Fluid Dynamics from Kinetic Equations

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LECTURE 3

HILBERT AND CHAPMAN-ENSKOG EXPANSIONS

Hilbert's asymptotic solution

•Start from the dimensionless Boltzmann equation in the compressible Euler scaling $\mathsf{St} = 1$ and $\pi\mathsf{Kn} = \epsilon$:

$$\partial_t F_\epsilon + v \cdot \nabla_x F_\epsilon = \frac{1}{\epsilon}\mathcal{B}(F_\epsilon, F_\epsilon)$$

•Hilbert's expansion is a method for constructing solutions of the scaled Boltzmann equation above in $C^\infty(\mathbf{R}_+ \times \mathbf{R}_x^3 \times \mathbf{R}_v^3)[[\epsilon]]$ (i.e. formal power series in ϵ with coefficients that are smooth in (t, x, v):

$$F_\epsilon(t, x, v) = \sum_{n \geq 0} \epsilon^n F_n(t, x, v)$$

•The convergence radius of the above power series may very well be 0.

The linearized collision operator

•The leading order of Hilbert's expansion should be a local Maxwellian (see lecture 1) whose parameters are governed by Euler's system.

•This suggests to study the linearization at a Maxwellian M of Boltzmann's collision integral

$$\begin{aligned}\mathcal{L}_M\phi &= -2M^{-1}\mathcal{B}(M, M\phi)\\ &= \iint_{\mathbf{R}^3\times\mathbf{S}^2}(\phi+\phi_*-\phi'-\phi'_*)|(v-v_*)\cdot\omega|d\omega M_* dv_*\end{aligned}$$

WLOG, assume that $M = M_{1,0,1}$ (the centered, reduced Gaussian)

• Translation/Scaling invariance of $\mathcal{B}$ Denote by τ the action of $\mathbf{R}^3$ on functions by translations, and by m that of $\mathbf{R}_+^*$ by scaling:

$$\tau_w\phi(v) := \phi(v-w)\,, \quad m_a\phi(v) = \frac{1}{a^3}\phi\left(\frac{1}{a}v\right)$$

Then

$$\mathcal{B}(\tau_w F, \tau_w F) = \tau_w \mathcal{B}(F,F)\,; \quad \mathcal{B}(m_a F, m_a F) = a m_a \mathcal{B}(F,F)$$

• In particular, since $M_{\rho,u,\theta} = \rho\tau_u m_{\sqrt{\theta}} M_{1,0,1}$, one has

$$\mathcal{L}_{M_{\rho,u,\theta}}(\tau_u m_{\sqrt{\theta}}\phi) = \rho\sqrt{\theta}\tau_u m_{\sqrt{\theta}}\mathcal{L}_{M_{1,0,1}}\phi$$

•Notice that the operator $\mathcal{L}_M$ takes the form

$$(\mathcal{L}_M\phi)(v) = \lambda_M(|v|)\phi(v) - (\mathcal{K}_M\phi)(v)$$

where $\lambda(|v|)$ is the collision frequency, while $\mathcal{K}_M$ is an integral operator

$$\lambda(|v|) = 2\pi\int_{\mathbf{R}^3} |v - v_*|M_* dv_*\,, \quad \mathcal{K}_M\phi = \mathcal{K}_{1,M} - \mathcal{K}_{2,M}$$

and where the operators $\mathcal{K}_{1,M}$ and $\mathcal{K}_{2,M}$ are defined by

$$\mathcal{K}_{1,M}\phi = 2\iint_{\mathbf{R}^3\times\mathbf{S}^2} \phi'|(v - v_*)\cdot\omega| d\omega M_* dv_*$$
$$\mathcal{K}_{2,M}\phi = 2\pi\int_{\mathbf{R}^3} \phi_*|v - v_*|M_* dv_*$$

Lemma. (Hilbert 1912) *The operator $\mathcal{K}_{1,M}$ is compact on $L^2(Mdv)$.*

•Since $\mathcal{K}_{2,M}$ is also compact on $L^2(Mdv)$, Hilbert's lemma implies that

