Quantitative Compactness Estimates for Conservation Laws

François Golse
Ecole Polytechnique, Paris
golse@math.polytechnique.fr

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MOTIVATION

•Consider a parabolic PDE of the form

$$\begin{cases} \partial_t u + \partial_x f(u) = & \epsilon \partial_x^2 u\,, \quad x \in \mathbf{R}\,,\ t > 0 \\ u\big|_{t=0} = & u^{in} \end{cases}$$

For each $\epsilon > 0$, the energy equality

$$\int_{\mathbf{R}} \tfrac{1}{2} u(t,x)^2 dx + \epsilon \int_0^t \int_{\mathbf{R}} \partial_x u(s,x)^2 dx ds = \int_{\mathbf{R}} \tfrac{1}{2} u^{in}(x)^2 dx$$

gives us a bound on u in $L^\infty_t(L^2_x) \cap L^2_t(\dot{H}^1_x)$ so that the solution map

$$L^2_x \ni u^{in} \mapsto u \in L^2_{loc}(dtdx)$$

is compact by Rellich's theorem.

•What remains of this compactness in the limit as $\epsilon \to 0^+$ — that is, for entropy solutions of the inviscid equation?

•Consider the conservation law

$$\left\{ \begin{aligned} \partial_t u + \partial_x f(u) = & \ 0\,, \quad x \in \mathbf{R}\,,\ t > 0 \\ u\Big|_{t=0} = & \ u^{in} \end{aligned} \right.$$

with strictly convex flux $f \in C^1(\mathbf{R})$ such that $f'(z) \to \pm\infty$ as $z \to \pm\infty$.

Two compactness results:

•(P.D. Lax, 1954) For each $t > 0$, the entropy solution dynamics

$$u^{in} \mapsto u(t, \cdot)$$

is compact from L^1_x into $L^1_{loc}(dx)$

•(L. Tartar, 1979) Compensated compactness (entropy bound + div-curl)
$\Rightarrow$ convergence of the vanishing viscosity method

Both arguments are based on the fact that

$$u_n \rightharpoonup u \quad \text{and} \quad F(u_n) \rightharpoonup F(u)$$

for some suitable class of nonlinearities F implies that

$$u_n \longrightarrow u \quad \text{STRONGLY}$$

QUESTION (P.D. LAX, 2002): can one transform such arguments into quantitative compactness or regularity estimates?

Part 1: ϵ-entropy estimate for scalar conservation laws

Joint work with C. De Lellis

Let $f \in C^2(\mathbf{R})$ with $f'' \geq a > 0$, and s.t.(WLOG) $f(0) = f'(0) = 0$.

$$\begin{cases} \partial_t u + \partial_x f(u) = 0\,, & x \in \mathbf{R}\,,\ t > 0 \\ u\big|_{t=0} = u^{in} \end{cases}$$

Entropy solution semigroup $S(t) : u^{in} \mapsto u(t, \cdot)$; it satisfies the

Lax-Oleinik one-sided estimate: $\partial_x \left(S(t) u^{in} \right) \leq \frac{1}{at}\,, \quad t > 0$

Definition (Kolmogorov-Tikhomirov, 1959) For $\epsilon > 0$, the ϵ-entropy of E precompact in the metric space (X, d) is :

$$H(E|X) = \log_2 N_\epsilon(E)$$

where $N_\epsilon(E)$ is the minimal number of sets in an ϵ-covering of E — i.e. a covering of E by sets of diameter $\leq 2\epsilon$ in X

Example: $H_\epsilon([0,1]^n|\mathbf{R}^n) \simeq n|\log_2 \epsilon|$

For each $R, m, t > 0$, the set $\left\{ u\Big|_{[-R,R]} \text{ s.t. } u \in S(t)\overline{B_{L^1(\mathbf{R})}(0,m)} \right\}$ is precompact in $L^1([-R,R])$ (P.D. Lax, 1954)

