The Lorentz gas

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Lecture 3

Work with E. Caglioti (CMP2003, CRAS2008)

- Coding particle trajectories with continued fractions

- An ergodic theorem for collision patterns

In order to analyze the Boltzmann-Grad limit of the periodic Lorentz gas, we need a convenient way to encode particle trajectories.

First problem: for a particle leaving the surface of an obstacle in a given direction, to find the position of its next collision with an obstacle

Second problem: average — in some sense to be defined — in order to eliminate the direction dependence

The transfer map

IMPACT PARAMETER

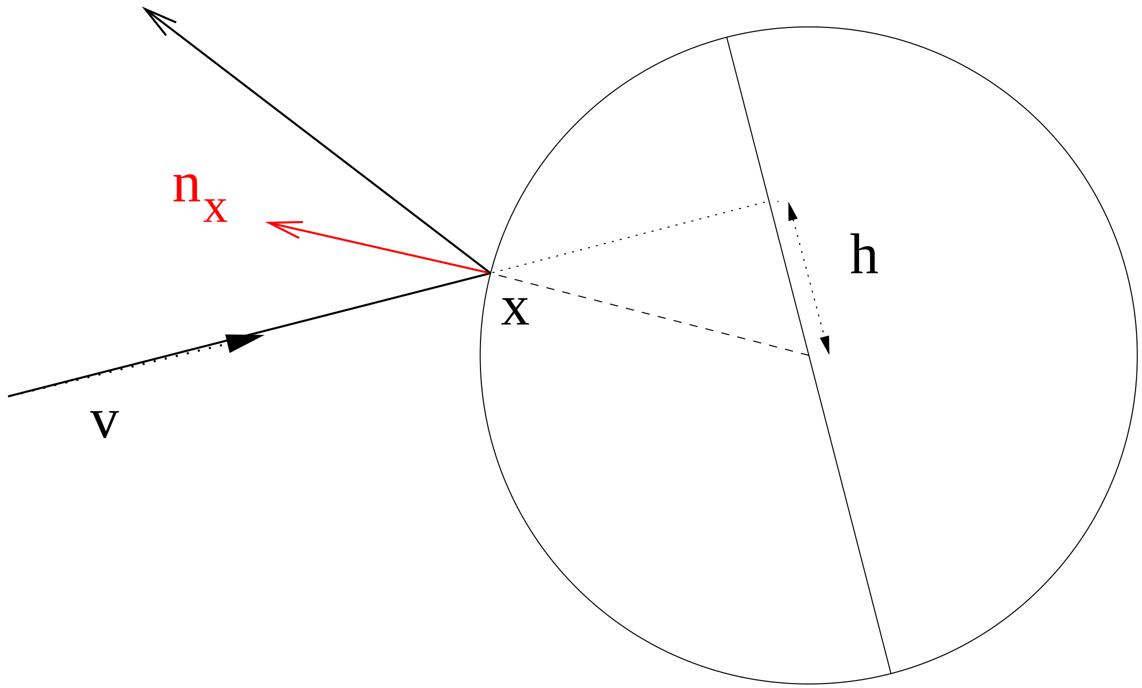
For a particle with velocity v located at the position x on the surface of an obstacle, we define its impact parameter by the formula

$$h_r(x, v) = \sin(\widehat{n_x, v})$$

Obviously

$$h_r(x, \mathcal{R}[n_x]v) = h_r(x, v)$$

where we recall the notation $\mathcal{R}[n]v = v - 2v \cdot nn$.



