From the kinetic theory of gases to continuum mechanics

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In memory of Carlo Cercignani (1939-2010)

The founding fathers



• J.C. Maxwell *On the Dynamical Theory of Gases*, Philosophical Trans. CLVII 1866

• *"Boltzmann's work on the principles of mechanics suggests the problem of developing mathematically the limiting processes which lead from the atomistic view to the laws of motion of continua"*

Hilbert's 6th problem, 1900

The Boltzmann equation

Unknown: the distribution function $F \equiv F(t,x,v) \geq 0$

If external forces are negligeable F satisfies the Boltzmann equation

$$\partial_t F + v \cdot \nabla_x F = \mathcal{C}(F)$$

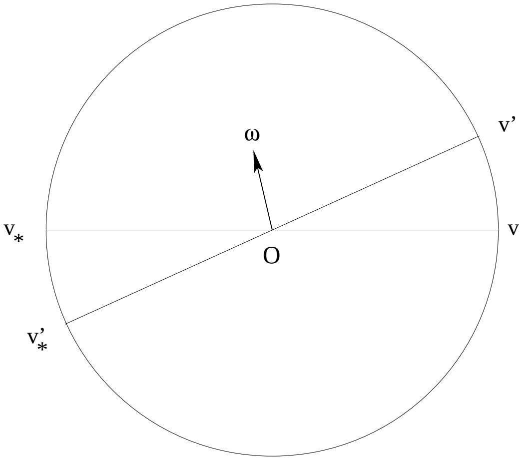
The collision integral is denoted by $\mathcal{C}(F)(t,x,v) := \mathcal{C}(F(t,x,\cdot))(v)$

$$\mathcal{C}(f)(v) = \frac{d^2}{2} \iint_{\mathbf{R}^3 \times \mathbf{S}^2} (f(v')f(v'_*) - f(v)f(v_*))|(v-v_*)\cdot\omega| dv_* d\omega$$

with the notation

$$\begin{cases} v' = v \ - (v - v_*)\cdot\omega\omega \\ v'_* = v_* + (v - v_*)\cdot\omega\omega \end{cases}$$

Geometry of collisions



Velocities v, v_*, v', v'_* in the center of mass reference frame

Local conservation laws

For all $f \equiv f(v)$ rapidly decaying as $|v| \to \infty$

$$\int_{\mathbf{R}^3} \mathcal{C}(f)dv = \int_{\mathbf{R}^3} \mathcal{C}(f)v_k dv = \int_{\mathbf{R}^3} \mathcal{C}(f)|v|^2 dv = 0\,, \quad k = 1,2,3$$

Thus, rapidly decaying solutions of the Boltzmann equation satisfy

$$\left\{\begin{aligned} \partial_t \int_{\mathbf{R}^3} Fdv + \operatorname{div}_x \int_{\mathbf{R}^3} vFdv &= 0 \quad \text{(mass)} \\ \partial_t \int_{\mathbf{R}^3} vFdv + \operatorname{div}_x \int_{\mathbf{R}^3} v \otimes vFdv &= 0 \quad \text{(momentum)} \\ \partial_t \int_{\mathbf{R}^3} \tfrac{1}{2}|v|^2 Fdv + \operatorname{div}_x \int_{\mathbf{R}^3} v\tfrac{1}{2}|v|^2 Fdv &= 0 \quad \text{(energy)} \end{aligned}\right.$$

Boltzmann's H Theorem and Maxwellian distributions

If $0 < f = O(|v|^{-m})$ for all $m > 0$ & $\ln f = O(|v|^n)$ for some $n > 0$ at ∞

$$\int_{\mathbf{R}^3} \mathcal{C}(f) \ln f dv \leq 0\,, \qquad = 0 \Leftrightarrow \mathcal{C}(f) = 0 \Leftrightarrow f \text{ Maxwellian}$$

i.e. there exists $\rho, \theta > 0$ and $u \in \mathbf{R}^3$ s.t.

$$f(v) = \mathcal{M}_{(\rho,u,\theta)}(v) := \frac{\rho}{(2\pi\theta)^{3/2}} \exp\left(-\frac{|v-u|^2}{2\theta}\right)$$

