The Lorentz gas

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Work with J. Bourgain, E. Caglioti, B. Wennberg

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Lecture 2

•Bourgain-G-Wennberg bounds on the distribution of free path lengths (1998)

•Non convergence to the Lorentz kinetic equation in the periodic case (2007)

•In the proof of Gallavotti's theorem for the case of a Poisson distribution of obstacles in space dimension $D = 2$, the probability that a strip of width $2r$ and length t does not meet any obstacle is e^{-2nrt}, where n is the parameter of the Poisson distribution — i.e. n is the average number of obstacles per unit volume. This justifies the loss term

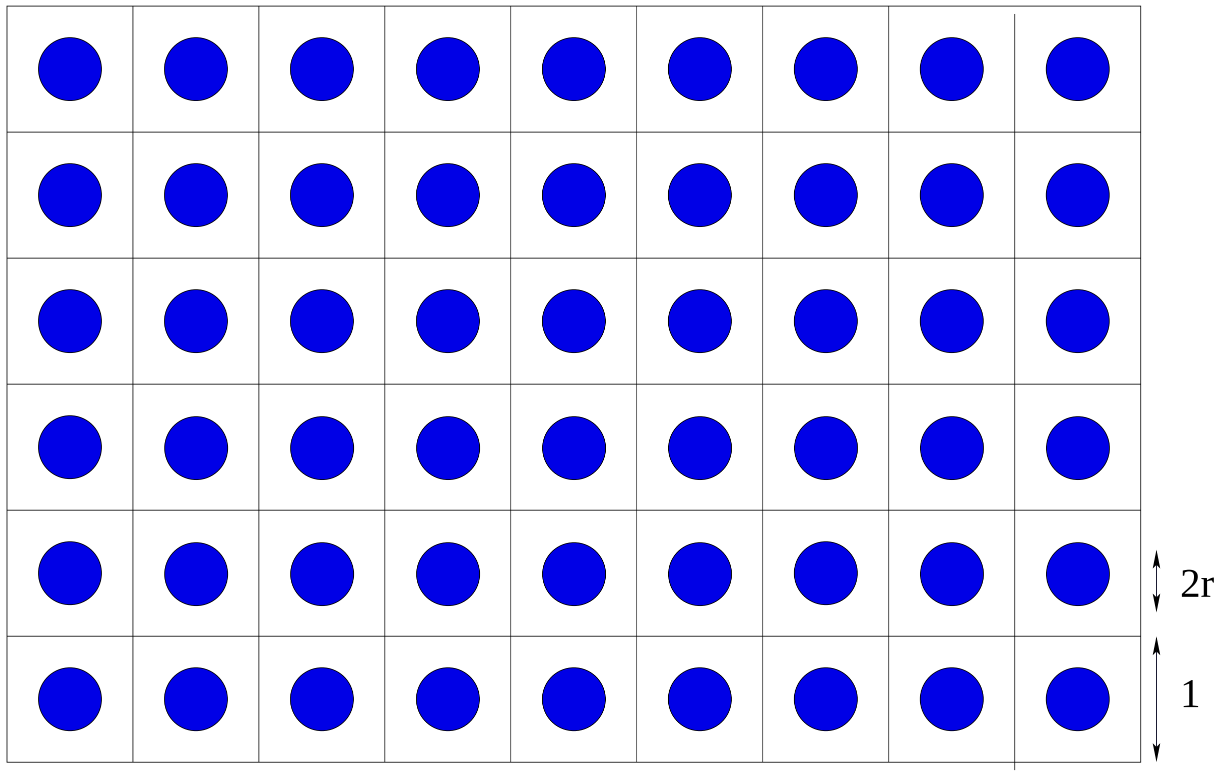
$$f^{in}(x - tv, v)e^{-\sigma t}$$

in the Duhamel series for the solution, or in

$$(\partial_t + v \cdot \nabla_x)f = -\sigma f + \sigma \int_0^{2\pi} f(t, x, R(\beta)v) \sin \tfrac{\beta}{2} \tfrac{d\beta}{4}$$

In the periodic case

FACT 1: there exist INFINITE strips which never meet any obstacles



FACT 2: the contribution of the 1-particle density leading to the loss term in the Lorentz kinetic equation is

$$f^{in}(x-tv,v)\mathbf{1}_{t<\tau_1(x,v,\vec{c})}$$

The analogous term in the periodic case is

$$f^{in}(x-tv,v)\mathbf{1}_{t<r^{D-1}\tau_r(x/r,-v)}$$

Passing to the L^∞ weak-* limit as $r \to 0$ involves the distribution of τ_r with (x,v) uniformly distributed in $(Z_r/\mathbf{Z}^D)\times\mathbf{S}^{D-1}$ — i.e. under the probability measure μ_r.

Santalò's formula gives the mean free path under the probability measure ν_r concentrated on the surface of the obstacles — it is IRRELEVANT for particles that have not yet encountered an obstacle.

