Fluid Dynamics from Kinetic Equations

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LECTURE 1

FROM BOLTZMANN TO EULER

Orders of magnitude, perfect gas

- For a monatomic gas at room temperature and atmospheric pressure, about 10^{20} gas molecules with radius $\simeq 10^{-8}$cm are to be found in any volume of 1cm^3

- Excluded volume (i.e. the total volume occupied by the gas molecules if tightly packed): $10^{20} \times \frac{4\pi}{3} \times (10^{-8})^{-3} \simeq 5 \cdot 10^{-4}cm^3 \ll 1$cm^3

EXCLUDED VOLUME NEGLIGEABLE $\Rightarrow$ PERFECT GAS

- Equation of state for a perfect gas:

$p = k\rho\theta$, where k =Boltzmann's constant$= 1.38 \cdot 10^{-23}$J/K

Notion of mean-free path

•Roughly speaking, the average distance between two successive collisions for any given molecule in the gas

•There are more than one precise mathematical definitions of that notion (for instance, one can use the empirical measure to compute the mean)

•Intuitively, the higher the gas density, the smaller the mean-free path; likewise, the bigger the molecules, the smaller the mean-free path; this suggests

$$\boxed{\text{mean-free path } \approx \frac{1}{\mathcal{N}\times\mathcal{A}}}$$

where $\mathcal{N}$ =number of gas molecules per unit volume and $\mathcal{A}$ =area of the section of any individual molecule

•For the same monatomic gas as before (at room temperature and atmospheric pressure), $\mathcal{N} = 10^{20}$ molecules/cm^3, while $\mathcal{A} = \pi \times (10^{-8})^2 \simeq 3 \cdot 10^{-16}$cm^2; hence the mean-free path is $\approx \frac{1}{3} \cdot 10^{-4}$cm $\ll 1$cm.

SMALL MEAN-FREE PATH REGIMES CAN OCCUR IN PERFECT GASES

•While keeping the same temperature, lower the pressure at 10^{-4}atm; then $\mathcal{N} = 10^{16}$ molecules/cm^3 and the mean-free path becomes $\approx \frac{1}{3}$cm which is comparable to the size of the 1cm^3 container

DEGREE OF RAREFACTION MEASURED BY KNUDSEN NUMBER

$$\mathrm{Kn} := \frac{\text{mean free path}}{\text{macroscopic length scale}}$$

Kinetic vs. fluid regimes

•Fluid regimes are characterized by $\boxed{\mathrm{Kn} \ll 1}$; the gas is in local thermodynamic equilibrium: its state is adequately described by:

$$\boxed{p \equiv p(t,x) \text{ pressure, } \theta \equiv \theta(t,x) \text{ temperature, } \vec{u} \equiv \vec{u}(t,x) \text{ velocity field}}$$

•Kinetic regimes are characterized by $\boxed{\mathrm{Kn} = O(1)}$; since the gas is more rarefied, there are not enough collisions per unit of time for a local thermodynamic equilibrium to be reached. However, also because of rarefaction, correlations are weak $\Rightarrow$ state of the gas is adequately described by

$$\boxed{F \equiv F(t,x,v) \text{ single-particle phase-space density}}$$

Macroscopic observables

•One calls F the “distribution function" or “number density"; $F(t,x,v)$ is the density (with respect to the Lebesgue measure $dxdv$) of particles which, at time t, are to be found at the position x with velocity v.

•Macroscopic quantities (observables) are computed by averaging the corresponding quantity for a single particle w.r.t. the measure $F(t,x,v)dxdv$:

$$\text{momentum} = \iint mvF(t,x,v)dxdv, \text{ energy} = \iint \frac{1}{2}m|v|^2F(t,x,v)dxdv$$

•Likewise, one can also define macroscopic densities (w.r.t. the Lebesgue measure dx):

$$\text{momentum density } = \iint mvF(t,x,v)dv$$

The Boltzmann equation

•The number density F is governed by the Boltzmann equation: in the absence of external force

$$\partial_t F + v \cdot \nabla_x F = \mathcal{B}(F, F)$$

where $\mathcal{B}(F, F)$ is the Boltzmann collision integral.

