ON THE STATISTICS OF FREE-PATH LENGTHS FOR THE PERIODIC LORENTZ GAS

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Consider the motion of a gas of point particles in a periodic array of spherical obstacles. Collisions involving two or more particles are neglected; only the collisions between the particles and the obstacles are taken into account. This talk reviews some results bearing on the distribution of free-path lengths for these particles, more precisely (1) upper and lower bounds for that distribution in any space dimension, and (2) the asymptotic evaluation of the tail of that distribution in the small obstacle limit, in space dimension two. Applications to kinetic theory are discussed.

1. Introduction

Almost 100 years ago, Lorentz[13] proposed the following linear kinetic equation to describe the motion of electrons in a metal:

$$(\partial_t + v \cdot \nabla_x + \tfrac{1}{m} F(t,x) \cdot \nabla_v) f(t,x,v) = N_{at} r_{at}^2 |v| \mathcal{C}(f(t,x,\cdot))(v) \tag{1}$$

where $f(t,x,v)$ is the (phase space) density of electrons which, at time t, are located at x and have velocity v. In Eq. (1), F is the electric force field, m the mass of the electron, while N_{at} and r_{at} designate respectively the number of metallic atoms per unit volume and the radius of each such atom. Finally $\mathcal{C}(f)$ is the collision integral: it acts on the velocity variable only, and is given, for all continuous $\phi \equiv \phi(v)$ by the formula

$$\mathcal{C}(\phi)(v) = \int_{|\omega|=1, v\cdot\omega>0} \big(\phi(v - 2(v\cdot\omega)\omega) - \phi(v)\big) \cos(v,\omega) d\omega\,. \tag{2}$$

In the case where $F \equiv 0$, Gallavotti[9,10] proved that Eq. (1) describes the Boltzmann-Grad limit of a gas of point particles undergoing elastic collisions on a random (Poisson) configuration of spherical obstacles. His result was successively strengthened by Spohn[15], and by Boldrighini-Bunimovich-Sinai[3].

The case of a periodic configuration of obstacles, perhaps closer to Lorentz' original ideas, leads to completely different results. It is the purpose of this talk to discuss some of these differences.

2. The periodic Lorentz gas

Let $D \in \mathbf{N}$, $D \geq 2$. For all $r \in (0, \frac{1}{2})$, let $Z_r = \{x \in \mathbf{R}^D \mid \mathrm{dist}(x, \mathbf{Z}^D) > r\}$ (the "billiard table"). The "free path length" or "(forward) exit time" for a particle starting from $x \in Z_r$

in the direction $v \in \mathbf{S}^{D-1}$ is defined as $\tau_r(x,v) = \inf\{t > 0 \,|\, x + tv \in \partial Z_r\}$. The function τ_r is then extended by continuity to $\{(x,v) \in \partial Z_r \times \mathbf{S}^{D-1} \,|\, v \cdot n_x \neq 0\}$, where n_x is the inward unit normal field on ∂Z_r. Finally, $\tau_r(x+k,v) = \tau_r(x,v)$ for each $(x,v) \in Z_r \times \mathbf{S}^{D-1}$ and $k \in \mathbf{Z}^D$: hence τ_r can be seen as a $[0,+\infty]$-valued function defined on $Y_r \times \mathbf{S}^{D-1}$ (and a.e. on $\overline{Y}_r \times \mathbf{S}^{D-1}$), where $Y_r = Z_r/\mathbf{Z}^D$.

If the components of $v \in \mathbf{S}^{D-1}$ are rationally independent — i.e. if $k \cdot v \neq 0$ for each $k \in \mathbf{Z}^D \setminus \{0\}$ — each orbit of the linear flow $x \mapsto x + tv$ is dense on $\mathbf{T}^D = \mathbf{R}^D/\mathbf{Z}^D$, and thus $\tau_r(x,v) < +\infty$ for each $x \in Z_r$.