Recall that

$$d\mu_r(x,v) = \frac{dxdv}{|Z_r/\mathbf{Z}^D||\mathbf{S}^{D-1}|}, \quad d\nu_r(x,v) = \frac{v\cdot n_x dxdv}{v\cdot n_x dxdv\text{-meas}(\Gamma_+^r)}$$

By the same lemma that implies Santalò's formula

$$\int \tau_r(x,v)d\mu_r(x,v) = \frac{1}{\ell}\int \tfrac{1}{2}\tau_r(x,v)^2 d\nu_r(x,v)$$

where

$$\ell = \frac{|Z_r/\mathbf{Z}^D||\mathbf{S}^{D-1}|}{v\cdot n_x dxdv\text{-meas}(\Gamma_+^r)}$$

Since τ_r is strongly oscillating (finite for irrational directions, possibly infinite for rational directions that become dense as $r \to 0^+$), it may happen that τ_r doesn't have a second moment under ν_r.

The distribution of free path lengths

With the notations

$$Z_r := \{x \in \mathbf{R}^D \,|\, \text{dist}(x, \mathbf{Z}^D) > r\}$$

and

$$\tau_r(x,v) = \inf\{t > 0 \,|\, x + tv \in \partial Z_r\}$$

define the (scaled) distribution under μ_r of free path lengths τ_r as

$$\Phi_r(t) := \mu_r(\{(x,v) \in (Z_r/\mathbf{Z}^D) \times \mathbf{S}^{D-1} \,|\, \tau_r(x,v) > t/r^{D-1}\})$$

Notice the scaling $t \mapsto t/r^{D-1}$: in accordance with Santalò's formula, the free path length τ_r is expected to be "of the order of $1/r^{D-1}$".

Theorem. (Bourgain-F.G.-Wennberg, 1998-2000) *In space dimension* $D \geq 2$*, there exists* $0 < C_D < C'_D$ *such that*

$$\frac{C_D}{t} \leq \Phi_r(t) \leq \frac{C'_D}{t} \quad \textit{whenever } t > 1 \textit{ and } 0 < r < \tfrac{1}{2}$$

•Proof of upper bound by Fourier series — reminiscent of Siegel's proof of the Minkowski convex body theorem

•Proof of lower bound: channel technique: see below. (The idea of channels had been used by Bleher for the diffusive scaling)

In particular

$$\int_{(Z_r/\mathbf{Z}^D)\times\mathbf{S}^{D-1}} \tau_r(x,v)\,d\mu_r(x,v) = +\infty$$

• Amplification: define

$$\phi_r(t|v) := \mu_r(\{x \in Z_r/\mathbf{Z}^D \mid \tau_r(x,v) > t/r^{D-1}\})$$

Theorem. (Caglioti-Golse 2003) *In space dimension $D = 2$*

$$\frac{1}{|\ln \epsilon|}\int_\epsilon^{1/4} \phi_r(t|v)\frac{dr}{r} \to \Phi(t) \text{ a.e. in } v \in \mathbf{S}^1$$

for each $t > 0$. Moreover, in the limit as $t \to +\infty$, one has

$$\Phi(t) = \frac{1}{\pi^2 t} + O\left(\frac{1}{t^2}\right).$$

Method of proof: continued fraction techniques, 3-length theorem

Proof of lower bound for $\Phi_r(t)$ in space dimension $D = 2$

Idea: as mentionned above, there are INFINITE strips included in Z_r — i.e. never meeting any obstacle. Call a channel any such open strip of maximum width, and let $\mathcal{C}_r$ be the set of all channels included in Z_r.

If $S \in \mathcal{C}_r$ and $x \in S$, define $\tau_S(x, v)$ the exit time from the channel:

$$\tau_S(x, v) = \inf\{t > 0 \,|\, x + tv \in \partial S\}\,, \quad (x, v) \in S \times \mathbf{S}^1$$

Obviously

$$\tau_r(x, v) \geq \sup\{\tau_S(x, v) \,|\, S \in \mathcal{C}_r \text{ s.t.} (x, v) \in S \times \mathbf{S}^1\}$$

so that

$$\Phi_r(t) \geq \mu_r \left(\bigcup_{S \in \mathcal{C}_r} \{(x, v) \in (S/\mathbf{Z}^2) \times \mathbf{S}^1 \,|\, \tau_S(x, v) > t/r\} \right)$$

Step 1: description of $\mathcal{C}_r$. Given $\omega \in \mathbf{S}^1$, let

$$\mathcal{C}_r(\omega) := \{\text{channels in } \mathcal{C}_r \text{ of direction } \omega\}\ ;$$

Lemma. *1) if $S \in \mathcal{C}_r(\omega)$, then $\mathcal{C}_r(\omega) := \{S + k \,|\, k \in \mathbf{Z}^2\}$;*

2) $\mathcal{C}_r(\omega) \neq \varnothing \Leftrightarrow \omega = \frac{(p,q)}{\sqrt{p^2+q^2}}$ with $(p,q) \in \mathbf{Z}^2 \setminus \{(0,0)\}$ such that

$$\text{g.c.d.}(p,q) = 1 \text{ and } \sqrt{p^2+q^2} < \frac{1}{2r}\ ;$$

Denote by $\mathcal{A}_r$ the set of all such $\omega \in \mathbf{S}^1$.