Solutions of the Boltzmann equation satisfy

$$\partial_t \int_{\mathbf{R}^3} F \ln F dv + \mathrm{div}_x \int_{\mathbf{R}^3} vF \ln F dv = \int_{\mathbf{R}^3} \mathcal{C}(f) \ln f dv \leq 0$$

Hilbert's expansion (1912)

Problem: study solutions of the Boltzmann equation that are slowly varying in the time and space variables

I.e. $F(t,x,v) = F_\epsilon(\epsilon t, \epsilon x, v)$, assuming $\partial_{\hat{t}} F_\epsilon\,,\ \nabla_{\hat{x}} F_\epsilon = O(1)$, with $(\hat{t},\hat{x}) = (\epsilon t, \epsilon x)$

Since F is a solution of the Boltzmann equation, one has

$$\partial_{\hat{t}} F_\epsilon + v \cdot \nabla_{\hat{x}} F_\epsilon = \frac{1}{\epsilon}\mathcal{C}(F_\epsilon)$$

Hilbert proposed to seek F_ϵ as a formal power series in ϵ with smooth coefficients:

$$F_\epsilon(\hat{t},\hat{x},v) = \sum_{n\geq 0} \epsilon^n F_n(\hat{t},\hat{x},v)$$

The compressible Euler limit

The leading order term in Hilbert's expansion is of the form

$$F_0(\hat{t},\hat{x},v)=\mathcal{M}_{(\rho,u,\theta)(\hat{t},\hat{x})}(v)$$

where (ρ,u,θ) is a solution of the compressible Euler system

$$\left\{\begin{aligned}\partial_{\hat{t}}\rho+\operatorname{div}_{\hat{x}}(\rho u)&=0\\ \rho(\partial_{\hat{t}}u+u\cdot\nabla_{\hat{x}}u)+\nabla_{\hat{x}}(\rho\theta)&=0\\ \partial_{\hat{t}}\theta+u\cdot\nabla_{\hat{x}}\theta+\tfrac{2}{3}\theta\operatorname{div}_{\hat{x}}u&=0\end{aligned}\right.$$

Proof by Caflisch (1980) using a truncated Hilbert expansion

Difficulties:

a) the truncated Hilbert expansion may be negative for some $\hat{t},\hat{x},v$
b) the k-th term in Hilbert's expansion is of order $F_k=O(|\nabla_{\hat{x}}^k F_0|)$
c) generic solutions of Euler's equations lose regularity in finite time

The Boltzmann equation with Maxwellian equilibrium at ∞

Notation: henceforth we set $M := \mathcal{M}_{(1,0,1)}$

Relative entropy

$$H(F|M) = \iint_{\mathbf{R}^3\times\mathbf{R}^3} \left[F \ln\left(\frac{F}{M}\right) - F + M \right] dxdv \quad (\geq 0)$$

Consider the Cauchy problem

$$\begin{cases} \partial_t F + v\cdot\nabla_x F = \mathcal{C}(F)\,, & (t,x,v) \in \mathbf{R}_+^* \times \mathbf{R}^3 \times \mathbf{R}^3 \\ F\big|_{t=0} = F^{in}\,, & F(t,x,v) \to M \quad \text{as } |x| \to +\infty \end{cases}$$

Weak formulation of $F \to M$ as $|x| \to \infty$: seek F such that

$$H(F|M) < +\infty$$

Renormalized solutions

Observation: for each $r > 0$, one has

$$\iint_{|x|+|v|\leq r} \frac{\mathcal{C}(F)}{\sqrt{1+F}} dvdx \leq C \iint_{|x|\leq r} (-\mathcal{C}(F) \ln F + (1+|v|^2)F) dxdv$$

Definition of renormalized solutions of the Boltzmann equation

$0 \leq F \in C(\mathbf{R}_+, L^1_{loc}(\mathbf{R}^3 \times \mathbf{R}^3))$ satisfying $H(F(t)|M) < +\infty$ and

$$M(\partial_t + v \cdot \nabla_x)\Gamma(F/M) = \Gamma'(F/M)\mathcal{C}(F)$$

in the sense of distributions, for each $\Gamma \in C^1(\mathbf{R}_+)$ s.t. $\Gamma'(Z) \leq \frac{C}{\sqrt{1+Z}}$