Theorem. (C. DeLellis, F.G. 2005) *For each* $\epsilon > 0$, *one has*

$$H_\epsilon\left(S(t)\overline{B_{L^1(\mathbf{R})}(0,m)} | L^1([-R,R])\right) \le \frac{C_1(t)}{\epsilon} + 2\log_2\left(\frac{C_2(t)}{\epsilon} + C_3(t)\right)$$

where

$$C_1(t) = \tfrac{32R^2}{at} + 32RM(t)\,, \quad C_3(t) = 3 + \frac{2tM(t)c_M(t)}{R + \sqrt{mat}}$$

$$C_2(t) = \tfrac{8R}{at}\left(R + \sqrt{mat} + 2tM(t)c_{M(t)}\right)$$

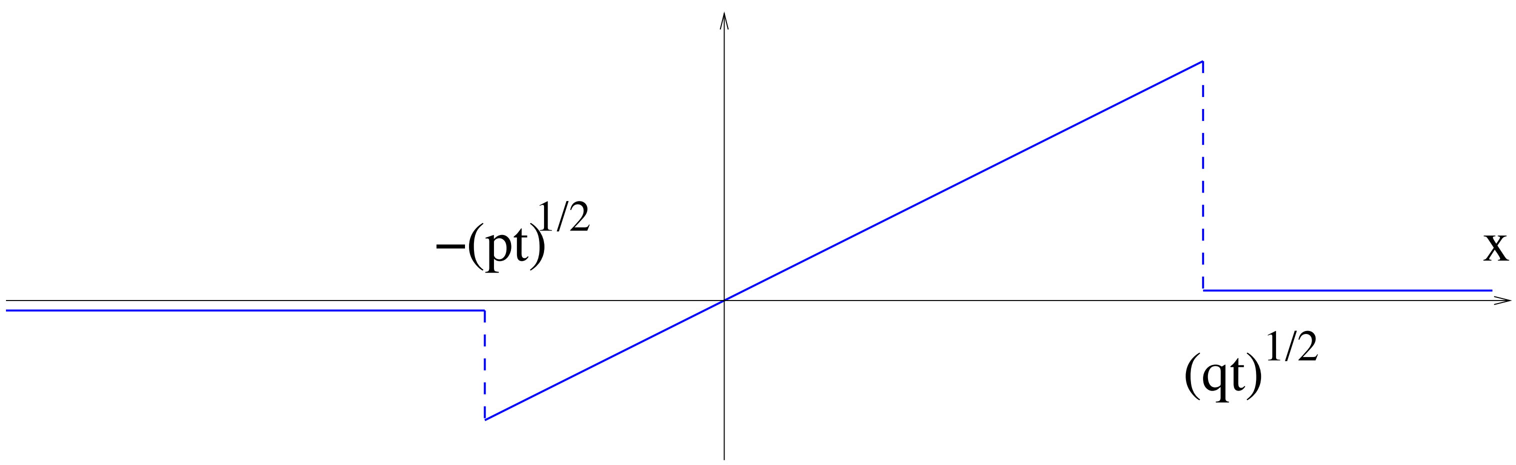
and with the notations $M(t) = \sqrt{\frac{4m}{at}}$ *and* $c_M = \sup_{|z|\le M} f''(z)$.

•(P.D. Lax, 1957) In the limit as $t \to +\infty$, one has $S(t)u^{in} - N_{p,q}(t) \to 0$ in $L^1(\mathbf{R})$ where $N_{p,q}$ is the N-wave

$$N_{p,q}(t) = \begin{cases} x/f''(0)t & \text{if } -\sqrt{pt} < x < \sqrt{qt} \\ 0 & \text{otherwise} \end{cases}$$

and where

$$p = -2f''(0)\inf_y \int_{-\infty}^{y} u^{in}, \quad q = 2f''(0)\sup_y \int_y^{\infty} u^{in}$$



•Hence, in the limit as $\epsilon \to 0^+$, one has

$$\overline{\lim_{t\to+\infty}}\text{(resp. } \underline{\lim_{t\to+\infty}})H_\epsilon(S(t)\overline{B_{L^1(\mathbf{R})}(0,m)}|L^1(\mathbf{R})) \sim 2|\log_2 \epsilon|$$