3) for $\omega \in \mathcal{A}_r$, the elements of $\mathcal{C}_r(\omega)$ are open strips of width

$$w(\omega, r) = \frac{1}{\sqrt{p^2+q^2}} - 2r$$

Proof: 1) is trivial.

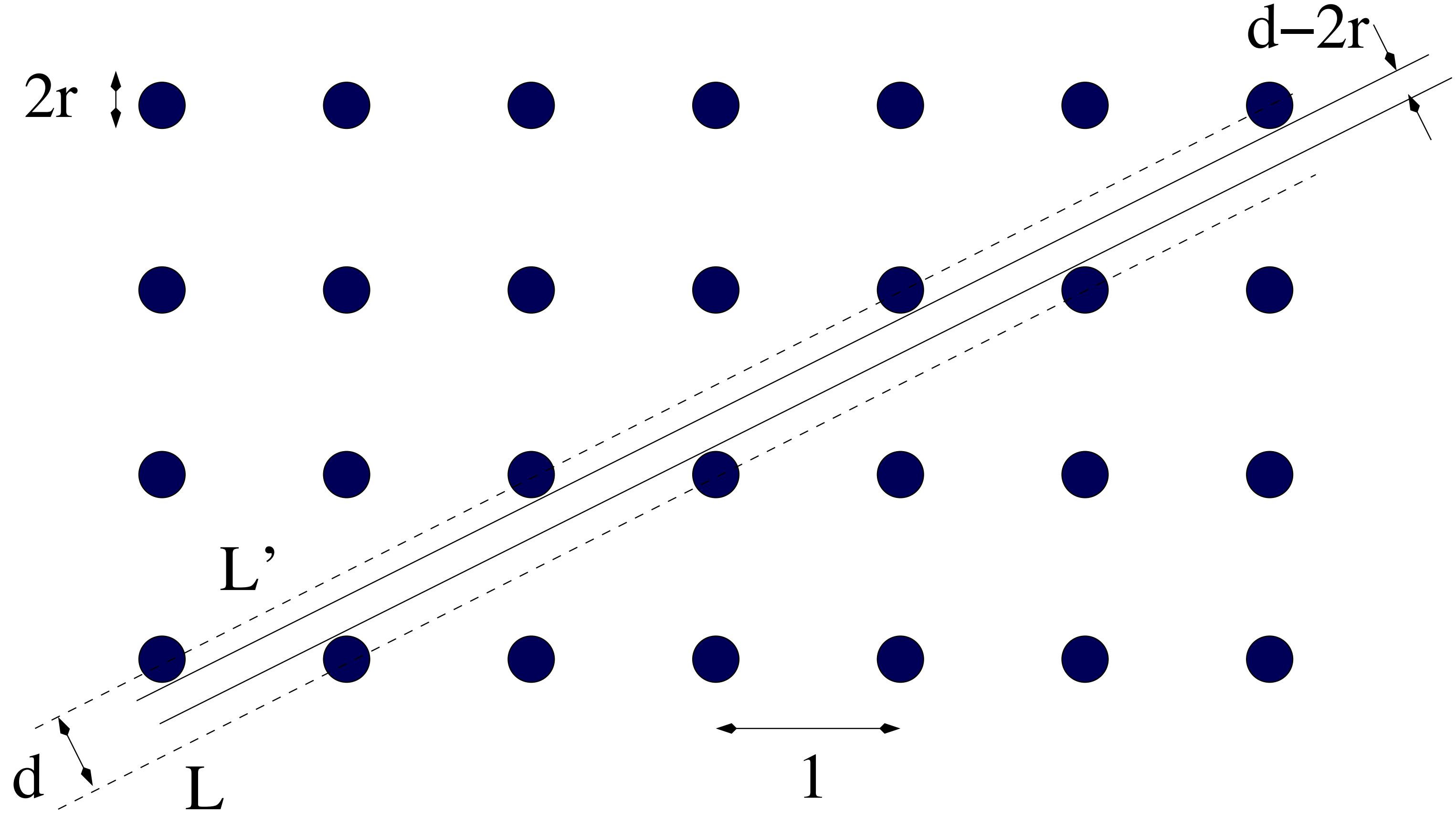
2) if L is an infinite line of direction $\omega \in \mathbf{S}^1$ such that ω_2/ω_1 is irrational, $L/\mathbf{Z}^2$ is an orbit of a linear flow on $\mathbf{T}^2$ with irrational slope $\Rightarrow L/\mathbf{Z}^2$ is dense in $\mathbf{T}^2 \Rightarrow L$ cannot be included in Z_r.