The impact parameter h corresponding with the collision point x at the surface of an obstacle, and a direction v

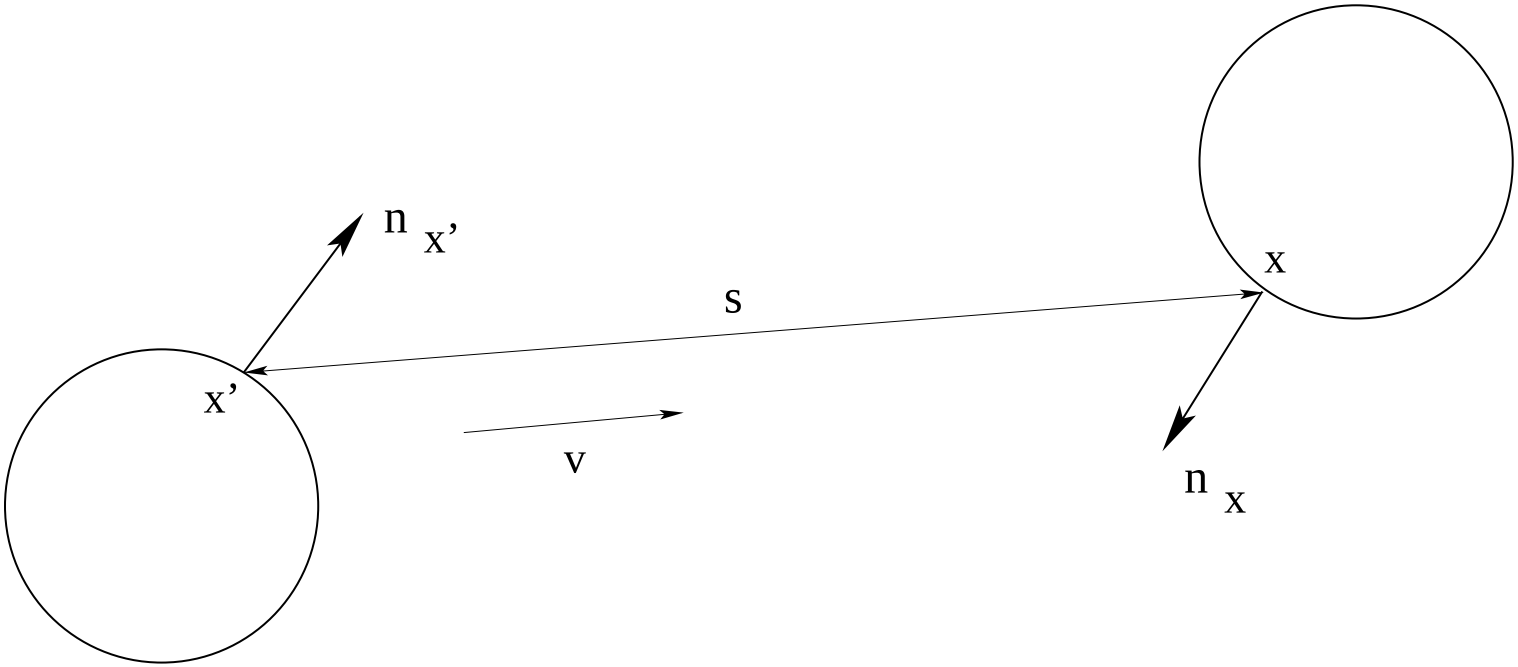
Transfer Map

For a particle leaving the surface of an obstacle in the direction v and with impact parameter h', define

$$T_r(h', v) = (s, h) \text{ with } \begin{cases} s = \text{ distance to the next collision point} \\ h = \text{ impact parameter at the next collision} \end{cases}$$

The particle trajectories are completely determined by the transfer map T_r and iterates thereof.

Therefore, a first step in finding the Boltzmann-Grad limit of the periodic, 2D Lorentz gas, is to compute the limit of T_r as $r \to 0^+$.