•Our bound on the ϵ-entropy does not capture this behavior; yet it shows that

$$\overline{\lim_{t\to+\infty}} H_\epsilon(S(t)\overline{B_{L^1(\mathbf{R})}(0,m)}|L^1([-R(t),R(t)])) = O(1)$$

as $\epsilon \to 0^+$ whenever $R(t) = o(\sqrt{t})$; consistent with the fact that the dependence of the N-wave in p, q can be seen only on intervals of length at least $O(\sqrt{t})$

Motivation: P.D. Lax advocated using ϵ-entropy estimates for defining a notion of resolving power of a numerical scheme for the conservation law

$$\partial_t u + \partial_x f(u) = 0$$

Part 2: regularity by compensated compactness

Scalar conservation laws in space dimension 1

Let $f \in C^2(\mathbf{R})$ with $f'' \geq a > 0$, and s.t.(WLOG) $f(0) = f'(0) = 0$;

$$\begin{cases} \partial_t u + \partial_x f(u) = 0\,, & x \in \mathbf{R}\,,\ t > 0 \\ u\big|_{t=0} = u^{in} & \end{cases}$$

An adaptation of Tartar's compensated compactness method leads to

Theorem. *For each* $u^{in} \in L^\infty(\mathbf{R})$ *s.t.* $u^{in}(x) = 0$ *a.e. in* $|x| \geq R$*, the entropy solution* $u \in B^{1/4,4}_{\infty,loc}(\mathbf{R}^*_+ \times \mathbf{R})$*, i.e.*

$$\int_0^\infty \int_{\mathbf{R}} \chi(t,x)^2 |u(t,x) - u(t+s,x+y)|^4 dxdt = O(|s| + |y|)$$

for each $\chi \in C^1_c(\mathbf{R}^*_+ \times \mathbf{R})$

•<u>DEGENERATE CONVEX FLUXES:</u> assume that $f \in C^2(\mathbf{R})$ satisfies

$$f''(v) > 0 \text{ for each } v \in \mathbf{R} \setminus \{v_1, \ldots, v_n\}$$
$$f''(v) \geq a_k |v - v_k|^{2\beta_k} \text{ for each } v \text{ near } v_k, \text{ for } k = 1, \ldots, n$$

for some $v_1, \ldots, v_n \in \mathbf{R}$ and $a_1, \beta_1, \ldots, a_n, \beta_n > 0$.

Theorem. *For each $u^{in} \in L^\infty(\mathbf{R})$ s.t. $u^{in}(x) = 0$ a.e. in $|x| \geq R$, the entropy solution $u \in B^{1/p,p}_{\infty,loc}(\mathbf{R}^*_+ \times \mathbf{R})$, with $p = 2 \max_{1 \leq k \leq n} \beta_k + 4$ i.e.*

$$\int_0^\infty \int_{\mathbf{R}} \chi(t,x)^2 |u(t,x) - u(t+s, x+y)|^p dxdt = O(|s| + |y|)$$

for each $\chi \in C^1_c(\mathbf{R}^*_+ \times \mathbf{R})$

Comparison with known results

•Lax-Oleinik estimate $\Rightarrow u \in BV_{loc}(\mathbf{R}_+^* \times \mathbf{R})$ (specific to scalar conservation laws, space dimension 1, and $f'' \geq a > 0$)

•Perthame-Jabin (2002) prove that $u \in W_{loc}^{s,p}(\mathbf{R}_+^* \times \mathbf{R})$ for $s < \frac{1}{3}$ and $1 \leq p < \frac{5}{2}$. Proof based on kinetic formulation + velocity averaging; generalizes to degenerate fluxes, higher space dimensions + one particular 2×2 system in space dimension 1 (isentropic Euler with $\gamma = 3$.)

•DeLellis-Westdickenberg (2003) prove that one cannot obtain better regularity than $B_\infty^{1/r,r}$ for $r \geq 3$ or $B_r^{1/3,r}$ for $1 \leq r < 3$ by using only the fact that the entropy production is a bounded Radon measure without using that it is a positive measure — as does the Perthame-Jabin, or our proof.

$\Rightarrow$ the compensated compactness method gives a regularity estimate in the DeLellis-Westdickenberg optimality class

Proof of regularity by compensated compactness

•Non degenerate case: $f'' \geq a > 0$ and (WLOG) $f(0) = f'(0) = 0$.