Assume that $\omega = \frac{(p,q)}{\sqrt{p^2+q^2}}$ with $(p, q) \in \mathbf{Z}^2 \setminus \{(0, 0)\}$ coprime, and let L, L' be two infinite lines with direction ω, with equations

$$qx - py = a \text{ and } qx - py = a' \text{ resp.}$$

Obviously

$$\text{dist}(L, L') = \frac{|a - a'|}{\sqrt{p^2 + q^2}}$$



A channel of direction $\omega = \frac{1}{\sqrt{5}}(2,1)$; minimal distance d between lines L and L' of direction ω through lattice points

If $L \cup L'$ is the boundary of a channel of direction $\omega = \frac{(p,q)}{\sqrt{p^2+q^2}} \in \mathcal{A}_0$ included in $\mathbf{R}^2 \setminus \mathbf{Z}^2$ — i.e. of an element of $\mathcal{C}_0(\omega)$, then L and L' intersect $\mathbf{Z}^2$ so that

$$a, a' \in p\mathbf{Z} + q\mathbf{Z} = \mathbf{Z}$$

Since $\mathrm{dist}(L, L') > 0$ is minimal, then $|a - a'| = 1$, so that

$$\mathrm{dist}(L, L') = \frac{1}{\sqrt{p^2 + q^2}}$$

Likewise, if $L \cup L' = \partial S$ with $S \in \mathcal{C}_r$, then L and L' are parallel infinite lines tangent to ∂Z_r, and the minimal distance between any such distinct lines is

$$\mathrm{dist}(L, L') = \frac{1}{\sqrt{p^2 + q^2}} - 2r$$

This entails 2) and 3) □

Step 2: the exit time from a channel.

Let $\omega = \frac{(p,q)}{\sqrt{p^2+q^2}} \in \mathcal{A}_r$ and let $S \in \mathcal{C}_r(\omega)$. Cut S into three parallel strips of equal width and call $\widehat{S}$ the middle one. For each $t > 1$ define

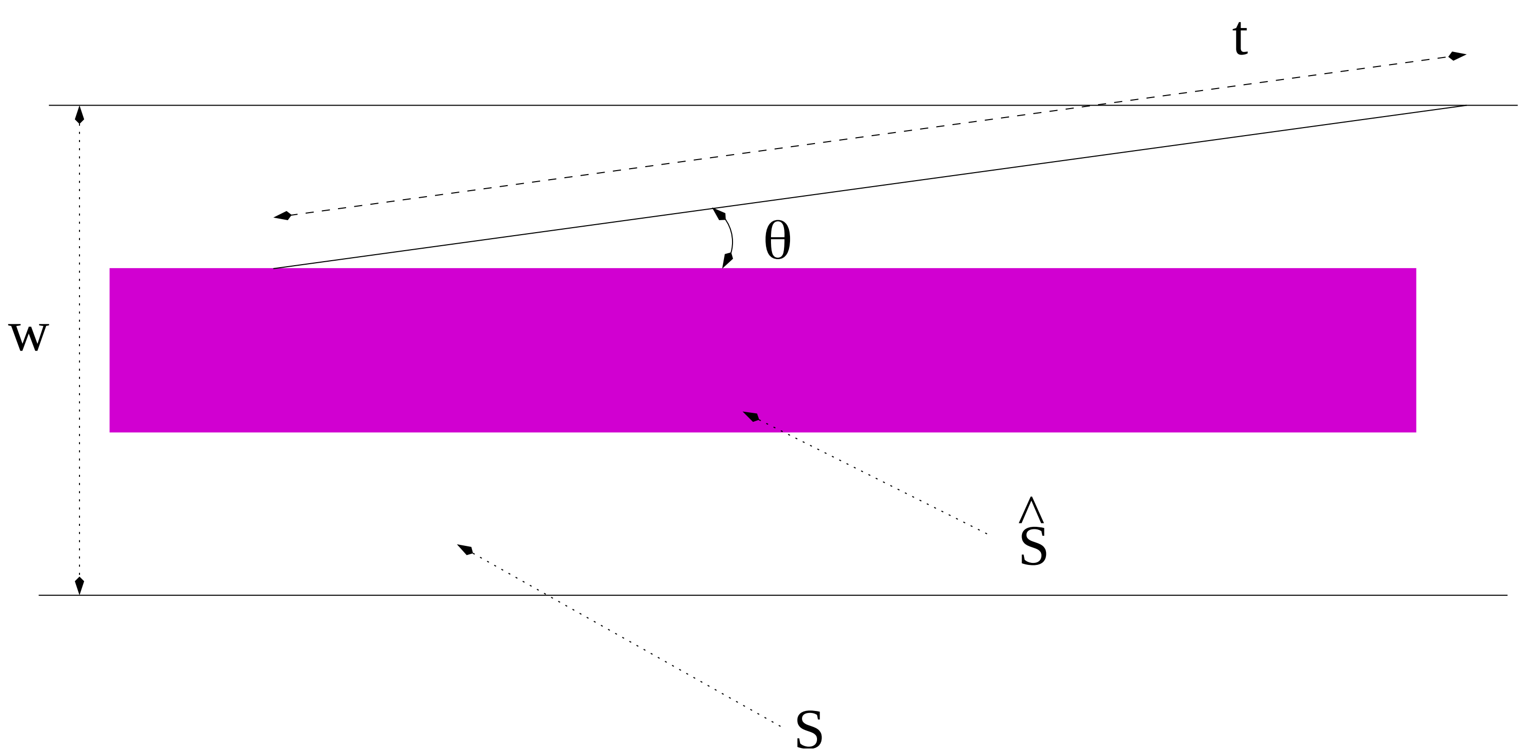
$$\theta \equiv \theta(\omega, r, t) := \text{arcsin}\left(\frac{rw(\omega, r)}{3t}\right)$$

