAL LIMITS

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I. INTRODUCTION TO THE BGL PROGRAM

The program laid out by Bardos, Golse, and Levermore [BGL93] is (1) to identify those fluid dynamical systems that can be obtained through a moment-based formal derivation like those outlined in the last lecture, and (2) to give a full mathematical justification of those formal derivations.

In order to carry out this program, we must make precise: (1) the notion of solution for the Boltzmann equation, and (2) the notion of solution for the fluid dynamical systems. Ideally, these solutions should be global while the bounds should be physically natural.

Global Solutions

We therefore work in the setting of DiPerna-Lions renormalized solutions for the Boltzmann equation, in the settings of L^2 solutions for the acoustic and Stokes systems, and in the setting of Leray solutions for the Navier-Stokes system. These theories have the virtues of considering physically natural classes of initial data, and consequently, of yielding global solutions. There is no such theory for the Euler system, so one must work in the setting of local classical soltions for now.

One of the central goals of the BGL program is to connect the DiPerna-Lions theory of renormalized solutions of the Boltzmann equation to the Leray theory of weak solutions of the incompressible Navier-Stokes system.

II. DIPERNA-LIONS SOLUTIONS FOR BOLTZMANN

The DiPerna-Lions theory gives the existence of a global weak solution to a class of formally equivalent initial-value problems that are obtained by multiplying the Boltzmann equation by $\Gamma'(G)$, where Γ' is the derivative of an admissible function Γ:

$$\Big(\tau_\epsilon \partial_t + v \cdot \nabla_x\Big)\Gamma(G) = \frac{1}{\epsilon}\,\Gamma'(G)\mathcal{Q}(G,G)\,,$$
$$G(v,x,0) = G^{in}(v,x) \geq 0\,.$$

This is the so-called renormalized Boltzmann equation. A differentiable function $\Gamma : [0,\infty) \to \mathbb{R}$ is called *admissible* if for some constant $C_\Gamma < \infty$ it satisfies

$$\Big|\Gamma'(Z)\Big| \leq \frac{C_\Gamma}{\sqrt{1+Z}} \quad \text{for every } Z \geq 0\,.$$

The solutions lie in $C([0,\infty); w\text{-}L^1(M\,\mathrm{d}v\,\mathrm{d}x))$, where the prefix “$w$-” on a space indicates that the space is endowed with its weak topology.

DiPerna-Lions Solutions

We say that $G \geq 0$ is a weak solution of the renormalized Boltzmann equation provided that it is initially equal to G^{in}, and that for every $Y \in L^\infty(\mathrm{d}v; C^1(\mathbb{T}^D))$ and every $[t_1, t_2] \subset [0, \infty)$ it satisfies

$$\begin{aligned} \tau_\epsilon \int_{\mathbb{T}^D} \langle \Gamma(G(t_2))\, Y \rangle \,\mathrm{d}x - \tau_\epsilon \int_{\mathbb{T}^D} \langle \Gamma(G(t_1))\, Y \rangle \,\mathrm{d}x & \\ - \int_{t_1}^{t_2} \int_{\mathbb{T}^D} \langle \Gamma(G)\, v \cdot \nabla_x Y \rangle \,\mathrm{d}x\,\mathrm{d}t & \\ = \frac{1}{\epsilon} \int_{t_1}^{t_2} \int_{\mathbb{T}^D} \left\langle \Gamma'(G) \mathcal{Q}(G, G)\, Y \right\rangle dx\, dt \,. & \end{aligned}$$

If G is such a weak solution of for one such Γ with $\Gamma' > 0$, and if G satisfies certain bounds, then it is a weak solution for every admissible Γ. Such solutions are called *renormalized solutions* of the Boltzmann equation.

DiPerna-Lions Theorem - 1

Theorem. 1 (DiPerna-Lions Renormalized Solutions) *Let b satisfy*

$$\lim_{|v|\to\infty} \frac{1}{1+|v|^2} \int_{\mathbb{S}^{D-1}\times K} b(\omega, v_* - v)\,\mathrm{d}\omega\,\mathrm{d}v_* = 0$$

for every compact $K \subset \mathbb{R}^D$.