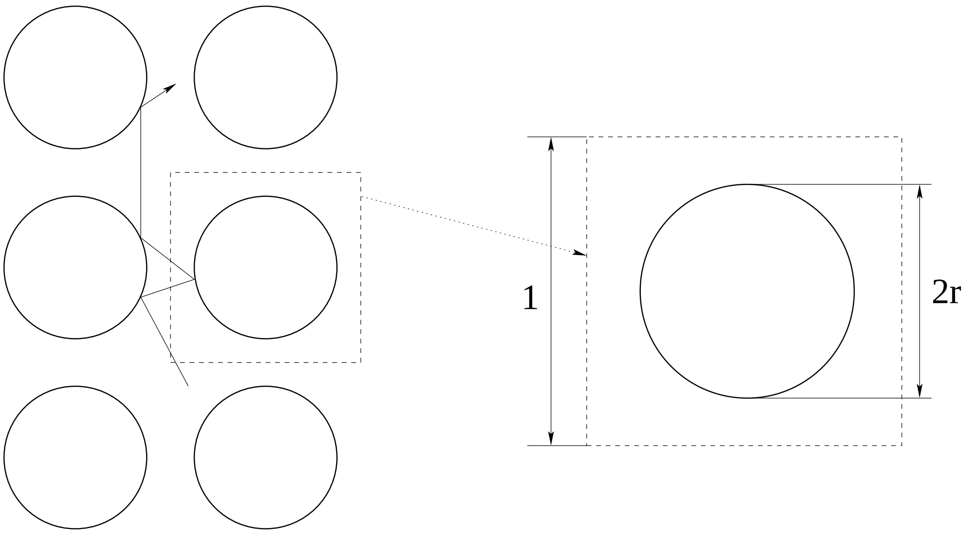

Figure 1. Z_r and the punctured torus Y_r

There are two different, natural phase spaces on which to study the free path length τ_r.

The first one is $\Gamma_r^+ = \{(x,v) \in \partial Z_r \times \mathbf{S}^{D-1} \,|\, v \cdot n_x > 0\}$ — or its quotient under the action of $\mathbf{Z}^D$-translations on space variables $\tilde{\Gamma}_r^+ = \Gamma_r/\mathbf{Z}^D$ — equipped with its Borel σ-algebra and the probability measure ν_r proportional to γ_r, where $d\gamma_r(x,v) = (v \cdot n_x) dS(x) dv$, with dS being the surface element on ∂Z_r.

The second one is $Y_r \times \mathbf{S}^{D-1}$, equipped with its Borel σ-algebra and the probability measure μ_r proportional to the Lebesgue measure on $Y_r \times \mathbf{S}^{D-1}$.

On the first phase space $\tilde{\Gamma}_r$, defining a notion of "mean free path" for the "Lorentz gas" — i.e. a gas of point particles undergoing elastic collisions with the periodic configuration of obstacles defined as the complement of Z_r — and evaluating the corresponding quantity is an easy matter. It is found that[a b]

$$\text{mean free path } = \mathbf{E}^{\nu_r}(\tau_r) = \frac{|Y_r||\mathbf{S}^{D-1}|}{\gamma_r(\tilde{\Gamma}_r)} = \frac{1}{|\mathbf{B}^{D-1}|r^{D-1}} - \frac{|\mathbf{B}^D|}{|\mathbf{B}^{D-1}|} r\,, \tag{3}$$

[a]If P is a probability measure and X a random variable on Ω, we denote by $\mathbf{E}^P(X)$ the expectation — i.e. the mean — of X with respect to P.

[b]If A is a measurable d-dimensional set in $\mathbf{R}^D$ ($d \leq D$), $|A|$ designates its d-dimensional volume.

where $\mathbf{B}^d$ is the d-dimensional unit ball. The explicit computation of $\mathbf{E}^{\nu_r}(\tau_r)$ (i.e. the second equality above) is credited to Santalo (see Ref. 14, p. 42). Observe that, in the limit as $r \to 0^+$ and in the case of space dimension $D = 3$, this evaluation of the mean free path coincides with the reciprocal of the factor

$$N_{at} r_{at}^2 \int_{|\omega|=1, v\cdot\omega>0} \cos(v,\omega) d\omega$$

appearing in Eq. (1).

