From the N-body problem to the cubic NLS equation

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●Formal derivation by N.N. Bogolyubov (1947)— see for instance Landau-Lifshitz vol. 9, §25

●More recently, there have been rigorous derivations of nonlinear PDEs in the single particle phase-space from the linear, N-body problem. See for instance the derivation of the Boltzmann equation for a hard sphere gas by Lanford (1975) and then Illner-Pulvirenti (1986).

●Derivation of the Schrödinger-Poisson equation from the quantum N-body problem with Coulomb potential: Bardos-G-Mauser (2000), Erdös-Yau (2001)

●Work in collaboration with Riccardo Adami et Sandro Teta (preprint June 2005); space dimension 1, global in time.

●More recent preprint by L. Erdös and H.-T. Yau (preprint August 2005); space dimension 3, global in time.

The N-body Schrödinger equation

• Unknown: the N-particle wave function

$$\Psi_N \equiv \Psi_N(t, X_N) \in \mathbf{C}\,, \quad X_N = (x_1, \ldots, x_N) \in \mathbf{R}^N$$
$$\int_{\mathbf{R}^N} |\Psi_N(t, X_N)|^2 dX_N = 1$$

• Hamiltonian

$$\mathcal{H}_N := -\tfrac{1}{2}\Delta_N + U_N = \sum_{k=1}^{N} -\tfrac{1}{2}\partial^2_{x_j} + \sum_{1\le k<l\le N} U(x_k - x_l)$$

where the potential $U(z) = U(-z)$ is real-valued, compactly supported (hence short-range), smooth and nonnegative.

•Hence the wave function Ψ_N satisfies

$$i\partial_t \Psi_N = \mathcal{H}_N \Psi_N\,, \quad \text{soit}$$

$$i\partial_t \Psi_N = \sum_{k=1}^{N} -\tfrac{1}{2}\partial_{x_j}^2 \Psi_N + \sum_{1\le k<l\le N} U(x_k - x_l)\Psi_N$$

•In the sequel, all particles considered are bosons, meaning that the wave function Ψ_N is symmetrical in the variables x_k (Bose statistics):

$$\Psi_N(t, x_1, \ldots, x_N) = \Psi_N(t, x_{\sigma(1)}, \ldots, x_{\sigma(N)}) \text{ pour tout } \sigma \in \mathfrak{S}_N\,.$$

One easily checks the following: if $\Psi_N\Big|_{t=0}$ is symmetrical in the x_ks and Ψ_N solves the N-body Schrödinger equation, then $\Psi_N(t,\cdot)$ also is symmetrical in the x_ks for each $t \in \mathbf{R}$.

Scaling

•We shall be using two different scaling assumptions, as follows:

a) a collective scaling of mean-field type:

$$U(x_k - x_l) := \frac{1}{N} V_N(x_k - x_l)$$

so that the interaction potential per particle is

$$\tfrac{1}{2} \sum_{k \neq l} \frac{1}{N} V_N(x_k - x_l) |\Psi_N(X_N)|^2 = O(1)$$

b) and an ultra-short range scaling

$$V_N(z) := N^\gamma V(N^\gamma z) \text{ with } 0 < \gamma < \tfrac{1}{2},$$

and V nonnegative, even and smooth

•The total energy of the system of particles considered is

$$\langle H_N \Psi_N | \Psi_N \rangle = \sum_{k=1}^{N} \frac{1}{2} \|\partial_{x_k} \Psi_N\|_{L^2}^2$$
$$+ \sum_{1 \leq k < l \leq N} N^{\gamma-1} \int V(N^\gamma(x_k - x_l)) |\Psi_N(X_N)|^2 dX_N$$

We shall be using only wave functions for which

$$\langle H_N \Psi_N | \Psi_N \rangle = O(N)$$

Example: an important example of such wave functions is the case of a tensor product

$$\Psi_N(X_N) := \prod_{k=1}^{N} \psi(x_k) \text{ avec } \psi \in H^1(\mathbf{R})$$

Density matrix, marginals

•The density matrix is the integral operator on $L^2(\mathbf{R}^N)$ whose kernel is

$$\rho_N(t, X_N, Y_N) := \Psi_N(t, X_N)\overline{\Psi_N(t, Y_N)}$$

a standard notation for this operator is $\rho_N(t) = |\Psi_N(t,\cdot)\rangle\langle\Psi_N(t,\cdot)|$; it is a rank-one orthogonal projection.