The transfer map

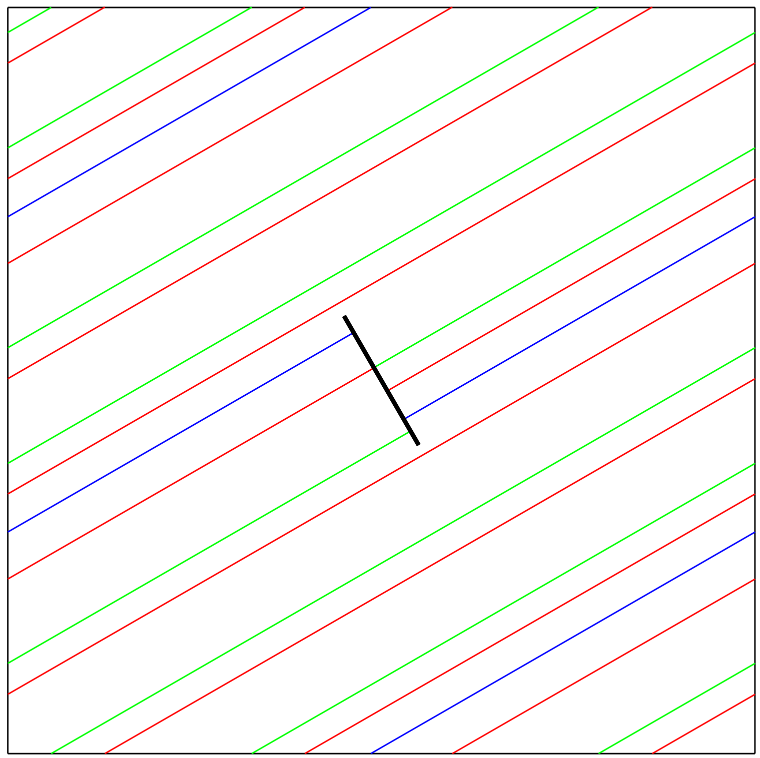
The 3-length theorem

QUESTION (R. THOM, 1989): on a flat 2-torus with a disk removed, consider a linear flow with irrational slope. What is the longest orbit?

Theorem. (Blank-Krikorian, 1993) *On a flat 2-torus with a vertical slit removed, consider a linear flow with irrational slope* $0 < \alpha < 1$*. The orbits have at most 3 different lengths — exceptionally 2, but generically 3.*

These lengths are expressed in terms of the continued fraction expansion of the slope α.

Three types of orbits

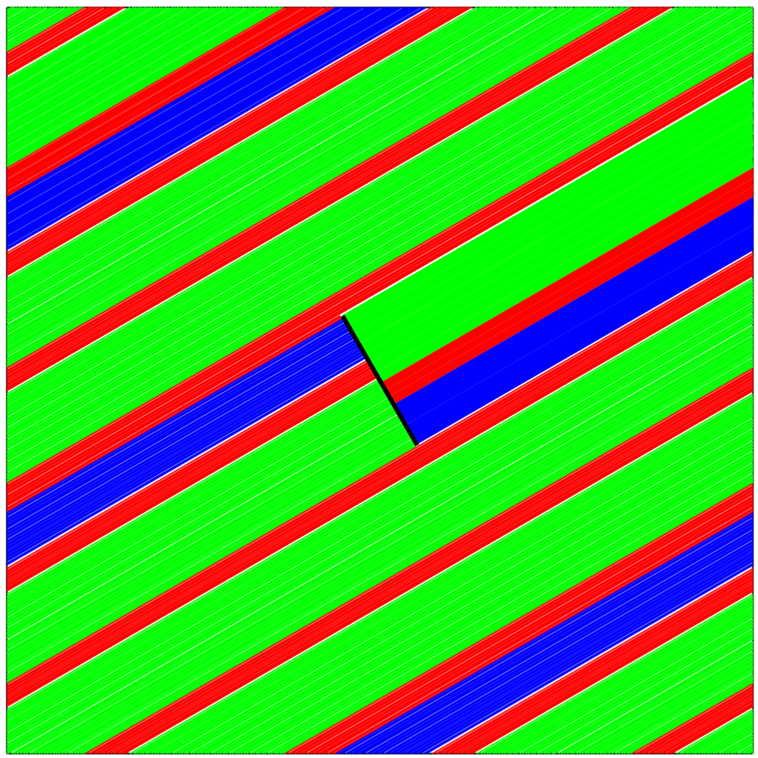


Idea (E. Caglioti, F.G. (2003)):

Orbits with the same lengths in the Blank-Krikorian theorem define a 3-term partition of the 2-torus into parallel strips, whose lengths and widths are computed exactly in terms of the continued fraction expansion of the slope.