•Because of rarefaction, collisions other than BINARY are neglected.

•At the kinetic level of description, the size of particles is neglected everywhere but in the expression of the mean-free path: collisions are LOCAL and INSTANTANEOUS

$\Rightarrow \quad \mathcal{B}(F, F)$ operates only on the v-variable in F

The collision integral (hard sphere gas)

•For a gas of hard spheres with radius r, Boltzmann's collision integral is

$$\mathcal{B}(F,F)(v)=\iint_{\mathbf{R}^3\times\mathbf{S}^2}\Big(F(v')F(v'_*)-F(v)F(v_*)\Big)\,r^2|(v-v_*)\cdot\omega|d\omega dv_*$$

where the velocities v' and v'_* are defined in terms of v, v_* and ω by

$$v' \equiv v'(v,v_*,\omega) = v-(v-v_*)\cdot\omega\omega$$
$$v'_* \equiv v'_*(v,v_*,\omega) = v_*+(v-v_*)\cdot\omega\omega$$

•Usual notation: F_*, F' and F'_* designate resp. $F(v_*)$, $F(v')$ and $F(v'_*)$

Pre- to post-collision relations

•Given any velocity pair $(v, v_*) \in \mathbf{R}^6$, the pair $(v'(v, v_*, \omega), v'_*(v, v_*, \omega))$ runs through the set of solutions to the system of 4 equations

$v' + v'_* = v + v_*$	conservation of momentum
$\lvert v'\rvert^2 + \lvert v'_*\rvert^2 = \lvert v\rvert^2 + \lvert v_*\rvert^2$	conservation of kinetic energy

as ω runs through $\mathbf{S}^2$.

•The geometric interpretation of these formulas is as follows: in the reference frame of the center of mass of the particle pair, the velocity pair before and after collisions is made of two opposite vectors, $\pm\frac{1}{2}(v' - v'_*)$ and $\pm\frac{1}{2}(v - v_*)$. Conservation of energy implies that $|v - v_*| = |v' - v'_*|$.

Geometric interpretation of collision relations

•Hence $v - v_*$ and $v' - v'_*$ are exchanged by some orthogonal symmetry, whose invariant plane is orthogonal to $\pm\omega$.

Symmetries of the collision integral

•The collision integrand is invariant if one exchanges v and v_*:

$$\int_{\mathbf{R}^3} \mathcal{B}(F,F)\phi dv = \iiint (F'F'_* - FF_*)\phi|(v-v_*)\cdot\omega| d\omega dv_* dv$$
$$= \iiint (F'F'_* - FF_*)\frac{\phi+\phi_*}{2}|(v-v_*)\cdot\omega| d\omega dv_* dv$$

•The collision integrand is changed into its opposite if, given $\omega \in \mathbf{S}^2$, one exchanges (v, v_*) and (v', v'_*) (in the center of mass reference frame, this is a symmetry, and thus an involution).

•Further, $(v, v_*) \mapsto (v', v'_*)$ is an isometry of $\mathbf{R}^6$ (conservation of kinetic energy), so that $\boxed{dvdv_* = dv'dv'_*}$. Finally $\boxed{(v-v_*)\cdot\omega = -(v'-v'_*)\cdot\omega}$.