We shall only use the fact that the entropy solution u satisfies

$$\begin{aligned} \partial_t u + \partial_x f(u) &= 0 \\ \partial_t \tfrac{1}{2}u^2 + \partial_x g(u) &= -\mu \end{aligned}$$

where

$$g(v) := \int_0^v w f'(w) dw \text{ and } \iint_{\mathbf{R}_+ \times \mathbf{R}} |\mu| \leq \int_{\mathbf{R}} \tfrac{1}{2} |u^{in}|^2 dx < \infty$$

Notation: henceforth, we denote

$$\tau_{(s,y)}\phi(t,x) = \phi(t-s, x-y)\,, \quad \text{and } J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Step 1: the div-curl argument. Set

$$B = \begin{pmatrix} u \\ f(u) \end{pmatrix}, \qquad E = (\tau_{(s,y)} - I) \begin{pmatrix} \frac{1}{2}u^2 \\ g(u) \end{pmatrix}$$

One has

$$E, B \in L^\infty_{t,x}, \qquad \mathrm{div}_{t,x} B = 0, \quad \mathrm{div}_{t,x} E = \mu - \tau_{(s,y)}\mu$$

In particular, there exists

$$\pi \in \mathrm{Lip}(\mathbf{R}^*_+ \times \mathbf{R}), \quad \text{s.t. } B = J\nabla_{t,x}\pi$$

Integrating by parts shows that

$$\begin{aligned}\int_0^\infty \int_{\mathbf{R}} \chi^2 E \cdot J(\tau_{(s,y)}B - B)dtdx &= -\int_0^\infty \int_{\mathbf{R}} \chi^2 E \cdot \nabla_{t,x}(\tau_{(s,y)}\pi - \pi)dtdx \\ &= \int_0^\infty \int_{\mathbf{R}} \nabla_{t,x}\chi^2 \cdot E(\tau_{(s,y)}\pi - \pi)dtdx \\ &+ \int_0^\infty \int_{\mathbf{R}} \chi^2(\tau_{(s,y)}\pi - \pi)(\mu - \tau_{(s,y)}\mu)\end{aligned}$$

Therefore, one has the upper bound

$$\int_0^\infty \int_{\mathbf{R}} \chi^2 E \cdot J(\tau_{(s,y)} B - B) dt dx$$
$$\leq \left(\|\nabla_{t,x} \chi^2\|_{L^1} \|E\|_{L^\infty} + 2\|\chi^2\|_{L^\infty} \iint |\mu| \right) \mathrm{Lip}(\pi)(|s| + |y|)$$

which leads to an estimate of the form

$$\int_0^\infty \int_{\mathbf{R}} \chi^2 \Big((\tau_{(s,y)} u - u)(\tau_{(s,y)} g(u) - g(u))$$
$$- \tfrac{1}{2} (\tau_{(s,y)} u^2 - u^2)(\tau_{(s,y)} f(u) - f(u)) \Big) dt dx \leq C(|s| + |y|)$$

Next we shall give a lower bound for the integrand in the left-hand side.

Remark here the div-curl argument reduces to a simple integration by parts, since $\mathrm{div}_{t,x} B = 0$.

Step 2: a pointwise inequality

Lemma. *For each $v, w \in \mathbf{R}$, one has ($f'' \geq a > 0$)*

$$(w-v)(g(w)-g(v)) - \tfrac{1}{2}(w^2-v^2)(f(w)-f(v)) \geq \tfrac{a}{12}|w-v|^4$$

<u>Proof:</u> WLOG, assume that $v < w$, and write

$$(w-v)(g(w)-g(v)) - \tfrac{1}{2}(w^2-v^2)(f(w)-f(v)) =$$
$$\int_v^w d\xi \int_v^w \zeta f'(\zeta)d\zeta - \int_v^w \xi d\xi \int_v^w f'(\zeta)d\zeta = \int_v^w \int_v^w (\zeta-\xi)f'(\zeta)d\xi d\zeta$$
$$= \tfrac{1}{2}\int_v^w \int_v^w (\zeta-\xi)(f'(\zeta)-f'(\xi))d\xi d\zeta \geq \tfrac{a}{2}\int_v^w \int_v^w (\zeta-\xi)^2 d\xi d\zeta$$