On the second phase space $Y_r \times \mathbf{S}^{D-1}$ — which is slightly more natural, at least for the kinetic equation (1) — the analogous definition of the mean free path fails because $\mathbf{E}^{\mu_r}(\tau_r) = +\infty$ — see below. In fact, as noticed in Dumas-Dumas-Golse[8]

Lemma 2.1. *Let $f \in C^1(\mathbf{R}_+)$ satisfy $f(0) = 0$. Then*

$$\gamma_r(\tilde{\Gamma}_r)\mathbf{E}^{\nu_r}(f(\tau_r)) = |Y_r||\mathbf{S}^{D-1}|\mathbf{E}^{\mu_r}(f'(\tau_r)) .$$

In the case where $f(z) = z$, this identity gives back Santalo's formula (3). In the case where $f(z) = \frac{1}{2}z^2$, it shows that $\mathbf{E}^{\mu_r}(\tau_r) = \frac{\gamma_r(\tilde{\Gamma}_r)}{2|Y_r||\mathbf{S}^{D-1}|}\mathbf{E}^{\nu_r}(\tau_r^2)$. As one can imagine, $\tau_r(x,v)$ is a wildly oscillating function. For one thing, it depends upon arithmetic characteristics of v — such as which Diophantine class v belongs to — that are known to be very unstable as v runs through $\mathbf{S}^{D-1}$. Hence it is not very surprising that τ_r has infinite moments of order higher than one.

3. Bounds on the distribution of free path lengths

Since $\mathbf{E}^{\mu_r}(\tau_r) = +\infty$, the next simple thing to compute is the distribution of τ_r under μ_r. With applications to kinetic theory in mind, it is in fact more natural to consider the following, slightly more general object:

$$\Phi_r^m(t) = m(v)d\mu_r(x,v)\text{-meas}(\{(x,v) \in Y_r \times \mathbf{S}^{D-1} \mid \tau_r(x,v) > t\})$$

where $m \in C(\mathbf{S}^{D-1})$, $m > 0$ and $\mathbf{E}^{\mu_r}(m) = 1$. Theorem 3.1 below shows that, although τ_r is not an element of $L^1(Y_r \times \mathbf{S}^{D-1}, \mu_r)$, it does not miss by much: in particular $\tau_r \in L^{1,\infty}(Y_r \times \mathbf{S}^{D-1}, \mu_r)$ (Marcinkiewicz' weak L^1 space[16]).

Theorem 3.1. *For each $m \in C(\mathbf{S}^{D-1})$ such that $m > 0$ and $\mathbf{E}^{\mu_r}(m) = 1$, there exists two positive constants C_m and C'_m such that, for each $r \in (0, \frac{1}{2})$ and each $t > 1/r^{D-1}$,*

$$\frac{C_m}{r^{D-1}t} \leq \Phi_r^m(t) \leq \frac{C'_m}{r^{D-1}t} .$$

A weaker variant of the upper bound was proved by Dumas-Dumas-Golse[7] for space dimension 2 (using an improvement by Dumas of his ergodization rate estimates in Ref. 6). These investigations suggested that $1/r^{D-1}$ was the right length scale for this problem. The upper bound for any $D \geq 2$ is proved in Bourgain-Golse-Wennberg[4] by a method based on Fourier series that is vaguely reminiscent of Siegel's proof of Minkowski's convex body theorem. In the case of space dimension $D = 2$, a proof of the lower bound is also to be found in Ref. 4. It is based on an entirely different argument, more precisely on the construction of obstacle-free channels of rational direction and on a careful estimate of the

