The periodic Lorentz gas in the Boltzmann-Grad limit

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Abstract. Consider the motion of a single point particle bouncing in a fixed system of spherical obstacles. It is assumed that collisions are perfectly elastic, and that the particle is subject to no external force between collisions, so that the particle moves at constant speed. This type of dynamical system belongs to the class of dispersing billiards, and is referred to as a "Lorentz gas". A Lorentz gas is called periodic when the obstacle centers form a lattice. Assuming that the initial position and direction of the particle are distributed under some smooth density with respect to the uniform measure, one seeks the evolution of that density under the dynamics defined by the particle motion in some large scale limit for which the number of collisions per unit of time is of the order of unity. This scaling limit is known as "the Boltzmann-Grad limit", and is the regime of validity for the Boltzmann equation in the kinetic theory of gases. Whether this evolution is governed in such a limit by a PDE analogous to the Boltzmann equation is a natural question, and the topic of this paper.

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1. Introduction

In 1905, H. Lorentz proposed the following linear kinetic equation to describe the motion of electrons in a metal [23]:

$$\partial_t f + v \cdot \nabla_x f + \tfrac{1}{m} F(t,x) \cdot \nabla_v f(t,x,v) = N_{\mathrm{at}} r_{\mathrm{at}}^2 |v| \mathcal{C}(f(t,x,\cdot))(v), \tag{1}$$

where the unknown $f(t,x,v)$ is the density of electrons which, at time t, are located at x and have velocity v. In (1), F is the electric force field, m the mass of the electron, while N_{at} and r_{at} designate respectively the number of metallic atoms per unit volume and the radius of each such atom. Finally $\mathcal{C}(f)$ is the collision integral: it acts on the velocity variable only, and is given, for each continuous $\phi \equiv \phi(v)$ by the formula

$$\mathcal{C}(\phi)(v) = \int_{|\omega|=1,\, v\cdot\omega>0} \big(\phi(v - 2(v\cdot\omega)\omega) - \phi(v)\big) \cos(v,\omega)\, d\omega. \tag{2}$$

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Although a microscopic model, this equation is only a statistical description of electron motion and by no means a first principle of electrodynamics. For instance, (1) only holds for probability densities f, and does not have distributional solutions of the form

$$f \equiv \delta_{(x(t),v(t))},$$

as one would expect in any situation where there is only one electron and its trajectory in phase space $(x(t), v(t))$ is known exactly (i.e. with probability 1). Obviously, this inconsistency comes from the Lorentz collision integral $\mathcal{C}$, and not from the electric force. Hence we shall assume throughout this lecture that the electric force

$$F \equiv 0$$

and restrict our attention to the collision integral.

Since the Lorentz equation is not itself a first principle of physics, it is natural to understand whether it can be derived from one such first principle. This question belongs to the class of problems known as "hydrodynamic limits" – although in the present case, the term "mesoscopic limit" would be more appropriate.

The interest of mathematicians in this type of question originates in Hilbert's attempts to justify rigorously the equations of fluid mechanics on the basis of the kinetic theory of gases, which he cited as an example in his 6th problem on the axiomatization of physics [19]. In [23], Lorentz himself established his model by analogy with the Boltzmann equation for a gas of hard spheres, and did not seek any rigorous derivation for it – avoiding in particular the rather subtle arguments proposed by L. Boltzmann as a justification for the equation bearing his name.

In this paper, we shall discuss whether the Lorentz equation (1) can be rigorously derived in some asymptotic limit from a very simple mechanistic model for electron motion known as the "Lorentz gas". Although not entirely satisfactory in the context of electrodynamics, this model is to the kinetic theory of electrons what molecular gas dynamics is to the kinetic theory of gases.