•For each $1 \leq k < N$, defined the k-particle marginal of ρ_N to be

$$\rho_{N:k}(t, X_k, Y_k) := \int_{\mathbf{R}^{N-k}} \rho_N(t, X_k, Z_{k+1}^N, Y_k, Z_{k+1}^N) dZ_{k+1}^N$$

where $Z_{k+1}^N := (z_{k+1}, \ldots, z_N)$. We denote by $\rho_{N:k}(t)$ the associated integral operator; it is a nonnegative, trace-class operator with trace equal to 1.

Theorem. *Let* $0 \le V \in C^\infty(\mathbf{R})$ *and* $\gamma \in (0, \frac{1}{2})$*; assume there exists* $M > 0$ *such that*

$$\Psi_N\Big|_{t=0} \equiv \prod_{k=1}^{N} \psi^{in}(x_k)\,, \quad \text{with } \left\langle (-\Delta_N)^n \Psi_N\Big|_{t=0} \Big| \Psi_N\Big|_{t=0} \right\rangle \le M^n N^n$$

for $n = 1, \ldots, N$*. Then, for all* $t \ge 0$*, the sequence of single-particle marginals*

$$\rho_{N:1}(t,x,y) \to \psi(t,x)\overline{\psi(t,y)} \text{ as } N \to \infty$$

in Hilbert-Schmidt norm, where ψ *solves*

$$i\partial_t\psi + \tfrac{1}{2}\partial_x^2\psi - \alpha|\psi|^2\psi = 0\,, \quad \text{with } \alpha := \int_{\mathbf{R}} V(x)dx$$

$$\psi\Big|_{t=0} = \psi^{in}$$

BBGKY hierarchy

•We shall be writing a sequence of equations satisfied by the sequence of marginals $\rho_{N:j}$, where $j = 1, \dots, N$. Start from the von Neumann equation satisfied by ρ_N:

$$i\partial_t \rho_N = [\mathcal{H}_N, \rho_N]$$

•In that equation, set $x_2 = y_2 = z_2, \dots, x_N = y_N = z_N$, and integrate in $z_2, \dots, z_N$:

$$\begin{aligned} &i\partial_t \rho_{N:1} + \tfrac{1}{2}(\partial^2_{x_1} - \partial^2_{y_1})\rho_{N:1} \\ &= (N-1)\int [U(x_1 - z) - U(y_1 - z)]\rho_{N:2}(t, x_1, z, y_1, z)dz \end{aligned}$$

We recall that $U(z) = N^{\gamma-1}V(N^\gamma z)$ avec $0 < \gamma < \frac{1}{2}$.

•For $j = 2, \ldots, N-1$, the analogous equation is

$$
\begin{aligned}
i\partial_t \rho_{N:j} + \tfrac{1}{2}\sum_{k=1}^{j} (\partial^2_{x_k} - \partial^2_{y_k})\rho_{N:j} \\
= (N-j)\sum_{k=1}^{j}\int [U(x_k - z) - U(y_k - z)]\rho_{N:j+1}(t, X_k, z, Y_k, z)dz \\
+ \sum_{1\le k<l\le j} [U(x_k - x_l) - U(y_k - y_l)]\rho_{N:j}(t, X_k, Y_k)
\end{aligned}
$$

•For $j = N$, this equation is nothing but the von Neumann equation for the N-body density matrix ρ_N.

•Conceptually, it is advantageous to deal with infinite hierarchies of equations: in the sequel, we set $\rho_{N:j} = 0$ whenever $j > N$.