The collision pattern for particles leaving the surface of one obstacle — and therefore the transfer map — can be explicitly determined in this way, for a.e. direction $v \in \mathbf{S}^1$.

Three types of orbits



THE CLASSICAL 3-LENGTH THEOREM

Conjectured by Steinhaus, proved by Vera Sòs in 1957

Theorem. *Let* $\alpha \in (0,1) \setminus \mathbf{Q}$ *and* $N \geq 1$. *The sequence*

$$\{n\alpha \,|\, 0 \leq n \leq N\}$$

defines $N+1$ *intervals on the circle of unit length* $\simeq \mathbf{R}/\mathbf{Z}$. *The lengths of these intervals take at most 3 different values.*

Continued fractions: crash course no. 1

•Assume $0 < v_2 < v_1$ and set $\alpha = v_2/v_1$, and consider the continued fraction expansion of α:

$$\alpha = [0; a_0, a_1, a_2, \dots] = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \dots}} .$$

Define the sequences of convergents $(p_n, q_n)_{n\geq 0}$ — meaning that

$$\frac{p_{n+2}}{q_{n+2}} = [0; a_0, \dots, a_n] , \quad n \geq 2$$

by the recursion formulas

$$\begin{aligned} p_{n+1} &= a_n p_n + p_{n-1} , \qquad & p_0 = 1 , \ p_1 = 0 , \\ q_{n+1} &= a_n q_n + q_{n-1} & q_0 = 0 , \ q_1 = 1 , \end{aligned}$$

Let d_n denote the sequence of errors

$$d_n = |q_n\alpha - p_n| = (-1)^{n-1}(q_n\alpha - p_n)\,, \quad n \geq 0$$

so that

$$d_{n+1} = -a_n d_n + d_{n-1}\,, \quad d_0 = 1\,,\ d_1 = \alpha\,.$$

The sequence d_n is decreasing and converges to 0, at least exponentially fast.