Remark Tartar uses the flux f as entropy, together with Cauchy-Schwarz

$$(w-v)(h(w)-h(v)) \geq (f(w)-f(v))^2 \quad \text{with } h(v) := \int_0^v f'(w)^2 dw$$

which is OK since he is aiming at proving compactness, not regularity

Step 3: conclusion Putting together the upper bound for the integral in Step 1 and the lower bound for the integrand of the left hand side obtained in Step 2, we find that

$$\tfrac{a}{12} \int_0^\infty \int_{\mathbf{R}} \chi^2 |\tau_{(s,y)} u - u|^4 dt dx \leq C(|s| + |y|)$$

which is the announced $B^{1/4,4}_{\infty,loc}$ estimate for the entropy solution u. $\square$

Remark Here we have used only one convex entropy $\frac{1}{2}u^2$. By using all Krushkov entropies, the compensated compactness argument above leads to the optimal regularity estimate in $B^{1/3,3}_{\infty,loc}$ (B. Perthame)

1D Isentropic Euler system, $1 < \gamma < 3$

Unknowns: $\rho \equiv \rho(t,x)$ (density) and $u \equiv u(t,x)$ (velocity field)

$$\partial_t \rho + \partial_x(\rho u) = 0$$
$$\partial_t(\rho u) + \partial_x\left(\rho u^2 + \kappa \rho^\gamma\right) = 0$$

•Hyperbolic system of conservation laws, with characteristic speeds

$$\lambda_+ := u + \theta\rho^\theta > u - \theta\rho^\theta =: \lambda_- \,, \quad \text{with } \theta = \sqrt{\kappa\gamma} = \tfrac{\gamma-1}{2}$$

•Along any C^1 solution (ρ, u), this system can be put in diagonal form

$$\partial_t w_+ + \lambda_+ \partial_x w_+ = 0\,,$$
$$\partial_t w_- + \lambda_- \partial_x w_- = 0\,,$$

where $w_\pm \equiv w_\pm(\rho, u)$ are the Riemann invariants

$$w_+ := u + \rho^\theta > u - \rho^\theta =: w_-$$

•R. DiPerna (1983) proved that, for each initial data (ρ^{in}, u^{in}) satisfying

$$(\rho^{in} - \bar{\rho}, u^{in}) \in C_c^2(\mathbf{R}) \text{ and } \rho^{in} > 0$$

there exists an entropy (weak) solution (ρ, u) of the isentropic Euler system that satisfies the L^∞ bound

$$0 \leq \rho \leq \rho^* = \sup_{x \in \mathbf{R}} \left(\tfrac{1}{2}(w_+(\rho^{in}, u^{in}) - w_-(\rho^{in}, u^{in})\right)^{1/\theta}$$

$$\inf_{x \in \mathbf{R}} w_-(\rho^{in}, u^{in}) =: u_* \leq u \leq u^* := \sup_{x \in \mathbf{R}} w_+(\rho^{in}, u^{in})$$

•DiPerna's argument applies to $\gamma = 1 + \frac{2}{2n+1}$, for each $n \geq 1$; improvements by G.Q. Chen and, more recently, by P.-L. Lions, B. Perthame, P. Souganidis and E. Tadmor, by using a kinetic formulation of Euler's system

•<u>Problem:</u> is there a regularizing effect for isentropic Euler? what is the regularity of entropy solutions?

Admissible solutions

•**Weak entropies:** an entropy ϕ for the isentropic Euler system is called a "weak entropy" if $\phi\Big|_{\rho=0} = 0$.

Example: the energy $\mathcal{E}$, with energy flux $\mathcal{G}$:

$$\begin{cases} \mathcal{E}(U) &= \frac{1}{2}\rho u^2 + \frac{\kappa}{\gamma-1}\rho^\gamma \\ \mathcal{G}(U) &= u(\mathcal{E}(U) + \kappa\rho^\gamma) \end{cases} \qquad \text{where } U = \begin{pmatrix} \rho \\ \rho u \end{pmatrix}$$