Theorem. *The operator $\mathcal{L}_M$ is a nonnegative, unbounded self-adjoint Fredholm operator on $L^2(Mdv)$ with domain $L^2(\lambda(|v|)^2 Mdv)$. Further, its nullspace is the set of collision invariants, i.e.*

$$\ker \mathcal{L}_M = \operatorname{span}\{1, v_1, v_2, v_3, |v|^2\}\,.$$

Moreover, there exists $c_0 > 0$ such that, for each $\phi \in L^2(\lambda(|v|)Mdv)$:

$$\phi \perp \ker \mathcal{L}_M \Rightarrow \int_{\mathbf{R}^3} \phi \mathcal{L}_M \phi M dv \ge c_0 \int_{\mathbf{R}^3} \phi^2 \lambda(|v|) M dv\,.$$

Finally, there exists $c_1 > 1$ such that

$$\tfrac{1}{c_1}(1+|v|) \le \lambda(|v|) \le c_1(1+|v|)$$

A nonlinear variant of Hilbert's lemma

Theorem. (P.-L. Lions 1993) *The gain term in Boltzmann's integral*

$$\mathcal{B}_+(F,F) = \iint F'F'_*|(v-v_*)\cdot\omega|d\omega dv_*$$

maps $L^2_{comp}(\mathbf{R}^3)$ *continuously into* $H^1(\mathbf{R}^3)$.

•Here is the very elegant proof found by Bouchut-Desvillettes: parametrize the solutions to the collision relations

$$v' + v'_* = v + v_*\,, \quad |v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2$$

as follows:

$$v' = \tfrac{1}{2}(v+v_*) + \tfrac{1}{2}|v-v_*|\sigma\,, \quad v'_* = \tfrac{1}{2}(v+v_*) + \tfrac{1}{2}|v-v_*|\sigma$$

where σ runs through $\mathbf{S}^2$.

The two parametrizations of the collision relations

•A straightforward change of variables shows that

$$\mathcal{B}_+(F,F)(v) = 2\iint F\left(\tfrac{v+v_*}{2}+\tfrac{|v-v_*|}{2}\sigma\right)F\left(\tfrac{v+v_*}{2}-\tfrac{|v-v_*|}{2}\sigma\right)|v-v_*|d\sigma dv_*$$

•Compute the Fourier transform of $\mathcal{B}_+(F,F)$ by the pre- to post-collision change of variables:

$$\widehat{\mathcal{B}(F,F)}(\xi) = 2\iiint FF_* e^{-i\xi\cdot\left(\frac{v+v_*}{2}+\frac{|v-v_*|}{2}\sigma\right)}|v-v_*|d\sigma dv_* dv$$

$$= 2\iint F(v)F(v_*)e^{-i\frac{v+v_*}{2}\xi\cdot v}\left(\int e^{-i\xi\cdot\frac{|v-v_*|}{2}\sigma}|v-v_*|d\sigma\right)|v-v_*|dv_* dv$$

•Compute the inner integral in spherical coordinates with polar axis $\mathbf{R}\xi$:

$$\int e^{-i\xi\cdot\frac{|v-v_*|}{2}\sigma}|v-v_*|d\sigma = 2\pi\int_0^\pi e^{-i\frac{|\xi||v-v_*|}{2}\cos\theta}\sin\theta d\theta$$

$$= \frac{8\pi}{|\xi||v-v_*|}\sin\frac{|\xi||v-v_*|}{2}$$

•Setting $z = \frac{v+v_*}{2}$ and $w = \frac{v-v*}{2}$

$$|\xi||\widehat{\mathcal{B}(F,F)}(\xi) = 64\pi \iint |\widehat{F(\cdot + w)F(}\cdot - w)(\xi)\,\mathrm{sin}(|\xi||w|)dw$$

By Cauchy-Schwarz and the Plancherel identity,

$$\begin{aligned} \||\xi||\widehat{\mathcal{B}(F,F)}\|^2_{L^2_\xi} &\leq 64\pi \int \frac{dw}{(1+|w|)^{3+0}} \\ &\times (2\pi)^3 \iint F(z+w)^2 F(z-w)^2 (1+|w|)^{3+0} dz dw \\ &\leq C \iint F(v)^2 F(v_*)^2 (1+|v-v_*|)^{3+0} dv dv_* \end{aligned}$$