•By passing to the limit (at the formal level) in the BBGKY hierarchy as $N \to \infty$ and for j fixed; remember that

$$U(z) = \frac{1}{N} V_N(z) \text{ and that } V_N \to \alpha\delta_{z=0} \text{ with } \alpha = \int V(x)dx$$

•Assuming that $\rho_{N:j} \to \rho_j$ for $N \to \infty$, we find that

$$i\partial_t \rho_j + \frac{1}{2}\sum_{k=1}^{j} (\partial_{x_k}^2 - \partial_{y_k}^2)\rho_j$$
$$= \alpha \sum_{k=1}^{j} \int [\delta(x_k - z) - \delta(y_k - z)]\rho_{j+1}(t, X_k, z, Y_k, z)dz$$

Unlike in the case of the BBGKY hierarchy, $j \geq 1$ is unlimited, so that this new hierarchy has infinitely many equations.

•Let ψ be a smooth solution of the cubic NLS equation

$$i\partial_t\psi + \frac{1}{2}\partial_x^2\psi = \alpha|\psi|^2\psi$$

Define then

$$\rho_j(t, X_j, Y_j) := \prod_{k=1}^{j} \psi(t, x_k)\overline{\psi(t, y_k)}$$

We find that

$$i\partial_t\rho_1 + \frac{1}{2}(\partial_x^2 - \partial_y^2)\rho_1 = \alpha(\rho_1(t, x, x) - \rho_1(t, y, y))\rho_1$$

•More generally, a straightforward computation shows that

the sequence ρ_j soves the inifinite hierarchy

•This suggests the following strategy, inspired from the derivation by Lanford of the Boltzmann equation from the classical N-body problem:

a) for the N-body Schrödinger equation, pick the initial data

$$\Psi_N\Big|_{t=0} := \prod_{k=1}^{N} \psi\Big|_{t=0}(x_k)\,;$$

b) show that the sequence of marginals $\rho_{N:j} \to \rho_j$ as $N \to \infty$ and for each fixed j in some suitable sense; next show that ρ_j solves the infinite hierarchy by passing to the limit in the BBGKY hierarchy;

c) prove that the infinite hierarchy has a unique solution which implies that

$$\rho_j(t, X_j, Y_j) = \prod_{k=1}^{j} \psi(t, x_k)\overline{\psi(t, y_k)}$$

where ψ is the solution of cubic NLS.

An abstract uniqueness argument

- Consider the infinite hierarchy of equations

$$u_n' + A_n u_n = L_{n,n+1} u_{n+1}\,, \quad u_n(0) = 0\,, \qquad n \geq 1$$

where u_n takes its values in a Banach space E_n; here the linear operator $L_{n,n+1}$ belongs to $\mathcal{L}(E_{n+1}, E_n)$ while A_n is the generator of a one-parameter group of isometries $U_n(t)$ on E_n.

- Defining $v_n(t) := U_n(-t)u_n(t)$, one sees that

$$v_n'(t) = U_n(-t)L_{n,n+1}U_{n+1}(t)v_{n+1}(t)\,,$$
$$u_n(0) = 0\,.$$

Lemma. *Assume there exists* $C > 0$ *and* $R > 0$ *s.t.*

$$\|L_{n,n+1}\|_{\mathcal{L}(E_{n+1},E_n)} \le Cn \text{ and } \|u_n(t)\|_{E_n} \le R^n$$

for each $n \ge 1$ *and each* $t \in [0, T]$.

Then $u_n \equiv 0$ *on* $[0, T]$ *for each* $n \ge 1$.

Proof: Consider the decreasing scale of Banach spaces

$$B_r := \left\{ v = (v_n)_{n\ge 0} \in \prod_{n\ge 1} E_n \mid \|v\|_r = \sum_{n\ge 1} r^n \|v_n\|_{E_n} < +\infty \right\}$$

and set

$$F(v) := (U_n(-t)L_{n,n+1}U_{n+1}(t)v_{n+1})_{n\ge 1} .$$

•A straightforward computation shows that

$$\|F(v)\|_{r_1} \leq C \sum_{n\geq 1} n r_1^n \|v_n\|_{E_n} \leq C \sum_{n\geq 1} \frac{r^{n+1}-r_1^{n+1}}{r-r_1} \|v_n\|_{E_n} \leq \frac{C\|v\|_r}{r-r_1}$$

We conclude by applying the abstract variant of the Cauchy-Kowalewski theorem proved by Nirenberg and Ovsyanikov.