•DiPerna's solutions are obtained from solutions of the parabolic system

$$\partial_t U_\epsilon + \partial_x F(U_\epsilon) = \epsilon \partial_{xx} U_\epsilon$$

in the limit as $\epsilon \to 0^+$. These solutions satisfy

$$\partial_t \mathcal{E}(U) + \partial_x \mathcal{G}(U) = -M \text{ with } M = \text{w-}\lim_{\epsilon \to 0} \epsilon D^2 \mathcal{E}(U_\epsilon) : \partial_x U_\epsilon^{\otimes 2} \geq 0$$

•Each weak entropy ϕ has its dissipation dominated by that of E:

$$|D^2\phi(U)| \leq C_{\phi,K} D^2\mathcal{E}(U) \text{ for } U \in K \text{ compact subset of } \mathbf{R}_+ \times \mathbf{R}$$

•Hence DiPerna solutions of Euler's system constructed as above satisfy, for each weak entropy ϕ, the entropy condition

$$\partial_t\phi(U) + \partial_x\psi(U) = -\mu[\phi]$$

where $\mu[\phi]$ is a bounded Radon measure verifying the bound

$$|\langle\mu[\phi], \chi\rangle| \leq C_{\phi,K}\langle M, \chi\rangle\,, \quad \chi \in C_c^\infty(\mathbf{R}_+ \times \mathbf{R})$$

where M is the energy dissipation.

Definition. *Let $\mathcal{O} \subset \mathbf{R}_+^* \times \mathbf{R}$ open. A weak solution $U = (\rho, \rho u)$ s.t.*

$$0 < \rho_* \leq \rho \leq \rho^* \quad \text{and} \quad u_* \leq u \leq u^* \text{ for } (t,x) \in \mathcal{O}$$

is called an <u>*admissible solution on $\mathcal{O}$*</u> *iff for each entropy ϕ,* <u>*weak or not*</u>*,*

$$\partial_t \phi(U) + \partial_x \psi(U) = -\mu[\phi]$$

is a Radon measure such that

$$\|\mu[\phi]\|_{\mathcal{M}_b(\mathcal{O})} \leq C(\rho_*, \rho^*, u_*, u^*) \|D^2\phi\|_{L^\infty([\rho_*,\rho^*]\times[u_*,u^*]} \int_{\mathcal{O}} M$$

•<u>Example:</u> any DiPerna solution whose viscous approximation U_ϵ satisfies the uniform lower bound

$$\rho_\epsilon \geq \rho_* > 0 \quad \text{on } \mathcal{O} \text{ for each } \epsilon > 0$$

is admissible on $\mathcal{O}$.

•Existence of admissible solutions in the large?

Theorem. *Assume that $\gamma \in (1,3)$ and let $\mathcal{O}$ be any open set in $\mathbf{R}_+^* \times \mathbf{R}$. Any admissible solution of Euler's system on $\mathcal{O}$ satisfies*

$$\iint_{\mathcal{O}} |(\rho,u)(t+s,x+y)-(\rho,u)(t,x)|^2 dxdt \leq \text{Const.}|\ln(|s|+|y|)|^{-2}$$

whenever $|s|+|y|<\frac{1}{2}$.

•In the special case $\gamma = 3$, the same method gives

Theorem. *Assume that $\gamma = 3$ and let $\mathcal{O}$ be any open set in $\mathbf{R}_+^* \times \mathbf{R}$. Any admissible solution of Euler's system on $\mathcal{O} \subset \mathbf{R}_+^* \times \mathbf{R}$ satisfies*

$$(\rho,u) \in B^{1/4,4}_{\infty,loc}(\mathcal{O})$$

•For $\gamma = 3$, by using the kinetic formulation and velocity averaging, one has (Lions-Perthame-Tadmor JAMS 1994, Jabin-Perthame COCV 2002)

$$\rho, \rho u \in W^{s,p}_{loc}(\mathbf{R}_+ \times \mathbf{R}) \text{ for all } s < \frac{1}{4},\ 1 \leq p \leq \frac{8}{5}$$

•The kinetic formulation for $\gamma \in (1, 3)$ is of the form

$$\partial_t \chi + \partial_x [(\theta \xi + (1-\theta) u(t,x))\chi] = \partial_{\xi\xi} m \qquad \text{with } m \geq 0$$
$$\text{and } \chi = [(w_+ - \xi)(\xi - w_-)]_+^{\lambda} \qquad \text{for } \lambda = \frac{3-\gamma}{2(\gamma-1)}$$