By induction, one verifies that

$$q_n d_{n+1} + q_{n+1} d_n = 1\,, \quad n \geq 0\,.$$

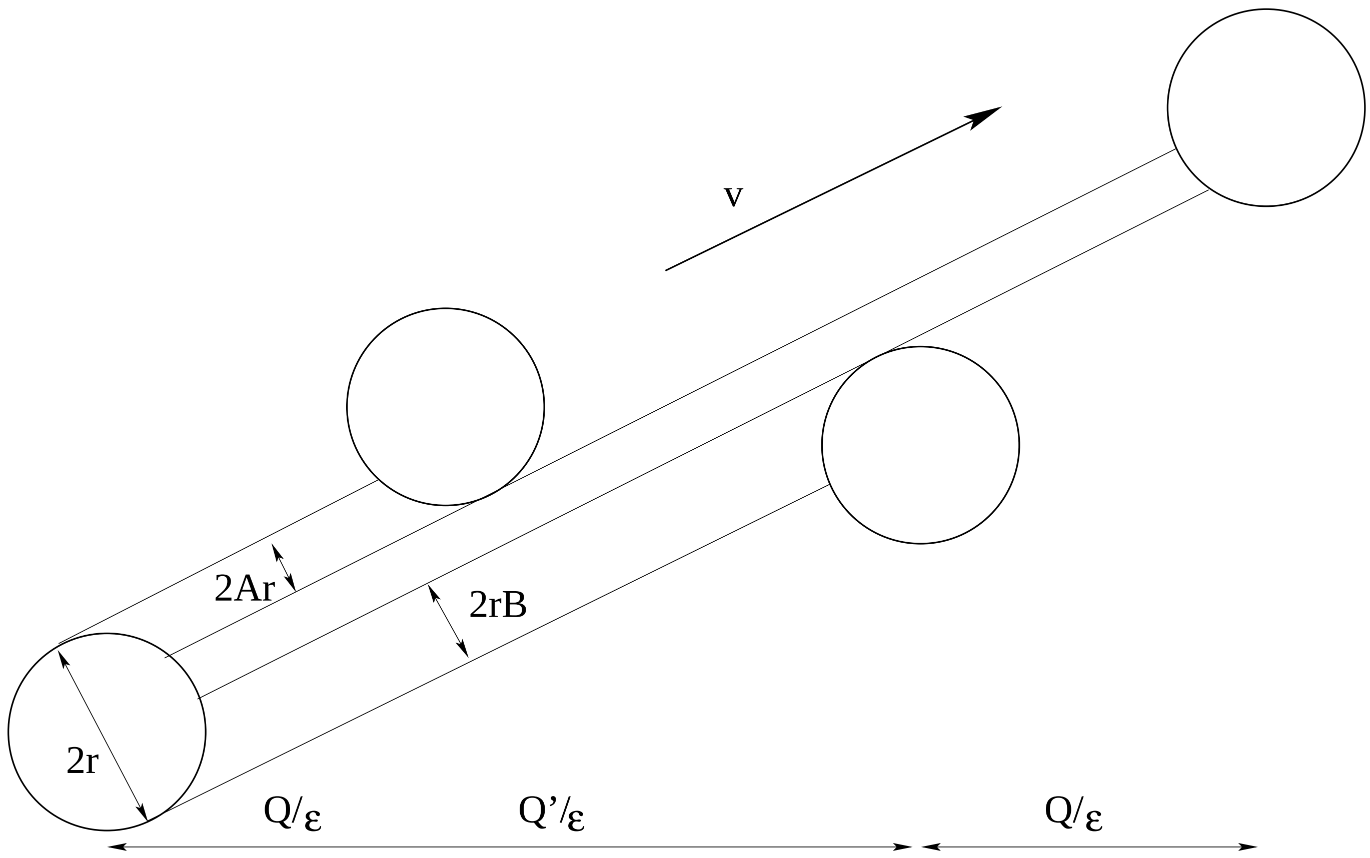
Notation: we write $p_n(\alpha), q_n(\alpha), d_n(\alpha)$ to indicate the dependence of these quantities in α.

Consequence of the 3-length theorem:

A particle leaving the surface of one obstacle in some irrational direction v will next collide with at most 3 — exceptionally 2 — other obstacles.

Any such collision pattern is completely determined by 4 parameters, computed in terms of the continued fraction expansion of v_2/v_1 — in the case where $0 < v_2 < v_1$, to which the general case can be reduced by obvious symmetry arguments.

COLLISION PATTERN SEEN FROM THE SURFACE OF ONE OBSTACLE



Here $\epsilon = 2r/v_1$

Collision Patterns

- Assume therefore $0 < v_2 < v_1$ with $\alpha = v_2/v_1 \notin \mathbf{Q}$.

- Set $\epsilon = 2r\sqrt{1+\alpha^2}$ and define

$$N(\alpha,\epsilon) = \inf\{n \geq 0 \,|\, d_n(\alpha) \leq \epsilon\}\,,$$
$$k(\alpha,\epsilon) = -\left\lfloor \frac{\epsilon - d_{N(\alpha,\epsilon)-1}(\alpha)}{d_{N(\alpha,\epsilon)}(\alpha)} \right\rfloor$$

- The parameters defining the collision pattern are

$$A(v,r) = 1 - \frac{d_{N(\alpha,\epsilon)}(\alpha)}{\epsilon}\,, \qquad Q(v,r) = \epsilon q_{N(\alpha,\epsilon)}(\alpha)$$
$$B(v,r) = 1 - \frac{d_{N(\alpha,\epsilon)-1}(\alpha)}{\epsilon} + \frac{k(\alpha,\epsilon) d_{N(\alpha,\epsilon)}(\alpha)}{\epsilon}\,, \quad \Sigma(v,r) = (-1)^{N(\alpha,\epsilon)}$$

Meaning of the parameter Σ: it determines the relative position of the closest and next to closest obstacles seen from the particle leaving the surface of the obstacle at the origin in the direction v.