Given any initial data G^{in} in the entropy class

$$E(M\mathrm{d}v\,\mathrm{d}x) = \left\{ G^{in} \geq 0 \,:\, H(G^{in}) < \infty \right\},$$

there exists at least one $G \geq 0$ in $C([0,\infty); w\text{-}L^1(M\mathrm{d}v\,\mathrm{d}x))$ that for every admissible function Γ is a weak solution of renormalized Boltzmann equation.

This solution satisfies a weak form of the local conservation law of mass

$$\tau_\epsilon \epsilon\, \partial_t \langle G \rangle + \nabla_x \cdot \langle v\, G \rangle = 0\,.$$

Moreover, there exists a martix-valued distribution W *such that* $W\,\mathrm{d}x$ *is nonnegative definite measure and* G *and* W *satisfy a weak form of the local conservation law of momentum*

$$\tau_\epsilon \partial_t \langle v\, G \rangle + \nabla_x \cdot \langle v \otimes v\, G \rangle + \nabla_x \cdot W = 0\,,$$

and for every $t > 0$*, the global energy equality*

$$\int_{\mathbb{T}^D} \langle \tfrac{1}{2}|v|^2 G(t) \rangle\, \mathrm{d}x + \int_{\mathbb{T}^D} \tfrac{1}{2}\, \mathrm{tr}(W(t))\, \mathrm{d}x = \int_{\mathbb{T}^D} \langle \tfrac{1}{2}|v|^2 G^{in} \rangle\, \mathrm{d}x\,,$$

and the global entropy inequality

$$H(G(t)) + \int_{\mathbb{T}^D} \tfrac{1}{2}\, \mathrm{tr}(W(t))\, \mathrm{d}x + \frac{1}{\epsilon\, \tau_\epsilon} \int_0^t R(G(s))\, \mathrm{d}s \le H(G^{in})\,.$$

DiPerna-Lions Theorem - 3

Remarks: DiPerna-Lions renormalized solutions are very weak — much weaker than standard weak solutions. They are not known to satisfy many properties that one would formally expect to be satisfied by solutions of the Boltzmann equation. In particular, the theory does not assert either the local conservation of momentum, the global conservation of energy, the global entropy equality, or even a local entropy inequality; nor does it assert the uniqueness of the solution.

III. LERAY SOLUTIONS FOR NAVIER-STOKES

The DiPerna-Lions theory has many similarities with the Leray theory of global weak solutions of the initial-value problem for Navier-Stokes type systems. For the Navier-Stokes system with mean zero initial data, we set the Leray theory in the following Hilbert spaces of vector- and scalar-valued functions:

$$
\begin{aligned}
\mathbf{H}_v &= \left\{ w \in L^2(dx; \mathbb{R}^D) \;:\; \nabla_x \cdot w = 0 \,, \int w \, dx = 0 \right\}, \\
\mathbf{H}_s &= \left\{ \chi \in L^2(dx; \mathbb{R}) \;:\; \int \chi \, dx = 0 \right\}, \\
\mathbf{V}_v &= \left\{ w \in \mathbf{H}_v \;:\; \int |\nabla_x w|^2 \, dx < \infty \right\}, \\
\mathbf{V}_s &= \left\{ \chi \in \mathbf{H}_s \;:\; \int |\nabla_x \chi|^2 \, dx < \infty \right\}.
\end{aligned}
$$

Let $\mathbf{H} = \mathbf{H}_v \oplus \mathbf{H}_s$ and $\mathbf{V} = \mathbf{V}_v \oplus \mathbf{V}_s$.