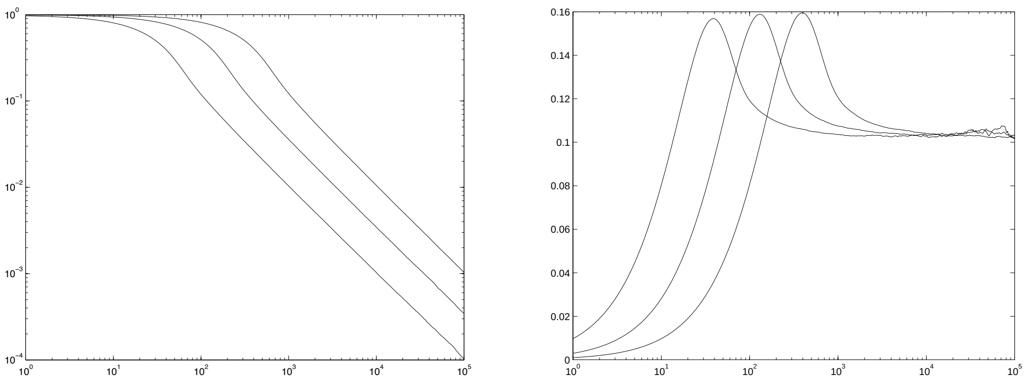

Figure 2. Left: Log-log-plot of $\Phi_r^1(t)$ *vs.* t, Right: Plot of $tr\Phi_r^1(t)$ *vs.* t, for $r = 0.01$, 0.03, and 0.001.

width thereof. Later, an argument of this type was extended to arbitrary space dimension by Golse-Wennberg[12].

The numerical computations above (taken from Golse-Wennberg[12]) suggest that $\Phi_r^1(t/r) \sim C/t$ as $t \to +\infty$ and $r \to 0^+$, with $0.1 < C < 0.11$ (inasmuch as the numerical evaluation of $tr\Phi_r^1(t)$ for $r \to 0^+$ and $t >> 1/r$ can be trusted).

4. Asymptotic evaluation of the distribution of free path lengths for $D = 2$

In the case of space dimension $D = 2$, one can consider sections of the linear flow on $\mathbf{T}^2$, which leads to studying iterates of a rotation on the unit circle. This suggests that the continued fraction expansion of the slope of the linear flow considered is the appropriate tool for evaluating τ_r.

Theorem 4.1. *Let $m \in L^\infty(\mathbf{S}^{D-1})$ satisfy $m \geq 0$ a.e. and $\mathbf{E}^{\mu_r}(m) = 1$. Then, as $t \to +\infty$*

$$\overline{\lim_{\epsilon \to 0^+}} \frac{1}{|\ln \epsilon|} \int_\epsilon^{1/4} \Phi_r^m\left(\frac{t}{r}\right) \frac{dr}{r} = \frac{1}{\pi^2 t} + O\left(\frac{1}{t^2}\right),$$

$$\underline{\lim_{\epsilon \to 0^+}} \frac{1}{|\ln \epsilon|} \int_\epsilon^{1/4} \Phi_r^m\left(\frac{t}{r}\right) \frac{dr}{r} = \frac{1}{\pi^2 t} + O\left(\frac{1}{t^2}\right).$$

This result was proved by Caglioti-Golse[5]. The proof uses essentially two different ideas, which are sketched below.

4.1. *A partition of* $\mathbf{T}^2$

In 1989, R. Thom posed the following problem: "To find the longest orbit of a linear flow with irrational slope on a flat torus with a disk removed". This problem was essentially solved by Blank-Krikorian[1], by the following construction.

For $R \in (0, 1)$, let $Y[R]$ be the flat torus with a vertical slit of length R removed: $Y[R] = \mathbf{T}^2 \setminus (\{0\} \times [0, R] \text{ mod. } 1)$. Let $v = (\cos\theta, \sin\theta)$ with $\theta \in (0, \frac{\pi}{4})$ such that $\tan\theta \notin \mathbf{Q}$. Call $[a_1, a_2, a_3, \ldots]$ with $a_n \in \mathbf{N}$, the continued fraction expansion of $\alpha = \tan\theta$, meaning that

$$\alpha = [a_1, a_2, a_3, \ldots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}}\,.$$

Call p_n/q_n its sequence of convergents (the integers p_n and q_n being co-prime), i.e.