The key idea is to view B_r as the analogue of the class of functions with holomorphic extension to a strip of width r. The estimate above is similar to Cauchy's inequality bearing on the derivative of a holomorphic function. Hence F behaves like a differential operator of order 1.

Interaction estimate

•The first difficulty is to find Banach spaces E_n such that the interaction term $L_{n,n+1}$ is bounded by $O(n)$.

•Set $S_j := (1 - \partial_{x_j})^{1/2}$; define

$$E_n := \{\rho_n \in \mathcal{L}(L^2(\mathbf{R}^n)) \mid S_1 \dots S_n \rho_n S_1 \dots S_n \text{ is Hilbert-Schmidt}\}$$

which is a Hilbert space for the norm

$$\|\rho_n\|_{E_n} := \|S_1 \dots S_n \rho_n S_1 \dots S_n\|_{\mathcal{L}^2}$$
$$= \left(\iint \left| \prod_{j=1}^{n} (1 - \partial_{x_j})^{1/2} (1 - \partial_{y_j})^{1/2} \rho_n(X_n, Y_n) \right|^2 dX_n dY_n \right)^{1/2}$$

Proposition. *Let $\rho \in E_{n+1}$ and U be a tempered distribution whose Fourier transform is bounded on $\mathbf{R}$. Let σ be the integral operator with kernel*

$$\sigma(X_n, Y_n) := \int U(x_1 - z)\rho(X_n, z, Y_n, z)dz$$

Then

$$\|\sigma\|_{E_n} \leq C\|\hat{U}\|_{L^\infty}\|\rho\|_{E_{n+1}}$$

• In the BBGKY hierarchy, the operator $L_{n,n+1}$ is the sum of $2n$ terms analogous to the one treated in the proposition above. Hence

$$\|L_{n,n+1}\|_{\mathcal{L}(E_{n+1},E_n)} \leq Cn\|V\|_{L^1}$$

Sketch of the proof: Do it for the limiting interaction $U = \delta_0$. Then

$$\sigma(X_n, Y_n) = \rho_{n+1}(X_n, Y_n, x_1, x_1)$$

If ρ_{n+1} was the $n+1$st fold tensor product of functions of a single variable, the inequality that we want to prove reduces to the fact that $H^1(\mathbf{R})$ is an algebra.

The same proof (in Fourier space variables) works for the restriction of functions of arbitrarily many variables to a subspace of arbitrary codimension, provided that cross-derivatives of these functions are bounded in L^2 — this is different from the trace problem for functions in $H^1(\mathbf{R}^n)$.

This proof extends to the case where $\widehat{U}$ is an arbitrary function in L^∞

An elementary computation shows that

$$\widehat{\sigma}(\Xi_n, H_n) = \iint \widehat{\rho}_{n+1}(\xi_1 - k, \Xi_2^n, k - l; H_n, l)\frac{dkdl}{4\pi^2}$$

●Set

$$\Gamma_n(\Xi_n) := \prod_{k=1}^{n} \sqrt{1 + \xi_k^2}$$

we seek to estimate

$$\|\sigma\|_{E_n}^2 = \iint \Gamma_n(\Xi_n)^2 \Gamma_n(H_n)^2 |\widehat{\sigma}(\Xi_n, H_n)|^2 \frac{d\Xi_n dH_n}{(2\pi)^{2n}}$$