Hence

$$\|\mathcal{B}(F,F)\|_{\dot{H}^1} \leq C \left\| F(1+|v|)^{\frac{3+0}{2}} \right\|^2_{L^2}$$

•Fredholm's alternative: Consider the (integral) equation $\mathcal{L}_M\phi = \psi$. Either

- $\psi \perp \ker \mathcal{L}_M$ $\Rightarrow$ there exists a unique solution $\phi_0 \perp \ker \mathcal{L}_M$ (denoted by $\phi_0 = cL_M^{-1}\psi$); all solutions are of the form $\phi_0 + n$ with $n \in \ker \mathcal{L}_M$;

- otherwise, there exists no solution ϕ to the above equation.

•Example: For $M = M_{1,0,1}$, consider the vector field B and the tensor field A defined by

$$A(v) = v^{\otimes 2} - \frac{1}{3}|v|^2 I\,, \quad B(v) = \frac{1}{2}v(|v|^2 - 5)$$

Notice that $A \perp \ker \mathcal{L}_M$, $B \perp \ker \mathcal{L}_M$ and $A \perp B$; there exist $\mathcal{L}_M^{-1}A \perp \ker \mathcal{L}_M$ and $\mathcal{L}_M^{-1}B \perp \ker \mathcal{L}_M$

• Rotational invariance of $\mathcal{B}$ Let $R \in O_3(\mathbf{R})$; it acts on functions f on $\mathbf{R}^3$, on vector fields U on $\mathbf{R}^3$, and on 2-contravariant tensors fields S on $\mathbf{R}^3$ as follows:

$$f_R(v) = f(R^T v)\,, \quad U_R(v) = RU(R^T v)\,, \quad S_R(v) = RS(R^T v)R^T$$

•The Boltzmann collision integral is rotationally invariant:

$$\mathcal{B}(F_R, F_R) = \mathcal{B}(F, F)_R\,, \text{ therefore } \mathcal{L}_{M_{1,0,1}}\phi_R = (\mathcal{L}_{M_{1,0,1}}\phi)_R$$

since $M_{1,0,1}$ is a radial function.

•One has $A_R = A$ and $B_R = B$; hence $(\mathcal{L}_M^{-1}A)_R = \mathcal{L}_M^{-1}A$ and $(\mathcal{L}_M^{-1}B)_R = \mathcal{L}_M^{-1}B$. Therefore, there exist $\alpha \equiv \alpha(|v|)$ and $\beta \equiv \beta(|v|)$ s.t.

$$\mathcal{L}_M^{-1}A(v) = \alpha(|v|)A(v)\,, \quad \mathcal{L}_M^{-1}B(v) = \beta(|v|)B(v)$$

The Hilbert expansion

•Seek a solution of

$$\partial_t F_\epsilon + v \cdot \nabla_x F_\epsilon = \frac{1}{\epsilon}\mathcal{B}(F_\epsilon, F_\epsilon)$$

in the form

$$F_\epsilon(t,x,v) = \sum_{n\geq 0} \epsilon^n F_n(t,x,v) \in C^\infty_{t,x,v}[[\epsilon]]$$

•Order 0: $\mathcal{B}(F_0, F_0) \equiv 0$, which implies that F_0 is a local Maxwellian

$$F_0(t,x,v) = M_{\rho_0(t,x),u_0(t,x),\theta_0(t,x)}(v)$$

•Order 1: one finds that

$$\partial_t F_0 + v \cdot \nabla_x F_0 = 2\mathcal{B}(F_0, F_1) = -M_{\rho_0,u_0,\theta_0}\mathcal{L}_{M_{\rho_0,u_0,\theta_0}}\left(\frac{F_1}{M_{\rho_0,u_0,\theta_0}}\right)$$

Once F_0 is known, one finds F_1 by solving the Fredholm integral equation above.