$$\frac{p_{n+1}}{q_{n+1}} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots \cfrac{1}{a_n}}}\,,$$

and let d_n be the sequence of errors defined as $d_n = |q_n\alpha - p_n|$. Consider then the following nested partition of $(0,1)$:

$$(0,1) = \bigcup_{n\geq 1} \bigcup_{1\leq k\leq a_n} I_{n,k}\,, \text{ with } I_{n,k} = [\sup(d_n, d_{n-1} - kd_n), d_{n-1} - (k-1)d_n)\,.$$

In Ref. 1, Blank and Krikorian proved the following

Proposition 4.2. *Assume that $R \in I_{n,k}$. Any orbit of the linear flow with slope $\tan\alpha$ on $Y[R]$ has length either q_n, or $q_{n-1} + kq_n$, or else $q_{n-1} + (k+1)q_n$.*

Following Blank and Krikorian, the shortest orbits are said to be "of type A", the longest ones "of type C", and the remaining orbits "of type B".

In Ref. 5, the proposition above was used to construct a partition of $Y[R]$ into three strips, each strip being the union of all orbits of type A, B, or C respectively. Call $\psi_R(t,v)$ the distribution of free path lengths in $Y[R]$ for particles moving in the direction v from a uniformly distributed starting point x. By using the partition of $Y[R]$ mentioned above, especially the width of each one of the three strips in that partition which can be easily expressed in terms of the sequence of errors d_n (see Ref. 5, p. 206), one arrives at an explicit formula for $\psi_R(t,v)$, whose graph is represented in figure 3 below.

For the problem that we consider, the only significant part in the graph below is the middle one — i.e. the contribution of orbits of type B only. More precisely

Lemma 4.3. *Let $r \in (0, \frac{1}{4})$, $\theta \in (0, \frac{\pi}{4})$ be such that $\tan\theta \notin \mathbf{Q}$, and $v = (\cos\theta, \sin\theta)$. Assume that $R = 2r/\cos\theta \in I_{n,k}$. Then*[c]

$$\left|\psi_R\left(\frac{t}{r}, v\right) - \left(1 - \frac{R}{d_n} - t\frac{d_{n-1}}{R}\right)_+\right| \leq \frac{4}{k}\mathbf{1}_{(t-2,+\infty)}(k)\,, \text{ for all } t > q_n R\,.$$

This is Lemma 4.2 in Ref. 5, to which we refer the reader interested in a complete proof.

4.2. *An ergodic lemma*

Given $\alpha \in (0,1)\setminus\mathbf{Q}$ and $\epsilon > 0$, define[d] $N(\alpha,\epsilon) = \inf\{n \in \mathbf{N} \,|\, d_n(\alpha) < \epsilon\}$. Define $\Delta_j(\alpha, x) = -x - \ln d_{N(\alpha, e^{-x})-j}(\alpha)$ for $j = 0, 1, \ldots$.

[c] If $x \in \mathbf{R}$, the notation x_+ designates $\sup(x,0)$; the notation $\mathbf{1}_A$ designates the indicator function of A.
[d] Whenever necessary, we specify the dependence upon α of the continued fraction expansion of α, denoting by $q_n(\alpha)$ the denominator of the n-th convergent, by $d_n(\alpha)$ the n-th error, and so on.