•Since

$$\Gamma_1(\xi_1) \le (\Gamma_1(\xi_1 - k) + \Gamma_1(k - l) + \Gamma_1(l)))$$

it follows that

$$\begin{aligned}\frac{1}{3}\Gamma_1(\xi_1)^2 \left| \iint \widehat{\rho}_{n+1}(\xi_1 - k, \Xi_2^n, k - l; H_n, l)\frac{dkdl}{4\pi^2}\right|^2 \\ \le \left| \iint \Gamma_1(\xi_1 - k)\widehat{\rho}_{n+1}(\xi_1 - k, \Xi_2^n, k - l; H_n, l)\frac{dkdl}{4\pi^2}\right|^2 \\ + \left| \iint \Gamma_1(k - l)\widehat{\rho}_{n+1}(\xi_1 - k, \Xi_2^n, k - l; H_n, l)\frac{dkdl}{4\pi^2}\right|^2 \\ + \left| \iint \Gamma_1(l)\widehat{\rho}_{n+1}(\xi_1 - k, \Xi_2^n, k - l; H_n, l)\frac{dkdl}{4\pi^2}\right|^2\end{aligned}$$

•By the Cauchy-Schwarz inequality

$$\left|\iint \Gamma_1(\xi_1-k)\widehat{\rho}_{n+1}(\xi_1-k,\Xi_2^n,k-l,H_n,l)\frac{dkdl}{4\pi^2}\right|^2$$
$$\leq C\iint|\widehat{\rho}_{n+1}(\xi_1-k,\Xi_2^n,k-l;H_n,l)|^2\frac{\Gamma_1(\xi_1-k)^2\Gamma_1(k-l)^2\Gamma_1(l)^2dkdl}{4\pi^2}$$

where

$$C:=\iint\frac{dkdl}{\Gamma_1(k-l)^2\Gamma_1(l)^2}<\infty\,.$$

•The two other terms are treated in the same manner. Therefore

$$\iint \Gamma_n(\Xi_n)^2\Gamma_n(H_n)^2|\widehat{\sigma}(\Xi_n,H_n)|^2\,\frac{d\Xi_n dH_n}{(2\pi)^{2n}}\leq C'\iint \Gamma_{n-1}(\Xi_2^n)^2\Gamma(H_n)^2$$
$$\times\iint|\widehat{\rho}_{n+1}(\xi_1-k,\Xi_2^n,k-l;H_n,l)|^2\,\frac{\Gamma_1(\xi_1-k)^2\Gamma_1(k-l)^2\Gamma_1(l)^2dkdl}{4\pi^2}\frac{d\Xi_n dH_n}{(2\pi)^{2n}}$$

with $C':=3C$. We conclude after changing variables:

$$(\xi_1-k,k-l,l)\to(\xi_1,\xi_{n+1},\eta_{n+1})$$

Growth estimate for $\|\rho_n\|_{E_n}$

Proposition. *Let* $0 \le V \in C_c^2(\mathbf{R})$ *and* $\gamma \in (0,1)$; *define*

$$H_N = -\tfrac{1}{2}\Delta_{X_N} + \sum_{1\le k<l\le N} N^{\gamma-1} V(N^\gamma(x_k - x_l))$$

Assume that, for each $n \ge 1$ *and each* $N \ge N_0(n)$,

$$\langle H_N^n \Psi_N^{in} | \Psi_N^{in} \rangle \le M^n N^n \text{ with } \Psi_N^{in}(X_n) = \prod_{k=1}^N \psi^{in}(x_k)\,.$$

Then, for each $M_1 > M$, *there exists* $N_1 = N_1(M_1, n)$ *such that*

$$\mathrm{trace}(S_1 \ldots S_n \rho_{N,n}(t) S_1 \ldots S_n) \le M_1^n$$

for each $t \ge 0$ *and each* $N \ge N_1$.

Sketch of the proof: This is a variant of an argument by Erdös et Yau for the existence of a solution to the infinite hierarchy in space dimension 3.

•The only estimate involving derivatives that is propagated by the N-body equation bears on

$$\langle H_N^n \Psi_N | \Psi_N \rangle$$

In this quantity, the typical term is

$$\int \left| \prod_{j_1 < \ldots < j_n} \partial_{x_{j_1}} \ldots \partial_{x_{j_1}} \Psi_N \right|^2 dX_N$$

In all the other terms, either one derivative bears on V, leading to a lesser order term, or there is a multiple derivative in one of the x_js, and there are less many of such terms.