Lemma. *1) if $x \in \widehat{S}$ and $v \in (R[-\theta]\omega, R[\theta]\omega)$, where $R[\theta]$ designates the rotation of an angle θ, then*

$$\tau_S(x, v) \geq t/r\,;$$

2) moreover

$$\mu_r((\widehat{S}/\mathbf{Z}^2) \times (R[-\theta]\omega, R[\theta]\omega)) = \tfrac{2}{3}w(\omega, r)\theta(\omega, r, t)$$



Exit time from the middle third $\widehat{S}$ of an infinite strip S of width w

Step 3: putting all channels together. Recall that we need to estimate

$$\mu_r\left(\bigcup_{S\in\mathcal{C}_r}\{(x,v)\in(S/\mathbf{Z}^2)\times\mathbf{S}^1\,|\,\tau_S(x,v)>t/r\}\right)$$

1) pick

$$\mathcal{A}_r\ni\omega=\frac{(p,q)}{\sqrt{p^2+q^2}}\neq\frac{(p',q')}{\sqrt{p'^2+q'^2}}=\omega'\in\mathcal{A}_r$$

Observe that

$$|\sin(\widehat{\omega,\omega'})|=\frac{|pq'-p'q|}{\sqrt{p^2+q^2}\sqrt{p'^2+q'^2}}\geq\frac{1}{\sqrt{p^2+q^2}\sqrt{p'^2+q'^2}}$$
$$\geq\max\left(\frac{2r}{\sqrt{p^2+q^2}},\frac{2r}{\sqrt{p'^2+q'^2}}\right)\geq\sin\theta(\omega,r,t)+\sin\theta(\omega',r,t)$$
$$\geq\sin(\theta(\omega,r,t)+\theta(\omega',r,t))$$

whenever $t>1$,

Then, whenever $S \in \mathcal{C}_r(\omega)$ and $S' \in \mathcal{C}_r(\omega')$

$$(\widehat{S} \times (R[-\theta]\omega, R[\theta]\omega)) \cap (\widehat{S}' \times (R[\theta']\omega', R[\theta']\omega'))) = \varnothing$$

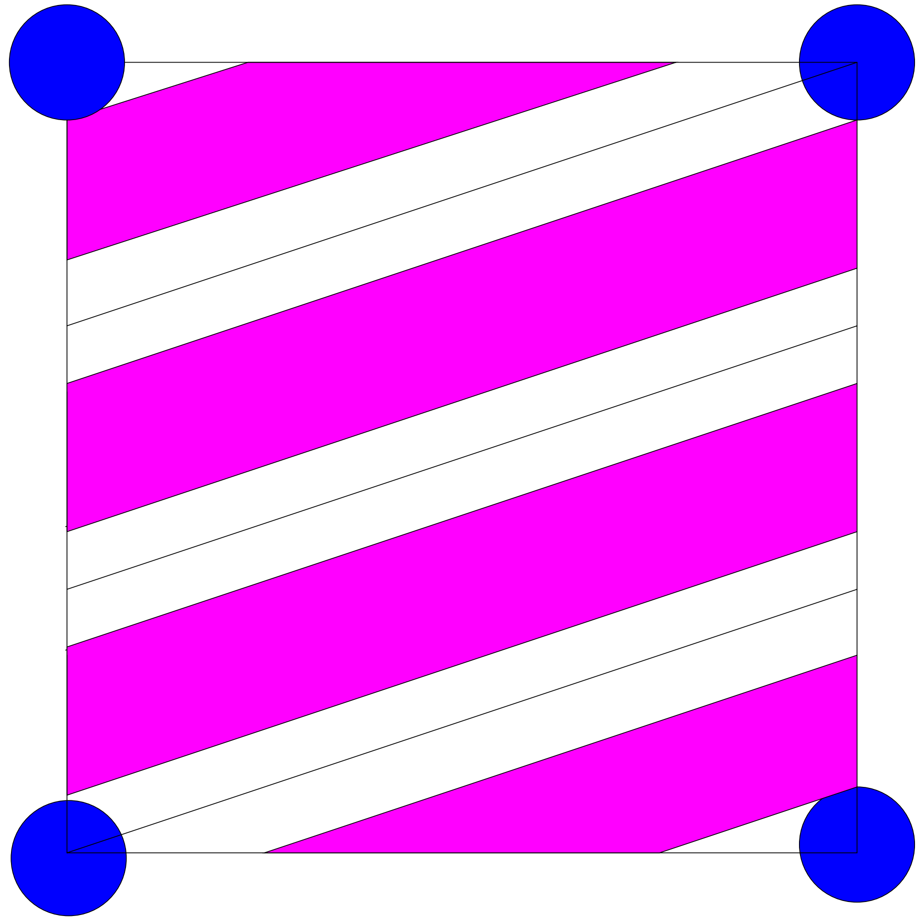
with $\theta = \theta(\omega, r, t)$, $\theta' = \theta'(\omega', r, t)$ and $R[\theta]$ =the rotation of an angle θ.