Leray Theorem

Theorem. 2 (Leray Solutions) *Given any initial data* $(u^{in}, \theta^{in}) \in \mathbf{H}$*, there exists at least one* $(u, \theta) \in C([0,\infty); w\text{-}\mathbf{H}) \cap L^2(dt; \mathbf{V})$ *that is a weak solution of the Navier-Stokes system in the sense that for every* $(w, \chi) \in \mathbf{H} \cap C^1(\mathbb{T}^D)$ *and every* $[t_1, t_2] \subset [0, \infty)$ *it satisfies*

$$\begin{aligned} \int w \cdot u(t_2)\, \mathrm{d}x - \int w \cdot u(t_1)\, \mathrm{d}x - \int_{t_1}^{t_2}\!\!\int \nabla_x w : (u \otimes u)\, \mathrm{d}x\, \mathrm{d}t \\ = -\nu \int_{t_1}^{t_2}\!\!\int \nabla_x w : \nabla_x u\, \mathrm{d}x\, \mathrm{d}t\,, \end{aligned}$$

$$\begin{aligned} \int \chi\, \theta(t_2)\, \mathrm{d}x - \int \chi\, \theta(t_1)\, \mathrm{d}x - \int_{t_1}^{t_2}\!\!\int \nabla_x \chi \cdot (u\, \theta)\, \mathrm{d}x\, \mathrm{d}t \\ = -\tfrac{2}{D+2}\, \kappa \int_{t_1}^{t_2}\!\!\int \nabla_x \chi \cdot \nabla_x \theta\, \mathrm{d}x\, \mathrm{d}t\,. \end{aligned}$$

Moreover, for every $t > 0$, (u, θ) satisfies the dissipation inequalities

$$\int \tfrac{1}{2}|u(t)|^2 \mathrm{d}x + \int_0^t \int \nu |\nabla_x u|^2 \mathrm{d}x\, \mathrm{d}s \leq \int \tfrac{1}{2}|u^{in}|^2 dx\,,$$
$$\int \tfrac{D+2}{4}|\theta(t)|^2 \mathrm{d}x + \int_0^t \int \kappa |\nabla_x \theta|^2 \mathrm{d}x\, \mathrm{d}s \leq \int \tfrac{D+2}{4}|\theta^{in}|^2 \mathrm{d}x\,.$$

Remarks: By arguing formally from the Navier-Stokes system, one would expect these inequalities to be equalities. However, that is not asserted by the Leray theory. Also, as was the case for the DiPerna-Lions theory, the Leray theory does not assert uniqueness of the solution.

A Variant of Leray Theory

Because the role of the above dissipation inequalities is to provide a-priori estimates, the existence theory also works if they are replaced by the single dissipation inequality

$$\int \tfrac{1}{2}|u(t)|^2 + \tfrac{D+2}{4}|\theta(t)|^2 dx + \int_0^t\!\!\int \nu|\nabla_x u|^2 + \kappa|\nabla_x \theta|^2 dx\, ds \\ \leq \int \tfrac{1}{2}|u^{in}|^2 + \tfrac{D+2}{4}|\theta^{in}|^2 dx\,.$$

It is this version of the Leray theory that we will obtain in the limit.

IV. VELOCITY AVERAGING

Both the DiPerna-Lions theory and the Leray theory have proofs based on compactness arguments. (This is why they do not yield uniqueness.) Both construct a sequence of approximate solutions and then use compactness to extract a converging subsequence. Both need compactness in a strong topology in order to pass to the limit in nonlinear terms. For the Leray theory strong compactness follows by showing compactness in $C([0,\infty); w\text{-}\mathbf{H})$ and in $w\text{-}L^2(\mathrm{d}t; w\text{-}\mathbf{V})$, and using the fact

$$C([0,\infty); w\text{-}\mathbf{H}) \cap w\text{-}L^2(\mathrm{d}t; w\text{-}\mathbf{V}) \longrightarrow L^2(\mathrm{d}t; \mathbf{H}) \quad \text{is continuous}\,.$$

The compactness in $w\text{-}L^2(\mathrm{d}t; w\text{-}\mathbf{V})$ follows from the dissipation estimate. The spatial regularity in the DiPerna-Lions theory required a new tool.