Symmetries of the collision integral 2

Theorem. *Assume that $F \in L^1(\mathbf{R}^3)$ is rapidly decaying at infinity, i.e.*

$$F(v) = O(|v|^{-n}) \text{ as } |v| \to +\infty \text{ for all } n \geq 0$$

while $\phi \in C(\mathbf{R}^3)$ has at most polynomial growth at infinity, i.e.

$$\phi(v) = O(1 + |v|^m) \text{ as } |v| \to +\infty \text{ for some } m \geq 0$$

Then, one has:

$$\int_{\mathbf{R}^3} \mathcal{B}(F,F)\phi dv = \iiint FF_* \frac{\phi + \phi_* - \phi' - \phi'_*}{2} |(v - v_*) \cdot \omega| d\omega dv_* dv$$

$$= \iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} (F'F'_* - FF_*) \frac{\phi + \phi_* - \phi' - \phi'_*}{4} |(v - v_*) \cdot \omega| d\omega dv_* dv$$

Collision invariants

• These are the functions $\phi \equiv \phi(v) \in C(\mathbf{R}^3)$ such that

$$\phi(v) + \phi(v_*) - \phi(v') - \phi(v'_*) = 0 \text{ for all } (v, v_*) \in \mathbf{R}^3 \text{ and } \omega \in \mathbf{S}^2$$

Theorem. *Any collision invariant is of the form*

$$\phi(v) = a + b_1 v_1 + b_2 v_2 + b_3 v_3 + c|v|^2, \quad a, b_1, b_2, b_3, c \in \mathbf{R}$$

• If ϕ is any collision invariant and $F \in L^1(\mathbf{R}^3)$ is rapidly decaying, then

$$\int_{\mathbf{R}^3} \mathcal{B}(F, F)\phi dv = 0$$

Local conservation laws

•In particular, if $F \equiv F(t,x,v)$ is a solution to the Boltzmann equation that is rapidly decaying in the v-variable

$$\int_{\mathbf{R}^3} \mathcal{B}(F,F)dv = \int_{\mathbf{R}^3} v_k \mathcal{B}(F,F)dv = \int_{\mathbf{R}^3} \tfrac{1}{2}|v|^2 \mathcal{B}(F,F)dv = 0$$

for $k = 1,2,3$.

•Therefore, one has the local conservation laws:

$$\begin{aligned} \partial_t \int_{\mathbf{R}^3} F dv + \mathrm{div}_x \int_{\mathbf{R}^3} vF dv &= 0\,, \quad \text{(mass)} \\ \partial_t \int_{\mathbf{R}^3} vF dv + \mathrm{div}_x \int_{\mathbf{R}^3} v^{\otimes 2} F dv &= 0\,, \quad \text{(momentum)} \\ \partial_t \int_{\mathbf{R}^3} \tfrac{1}{2}|v|^2 F dv + \mathrm{div}_x \int_{\mathbf{R}^3} v\tfrac{1}{2}|v|^2 F dv &= 0\,, \quad \text{(energy)} \end{aligned}$$

Boltzmann's H Theorem

• Assume that $0 < F \in L^1(\mathbf{R}^3)$ is rapidly decaying and such that ln F has polynomial growth at infinity. Then

$$\int_{\mathbf{R}^3} \mathcal{B}(F,F) \ln F dv =$$
$$-\frac{1}{4} \iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} (F'F'_* - FF_*) \ln \left(\frac{F'F'_*}{FF_*} \right) |(v - v_*) \cdot \omega| d\omega dv dv_* \leq 0$$

• The following conditions are equivalent:

$$\int_{\mathbf{R}^3} \mathcal{B}(F,F) \ln F dv = 0 \; \Leftrightarrow \; \mathcal{B}(F,F) = 0 \text{ a.e. } \; \Leftrightarrow F \text{ is a Maxwellian}$$

i.e. $F(v)$ is of the form

$$F(v) = M_{\rho,u,\theta}(v) := \frac{\rho}{(2\pi\theta)^{3/2}} e^{-\frac{|v-u|^2}{2\theta}} \text{ for some } \rho, \theta > 0 \text{ and } u \in \mathbf{R}^3$$