2) moreover, if $\omega = \frac{(p,q)}{\sqrt{p^2+q^2}} \in \mathcal{A}_r$ then

$$|\widehat{S}/\mathbf{Z}^2| = \frac{1}{3} w(\omega, r)\sqrt{p^2 + q^2}\,,$$

while

$$\#\{S/\mathbf{Z}^2 \,|\, S \in \mathcal{C}_r(\omega)\} = 1\,.$$



A channel modulo $\mathbb{Z}^2$

CONCLUSION

Therefore, whenever $t > 1$

$$\bigcup_{S \in \mathcal{C}_r} (\widehat{S}/\mathbf{Z}^2) \times (R[-\theta]\omega, R[\theta]\omega) \subset \bigcup_{S \in \mathcal{C}_r} \{(x, v) \in (S/\mathbf{Z}^2) \times \mathbf{S}^1 \,|\, \tau_S(x, v) > t/r\}$$

and the left-hand side is a disjoint union.

Hence

$$\mu_r\left(\bigcup_{S\in\mathcal{C}_r}\{(x,v)\in(S/\mathbf{Z}^2)\times\mathbf{S}^1\mid\tau_S(x,v)>t/r\}\right)$$

$$\geq\sum_{\omega\in\mathcal{A}_r}\mu_r((\widehat{S}/\mathbf{Z}^2)\times(R[-\theta]\omega,R[\theta]\omega))$$

$$=\sum_{\substack{g.c.d.(p,q)=1\\ p^2+q^2<1/4r^2}}\frac{1}{3}w(\omega,r)\sqrt{p^2+q^2}\,2\theta(\omega,r,t)$$

$$=\sum_{\substack{g.c.d.(p,q)=1\\ p^2+q^2<1/4r^2}}\frac{2}{3}\sqrt{p^2+q^2}\,w(\omega,r)\arcsin\left(\frac{rw(\omega,r)}{3t}\right)$$

$$\geq\sum_{\substack{g.c.d.(p,q)=1\\ p^2+q^2<1/4r^2}}\frac{2}{3}\sqrt{p^2+q^2}\frac{rw(\omega,r)^2}{3t}$$

Now

$$\sqrt{p^2+q^2} < 1/4r \Rightarrow w(\omega,r) = \frac{1}{\sqrt{p^2+q^2}} - 2r \geq \frac{1}{2\sqrt{p^2+q^2}}$$

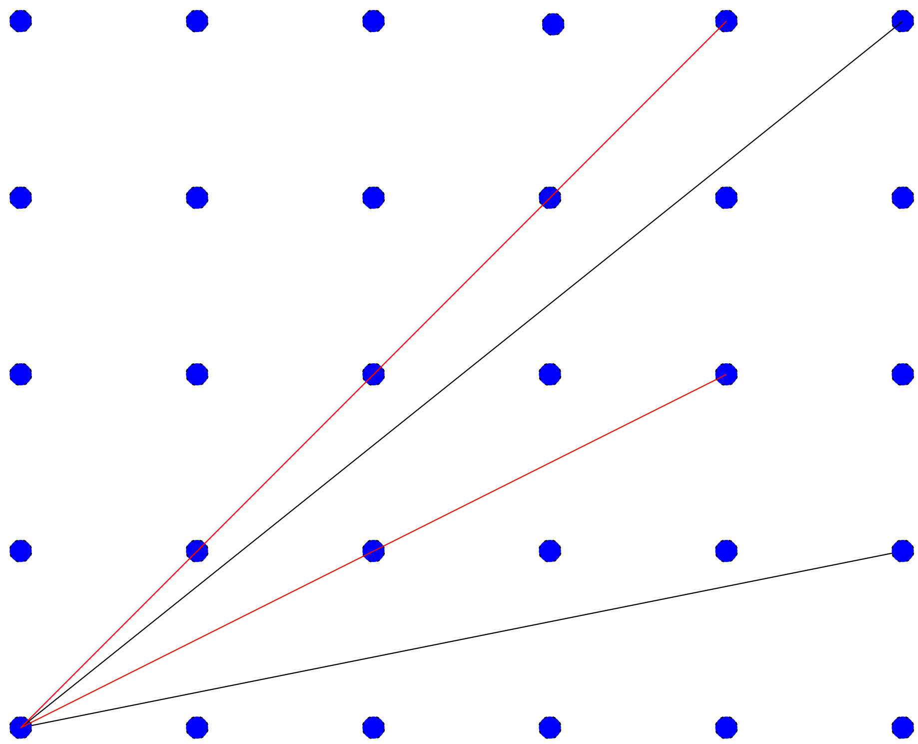
so that

$$\Phi_r(t) \geq \sum_{\substack{g.c.d.(p,q)=1 \\ p^2+q^2<1/16r^2}} \frac{2}{3}\sqrt{p^2+q^2}\frac{rw(\omega,r)^2}{3t}$$

$$\geq \frac{r^2}{18t} \sum_{\substack{g.c.d.(p,q)=1 \\ p^2+q^2<1/16r^2}} \left[\frac{1}{r\sqrt{p^2+q^2}}\right]$$