(The case represented on the figure corresponds with $\Sigma = +1$.)

Computation of Q': the parameter Q' is not independent from A, B, Q, since one must have

$$AQ + BQ' + (1 - A - B)(Q + Q') = 1$$

(each term in this sum corresponding to the surface of one of the three strips in the 3-term partition of the 2-torus).

$$Q'(v,r) = \frac{1 - Q(v,r)(1 - B(v,r))}{1 - A(v,r)}$$

Approximation of the transfer map

• For each $(A, B, Q, \Sigma) \in \mathbf{K} :=]0,1[^3 \times \{\pm 1\}$, we set

$$\begin{aligned}
\mathbf{T}_{A,B,Q,\Sigma}(h') &= (Q, h' - 2\Sigma(1-A)) \\
\text{if} &\quad 1 - 2A < \Sigma h' \leq 1 \\
\mathbf{T}_{A,B,Q,\Sigma}(h') &= (Q', h' + 2\Sigma(1-B)) \\
\text{if} &\quad -1 \leq \Sigma h' < -1 + 2B \\
\mathbf{T}_{A,B,Q,\Sigma}(h') &= (Q' + Q, h' + 2\Sigma(A-B)) \\
\text{if} &\quad -1 + 2B \leq \Sigma h' \leq 1 - 2A
\end{aligned}$$

Proposition. (E. Caglioti, F.G., 2007) *One has*

$$T_r(h', v) = \mathbf{T}_{(A,B,Q,\Sigma)(v,r)}(h') + (O(r^2), 0)$$

in the limit as $r \to 0^+$.

INTERMISSION

•We have solved problem 1: to find a convenient way of coding the billiard flow in the periodic case and for space dimension 2, for a.e. fixed direction

•It remains to solve problem 2: find a convenient way of averaging the computation above so as to get rid of the direction dependence

Continued fractions: crash course no. 2

Consider the Gauss map

$$T: (0,1)\setminus\mathbf{Q} \ni x \mapsto Tx = \frac{1}{x} - \left[\frac{1}{x}\right] \in (0,1)\setminus\mathbf{Q}$$

Invariant probability measure (found by Gauss):

$$dg(x) = \frac{1}{\ln 2}\frac{dx}{1+x}$$

The Gauss map T is ergodic: by Birkhoff's theorem, for $f \in L^1(0,1;dg)$

$$\frac{1}{N}\sum_{k=0}^{N-1} f(T^k x) \to \int_0^1 f(z)dg(z) \text{ a.e. in } x \in (0,1)$$

•For

$$\alpha = [0; a_0, a_1, a_2, \ldots] = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \ldots}} \in (0,1) \setminus \mathbf{Q}$$

one has

$$a_k(\alpha) = \left[\frac{1}{T^k \alpha}\right] \Rightarrow a_k(T\alpha) = a_{k+1}(\alpha)\,, \quad k \geq 0$$