•Set n and $C \in]0,1[$; one shows the existence of $N_0(C,n)$ st. for each $N \geq N_0$ and each $\Psi \in D(H_N)$

$$\langle (N+H_N)^n \Psi | \Psi \rangle \geq C^n N^n \langle \Psi | S_1^2 \dots S_n^2 \Psi \rangle$$

This result is trivial for $n = 0, 1$ (since $V \geq 0$ and $\Psi_N(t, X_N)$ is symmetrical in the x_js).

The general case follows by induction on n: assuming the inequality proved for $k = 0, \dots, n$ we prove it for $n+2$.

•Write

$$H_N + N = \sum_{k=1}^{N} S_k^2 + \sum_{1 \leq k < l \leq N} N^{\gamma-1} V(N^\gamma (x_k - x_l))$$

•Define

$$H_{n+1,N} := H_N + N - \sum_{k=n+1}^{N} S_k^2 \geq 0\,.$$

•Then

$$\langle\Psi|(N + H_N)S_1^2 \ldots S_n^2(N + H_N)\Psi\rangle =$$
$$\sum_{n<j_1,j_2\leq N} \langle\Psi|S_{j_1}^2 S_1^2 \ldots S_n^2 S_{j_2}^2\Psi\rangle + \sum_{n<j_1\leq N} 2\Re\langle\Psi|S_{j_1}^2 S_1^2 \ldots S_n^2 H_{n+1,N}\Psi\rangle$$
$$+\langle\Psi|H_{n+1,N}S_1^2 \ldots S_n^2 H_{n+1,N}\Psi\rangle$$

•Since

$$H_{n+1,N}S_1^2 \ldots S_n^2 H_{n+1,N} \geq 0$$

the last term in the r.h.s. is disposed of.

•Recall that Ψ_N is symmetrical in the space variables, i.e.

$$\Psi_N(t, x_{\sigma(1)}, \ldots, x_{\sigma(N)}) = \Psi_N(t, x_1, \ldots, x_N) \text{ for each } \sigma \in \mathfrak{S}_N$$

•Hence, denoting by W_{kl} the multiplication by $N^\gamma V(N^\gamma(x_k - x_l))$ acting on $L^2(\mathbf{R}^N)$, one has

$$\begin{aligned}
&\langle \Psi | (N + H_N) S_1^2 \ldots S_n^2 (N + H_N) \Psi \rangle \\
&\geq (N-n)(N-n-1) \langle \Psi | S_1^2 \ldots S_n^2 S_{n+1}^2 S_{n+2}^2 \Psi \rangle \\
&+ (2n+1)(N-n) \langle \Psi | S_1^4 S_2^2 \ldots S_{n+1}^2 \Psi \rangle \\
&+ \frac{n(n+1)(N-n)}{N} \Re \langle \Psi | W_{12} S_1^2 \ldots S_n^2 S_{n+1}^2 \Psi \rangle \\
&+ \frac{(n+1)(N-n)(N-n-1)}{N} \Re \langle \Psi | W_{1,n+2} S_1^2 \ldots S_n^2 S_{n+1}^2 \Psi \rangle
\end{aligned}$$

•By Sobolev embedding, one has the following obvious inequality

$$W(x-y) \le \|W\|_{L^1}(1-\partial_{xx})$$

•Hence all the terms involving V are of a lesser order:

$$\begin{aligned} 2\Re\langle\Psi|W_{12}S_1^2\dots S_n^2 S_{n+1}^2\Psi\rangle \ge \\ -\|V''\|_{L^1\cap L^\infty}(N^{2\gamma}\langle\Psi|S_1^2\dots S_{n+1}^2\Psi\rangle \\ +N^\gamma\langle\Psi|S_1^4 S_2^2\dots S_{n+1}^2\Psi\rangle) \end{aligned}$$

and similarly

$$\begin{aligned} 2\Re\langle\Psi|W_{1,n+2}S_1^2\dots S_n^2 S_{n+1}^2\Psi\rangle \ge \\ -\|V'\|_{L^1}N^\gamma\langle\Psi|S_1^2\dots S_n^2 S_{n+1}^2 S_{n+2}^2\Psi\rangle \end{aligned}$$