This gives the desired conclusion since

$$\sum_{\substack{g.c.d.(p,q)=1 \\ p^2+q^2<1/16r^2}} \left[\frac{1}{r\sqrt{p^2+q^2}}\right] = \sum_{p^2+q^2<1/16r^2} 1 \sim \frac{\pi}{16r^2}.\square$$



Black lines issued from the origin terminate at integer points with coprime coordinates; red lines terminate at integer points whose coordinates are not coprime

Non convergence to the Lorentz equation

For $0 < \epsilon = 1/n < 1/2$ with $n \in \mathbf{N}$, define

$$Y_\epsilon = \{x \in \mathbf{T}^D \,|\, \text{dist}(x, \epsilon^{D-1}\mathbf{Z}^D) > \epsilon^D\} = \epsilon^{D-1}(Z_\epsilon/\mathbf{Z}^D)$$

For each $f^{in} \in C(\mathbf{T}^D \times \mathbf{S}^{D-1})$, let f_ϵ be the solution of

$$\begin{aligned} \partial_t f_\epsilon + v \cdot \nabla_x f_\epsilon = 0\,, \quad & (x,v) \in Y_\epsilon \times \mathbf{S}^{D-1} \\ f_\epsilon(t,x,v) = f_\epsilon(t,x,\mathcal{R}[n_x]v)\,, \quad & (x,v) \in \partial Y_\epsilon \times \mathbf{S}^{D-1} \\ f_\epsilon\Big|_{t=0} = f^{in}\,, & \end{aligned}$$

where n_x is unit normal vector to ∂Y_ϵ at the point x, pointing towards the interior of Y_ϵ. By the method of characteristics

$$f_\epsilon(t,x,v) = f^{in}\left(\epsilon^{D-1} X_\epsilon\left(-\tfrac{t}{\epsilon^{D-1}}; \tfrac{x}{\epsilon^{D-1}}, v\right); V_\epsilon\left(-\tfrac{t}{\epsilon^{D-1}}; \tfrac{x}{\epsilon^{D-1}}, v\right)\right)$$

where (X_ϵ, V_ϵ) is the billiard flow in Z_ϵ.

Theorem. (F.G., 2007) *There exist initial data* $f^{in} \equiv f^{in}(x) \in C(\mathbf{T}^D)$ *such that no subsequence of* f_ϵ *converges for the weak-* topology of* $L^\infty(\mathbf{R}_+ \times \mathbf{T}^D \times \mathbf{S}^{D-1})$ *to the solution* f *of a linear Boltzmann equation of the form*

$$(\partial_t + v \cdot \nabla_x) f(t,x,v) = \sigma \int_{\mathbf{S}^{D-1}} p(v,v')(f(t,x,v') - f(t,x,v))dv'$$
$$f\Big|_{t=0} = f^{in},$$

where $\sigma > 0$ *and* $0 \le p \in L^2(\mathbf{S}^{D-1} \times \mathbf{S}^{D-1})$ *satisfies*

$$\int_{\mathbf{S}^{D-1}} p(v,v')dv' = \int_{\mathbf{S}^{D-1}} p(v',v)dv' = 1 \text{ a.e. in } v \in \mathbf{S}^{D-1}.$$

In particular, the Lorentz kinetic model cannot govern the Boltzmann-Grad limit of the particle density

STEP 1: SPECTRAL ARGUMENT FOR THE LINEAR BOLTZMANN EQUATION

With $\sigma > 0$ and p as above, consider the unbounded operator A on $L^2(\mathbf{T}^D \times \mathbf{S}^{D-1})$ defined by

$$(A\phi)(x,v) = -v \cdot \nabla_x \phi(x,v) - \sigma\phi(x,v) + \sigma \int_{\mathbf{S}^{D-1}} p(v,v')\phi(x,v')dv' \,,$$
$$D(A) = \{\phi \in L^2(\mathbf{T}^D \times \mathbf{S}^{D-1}) \,|\, v \cdot \nabla_x \phi \in L^2(\mathbf{T}^D \times \mathbf{S}^{D-1})\} \,.$$

Theorem. (Ukai-Point-Ghidouche, 1979) *There exists positive constants C and γ such that*

$$\|e^{tA}\phi - \langle\phi\rangle\|_{L^2(\mathbf{T}^D \times \mathbf{S}^{D-1})} \le Ce^{-\gamma t}\|\phi\|_{L^2(\mathbf{T}^D \times \mathbf{S}^{D-1})} \,, \quad t \ge 0 \,,$$

for each $\phi \in L^2(\mathbf{T}^D \times \mathbf{S}^{D-1})$, where

$$\langle\phi\rangle = \frac{1}{|\mathbf{S}^{D-1}|} \iint_{\mathbf{T}^D \times \mathbf{S}^{D-1}} \phi(x,v)dxdv \,.$$

STEP 2: COMPARISON WITH THE CASE OF ABSORBING OBSTACLES

Assume that $f^{in} \equiv f^{in}(x) \geq 0$ on $\mathbf{T}^D$. Then

$$f_\epsilon(t,x,v) \geq g_\epsilon(t,x,v) = f^{in}(x-tv)\mathbf{1}_{Y_\epsilon}(x)\mathbf{1}_{\epsilon^{D-1}\tau_\epsilon(x/\epsilon^{D-1},v)>t}$$

Indeed, g is the density of particles with the SAME initial data as f, but assuming that each particle DISAPPEAR when colliding with an obstacle instead of being REFLECTED.