In other words,

$$T[0; a_0, a_1, a_2, \ldots] = [0; a_1, a_2, a_3 \ldots]$$

• The recursion relation on the error terms

$$d_{n+1}(\alpha) = -a_n(\alpha)d_n(\alpha) + d_{n-1}(\alpha)\,, \quad d_0(\alpha) = 1\,,\ d_1(\alpha) = \alpha$$

$$\Rightarrow \alpha d_n(T\alpha) = d_{n+1}(\alpha)\,, \quad n \geq 0$$

so that

$$d_n(\alpha) = \prod_{k=0}^{n-1} T^k\alpha\,, \quad n \geq 0$$

•Unfortunately, the dependence of $q_n(\alpha)$ in α is more complicated. One way around this: starting from the relation

$$q_{n+1}(\alpha)d_n(\alpha) + q_n(\alpha)d_{n+1}(\alpha) = 1$$

we see that

$$\begin{aligned} q_n(\alpha)d_{n-1}(\alpha) &= \sum_{j=1}^{n} (-1)^{n-j} \frac{d_n(\alpha)d_{n-1}(\alpha)}{d_j(\alpha)d_{j-1}(\alpha)} \\ &= \sum_{j=1}^{n} (-1)^{n-j} \prod_{k=j}^{n-1} T^{k-1}\alpha T^k \alpha \end{aligned}$$

- Besides, since

$$\theta \cdot T\theta < \frac{1}{2}$$

one can truncate the summation above at the cost of some exponentially small error term

$$\left| q_n(\alpha) d_{n-1}(\alpha) - \sum_{j=n-l}^{n} (-1)^{n-j} \frac{d_n(\alpha) d_{n-1}(\alpha)}{d_j(\alpha) d_{j-1}(\alpha)} \right|$$

$$= \left| q_n(\alpha) d_{n-1}(\alpha) - \sum_{j=n-l}^{n} (-1)^{n-j} \prod_{k=j}^{n-1} T^{k-1}\alpha T^k \alpha \right| \leq 2^{-l}$$

QUESTION: HOW TO AVERAGE TO GET RID OF DIRECTION DEPENDENCE?

•obvious idea: average over direction. Not trivial (see lecture 4)

•less obvious idea: average over obstacle radius!

Intuition: look at the relation for the d_ns, and the definition of $N(\alpha, \epsilon)$:

$$\alpha d_{n-1}(T\alpha) = d_n(\alpha)\,, \quad N(\alpha,\epsilon) = \inf\{n \geq 1 \,|\, d_n(\alpha) \leq \epsilon\}$$

Therefore, the problem is (almost) invariant if one changes

$$r \mapsto r/\alpha\,, \quad \alpha \mapsto T\alpha\,, \quad N(\alpha,\epsilon) \mapsto N(\alpha,\epsilon) - 1 = N(T\alpha, \epsilon/\alpha)$$

$\Rightarrow$ Cesaro average for the scale invariant measure $\frac{dr}{r}$ on $\mathbf{R}_+^*$

An ergodic theorem

Lemma. (E. Caglioti, F.G. 2003) *For* $\alpha \in (0,1) \setminus \mathbf{Q}$*, set*

$$N(\alpha,\epsilon) = \inf\{n \geq 0 \,|\, d_n(\alpha) \leq \epsilon\}$$

For each $m \geq 0$ *and each* $f \in C(\mathbf{R}_+^{m+1})$*, one has*

$$\frac{1}{|\ln \eta|} \int_\eta^{1/4} f\left(\frac{d_{N(\alpha,\epsilon)}(\alpha)}{\epsilon}, \ldots, \frac{d_{N(\alpha,\epsilon)-m}(\alpha)}{\epsilon}\right) \frac{d\epsilon}{\epsilon} \to L_m(f)$$

a.e. in $\alpha \in (0,1)$ *as* $\eta \to 0^+$*, where the limit* $L_m(f)$ *is independent of* α.