An Example

Being hyperbolic, the transport operator $v \cdot \nabla_x$ propagates singularities along characteristics. Therefore, at first sight it seems hopeless that one might obtain any regularizing effect from the free streaming part of the Boltzmann equation — or of any other similar kinetic model. One can think of the following elementary example:

Example. Let $f \in L^2(\mathrm{d}t)$; define $F(v, x) = f(v_2 x_1 - v_1 x_2)$ for almost every $x, v \in \mathbb{R}^2$. Clearly, $F \in L^2_{loc}(\mathrm{d}v\,\mathrm{d}x)$ and $v \cdot \nabla_x F = 0$. However, because f can be any function in $L^2(\mathrm{d}t)$, $F \notin H^s_{loc}(\mathrm{d}v\,\mathrm{d}x)$ for any $s > 0$, although $v \cdot \nabla_x F \in L^2_{loc}(\mathrm{d}v\,\mathrm{d}x)$.

Regularity by Averaging

The key to obtaining regularizing effects from the transport operator $v \cdot \nabla_x$ is to seek those effects not on the kinetic density itself, but on velocity averages thereof — in other words, on the macroscopic densities. Here is the prototype of all Velocity Averaging results. Given $G \in L^2(\mathrm{d}v\,\mathrm{d}x)$ let F solve

$$F + v \cdot \nabla_x F = G\,.$$

Clearly $G \in L^2(\mathrm{d}v\,\mathrm{d}x)$. Let $\phi \in L^2(\mathrm{d}v)$ and define

$$\rho(x) = \int_{\mathbb{R}^D} \phi(v)\, F(v,x)\,\mathrm{d}v\,.$$

Velocity Averaging states that ρ gains regularity. This is easily shown as follows.

Regularity by Averaging

Let $\widehat{F}$, $\widehat{G}$, and $\widehat{\rho}$ denote respectively the Fourier transforms of F, G, and ρ in the x variable. Then

$$\widehat{\rho}(\xi) = \int_{\mathbb{R}^D} \frac{\phi(v)\,\widehat{G}(v,\xi)}{1 + iv\cdot\xi}\,\mathrm{d}v\,.$$

We need to study how $\widehat{\rho}(\xi)$ decays for $|\xi|$ large. By the Cauchy-Schwarz inequality

$$|\widehat{\rho}(\xi)|^2 \leq \frac{1}{m(\xi)} \int_{\mathbb{R}^D} |\widehat{G}(v,\xi)|^2 \mathrm{d}v\,,$$

with

$$\frac{1}{m(\xi)} = \int_{\mathbb{R}^D} \frac{|\phi(v)|^2}{1 + |v\cdot\xi|^2}\,\mathrm{d}v\,.$$

We must show that $m(\xi) \to \infty$ as $\xi \to \infty$.

We express $1/m(\xi)$ as

$$\frac{1}{m(\xi)} = \int_{\mathbb{R}^D} \frac{|\phi(v)|^2}{1 + |v \cdot \omega|^2 |\xi|^2} \, \mathrm{d}v \,,$$

where $\omega = \xi/|\xi| \in \mathbb{S}^{D-1}$ for every nonzero $\xi \in \mathbb{R}^D$. The integral is a decreasing family indexed by $|\xi|$ of continuous functions of ω. This family vanishes pointwise in ω as $|\xi| \to \infty$ by dominated convergence. By Dini's theorem, it vanishes uniformly over $\omega \in \mathbb{S}^{D-1}$, and therefore $m(\xi) \to \infty$ as $|\xi| \to \infty$. Because the family

$$\int_{\mathbb{R}^D} |\widehat{\rho}(\xi)|^2 \, m(\xi) \, \mathrm{d}\xi \leq \iint_{\mathbb{R}^D \times \mathbb{R}^D} |\widehat{G}(v, \xi)|^2 \, \mathrm{d}v \, \mathrm{d}\xi$$

is bounded by Plancherel's theorem, ρ is relatively compact in $L^2_{loc}(\mathrm{d}x)$ (by a variant of Rellich's compactness theorem).