2. The Lorentz gas

Let[1] $\vec{C} \subset \mathbb{R}^{\mathrm{D}}$ (the dimensions of interest being $\mathrm{D} = 2$ or $\mathrm{D} = 3$) satisfy the condition

$$d(\vec{C}) := \inf_{c,c' \in \vec{C}} |c - c'| > 0. \tag{3}$$

Pick $r \in \left(0, \frac{1}{2} d(\vec{C})\right)$, and consider the motion of a point particle moving at a constant velocity in the domain outside the union of fixed balls of radius r centered at the elements of $\vec{C}$, henceforth denoted

$$Z_r[\vec{C}] := \{x \in \mathbb{R}^{\mathrm{D}} \mid \operatorname{dist}(x, \vec{C}) > r\}. \tag{4}$$

[1] We designate by $\vec{C}$ the set of obstacle centers, to avoid confusion with several constants denoted by C in the sequel.

It is assumed that each collision between the particle and any of the balls is perfectly elastic. Put in other words, denoting by z the collision point and by n_z the inward unit normal to $\partial Z_r[\vec{C}]$ at the point z, the pre- and postcollisional velocities v_- and v_+ of the particle are related by the Descartes law of specular reflection

$$v_+ = v_- - 2(v_- \cdot n_z)n_z.$$

Obviously, the speed of the particle (i.e. the Euclidian norm of its velocity vector) is invariant under this law of reflection, so that we can assume without loss of generality that this speed is $|v| = 1$.

Assuming that the position and the velocity of the particle are respectively x and v at time $t = 0$, we denote by $X_r(t, x, v; \vec{C})$ and $V_r(t, x, v; \vec{C})$ respectively the position and the velocity of the particle at time t. They satisfy the differential equations

$$\begin{aligned} \dot{X}_r &= V_r \quad \text{if } \operatorname{dist}(X(t), \vec{C}) > r, \\ \dot{V}_r &= 0 \quad\ \ \text{if } \operatorname{dist}(X(t), \vec{C}) > r, \end{aligned} \tag{5}$$

while

$$\begin{aligned} X_r(t+0) &= X_r(t-0) && \text{if } \operatorname{dist}(X_r(t-0), \vec{C}) = r, \\ V_r(t+0) &= \mathcal{R}[n_{X_r(t-0)}]V_r(t-0) && \text{if } \operatorname{dist}(X_r(t-0), \vec{C}) = r, \end{aligned} \tag{6}$$

where $\mathcal{R}[n]$ designates the specular reflection

$$\mathcal{R}[n]v = v - 2(v \cdot n)n.$$

The dynamical system (X_r, V_r) is referred to as *the Lorentz gas* in the configuration of spherical obstacles of radius r centered at the points of $\vec{C}$.

Let $f^{\text{in}} \equiv f^{\text{in}}(x, v)$ be a probability density on the single-particle phase-space, i.e. a nonnegative measurable function defined a.e. on $Z_r[\vec{C}] \times \mathbb{S}^{\mathrm{D}-1}$ such that

$$\iint_{Z_r[\vec{C}] \times \mathbb{S}^{\mathrm{D}-1}} f^{\text{in}}(x, v)\, dx dv = 1.$$

Define $f_r \equiv f_r(t, x, v; \vec{C})$ to be the density with respect to $dxdv$ of the image measure of $f^{\text{in}}(x, v)dxdv$ under the flow (X_r, V_r), i.e.

$$f_r(t, x, v; \vec{C}) = f^{\text{in}}(X_r(-t, x, v; \vec{C}), V_r(-t, x, v; \vec{C})). \tag{7}$$

A natural question is whether $f_r(t, x, v; \vec{C})$ converges to a solution of the kinetic equation (1) with $F \equiv 0$ in the vanishing r limit, and under some appropriate scaling assumption on the obstacle configuration $\vec{C}$.