•Compatibility condition at order 1: in order for this Fredholm integral equation to have a solution, one must verify the compatibility condition

$$M^{-1}_{\rho_0,u_0,\theta_0}(\partial_t + v\cdot\nabla_x)F_0 \perp \operatorname{ker}\mathcal{L}_{M_{\rho_0,u_0,\theta_0}}$$

i.e.

$$\partial_t \int \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} M_{\rho_0,u_0,\theta_0} dv + \operatorname{div}_x \int v \otimes \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} M_{\rho_0,u_0,\theta_0} dv = 0$$

This compatibility condition means that (ρ_0, u_0, θ_0) solves the compressible Euler system.

•Assuming that (ρ_0, u_0, θ_0) solves the compressible Euler system, there exists a unique solution F_1^0 to the Fredholm equation

$$\partial_t F_0 + v \cdot \nabla_x F_0 = 2\mathcal{B}(F_0, F_1^0) \quad \text{s.t.} \int \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} F_1^0 dv \equiv 0$$

•Therefore F_1 (the first order term in Hilbert's expansion) is of the form

$$F_1(t,x,v) = F_1^0(t,x,v) + M_{(\rho_0,u_0,\theta_0)(t,x)}(a(t,x) + b(t,x)\cdot v + c(t,x)|v|^2)$$

with

$$F_1^0 = -M_{1,u_0,\theta_0}\left(\alpha(\theta,|V|)A(V) : D(u_0) + 2\beta(\theta,|V|)B(V)\cdot\nabla_x\sqrt{\theta_0}\right)$$

(see Chapman-Enskog expansion below) where

$$V = \frac{v - u_0}{\sqrt{\theta}_0}, \quad D(u) = \nabla_x u + (\nabla_x u)^T - \tfrac{2}{3}(\operatorname{div}_x u)I$$

but a, b and c remain undetermined so far.

•Order 2: one finds

$$\partial_t F_1 + v \cdot \nabla_x F_1 - \mathcal{B}(F_1, F_1) = 2\mathcal{B}(F_0, F_2)$$

which is another Fredholm integral equation for the unknown F_2. For this equation to have a solution, one must verify the compatibility conditions

$$\partial_t \int \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} F_1 dv + \mathrm{div}_x \int v \otimes \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} F_1 dv = 0$$

These 5 compatibility conditions are 5 PDEs for the five unknown functions a, b and c.

•Order n: one finds

$$\partial_t F_n + v \cdot \nabla_x F_n - \sum_{\substack{k+l=n \\ 1 \le k,l, \le n}} \mathcal{B}(F_k, F_l) = 2\mathcal{B}(F_0, F_{k+1})$$

which is the same Fredholm equation as above.

•Here again, the compatibility condition reduces to

$$\partial_t \int \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} F_n dv + \mathrm{div}_x \int v \otimes \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} F_n dv = 0$$

•More generally, the compatibility condition at order $n+1$ (to guarantee the existence of F_{n+1}) provides the equations satisfied by that part of F_n which belongs to the nullspace of $\mathcal{L}_{M_{\rho_0,u_0,\theta_0}}$.

The Chapman-Enskog expansion

•Seek a solution of

$$\partial_t F_\epsilon + v \cdot \nabla_x F_\epsilon = \frac{1}{\epsilon}\mathcal{B}(F_\epsilon, F_\epsilon)$$

in the form of a formal power series

$$F_\epsilon(t,x,v) = \sum_{n\geq 0} \epsilon^n F^{(n)}[\vec{P}(t,x)](v)$$

parametrized by the vector $\vec{P}$ of conserved densities of F_ϵ.

•Notation: $F^n[\vec{P}(t,x)](v)$ designates any quantity that depends smoothly on $\vec{P}$ and any finite number of its derivatives with respect to the x-variable at the same point (t,x), and on the v-variable.

•$F^n[\vec{P}(t,x)](v)$ doesn't contain time-derivatives of $\vec{P}$: the game is to eliminate $\partial_t \vec{P}$ in favor of x-derivatives via conservation laws satisfied by $\vec{P}$.