Implications of conservation laws + H Theorem

•If $F \equiv F(t,x,v) > 0$ is a solution to the Boltzmann equation that is rapidly decaying and such that $\ln F$ has polynomial growth in the v-variable, then

$$
\begin{aligned}
\partial_t \int_{\mathbf{R}^3} F dv + \mathrm{div}_x \int_{\mathbf{R}^3} vF dv &= 0\,, \quad \text{(mass)}\\
\partial_t \int_{\mathbf{R}^3} vF dv + \mathrm{div}_x \int_{\mathbf{R}^3} v^{\otimes 2} F dv &= 0\,, \quad \text{(momentum)}\\
\partial_t \int_{\mathbf{R}^3} \tfrac{1}{2}|v|^2 F dv + \mathrm{div}_x \int_{\mathbf{R}^3} v\tfrac{1}{2}|v|^2 F dv &= 0\,, \quad \text{(energy)}\\
\partial_t \int_{\mathbf{R}^3} F \ln F dv + \mathrm{div}_x \int_{\mathbf{R}^3} vF \ln F dv &\le 0\,, \quad \text{(entropy)}
\end{aligned}
$$

The last differential inequality bearing on the entropy density is reminiscent of the Lax-Friedrichs entropy condition that selects admissible solutions of hyperbolic systems of conservation laws.

Dimensionless form of the Boltzmann equation

•Choose macroscopic scales of time T and length L, and a reference temperature Θ; this defines 2 velocity scales:

$$V = \frac{L}{T} \text{ (macroscopic velocity) },\qquad \text{and } c = \sqrt{\Theta} \text{ (thermal speed)}$$

Finally, set $\mathcal{N}$ to be the total number of particles.

•Define dimensionless time, position, and velocity variables by

$$\hat{t} = \frac{t}{T}, \quad \hat{x} = \frac{x}{L}, \quad \hat{v} = \frac{v}{c}$$

and a dimensionless number density

$$\hat{F}(\hat{t}, \hat{x}, \hat{v}) = \frac{L^3 c^3}{\mathcal{N}} F(t, x, v)$$

Dimensionless form of the Boltzmann equation 2

- One finds that

$$\frac{L}{cT}\partial_{\hat{t}}\hat{F} + \hat{v}\cdot\nabla_{\hat{x}}\hat{F} = \frac{\mathcal{N}r^2}{L^2}\iint(\hat{F}'\hat{F}'_* - \hat{F}\hat{F}_*)|(\hat{v}-\hat{v}_*)\cdot\omega|d\omega d\hat{v}_*$$

- The pre-factor multiplying the collision integral is

$$L \times \frac{\mathcal{N}r^2}{L^3} = \frac{L}{\pi \times \text{mean free path}} = \frac{1}{\pi \mathsf{Kn}}$$

- The pre-factor multiplying the time derivative is

$$\frac{\frac{1}{T}\times L}{c} = \mathsf{St}\,, \qquad \text{(kinetic Strouhal number)}$$

$$\mathsf{St}\partial_{\hat{t}}\hat{F} + \hat{v}\cdot\nabla_{\hat{x}}\hat{F} = \frac{1}{\pi\mathsf{Kn}}\iint_{\mathbf{R}^3\times\mathbf{S}^2}(\hat{F}'\hat{F}'_* - \hat{F}\hat{F}_*)|(\hat{v}-\hat{v}_*)\cdot\omega|d\omega d\hat{v}_*$$

Compressible Euler scaling

•This scaling limit corresponds to $\boxed{\mathrm{St} = 1}$ and $\boxed{\pi \mathrm{Kn} =: \epsilon \ll 1}$, leading to the singular perturbation problem

$$\boxed{\partial_t F_\epsilon + v \cdot \nabla_x F_\epsilon = \frac{1}{\epsilon}\mathcal{B}(F_\epsilon, F_\epsilon) = \frac{1}{\epsilon}\iint (F'_\epsilon F'_{\epsilon *} - F_\epsilon F_{\epsilon *})|(v - v_*)\cdot\omega| d\omega d\hat{v}_*}$$

•One expects that, as $\epsilon \to 0$, $F_\epsilon \to F$ and $\mathcal{B}(F_\epsilon, F_\epsilon) \to \mathcal{B}(F, F) = 0$; hence $F(t, x, \cdot)$ is a Maxwellian for all (t, x), i.e.

$$\boxed{F(t, x, v) = M_{\rho(t,x),u(t,x),\theta(t,x)}(v) = \frac{\rho(t, x)}{(2\pi\theta(t, x))^{3/2}} e^{-\frac{|v-u(t,x)|^2}{2\theta(t,x)}}}$$

In other words, F is a local equilibrium.