Observe that (5) is the system of ordinary differential equations defining the characteristics of the free transport equation in $Z_r[\vec{C}] \times \mathbb{S}^{\mathrm{D}-1}$; therefore the density f_r is the solution of

$$\begin{aligned} \partial_t f_r + v \cdot \nabla_x f_r &= 0, && x \in Z_r[\vec{C}],\ |v| = 1,\ t > 0, \\ f_r(t, x, \mathcal{R}[n_x]v) &= f_r(t, x, v), && x \in \partial Z_r[\vec{C}],\ |v| = 1,\ t > 0, \\ f_r(0, x, v) &= f^{\text{in}}(x, v), && x \in Z_r[\vec{C}],\ |v| = 1. \end{aligned} \tag{8}$$

Hence the question above can be viewed as a some kind of homogenization problem for the transport equation. Analogous homogenization problems for the diffusion (Laplace) equation have been thoroughly studied – the work of Hruslov [20] is one of the first references on this topic; see also the lucid presentation of this class of problems in [10].

G. Gallavotti considered in [14] the case of random configurations $\vec{C}$ of obstacles; specifically, the points in $\vec{C}$ are independent and identically distributed, under Poisson's law with density $N_{\rm at}$. The radius of the obstacles is $r > 0$; it is assumed that $N_{\rm at} \to +\infty$ while $r \to 0$ so that $N_{\rm at} r^2 \to \sigma$. He proved that, in this limit, the expectation of $f_r(t, x, v; \vec{C})$ converges to the solution of (1) with initial data $f^{\rm in}$ and with $F \equiv 0$. His analysis is written in detail on pp. 48–55 in [15]. Later on, his result was strengthened in [29] by H. Spohn, who considered slightly more general distributions of obstacles. The a.s. convergence of $f_r(t, x, v; \vec{C})$ in $\vec{C}$ was proved by C. Boldrighini, L. A. Bunimovich and Ya. G. Sinai [5].

Obviously, the case of a Poisson distribution of obstacles is very natural in the context of the kinetic theory of (neutral) gases. For instance, one could think of a mixture of two hard sphere gases, one with light molecules, the other one with heavy molecules in equilibrium. If the concentration of the light gas is small, collisions between light molecules can be neglected; only binary collisions involving one molecule of each type are considered. This is essentially[2] the microscopic model studied in [14], [15]. For other applications (such as the motion of electrons in a metal) it may be useful to know what happens for other distributions of obstacles. In this paper, we shall discuss the case of a periodic distribution of obstacles.

3. The distribution of free path lengths

From now on, we shall restrict our attention to the case of a periodic Lorentz gas with spherical obstacles of radius $r \in \left(0, \frac{1}{2}\right)$ centered at the integer points, i.e. $\vec{C} = \mathbb{Z}^{\rm D}$. Since the configuration of obstacle centers is thus fixed, we shall henceforth abbreviate the notation introduced above by setting $X_r(t, x, v) := X_r(t, x, v; \mathbb{Z}^{\rm D})$, $V_r(t, x, v) := V_r(t, x, v; \mathbb{Z}^{\rm D})$, while $Z_r := Z_r[\mathbb{Z}^{\rm D}]$ and $f_r(t, x, v) := f_r(t, x, v, \mathbb{Z}^{\rm D})$.

In view of the probabilistic interpretation of the kinetic equation (1) and of the definition of the Boltzmann-Grad scaling, one expects that the free path lengths should play an important role in studying the periodic Lorentz gas above in that limit.

Definition 3.1. For $x \in Z_r$ and $v \in \mathbb{S}^{\rm D-1}$, the free path length (or forward exit time) for a particle starting at the position x in the direction v is

$$\tau_r(x, v) = \inf\{t > 0 \mid x + tv \in \partial Z_r\}.$$

[2] Except for the fact that heavy molecules may overlap in Gallavotti's model, while this cannot occur for real hard spheres: see condition (3).

For each $v \in \mathbb{S}^{D-1}$, the function $x \mapsto \tau_r(x, v)$ has a unique continuous extension to $Z_r \cup \{x \in \partial Z_r \mid v \cdot n_x \neq 0\}$ for which we shall abuse the notation $\tau_r(x, v)$.