Because of the presence of $u(t,x)$ in the advection velocity — which is just bounded, not smooth — classical velocity averaging lemmas (Agoshkov, G-Lions-Perthame-Sentis, DiPerna-Lions-Meyer, ...) do not apply in this case

Main ideas in the proof

Step 1: the div-curl bilinear estimate A variant of Murat-Tartar div-curl lemma is the following bilinear estimate

$$\left| \iint \chi^2 E \cdot JB dt dx \right| \leq \text{harmless localization terms}$$
$$+ \|\chi E\|_{L^p} \|\chi \operatorname{div}_{t,x} E\|_{W^{-1,p'}} + \|\chi B\|_{L^p} \|\chi \operatorname{div}_{t,x} B\|_{W^{-1,p'}}$$

where J is the rotation of an angle $\frac{\pi}{2}$ and $p \in (1, \infty)$. Apply this with

$$E = (\tau_{(s,y)} - I) \begin{pmatrix} \phi_1(\rho, u) \\ \psi_1(\rho, u) \end{pmatrix} \qquad B = (\tau_{(s,y)} - I) \begin{pmatrix} \phi_2(\rho, u) \\ \psi_2(\rho, u) \end{pmatrix}$$

where (ϕ_1, ψ_1) and (ϕ_2, ψ_2) are two entropy pairs, while (ρ, u) is an admissible solution of isentropic Euler on $\mathcal{O}$, and $\operatorname{supp}(\chi)$ is a compact subset of $\mathcal{O}$

The admissibility condition implies that

$$\mathrm{div}_{t,x} E = -(\tau_{(s,y)} - I)\mu[\phi_1]\,, \quad \mathrm{div}_{t,x} B = -(\tau_{(s,y)} - I)\mu[\phi_2]$$

with

$$\|\mu[\phi_j]\|_{\mathcal{M}_b(\mathcal{O})} \leq C\|D^2\phi_j\|_{L^\infty([\rho_*,\rho^*]\times[u_*,u^*])}$$

where $0 < \rho_* \leq \rho \leq \rho^*$ and $u_* \leq u \leq u^*$ on $\mathcal{O}$. By Sobolev embedding $W^{r,p}(\mathbf{R}^2) \subset C(\mathbf{R}^2)$ for $r > \frac{2}{p}$; by duality

$$\|\chi\,\mathrm{div}_{t,x} E\|_{W^{-1,p'}} \leq C_r\|D^2\phi_j\|_{L^\infty([\rho_*,\rho^*]\times[u_*,u^*])}(|s| + |y|)^{1-r}$$

and likewise for B, so that

$$\left|\iint \chi^2 E\cdot JBdtdx\right| \leq C_r\|D^2\phi_j\|_{L^\infty([\rho_*,\rho^*]\times[u_*,u^*])}(|s| + |y|)^{1-r}$$

CONCLUSION:

Div-curl$\Rightarrow$ upper bound for integral of Tartar's equation

Step 2: the Tartar equation for Lax entropies Define

$$T[\phi_1,\phi_2](U,V) := (\phi_1(V)-\phi_1(U))(\psi_2(V)-\psi_2(U))$$
$$-(\psi_1(V)-\psi_1(U))(\phi_2(V)-\phi_2(U))$$

for two entropy pairs (ϕ_1,ψ_1) and (ϕ_2,ψ_2), so that

$$E\cdot JB = T[\phi_1,\phi_2](\tau_{(s,y)}(\rho,u),(\rho,u))$$

Therefore, for each $\chi\in C_c^1(\mathcal{O})$, step 1 leads to an upper bound for

$$\iint_{\mathbf{R}_+^*\times\mathbf{R}}\chi^2 T[\phi_1,\phi_2](\tau_{(s,y)}(\rho,u),(\rho,u))dtds = \iint_{\mathbf{R}_+^*\times\mathbf{R}}\chi^2 E\cdot JBdtds$$
$$\leq C_r\|D^2\phi_j\|_{L^\infty([\rho_*,\rho^*]\times[u_*,u^*])}(|s|+|y|)^{1-r}$$

As in case of a scalar conservation law, we need a lower bound of that same quantity.