The DiPerna-Lions existence theorem

Thm. (DiPerna, P.-L. Lions, Masmoudi)

For each measurable $F^{in} \geq 0$ a.e. such that $H(F^{in}|M) < +\infty$, there exists a renormalized solution relative to M of the Boltzmann equation with initial data F^{in}. It satisfies

$$\begin{cases} \partial_t \int_{\mathbf{R}^3} F dv + \operatorname{div}_x \int_{\mathbf{R}^3} vF dv = 0 \\ \partial_t \int_{\mathbf{R}^3} vF dv + \operatorname{div}_x \int_{\mathbf{R}^3} v \otimes vF dv + \operatorname{div}_x m = 0 \end{cases}$$

with $m = m^T \geq 0$ and the entropy inequality

$$H(F(t)|M) + \int_{\mathbf{R}^3} \operatorname{tr} m(t) - \int_0^t \iint_{\mathbf{R}^3 \times \mathbf{R}^3} \mathcal{C}(F) \ln F ds dx dv \leq H(F^{in}|M)$$

Acoustic limit

Thm. [G.-Levermore, CPAM 2002]

Let F_ϵ be renormalized solutions of the Boltzmann equation with

$$F_\epsilon\Big|_{t=0} = \mathcal{M}_{(1+\delta_\epsilon\rho^{in}(\epsilon x),\delta_\epsilon u^{in}(\epsilon x),1+\delta_\epsilon\theta^{in}(\epsilon x))}$$

for $\rho^{in}, u^{in}, \theta^{in} \in L^2(\mathbf{R}^3)$ and $\delta_\epsilon |\ln \delta_\epsilon|^{1/2} = o(\sqrt{\epsilon})$. When $\epsilon \to 0$

$$\frac{1}{\delta_\epsilon}\int_{\mathbf{R}^3}\left(F_\epsilon\left(\frac{t}{\epsilon},\frac{x}{\epsilon},v\right) - M\right)(1, v, \tfrac{1}{3}|v|^2 - 1)dv \to (\rho, u, \theta)(t,x)$$

in $L^1_{loc}(dtdx)$ where

$$\left\{\begin{array}{ll} \partial_t\rho + \operatorname{div}_x u = 0\,, & \rho\Big|_{t=0} = \rho^{in}\,, \\ \partial_t u + \nabla_x(\rho+\theta) = 0\,, & u\Big|_{t=0} = u^{in}\,, \\ \tfrac{3}{2}\partial_t\theta + \operatorname{div}_x u = 0\,, & \theta\Big|_{t=0} = \theta^{in}\,. \end{array}\right.$$

Incompressible Euler limit

Thm. [StRaymond, ARMA2003]

Let $u^{in} \in H^3(\mathbf{R}^3)$ s.t. $\operatorname{div} u^{in} = 0$ and $u \in C([0,T]; H^3(\mathbf{R}^3))$ satisfy

$$\begin{cases} \partial_t u + u \cdot \nabla_x u + \nabla_x p = 0\,, \quad \operatorname{div}_x u = 0 \\ u\big|_{t=0} = u^{in} \end{cases}$$

Let F_ϵ be renormalized solutions of the B. equation with initial data

$$F_\epsilon\big|_{t=0} = \mathcal{M}_{(1,\delta_\epsilon u^{in}(\epsilon x),1)}$$

for $\delta_\epsilon = \epsilon^\alpha$ with $0 < \alpha < 1$. Then, in the limit as $\epsilon \to 0$, one has

$$\frac{1}{\delta_\epsilon}\int_{\mathbf{R}^3} vF_\epsilon\left(\frac{t}{\epsilon\delta_\epsilon}, \frac{x}{\epsilon}, v\right) dv \to u(t,x) \text{ in } L^\infty([0,T]; L^1_{loc}(\mathbf{R}^3))$$

Stokes limit

Thm. [G.-Levermore CPAM2002]