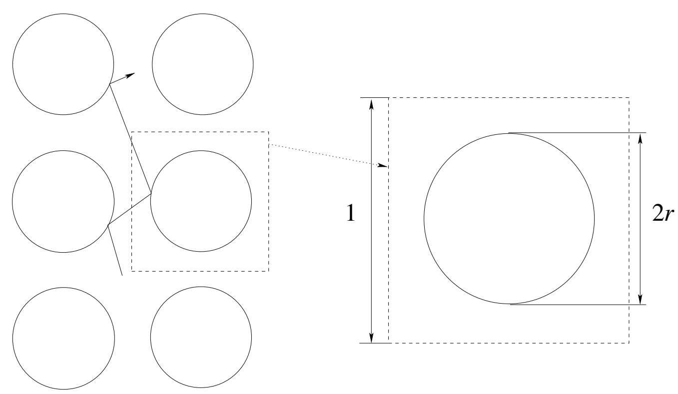

Figure 1. The periodic Lorentz gas.

Notice that $\tau_r(x + k, v) = \tau_r(x, v)$ for each $(x, v) \in Z_r \times \mathbb{S}^{D-1}$ and $k \in \mathbb{Z}^D$: hence τ_r can be seen as a $[0, +\infty]$-valued function defined on $Y_r \times \mathbb{S}^{D-1}$ and a.e. on $\bar{Y}_r \times \mathbb{S}^{D-1}$, with $Y_r = Z_r/\mathbb{Z}^D$.

If the components of $v \in \mathbb{S}^{D-1}$ are rationally independent – i.e. if $k \cdot v \neq 0$ for each $k \in \mathbb{Z}^D \setminus \{0\}$ – each orbit of the linear flow $x \mapsto x + tv$ is dense on $\mathbb{T}^D = \mathbb{R}^D/\mathbb{Z}^D$, so that $\tau_r(x, v)$ is finite for each $x \in Z_r$.

There are two different, natural phase spaces on which to study the free path length τ_r.

The first one is $\Gamma_r^+ = \{(x, v) \in \partial Z_r \times \mathbb{S}^{D-1} \mid v \cdot n_x > 0\}$ – or its quotient under the action of $\mathbb{Z}^D$-translations on space variables $\tilde{\Gamma}_r^+ = \Gamma_r/\mathbb{Z}^D$ – equipped with its Borel σ-algebra and the probability measure ν_r proportional to γ_r, where

$$d\gamma_r(x, v) = (v \cdot n_x) ds(x) dv,$$

ds being the surface element on ∂Z_r.

The second one is $Y_r \times \mathbb{S}^{D-1}$, equipped with its Borel σ-algebra and the probability measure μ_r proportional to the uniform measure on $Y_r \times \mathbb{S}^{D-1}$. The measure μ_r is obviously invariant under the flow (X mod. $\mathbb{Z}^D$, V) of the Lorentz gas.

Hence, there are two natural notions of a mean free path for the Lorentz gas:

$$\int_{\tilde{\Gamma}_r^+} \tau_r(x, v)\, d\nu_r(x, v) \quad \text{and} \quad \int_{Y_r \times \mathbb{S}^{D-1}} \tau_r(x, v)\, d\mu_r(x, v). \tag{9}$$

3.1. Santalò's formula for the mean free path. In [26], L. Santalò proposed the following simple and elegant explicit expression[3] for the first notion of mean free path.

$$\int_{\tilde{\Gamma}_r^+} \tau_r(x,v)\,d\nu_r(x,v) = \frac{|Y_r|\,|\mathbb{S}^{\mathrm{D}-1}|}{\gamma_r(\tilde{\Gamma}_r^+)} = \frac{1-|\mathbb{B}^{\mathrm{D}}|r^{\mathrm{D}}}{|\mathbb{B}^{\mathrm{D}-1}|r^{\mathrm{D}-1}} \qquad (\textit{Santalò's formula})$$

where $\mathbb{B}^{\mathrm{d}}$ is the d-dimensional unit ball (for the Euclidian norm).