•Use Lax entropies in Riemann invariant coordinates

$$\phi_\pm(w,k) = e^{kw_\pm}\left(A_0^\pm(w) + \frac{A_1^\pm(w)}{k} + \dots\right), \quad k \to \pm\infty$$
$$\psi_\pm(w,k) = e^{kw_\pm}\left(B_0^\pm(w) + \frac{B_1^\pm(w)}{k} + \dots\right), \quad w = (w_+, w_-)$$

•Such entropies exist for all strictly hyperbolic systems (Lax 1971): hence the need for the lower bound $\rho \geq \rho_* > 0$

•**Leading order term in Tartar's equation:** as $k \to +\infty$

$$T[\phi_+(\cdot,k),\phi_+(\cdot,-k)](U,V) = 2A_0^+(w(U))A_0^+(w(V))$$
$$\times(\lambda_+(U) - \lambda_+(V))\sinh(k(w_+(U) - w_+(V))) + \dots$$
$$T[\phi_+(\cdot,k),\phi_+(\cdot,-k)](U,V) = 2A_0^-(w(U))A_0^-(w(V))$$
$$\times(\lambda_-(U) - \lambda_-(V))\sinh(k(w_-(U) - w_-(V))) + \dots$$

At this point, we use two important features of Euler's isentropic system.

•Fact no.1: with $\theta = \frac{\gamma-1}{2}$,

$$\begin{pmatrix}\lambda_+\\ \lambda_-\end{pmatrix} = \mathcal{A}\begin{pmatrix}w_+\\ w_-\end{pmatrix} \text{ with } \mathcal{A} = \frac{1}{2}\begin{pmatrix}1+\theta & 1-\theta\\ 1-\theta & 1+\theta\end{pmatrix}$$

and for $\gamma \in (1,3)$ one has $\theta \in (0,1)$, leading to the coercivity estimate

$$\begin{pmatrix}\sinh(a)\\ \sinh(b)\end{pmatrix} \cdot \mathcal{A}\begin{pmatrix}a\\ b\end{pmatrix} \geq \theta\,(a\sinh(a) + b\sinh(b)) + (1-\theta) \times \text{ positive}$$

Suggests a lower bound on

$$a^2 T[\phi_+(\cdot,k),\phi_+(\cdot,-k)](U,V) + b^2 T[\phi_+(\cdot,k),\phi_+(\cdot,-k)](U,V)$$

provided that the leading order terms in Lax entropies are proportional:

$$aA_0^+(w) = bA_0^-(w)$$

•Fact no.2: Euler's isentropic system satisfies the relation

$$\partial_+ \left(\frac{\partial_- \lambda_+}{\lambda_+ - \lambda_-} \right) = \partial_- \left(\frac{\partial_+ \lambda_-}{\lambda_- - \lambda_+} \right)$$

Hence there exists a function $\Lambda \equiv \Lambda(w_+, w_-)$ such that

$$(\partial_+ \Lambda, \partial_- \Lambda) = \left(\frac{\partial_+ \lambda_-}{\lambda_- - \lambda_+}, \frac{\partial_- \lambda_+}{\lambda_+ - \lambda_-} \right)$$

so that one can take

$$A_0^+(w_+, w_-) = A_0^-(w_+, w_-) = e^{\Lambda(w_+, w_-)}$$

Here we choose

$$A_0(w_+, w_-) = (w_+ - w_-)^{\frac{1-\theta}{2\theta}}$$

Final remarks

•At variance with the original DiPerna argument (1983) for genuinely non-linear 2×2 system, the proof above is based on the leading order term in the Tartar equation — whereas DiPerna's argument uses the next to leading order term of the same equation

•Not all Lax entropies are convex, or weak entropies — i.e. vanish for $\rho = 0$. In order to control the entropy production

$$\partial_t \phi_\pm(w,k) + \partial_x \psi_\pm(w,k) =: -\mu_\pm^k$$

one needs locally admissible solutions

•Perhaps one can use only weak entropies — as in the original proof of compactness by DiPerna. This would require refining significantly the present argument.