Let F_ϵ be renormalized solutions of the Boltzmann equation with

$$F_\epsilon\Big|_{t=0} = \mathcal{M}_{(1-\delta_\epsilon\theta^{in}(\epsilon x),\delta_\epsilon u^{in}(\epsilon x),1+\delta_\epsilon\theta^{in}(\epsilon x))}$$

where $\delta_\epsilon|\ln\delta_\epsilon| = o(\epsilon)$ and $(u^{in},\theta^{in}) \in L^2 \times L^\infty(\mathbf{R}^3)$ s.t. $\operatorname{div}_x u^{in} = 0$. Then, in the limit as $\epsilon \to 0$, one has

$$\frac{1}{\delta_\epsilon}\int_{\mathbf{R}^3}\left(F_\epsilon\left(\frac{t}{\epsilon^2},\frac{x}{\epsilon},v\right) - M\right)(v,\tfrac{1}{3}|v|^2 - 1)dv \to (u,\theta)(t,x) \text{ in } L^1_{loc}$$

where
$$\left\{\begin{array}{lll}\partial_t u + \nabla_x p = \nu\Delta_x u\,, & \operatorname{div}_x u = 0 & u\Big|_{t=0} = u^{in}\\ \frac{5}{2}\partial_t\theta = \kappa\Delta_x\theta\,, & & \theta\Big|_{t=0} = \theta^{in}\end{array}\right.$$

Viscosity/Heat conductivity

•The viscosity and heat conductivity are given by the formulas

$$\nu = \tfrac{1}{5}\mathcal{D}^*(v \otimes v - \tfrac{1}{3}|v|^2 I)\,, \qquad \kappa = \tfrac{2}{3}\mathcal{D}^*(\tfrac{1}{2}(|v|^2 - 5)v)$$

where $\mathcal{D}$ is the Dirichlet form of the linearized collision operator

$$\mathcal{D}(\Phi) = \tfrac{1}{2}\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} |\Phi + \Phi_* - \Phi' - \Phi'_*|^2 |(v - v_*)\cdot\omega| M M_* \, dv\, dv_*\, d\omega$$

•P.-L. Lions and N. Masmoudi (ARMA 2000) proved a version of the above theorem without deriving the heat equation for θ.

Navier-Stokes limit

Thm. [G-StRaymond Invent. Math. 2004 & JMPA2009

Let F_ϵ be renormalized solutions of the Boltzmann equation with

$$F_\epsilon\Big|_{t=0} = \mathcal{M}_{(1-\epsilon\theta^{in}(\epsilon x),\epsilon u^{in}(\epsilon x),1+\epsilon\theta^{in}(\epsilon x))}$$

where $(u^{in},\theta^{in}) \in L^2 \times L^\infty(\mathbf{R}^3)$ s.t. $\operatorname{div}_x u^{in} = 0$. For some $\epsilon_n \to 0$,

$$\frac{1}{\epsilon_n}\int_{\mathbf{R}^3}\left(F_{\epsilon_n}\left(\frac{t}{\epsilon_n^2},\frac{x}{\epsilon_n},v\right)-M\right)(v,\tfrac{1}{3}|v|^2-1)dv \rightharpoonup (u,\theta)(t,x) \text{ in } L^1_{loc}$$

where (u,θ) is a "Leray solution" with initial data (u^{in},θ^{in}) of

$$\left\{\begin{array}{ll}\partial_t u + \operatorname{div}_x(u\otimes u) + \nabla_x p = \nu\Delta_x u\,, & \operatorname{div}_x u = 0\\ \frac{5}{2}(\partial_t\theta + \operatorname{div}_x(u\theta)) = \kappa\Delta_x\theta & \end{array}\right.$$

Leray solution of the Navier-Stokes-Fourier system

Solution $(u,\theta)\in C(\mathbf{R}_+;w{-}L^2(\mathbf{R}^3))$ in the sense of distributions s.t.

Leray inequality

$$\tfrac{1}{2}\int_{\mathbf{R}^3}(|u|^2+\tfrac{5}{2}|\theta|^2)(t,x)dx+\int_0^t\int_{\mathbf{R}^3}(\nu|\nabla_x u|^2+\kappa|\nabla_x\theta|^2)dxds$$
$$\leq\tfrac{1}{2}\int_{\mathbf{R}^3}(|u^{in}|^2+\tfrac{5}{2}|\theta^{in}|^2)(t,x)dx$$

•Program started by Bardos-G.-Levermore (CPAM 1993); partial results by Lions-Masmoudi (ARMA 20001); weak cutoff potentials (hard and soft) by Levermore-Masmoudi (ARMA 2010)
•Smooth solutions: small data Bardos-Ukai (M3AS 1991); short time convergence DeMasi-Esposito-Lebowitz (CPAM 1990).