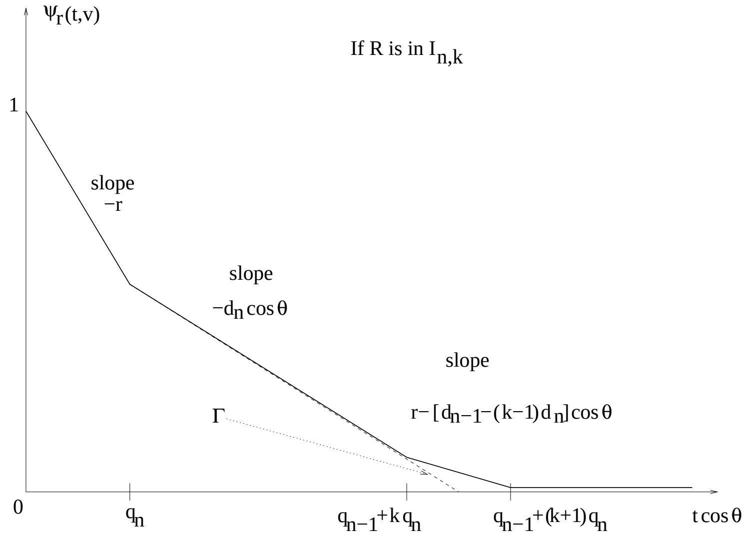

Figure 3. Graph of $t \mapsto \psi_R(t, v)$ for $R \in I_{n,k}$.

Lemma 4.4. *Let f be a bounded nonnegative measurable function on $\mathbf{R}^2$. For each $x^* \in \mathbf{R}$ and a.e. $\alpha \in (0, 1)$, one has*

$$\frac{1}{|\ln \epsilon|} \int_{x^*}^{|\ln \epsilon|} f(\Delta_0(\alpha, x), \Delta_1(\alpha, x))dx \to \tfrac{12}{\pi^2} \int_0^1 \frac{F(\theta)d\theta}{1+\theta}$$

as $\epsilon \to 0^+$, where

$$F(\theta) = \int_0^{|\ln(\theta)|} f(|\ln(\theta)| - y, -y)dy\,.$$

This result was proved by Caglioti-Golse[5], using that the Gauss map $T : (0, 1) \ni x \mapsto 1/x - [1/x] \in (0, 1)$ is ergodic with invariant measure $\frac{1}{\ln 2}\frac{dx}{1+x}$. See Ref. 5, pp. 209-210 for a complete proof of this result.

The key argument in the proof of Theorem 4.1 is to observe that, by Lemma 4.3, for each $t > 2$, $\psi_R\left(\frac{t}{r}, v\right) \simeq (1 - e^{\Delta_1(\alpha,-x)} - te^{-\Delta_0(\alpha,-x)})_+$, with $x = -\ln R$, up to an error of order $4/k$ as $k \geq t - 2$. Applying Lemma 4.4 to $f(z_1, z_2) = (1 - e^{z_2} - te^{-z_1})_+$ leads to the asymptotic estimates stated in Theorem 4.1.

Let us conclude this section with a few remarks on Theorem 4.1. As shown above, the proof is based upon comparing the radius r of the obstacles with the sequence of errors $d_n(\alpha)$. In view of the elementary formula $d_n(\alpha) = \alpha d_{n-1}(T\alpha)$, one sees that the exit time problem for a linear flow with slope α and obstacle size r is mapped to the same problem with slope $T\alpha$ and obstacle size r/α. Hence it is natural to consider averages for the Haar measure $\frac{dr}{r}$ on the multiplicative group $\mathbf{R}_+^*$ in the statement of Theorem 4.1.

Following the result by Caglioti-Golse[5], Boca-Zaharescu[2] have recently proved that, for $m \equiv 1$, $\Phi_r^1(t/r)$ converges to a limit as $r \to 0^+$ for all $t > 0$. Their proof is based on the same partition as in Ref. 5, but uses Farey fractions and Kloosterman sums instead of the analysis presented above.

5. Applications to kinetic theory

Set $D = 2$. Define $\Omega_\epsilon = \{\epsilon z \,|\, z \in Z_\epsilon\}$, and consider the transport equation

$$\begin{aligned} \partial_t g_\epsilon + v \cdot \nabla_x g_\epsilon = 0 \qquad & \text{on } \Omega_\epsilon \times \mathbf{S}^1, \qquad g_\epsilon\Big|_{t=0} = f^{in}\Big|_{\Omega_\epsilon \times \mathbf{S}^1}, \\ g_\epsilon(t,x,v) = 0 \qquad & \text{for } x \in \partial\Omega_\epsilon\,,\ v \cdot n_x > 0 \end{aligned} \tag{4}$$

with unknown is $g_\epsilon(t,x,v)$. Here, n_x is the inward unit normal at the point $x \in \partial\Omega_\epsilon$ and f^{in} is a given, nonnegative function of $C_c(\mathbf{R}^2 \times \mathbf{S}^1)$. Physically, this is the variant of the periodic Lorentz gas with scatterers replaced by holes (or traps) where impinging particles fall instead of bouncing back.