Regularity by Averaging in L^p

Because the operator $(I + v \cdot \nabla_x)^{-1}$ (which maps G on the solution F of $F + v \cdot \nabla_x F = G$) is a contraction mapping on both $L^1(\mathrm{d}v\, \mathrm{d}x)$ and $L^\infty(\mathrm{d}v\, \mathrm{d}x)$, by interpolation the Velocity Averaging result above also holds in L^p for every $p \in (1, \infty)$ by interpolation. However, it fails in L^1, as the following example shows. (It also fails in L^∞).

Example. Consider G_ϵ, a bounded family of $L^1(\mathrm{d}v\, \mathrm{d}x)$ such that $G_\epsilon \to \delta(v - v^*) \otimes \delta(x)$ weakly, where $|v^*| = 1$. For each ϵ, let F_ϵ be the solution to $F_\epsilon + v \cdot \nabla_x F_\epsilon = G_\epsilon$, which is given by

$$F_\epsilon(v, x) = \int_0^\infty e^{-t} G_\epsilon(v, x - vt)\, \mathrm{d}t\,.$$

Then both F_ϵ and $v \cdot \nabla_x F_\epsilon$ are bounded in $L^1(\mathrm{d}v\,\mathrm{d}x)$ and

$$\rho_\epsilon(x) = \int_{\mathbb{R}^D} F_\epsilon(v,x)\,\mathrm{d}v = \int_0^\infty \int_{\mathbb{R}^D} e^{-t} G_\epsilon(v, x - vt)\,\mathrm{d}v\,\mathrm{d}t$$

so that, for any test function $\psi \in C_c(\mathbb{R}^D)$,

$$\int_{\mathbb{R}^D} \rho_\epsilon(x)\,\psi(x)\,\mathrm{d}x \to \int_0^\infty e^{-t}\psi(v^* t) dt\,,$$

as $\epsilon \to 0$. Hence, ρ_ϵ converges weakly to a density supported on the half-line $v^*\mathbb{R}_+ \subset \mathbb{R}^D$. This means that the family ρ_ϵ is not relatively compact in $L^1_{loc}(\mathrm{d}x)$.

Regularity by Averaging in L^1

The last example rests on the possible build-up of concentrations in F_ϵ and $v \cdot \nabla_x G_\epsilon$. If such concentrations are ruled out, the same interpolation argument as above entails the following L^1 variant of Velocity Averaging.

Theorem. 3 (Golse-Lions-Perthame-Sentis) *Let F_ϵ be a family of measurable functions on $\mathbb{R}^D \times \mathbb{R}^D$ such that, for each compact subset K of $\mathbb{R}^D$, both families F_ϵ and $v \cdot \nabla_x F_\epsilon$ are uniformly integrable on $\mathbb{R}^D \times K$. Then the family ρ_ϵ defined by*

$$\rho_\epsilon(x) = \int_{\mathbb{R}^D} F_\epsilon(v, x)\,\mathrm{d}v$$

is relatively compact in $L^1_{loc}(\mathrm{d}x)$.

V. SURVEY OF THE BGL PROGRAM

The main result of [BGL93] for the Navier-Stokes limit is to recover the motion equation for a discrete-time version of the Boltzmann equation assuming the DiPerna-Lions solutions satisfy the local conservation of momentum and with the aid of a mild compactness assumption.

This result falls short of the goal in five respects.

- First, the heat equation was not treated because the v^3 terms in the heat flux could not be controlled.

- Second, local momentum conservation was assumed because DiPerna-Lions solutions are not known to satisfy the local conservation law of momentum (or energy) that one would formally expect.

- Third, unnatural technical assumptions were made on the Boltzmann collision kernel.