•That $\vec{P}$ is the vector of conserved densities of F_ϵ means that

$$\int F^{(0)}[\vec{P}](v) \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} dv = \vec{P}\,, \quad \int F^{(n)}[\vec{P}](v) \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} dv = \vec{0}\,,\ n \geq 1$$

•These conserved densities satisfy a formal system of conservation laws

$$\partial_t \vec{P} = \sum_{n\geq 0} \epsilon^n \,\mathrm{div}_x\, \Phi^{(n)}[\vec{P}]$$

where the formal fluxes are obtained from the local conservation laws:

$$\Phi^{(n)}[\vec{P}] = -\int v \otimes \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} F^{(n)}[\vec{P}](v) dv$$

•Order 0: one has

$$\mathcal{B}(F^{(0)}[\vec{P}], F^{(0)}[\vec{P}]) = 0\,, \text{ and thus } F^{(0)}[\vec{P}] = M_{\rho,u,\theta}$$

here

$$\vec{P} = \begin{pmatrix} \rho \\ \rho u \\ \rho(\frac{1}{2}|u|^2 + \frac{3}{2}\theta) \end{pmatrix}\,, \quad \Phi^{(0)}[\vec{P}] = -\begin{pmatrix} \rho u \\ \rho u^{\otimes 2} + \rho\theta I \\ \rho u(\frac{1}{2}|u|^2 + \frac{5}{2}\theta) \end{pmatrix}$$

Hence the formal conservation law at order 0 is

$$\partial_t \vec{P}^0 = \operatorname{div}_x \Phi^{(0)}[\vec{P}^0] \text{ mod. } O(\epsilon) \Leftrightarrow \text{ Euler system}$$

•Euler's system can be recast as

$$\begin{aligned} \partial_t \rho^0 + u^0 \cdot \nabla_x \rho^0 + \rho^0 \operatorname{div}_x u^0 &= 0 \\ \partial_t u^0 + (u^0 \cdot \nabla_x) u^0 + \frac{1}{\rho^0} \nabla_x (\rho^0 \theta^0) &= 0 \\ \partial_t \theta^0 + u^0 \cdot \nabla_x \theta^0 + \tfrac{2}{3} \theta^0 \operatorname{div}_x u^0 &= 0 \end{aligned}$$

•Order 1: one has

$$(\partial_t + v\cdot\nabla_x)F^{(0)}[\vec{P}^1] = 2\mathcal{B}(F^{(0)}[\vec{P}^1], F^{(1)}[\vec{P}^1])$$

using the formal conservation at order 0, eliminate $\partial_t F^{(0)}[\vec{P}^1]$ and replace it with x-derivatives of $F^{(0)}[\vec{P}^1]$:

$$(\partial_t + v\cdot\nabla_x)M_{\rho^1,u^1,\theta^1} = M_{\rho^1,u^1,\theta^1}\left(A(V):D(u^1) + 2B(V)\cdot\nabla_x\sqrt{\theta^1}\right) + O(\epsilon)$$

with the notations

$$V = \frac{v-u^1}{\sqrt{\theta^1}}, \quad A(V) = V^{\otimes 2} - \tfrac{1}{3}|v|^2 I, \quad B(V) = \tfrac{1}{2}V(|V|^2-5)$$

and where $D(u)$ is the traceless part of the deformation tensor of u:

$$D(u) = \tfrac{1}{2}\left(\nabla_x u + (\nabla_x u)^T - \tfrac{2}{3}\,\mathrm{div}_x\, u I\right)$$

•Therefore, $F^{(1)}[\vec{P}^1]$ is determined by the conditions

$$A(V) : D(u^1) + 2B(V) \cdot \nabla_x \sqrt{\theta^1} = -\mathcal{L}_{M_{\rho^1,u^1,\theta^1}} \left(\frac{F^{(1)}[\vec{P}^1]}{M_{\rho^1,u^1,\theta^1}} \right)$$

$$\int F^{(1)}[\vec{P}^1](v) \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} dv = 0$$

•By Hilbert's theorem, $\mathcal{L}_M$ is a Fredholm operator on $L^2(M dv)$; therefore

$$F^{(1)}[\vec{P}^1](v) = -M_{1,u^1,\theta^1} \Big(\alpha(\theta^1, |V|) A(V) : D(u^1) + 2\beta(\theta^1, |V|) B(V) \cdot \nabla_x \sqrt{\theta^1} \Big)$$