Obviously, $\|g_\epsilon\|_{L^\infty_{t,x,v}} = \|f^{in}\|_{L^\infty_{x,v}}$. Reasoning as in Ref. 13 suggests that $g_\epsilon \to g$ in $L^\infty_{t,x,v}$ weak-*, where g solves the uniformly damped transport equation

$$\partial_t g + v \cdot \nabla_x g + g = 0 \text{ on } \mathbf{R}^*_+ \times \mathbf{R}^2 \times \mathbf{S}^1\,, \quad g_{|t=0} = f^{in}\,, \tag{5}$$

but this is ruled out by Theorem 2.1 of Ref. 12. Theorem 4.1 suggests instead that the resulting damping rate should vanish in the limit as $t \to +\infty$. The following result was proved by Caglioti-Golse[5]:

Theorem 5.1. *Let $f^{in} \geq 0$ be a continuous bounded function on $\mathbf{R}^2 \times \mathbf{S}^1$ and let g_ϵ be, for each $\epsilon \in (0, \frac{1}{4})$, the solution of (4). Then, for each nonnegative, compactly supported, C^1 test function χ, one has*

$$\overline{\lim_{\epsilon \to 0}} \iint \left(\frac{1}{|\ln\epsilon|} \int_\epsilon^{1/4} g_r(t,x,v) \frac{dr}{r} \right) \chi(x,v)dxdv = \iint g(t,x,v)\chi(x,v)dxdv + O\left(\frac{1}{t^2}\right)$$

$$\underline{\lim}_{\epsilon \to 0} \iint \left(\frac{1}{|\ln\epsilon|} \int_\epsilon^{1/4} g_r(t,x,v) \frac{dr}{r} \right) \chi(x,v)dxdv = \iint g(t,x,v)\chi(x,v)dxdv + O\left(\frac{1}{t^2}\right)$$

as $t \to +\infty$, where

$$g(t,x,v) = \frac{1}{\pi^2 t} f^{in}(x - tv, v)\,. \tag{6}$$

In particular, g satisfies, in the sense of distributions, the transport equation

$$\partial_t g + v \cdot \nabla_x g + \frac{1}{t} g = 0\,, \quad (t,x,v) \in (0,+\infty) \times \mathbf{R}^2 \times \mathbf{S}^1\,. \tag{7}$$

In fact, Theorem 3.1 also rules out the possibility that the *original* periodic Lorentz gas (with reflecting instead of absorbing obstacles) may be described by the kinetic model (1) in the Boltzmann-Grad limit — i.e. in the same scaling limit as above. Indeed, let $f_\epsilon(t,x,v)$ be the solution to

$$\begin{aligned} \partial_t f_\epsilon + v \cdot \nabla_x f_\epsilon = 0 \qquad & \text{on } \Omega_\epsilon \times \mathbf{S}^1, \qquad f_\epsilon\Big|_{t=0} = f^{in}\Big|_{\Omega_\epsilon \times \mathbf{S}^1}, \\ f_\epsilon(t,x,v) = f_\epsilon(t,x,v - 2(v\cdot n_x)n_x) \qquad & \text{for } x \in \partial\Omega_\epsilon\,. \end{aligned} \tag{8}$$

Theorem 5.2. *There exist initial data f^{in} that are continuous on $\mathbf{T}^2 \times \mathbf{S}^1$ and such that, for ϵ of the form $\epsilon_n = 1/n$ with $n \geq 3$, neither f_{ϵ_n} nor any subsequence thereof converge in $L^\infty_{t,x,v}$ weak-* to the solution of*

$$(\partial_t + v \cdot \nabla_x) f(t,x,v) = \mathcal{C}(f(t,x,\cdot))(v)\,, \quad f\Big|_{t=0} = f^{in}$$

with collision integral $\mathcal{C}$ defined in (2).