Growth estimate for initial data

•We start from an initial data of the form

$$\Psi_N^{in}(X_N) = \prod_{j=1}^{N} \psi^{in}(x_j)$$

that satisfies

$$\langle \Psi_N^{in} | (-\Delta_N)^n \Psi_N^{in} \rangle \leq M^n N^n \,.$$

•We prove by induction that, if $V \in C_c^\infty(\mathbf{R})$ and $\gamma \in (0, \frac{1}{2})$, one has

$$(-\tfrac{1}{2}\Delta_N + U_N)^n \leq C^n (N - \Delta_N)^n$$

for each $n \geq 1$ and $N \geq N_*(n)$.

• Hence

$$\langle \Psi_N^{in} | (-\tfrac{1}{2}\Delta_N + U_N)^n \Psi_N^{in} \rangle \leq 2^{n-1} C^{n-1} M^n N^n$$

for each $n \geq 1$ and $N \geq N_*(n)$.

• To compare powers of the Hamiltonian with powers of the kinetic energy, it suffices to show that

$$U_N (N - \Delta_N)^{2n} U_N \leq (C' + C''(n) N^{(4\gamma-2)n})(N - \Delta_N)^{2n+2}$$

which is done by induction. One has to be careful only with the case $n = 0$ that sets the constant C' uniformly in n.

• The above computation where the condition $\gamma \in (0, \frac{1}{2})$ comes from.

Passing to the limit

•Let Ψ_N be the solution of the N-body Schrödinger equation with factorized initial data; let ρ_N be the density matrix and $\rho_{N:n}$ its n-th marginal.

•The sequence $((\rho_{N:n})_{n\geq 0})_{N\geq 0}$ is bounded in the product space

$$\prod_{n\geq 1} L^\infty(\mathbf{R}, E_n)$$

(each factor being endowed with the weak-* toplogy)

•On the other hand, if $(\rho_{N_j:n})_{n\geq 0}$ converges to $(\rho_n)_{n\geq 0}$ in that topology, the limit soves the infinite hierarchy in the sense of distributions. Notice in particular that

$$|\rho_n(t, X_n, Y_n)| \leq C\,\mathrm{trace}(S_1 \dots S_n \rho_n(t) S_1 \dots S_n)$$

so that $\rho_n \in L^\infty_{t,X_n,Y_n}$.

•The recollision term is estimated as follows

$$N^{\gamma-1}\int V(N^\gamma(x_1-x_2))\rho_{N:n}(t,X_n,Y_n)\phi(t,X_n,Y_n)dX_ndY_ndt$$
$$\leq CN^{\gamma-1}\|V\|_{L^\infty}\|\rho_{N:n}\|_{L^\infty(E_n)}\|\phi\|_{L^1}\to 0$$

as $N\to\infty$ and there are $2n(n-1)$ such terms in the n-th equation of the infinite hierarchy.

•As for the interaction term, remember that E_n is a Hilbert space, so that the convergence is weak and not only weak-* in the space variables. Since the linear interaction operator $L_{n,n+1}$ is norm-continuous from E_{n+1} to E_n, it is weakly continuous from E_{n+1} to E_n.

NB. The convergence to a solution of the infinite hierarchy follows from a careful analysis involving the conservation of energy. The interaction operator $L_{n,n+1}$ essentially reduces to taking the restriction of $\rho_{N:n+1}$ to a (linear) subspace of codimension 2. But

•$\rho_{N:n+1}$ is a trace-class operator, which allows taking the restriction to $x_{n+1} = y_{n+1}$, with a H^1 estimate that follows from the conservation of energy;

•this bound allows in turn taking the further restriction $x_{n+1} = x_1$ because H^1 functions have $H^{1/2} \subset L^2$ restrictions to hypersurfaces.

See Adami-Bardos-G.-Teta, Asympt. Anal. (2004).

•Analogous result in space dimension 3 (preprint by Erdös-Yau, 2004).