•<u>Problem:</u> to find the governing equations for $\rho(t, x)$, $u(t, x)$ and $\theta(t, x)$.

Formal Euler limit by the moment method

•Assume that F_ϵ is rapidly decaying and such that $\ln F_\epsilon$ has polynomial growth for large v's; assume further that $F_\epsilon \longrightarrow F$, and that the decay properties above are uniform in this limit.

•H Theorem implies that F is a local Maxwellian $M_{\rho,u,\theta}$:

$$\int_0^{+\infty}\iint \mathcal{B}(F_\epsilon, F_\epsilon) \ln F_\epsilon dv dx dt = \epsilon \iint F_\epsilon \ln F_\epsilon\Big|_{t=0} dx dv - \epsilon \varliminf_{t\to+\infty} \iint F_\epsilon \ln F_\epsilon\Big|_t dx dv \to 0$$

as $\epsilon \longrightarrow 0$; hence

$$\int_0^{+\infty}\iint \mathcal{B}(F, F) \ln F dv dx dt = 0$$

•Passing to the limit in the local conservation laws + the entropy differential inequality leads to the system of conservation laws for (ρ, u, θ)

$$\partial_t \int_{\mathbf{R}^3} M_{\rho,u,\theta} dv + \mathrm{div}_x \int_{\mathbf{R}^3} v M_{\rho,u,\theta} dv = 0$$
$$\partial_t \int_{\mathbf{R}^3} v M_{\rho,u,\theta} dv + \mathrm{div}_x \int_{\mathbf{R}^3} v^{\otimes 2} M_{\rho,u,\theta} dv = 0$$
$$\partial_t \int_{\mathbf{R}^3} \tfrac{1}{2}|v|^2 M_{\rho,u,\theta} dv + \mathrm{div}_x \int_{\mathbf{R}^3} v \tfrac{1}{2}|v|^2 M_{\rho,u,\theta} dv = 0$$

as well as the differential inequality

$$\partial_t \int_{\mathbf{R}^3} M_{\rho,u,\theta} \ln M_{\rho,u,\theta} dv + \mathrm{div}_x \int_{\mathbf{R}^3} v M_{\rho,u,\theta} \ln M_{\rho,u,\theta} dv \le 0$$

• The following formulas for the moments of a Maxwellian

$$\int M_{\rho,u,\theta} dv = \rho\,, \quad \int v M_{\rho,u,\theta} dv = \rho u\,,$$

$$\int v^{\otimes 2} M_{\rho,u,\theta} dv = \rho(u^{\otimes 2} + \theta I)\,, \quad \int \tfrac{1}{2}|v|^2 M_{\rho,u,\theta} dv = \tfrac{1}{2}\rho(|u|^2 + 3\theta)$$

$$\int v \tfrac{1}{2}|v|^2 M_{\rho,u,\theta} dv = \tfrac{1}{2}\rho u(|u|^2 + 5\theta)$$

and for its entropy and entropy flux

$$\int M_{\rho,u,\theta} \ln M_{\rho,u,\theta} dv = \rho \ln \left(\frac{\rho}{(2\pi\theta)^{3/2}} \right) - \tfrac{3}{2}\rho$$

$$\int v M_{\rho,u,\theta} \ln M_{\rho,u,\theta} dv = \rho u \ln \left(\frac{\rho}{(2\pi\theta)^{3/2}} \right) - \tfrac{3}{2}\rho u$$

show that (ρ, u, θ) is an admissible solution of Euler's system.