Pick f^{in} independent of v: $f^{in} \equiv \rho^{in}(x)$. By the maximum principle, $f_{\epsilon_n}(t,x,v) \geq g_{\epsilon_n}(t,x,v) = \rho^{in}(x-tv)\mathbf{1}_{(t/\epsilon_n,+\infty)}(\tau_{\epsilon_n}(x/\epsilon_n,-v))$. If $f_{\epsilon_n} \to f$ in $L^\infty_{t,x,v}$ weak-*, Theorem 3.1 and the same arguments as in the proof of Theorem 5.1 (see Ref. 5, pp. 217–218) imply that

$$f(t,x,v) \geq \frac{C_1}{t}\rho^{in}(x-vt)\,, \qquad t>1\,. \tag{9}$$

If f were the solution to (1) with $F \equiv 0$ and initial data $f(0,x,v) = \rho^{in}(x)$, it would satisfy

$$\left\| f(t,\cdot,\cdot) - \int_{\mathbf{T}^2} \rho^{in}(z)dz \right\|_{L^2(\mathbf{T}^2\times\mathbf{S}^1)} \leq Ae^{-\alpha t}\|\rho^{in}\|_{L^2(\mathbf{T}^2)} \tag{10}$$

for some constants $A>0$ and $\alpha>0$ independent of the choice of ρ^{in}. (This result was proved by Ghidouche-Point-Ukai[11] in the case of the linearized Boltzmann equation — see Theorem 1 (iii), p. 207 of Ref. 11; adapting it to Eq. (1) is obvious.) But (9) and (10) are clearly incompatible, since one can choose $\|\rho^{in}\|_{L^2} = 1$ with $\int_{\mathbf{T}^2}\rho^{in}dx$ arbitrarily small.

6. Conclusions

Because of the presence of too many long collision-free trajectories with near rational slopes, the Boltzmann-Grad limit of the Lorentz gas is not described by the kinetic equation (1). Whether the precise asymptotic result in Theorem 4.1 could lead to a positive result on this limit, as it does in the case of absorbing obstacles (see Theorem 5.1) remains an interesting open problem. Also, it would be interesting to extend Theorem 4.1 to space dimensions higher than 2; however, this could be hard since the current proof is based on continued fractions (the same can be said of Ref. 2 which uses Farey fractions instead).

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References

1. S. Blank, N. Krikorian, *Internat. J. of Math.* **4**, 721 (1993).
2. F. Boca, A. Zaharescu, preprint arXiv math.NT/0301270.
3. C. Boldrighini, L. Bunimovich, Ya.G. Sinai, *J. Statist. Phys.* **32**, 477 (1983).
4. J. Bourgain, F. Golse, B. Wennberg, *Commun. Math. Phys.* **190**, 491 (1998).
5. E. Caglioti, F. Golse, *Commun. Math. Phys.* **236**, 119 (2003).
6. H.S. Dumas, *J. Dynam. Differential Equations* **3**, 593 (1991).
7. H.S. Dumas, L. Dumas, F. Golse, *J. Statist. Phys.* **82**, 1385 (1996).
8. H.S. Dumas, L. Dumas, F. Golse, *J. Statist. Phys.* **87**, 943 (1997).
9. G. Gallavotti, *Phys. Rev. (2)* **185**, 308 (1969).
10. G. Gallavotti, Nota int. no. 358, Istit. di Fisica, Università di Roma, (1972).
11. H. Ghidouche, N. Point, S. Ukai, *J. de Math. Pures et Appl. (9)* **57**, 203 (1978)
12. F. Golse, B. Wennberg, M2AN Modél. Math. et Anal. Numér. **34**, 1151 (2000).
13. H. Lorentz, *Arch. Néerl.* **10**, 336 (1905).
14. L. Santalo, *Integral geometry and geometric probability*, Addison-Wesley (1976).

15. H. Spohn, *Commun. Math. Phys.* **60**, 277 (1978).
16. E. Stein *Harmonic analysis*, Princeton University Press (1993).