Boltzmann equation $\mathrm{Kn} = \epsilon \ll 1$		
von Karman relation $\mathrm{Ma}/\mathrm{Kn} = \mathrm{Re}$		
Ma	Sh	Hydrodynamic limit
$\delta_\epsilon \ll 1$	1	Acoustic system
$\delta_\epsilon \ll \epsilon$	ϵ	Stokes system
$\delta_\epsilon \gg \epsilon$	δ_ϵ	Incompressible Euler equations
ϵ	ϵ	Incompressible Navier-Stokes equations

Using the local conservation laws

Formal argument: Boltzmann equation $\Rightarrow$ local conservation laws

$$\partial_t \int_{\mathbf{R}^3} F_\epsilon \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} dv + \mathrm{div}_x \int_{\mathbf{R}^3} F_\epsilon v \otimes \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} dv = 0$$

Assuming that $F_\epsilon \to F$, Boltzmann's H Theorem implies

$$\int_0^\infty \iint \mathcal{C}(F) \ln F dx dv dt = 0 \quad \Rightarrow \quad F \equiv \mathcal{M}_{(\rho,u,\theta)(t,x)}(v)$$

Implies closure relations expressing

$$\int_{\mathbf{R}^3} F_\epsilon \begin{pmatrix} v \otimes v \\ \frac{1}{2} v|v|^2 \end{pmatrix} dv \quad \text{in terms of} \quad \int_{\mathbf{R}^3} F_\epsilon \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} dv$$

Vanishing of conservation defects

Difficulty: instead of the usual local conservation laws of mass, renormalized solutions of the Boltzmann equation satisfy

$$\partial_t \int_{\mathbf{R}^3} \Gamma\left(\frac{F_\epsilon}{M}\right) \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} M dv + \operatorname{div}_x \int_{\mathbf{R}^3} \Gamma\left(\frac{F_\epsilon}{M}\right) \begin{pmatrix} v \\ v \otimes v \\ \frac{1}{2}|v|^2 v \end{pmatrix} M dv$$
$$= \int_{\mathbf{R}^3} \Gamma'\left(\frac{F_\epsilon}{M}\right) \mathcal{C}(F_\epsilon) \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} dv$$

Problem: prove a) that r.h.s. $\to 0$ and b) that one recovers the usual conservation laws in the hydrodynamic limit $\epsilon \to 0$

Strong compactness tools

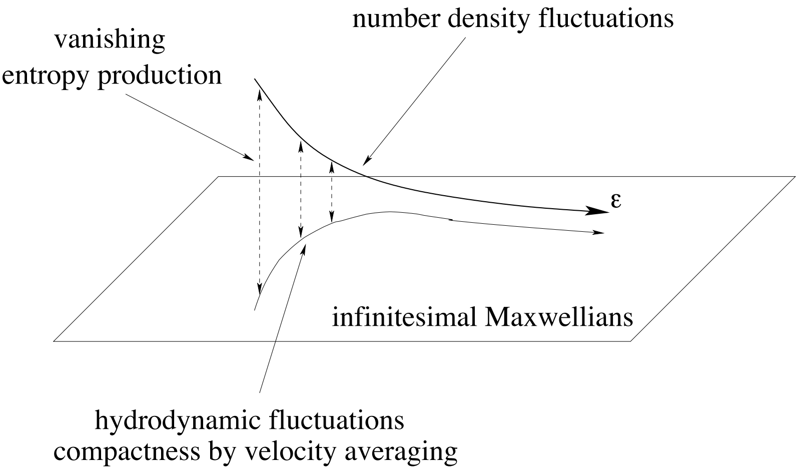
For the Navier-Stokes limit, STRONG compactness of number density fluctuations is needed to pass to the limit in nonlinearities

Thm. [F.G.-L. Saint-Raymond, CRAS 2002]