Proposition. *Let* $\mathbf{K} = [0,1]^3 \times \{\pm 1\}$*. For each* $F \in C(\mathbf{K})$*, there exists* $\mathcal{L}(F) \in \mathbf{R}$ *independent of* v *such that*

$$\frac{1}{\ln(1/\eta)} \int_\eta^{1/2} F(A(v,r), B(v,r), Q(v,r), \Sigma(v,r)) \frac{dr}{r} \to \mathcal{L}(F)$$

for a.e. $v \in \mathbf{S}^1$ *such that* $0 < v_2 < v_1$ *in the limit as* $\eta \to 0^+$*.*

Method of proof: a) eliminate the Σ dependence by considering

$$F(A,B,Q,\Sigma) = F_+(A,B,Q) + \Sigma F_-(A,B,Q)$$

Hence it suffices to consider the case where $F \equiv F(A,B,Q)$.

b) setting $\alpha = v_2/v_1$ and $\epsilon = 2r/v_1$, we recall that

$$A(v,r) \text{ is a function of } \frac{d_{N(\alpha,\epsilon)}(\alpha)}{\epsilon}$$

$$B(v,r) \text{ is a function of } \frac{d_{N(\alpha,\epsilon)}(\alpha)}{\epsilon} \text{ and } \frac{d_{N(\alpha,\epsilon)-1}(\alpha)}{\epsilon}$$

c) as for the Q dependence, proceed as follows: in $F(A,B,Q)$, replace $Q(v,r)$ with

$$\frac{\epsilon}{d_{N(\alpha,\epsilon)-1}} \sum_{j=N(\alpha,\epsilon)-l}^{N(\alpha,\epsilon)} (-1)^{N(\alpha,\epsilon)-j} \frac{d_{N(\alpha,\epsilon)}(\alpha) d_{N(\alpha,\epsilon)-1}(\alpha)}{d_j(\alpha) d_{j-1}(\alpha)}$$

at the expense of an error term of the order

$$O(\text{modulus of continuity of } F(2^{-m})) \to 0 \text{ as } l \to 0$$

uniformly as $\epsilon \to 0^+$.

•This substitution leads to an integrand of the form

$$f\left(\frac{d_{N(\alpha,\epsilon)}(\alpha)}{\epsilon},\ldots,\frac{d_{N(\alpha,\epsilon)-m-1}(\alpha)}{\epsilon}\right)$$

to which we apply the ergodic lemma above: its Cesaro mean converges, in the small radius limit, to some limit $\mathcal{L}_m(F)$ independent of α.

•By uniform continuity of F, one finds that

$$|\mathcal{L}_m(F)-\mathcal{L}_{m'}(F)| = O(\text{modulus of continuity of } F(2^{-m\vee m'}))$$

so that $\mathcal{L}_m(F)$ is a Cauchy sequence as $m\to\infty$. Hence

$$\mathcal{L}_m(F)\to\mathcal{L}(F) \text{ as } m\to\infty$$

and with the error estimate above for the integrand, for $\eta\to 0^+$

$$\frac{1}{\ln(1/\eta)}\int_\eta^{1/2} F(A(v,r),B(v,r),Q(v,r),\Sigma(v,r))\frac{dr}{r}\to\mathcal{L}(F)$$

Application to the transfer map

•With the ergodic theorem above, and the explicit approximation of the transfer expressed in terms of the parameters (A, B, Q, Σ) that determine collision patterns in any given direction v, we easily arrive at the following

Theorem. (E. Caglioti, F.G. 2007) *For each $h' \in [-1, 1]$, there exists a probability density $P(s, h|h')$ on $\mathbf{R}_+ \times [-1, 1]$ such that, for each $f \in C(\mathbf{R}_+ \times [-1, 1])$,*

$$\frac{1}{|\ln \eta|} \int_{\eta}^{1/4} f(T_r(h', v)) \frac{dr}{r} \to \int_0^{\infty} \int_{-1}^{1} f(s, h) P(s, h|h') ds dh$$

a.e. in $v \in \mathbf{S}^1$ as $\eta \to 0$

In other words, the transfer map converges in distribution and in the sense of Cesaro, in the small radius limit, to a transition probability $P(s, h|h')$ that is independent of v.

PROBLEM:

a) compute the transition probability $P(s, h|h')$ explicitly, and discuss its properties;

b) explain the role of this transition probability in the Boltzmann-Grad limit of the periodic Lorentz gas dynamics