- Fourth, the discrete-time case was treated in order to avoid having to control the time regularity of the acoustic modes.

- Finally, a mild compactness assumption was required to pass to the limit in certain nonlinear terms.

In recent works all of these shortcomings have been overcome.

Bardos-Golse-Levermore

Bardos, Golse, and Levermore [BGL98] recover the acoustic and the Stokes limits for the Boltzmann equation for cutoff collision kernels that arise from Maxwell potentials. In doing so, they control the energy flux and establish the local conservation laws of momentum and energy in the limit. The scaling they used was not optimal, essentially requiring

$$\frac{\delta_\epsilon}{\epsilon} \to 0 \quad \text{rather than} \quad \delta_\epsilon \to 0 \quad \text{for the acoustic limit}\,,$$
$$\frac{\delta_\epsilon}{\epsilon^2} \to 0 \quad \text{rather than} \quad \frac{\delta_\epsilon}{\epsilon} \to 0 \quad \text{for the Stokes limit}\,.$$

Lions-Masmoudi - 1

Lions and Masmoudi [LM00] recover the Navier-Stokes motion equation with the aid of only the local conservation of momentum assumption and the nonlinear compactness assumption that where made in [BGL93]. However, they do not recover the heat equation and they retain the same unnatural technical assumptions made in [BGL93] on the collision kernel.

There were two key new ingredients in their work. First, they were able to control the time regularity of the acoustic modes. Second, they were able to prove that the contribution of the acoustic modes to the limiting motion equation is just an extra gradiant term that can be incorporated into the pressure term.

Lions-Masmoudi - 2

They also recover the Stokes motion equation without the local conservation of momentum assumption and with essentially optimal scaling. However, they do not recover the heat equation and they retain the same unnatural technical assumptions made in [BGL93] on the collision kernel.

There are two reasons they do not recover the heat equation. First, it is unknown whether or not DiPerna-Lions solutions satisfy a local energy conservation law. Second, even if local energy conservation were assumed, the techniques they used to control the momentum flux would fail to control the heat flux.

Golse-Levermore - 1

Golse and Levermore [GL01] recover the acoustic and Stokes systems. They make natural assumptions on the collision kernel that include those classically derived from hard potentials.

For the Stokes limit they recover both the motion and heat equations with a near optimal scaling.

For the acoustic limit the scaling they used was not optimal, essentially requiring

$$\frac{\delta_\epsilon}{\epsilon^{\frac{1}{2}}} \longrightarrow 0 \qquad \text{rather than} \qquad \delta_\epsilon \longrightarrow 0\,.$$

Golse-Levermore - 2

There were two key new ingredients in this work. First, they control the local momentum and energy conservation defects of the DiPerna-Lions solutions with dissipation rate estimates that allowed them to recover these local conservation laws in the limit. Second, they also control the heat flux with dissipation rate estimates.

Because they treat the linear Stokes case, they do not face the need either to control the acoustic modes or for a compactness assumption, both of which are used to pass to the limit in the nonlinear terms in [LM00].

Golse-Saint Raymond

Without making any nonlinear compactness hypothesis, Golse-Saint Raymond [GSR03] recover the Navier-Stokes system for the Boltzmann equation with cutoff collision kernels that arise from Maxwell potentials. Their major breakthrough was the development of a new L^1 averaging lemma to prove the compactness assumption. This was extracted from Saint Raymond [SR98] where she recovered the Navier-Stokes limit for the BGK model. Their proof also employs key elements from [LM00] and [GL01].

Recently they have extend their result to the hard sphere collision kernel. It is this most recent result that will be described in the next lecture.

Some Major Open Problems in the Program:

- the acoustic limit with optimal scaling ($\delta_\epsilon \longrightarrow 0$);
- all limits for collision kernels from soft potentials;
- dominant-balance Stokes, Navier-Stokes, and Euler limits;
- uniform in time results (compressible Stokes system).