Assume that $f_n \equiv f_n(x,v)$ and $v \cdot \nabla_x f_n$ are bounded in $L^1(\mathbf{R}^N_x \times \mathbf{R}^N_v)$ while f_n is bounded in $L^1(\mathbf{R}^N_x; L^p(\mathbf{R}^N_v))$ for some $p > 1$. Then
a) f_n is weakly relatively compact in $L^1_{loc}(\mathbf{R}^N_x \times \mathbf{R}^N_v)$; and
b) for each $\phi \in C_c(\mathbf{R}^N)$ the sequence of velocity averages

$$\int_{\mathbf{R}^N} f_n(x,v)\phi(v)dv$$

is strongly relatively compact in $L^1_{loc}(\mathbf{R}^N)$


number density fluctuations
vanishing
entropy production
ε
infinitesimal Maxwellians
hydrodynamic fluctuations
compactness by velocity averaging

The relative entropy method: principle

Fact: In inviscid hydrodynamic limits, entropy production does **not** balance streaming $\Rightarrow$ velocity averaging fails

Idea: use regularity of the solution of the target equation + relaxation towards local equilibrium to prove compactness of fluctuations

Starting point: for u a smooth solution of the target equations — e.g. the incompressible Euler equations — study the evolution of

$$Z_\epsilon(t) = \frac{1}{\delta_\epsilon^2} H(F_\epsilon | \mathcal{M}_{(1,\delta_\epsilon u(\epsilon\delta_\epsilon t,\epsilon x),1)})$$

Idea of H.T. Yau (for Ginzburg-Landau lattice models 1993); later adapted to Boltzmann (F.G. 2000, Lions-Masmoudi 2001)

At the formal level, assuming the incompressible Euler scaling

$$\dot{Z}_\epsilon(t) = -\frac{1}{\delta_\epsilon^2}\int_{\mathbf{R}^3} \nabla_x u : \int_{\mathbf{R}^3} (v-\delta_\epsilon u)^{\otimes 2} F_\epsilon dvdx + \frac{1}{\delta_\epsilon}\int_{\mathbf{T}^3} \nabla_x p \cdot \int_{\mathbf{R}^3} (v-\delta_\epsilon u) F_\epsilon dvdx$$

The second term in the r.h.s. vanishes with ϵ since

$$\frac{1}{\delta_\epsilon}\int_{\mathbf{R}^3} vF_\epsilon\left(\frac{t}{\delta_\epsilon \epsilon}, \frac{x}{\epsilon}, v\right) dv \to \text{ divergence free field.}$$

Key idea: estimate the first term in the r.h.s. by Z_ϵ plus $o(1)$:

$$\frac{1}{\delta_\epsilon^2}\iint_{\mathbf{R}^3\times\mathbf{R}^3} \left|\nabla_x u : (v-\delta_\epsilon u)^{\otimes 2} F_\epsilon\right| dvdxds \le CZ_\epsilon(t) + o(1)$$

and conclude with Gronwall's inequality.

Conclusion

•**Global derivations** of fluid dynamic regimes from the Boltzmann equation without unphysical assumptions on the size or regularity of the data have been established by using
a) relative entropy and entropy production estimates, and
b) functional analytic methods in Lebesgue (L^p) spaces.

•At present, these methods leave aside the compressible Euler limit of the Boltzmann equation, or the asymptotic regime leading to the compressible Navier-Stokes equations. . .

•. . . as well as the case of the **steady Boltzmann equation** with some prescribed, external forcing term

Other open problems

•Even in weakly nonlinear regimes at the kinetic level, the relative entropy is not the solution to all problems. In several asymptotic regimes of the Boltzmann equation, the leading order and next to leading order fluctuations of the distribution function may interact to produce $O(1)$ effects in the fluid dynamic regime:

a) ghost effects (Sone 1996 H. Grad Lecture, Aoki & Kyoto school)

b) Navier-Stokes limit recovering viscous heating (Bobylev 1995, Bardos-Levermore-Ukai-Yang 2009)

c) hydrodynamic limits for thin layers of fluid (G. 2010)

Carlo Cercignani (1939-2010)

First H. Grad lecture “Mathematics and the Boltzmann equation"

Carlo Cercignani, la Sorbonne, 1992

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