•Hence the first order flux in the formal conservation law is

$$\Phi^{(1)}[\vec{P}^1] = \begin{pmatrix} 0 \\ \mu(\theta^1)D(u^1) \\ \mu(\theta^1)D(u^1)\cdot u^1 + \kappa(\theta^1)\nabla_x\theta^1 \end{pmatrix}$$

•Therefore, the formal conservation law at first order is

$$\partial_t \vec{P}^1 = \mathrm{div}_x\, \Phi^{(0)}[\vec{P}^1] + \epsilon\, \mathrm{div}_x\, \Phi^{(1)}[\vec{P}^1]$$

i.e. the compressible Navier-Stokes system with $O(\epsilon)$ dissipation terms

$$\begin{aligned} \partial_t\rho^1 + \mathrm{div}_x(\rho^1 u^1) &= 0 \\ \partial_t(\rho^1 u^1) + \mathrm{div}_x(\rho^1 (u^1)^{\otimes 2}) + \nabla_x(\rho^1\theta^1) &= \epsilon\,\mathrm{div}_x(\mu D(u^1)) \\ \partial_t\left(\rho(\tfrac{1}{2}|u^1|^2 + \tfrac{3}{2}\theta^1)\right) + \mathrm{div}_x\left(\rho^1 u^1(\tfrac{1}{2}|u^1|^2 + \tfrac{5}{2}\theta^1)\right) &= \epsilon\,\mathrm{div}_x(\kappa\nabla_x\theta^1) \\ &\quad + \epsilon\,\mathrm{div}_x(\mu D(u^1)\cdot u^1) \end{aligned}$$

•The viscosity and heat conduction coefficients are computed as follows:

$$\theta \int \alpha(\theta, V) A_{ij}(V) A_{kl}(V) M_{1,u,\theta} dv = \mu(\theta)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \tfrac{2}{3}\delta_{ij}\delta_{kl})$$

$$\theta \int \beta(\theta, V) B_i(V) B_j(V) M_{1,u,\theta} dv = \kappa(\theta)\delta_{ij}$$

or, in other words

$$\mu(\theta) = \tfrac{2}{15}\theta \int_0^{+\infty} \alpha(\theta, r) r^6 \frac{e^{-r^2/2} dr}{\sqrt{2\pi}}$$

$$\kappa(\theta) = \tfrac{1}{6}\theta \int_0^{+\infty} \beta(\theta, r) r^4 (r^2 - 5)^2 \frac{e^{-r^2/2} dr}{\sqrt{2\pi}}$$

•In the hard sphere case, one finds that

$$\mu(\theta) = \mu_0 \sqrt{\theta}, \quad \kappa(\theta) = \kappa_0 \sqrt{\theta}$$

Hilbert vs. Chapman-Enskog

•Hilbert's expansion more systematic? Chapman-Enskog expansion requires knowing in advance that one gets a system of local conservation laws at any order in ϵ.

•Chapman-Enskog expansion=reshuffling terms in Hilbert expansion? Not really: in the case of a boundary-value problem, Hilbert's expansion leads to a set of boundary conditions for (ρ_0, u_0, θ_0) that is adapted to the compressible Euler system, i.e. to a hyperbolic system.

•This is in general not consistent with the boundary conditions adapted to the compressible Navier-Stokes system, which is (degenerate) parabolic. (For instance: there may be a viscous boundary layer of thickness $O(\sqrt{\epsilon})$).

Deficiencies in both expansions

•Truncated Hilbert or Chapman-Enskog expansions are polynomials in v, and thus may not be nonnegative for all t, x and v. See a proof by Caflisch (CPAM 1980) of the compressible Euler limit; lack of positivity may be cured by suitable initial layers, as constructed by Lachowicz (M2AS 1987).

•Hydrodynamic equations may develop singularities in finite time (as in the case of the compressible Euler system) — or it may be unknown whether the solution remains smooth for all times (as in the case of 3D incompressible Navier-Stokes). Truncated expansions cannot provide a justification of the hydrodynamic limit past the time of appearance of a singularity in the limiting solution.