Then

$$\mathbf{1}_{Y_\epsilon}(x) \to 1 \text{ a.e. on } \mathbf{T}^D \text{ and } |\mathbf{1}_{Y_\epsilon}(x)| \leq 1$$

while, after extracting a subsequence if needed,

$$\mathbf{1}_{\epsilon^{D-1}\tau_\epsilon(x/\epsilon^{D-1},v)>t} \rightharpoonup \Psi(t,v) \text{ in } L^\infty(\mathbf{R}_+ \times \mathbf{T}^D \times \mathbf{S}^{D-1}) \text{ weak-*}$$

Therefore, if f is a weak-* limit point of f_ϵ in $L^\infty(\mathbf{R}_+ \times \mathbf{T}^D \times \mathbf{S}^{D-1})$ as $\epsilon \longrightarrow 0$

$$f(t,x,v) \geq f^{in}(x-tv)\Psi(t,v)$$

STEP 3: USING THE BGW LOWER BOUND ON THE DISTRIBUTION OF τ_r

Therefore, denoting dv the uniform probability measure on $\mathbf{S}^{D-1}$

$$\begin{aligned}\iint_{\mathbf{T}^D\times\mathbf{S}^{D-1}} f(t,x,v)^2 dxdv &\geq \iint_{\mathbf{T}^D\times\mathbf{S}^{D-1}} f^{in}(x-tv)^2\Psi(t,v)^2 dxdv \\ &= \int_{\mathbf{T}^D} f^{in}(y)^2 dy \int_{\mathbf{S}^{D-1}} \Psi(t,v)^2 dv \\ &\geq \|f^{in}\|^2_{L^2(\mathbf{T}^D)} \left(\int_{\mathbf{S}^{D-1}} \Psi(t,v)dv\right)^2 \\ &= \|f^{in}\|^2_{L^2(\mathbf{T}^D)}\Phi(t)^2\end{aligned}$$

By the BGW lower bound on the distribution Φ of free path lengths

$$\|f(t,\cdot,\cdot)\|_{L^2(\mathbf{T}^D\times\mathbf{S}^{D-1})} \geq \frac{C_D}{t}\|f^{in}\|_{L^2(\mathbf{T}^D)}\,, \quad t > 1\,.$$

On the other hand, by the spectral estimate, if f is a solution of the linear Boltzmann equation, one has

$$\|f(t,\cdot,\cdot)\|_{L^2(\mathbf{T}^D\times\mathbf{S}^{D-1})} \leq \int_{\mathbf{T}^D} f^{in}(y)dy + Ce^{-\gamma t}\|f^{in}\|_{L^2(\mathbf{T}^D)}$$

so that

$$\frac{C_D}{t} \leq \frac{\|f^{in}\|_{L^1(\mathbf{T}^D)}}{\|f^{in}\|_{L^2(\mathbf{T}^D)}} + Ce^{-\gamma t}$$

for each $t > 1$.

STEP 4: CHOICE OF INITIAL DATA

Pick ρ to be a bump function supported near $x = 0$ and such that

$$\int \rho(x)^2 dx = 1 \, .$$

Take f^{in} to be $x \mapsto \lambda^{D/2}\rho(\lambda x)$ periodicized, so that

$$\int_{\mathbf{T}^D} f^{in}(x)^2 dx = 1 \, , \text{ while } \int_{\mathbf{T}^D} f^{in}(y) dy = \lambda^{-D/2} \int \rho(x) dx \, .$$

For such initial data, the inequality above becomes

$$\frac{C_D}{t} \leq \lambda^{-D/2} \int \rho(x) dx + Ce^{-\gamma t}$$

Conclude by choosing λ so that

$$\lambda^{-D/2} \int \rho(x) dx < \sup_{t>1} \left(\frac{C_D}{t} - Ce^{-\gamma t} \right) > 0$$

Remarks:

•same result (and same proof) for any smooth obstacle shape included in a shell $\{x \in \mathbf{R}^D \,|\, C\epsilon^D < \text{dist}(x, \epsilon^{D-1}\mathbf{Z}^D) < C'\epsilon^D\}$

•same result (and same proof) if the specular reflection boundary condition is replaced by more general boundary conditions (absorption, diffuse reflection, accomodation...)

BUT introducing some stochasticity in the periodic problem can lead to a BG limit that is described by the Lorentz kinetic model.

Example (B. Wennberg and V. Ricci, 2004) in space dimension 2, take obstacles that are disks of radius r centered at the points of $r^{1/(2-\eta)}\mathbf{Z}^2$, assuming that $0 < \eta < 1$. Santalò's formula suggests that the free-path lengths scale like $r^{\eta/(2-\eta)} \to 0$.

Suppose the obstacles are removed independently with large probability — specifically, with probability $p = 1 - r^{\eta/(2-\eta)}$. In that case, the Lorentz equation governs the 1-particle density in the BG limit $r \to 0^+$.