For $\mathrm{D}=3$, one finds

$$\int_{\tilde{\Gamma}_r^+} \tau_r(x,v)\,d\nu_r(x,v) = \frac{1-\frac{4}{3}\pi r^3}{\pi r^2}.$$

With $N_{\mathrm{at}}=1$ and $|v|=1$, this is indeed equivalent in the vanishing r limit to the reciprocal of the factor

$$N_{\mathrm{at}} r^2 |v| \int_{|\omega|=1,\, v\cdot\omega>0} \cos(v,\omega)\,d\omega = \pi r^2$$

that appears in (1). However encouraging, this by itself is not enough to justify the relevance of (1) in the description of the Boltzmann-Grad limit of the periodic Lorentz gas (see the discussion in Section 4 below).

Here is a quick proof of Santalò's formula.

Lemma 3.2 (Dumas–Dumas–Golse [13]). *Let $f \in C^1(\mathbb{R}_+)$ be such that $f(0)=0$. Then one has*

$$\gamma_r(\tilde{\Gamma}_r^+)\int_{\Gamma_r^+} f(\tau_r(x,v))d\nu_r(x,v) = |Y_r|\,|\mathbb{S}^{\mathrm{D}-1}|\int_{Y_r\times\mathbb{S}^{\mathrm{D}-1}} f'(\tau_r(x,v))\,d\mu_r(x,v)$$

This lemma entails Santalò's formula by letting $f(z)=z$, since the integral on the right-hand side of the identity above is equal to 1.

Proof. For each $x \in Z_r$, one has $\tau_r(x+tv,v) = \tau_r(x,v)+t$ for all t near 0. Differentiating in t shows that

$$\begin{aligned} v\cdot\nabla_x\tau_r &= 1, \quad x\in Y_r,\ |v|=1,\\ \tau_r\big|_{\tilde{\Gamma}_r^+} &= 0. \end{aligned}$$

Multiplying each side of the first equality by $f'(\tau_r(x,v))$ and integrating for the uniform measure gives

$$\int_{Y_r\times\mathbb{S}^{\mathrm{D}-1}} \mathrm{div}_x(vf(\tau_r(x,v)))\,dxdv = |Y_r|\,|\mathbb{S}^{\mathrm{D}-1}|\int_{Y_r\times\mathbb{S}^{\mathrm{D}-1}} f'(\tau_r(x,v))\,d\mu_r(x,v).$$

We conclude by applying Green's formula to the integral on the left-hand side. □

[3]If A is a d-dimensional measurable subset of $\mathbb{R}^{\mathrm{D}}$ (with $\mathrm{d}\le\mathrm{D}$), the notation $|A|$ denotes its d-dimensional volume.

3.2. Bounds on the distribution of free path lengths. For each point of the form $x = \frac{1}{2}(1, \ldots, 1) \in Z_r$, the free path length $\tau_r(x, v)$ is infinite for some $v \in \mathbb{Q}^{\mathrm{D}}$, while it is finite whenever the components of v are not rationally dependent. This suggests the presence of tremendous oscillations in the graph of the function τ_r.

Therefore, it becomes interesting to study the distribution of values of $\tau_r(x, v)$. We shall do so in the phase space $Y_r \times \mathbb{S}^{\mathrm{D}-1}$ equipped with the probability measure μ_r. On the other hand, Santalò's formula suggests that the appropriate scale to measure the free path length is the reciprocal of $r^{\mathrm{D}-1}$. Hence we consider

$$\Phi_r(t) = \mu_r\Big(\Big\{(x, v) \in Y_r \times \mathbb{S}^{\mathrm{D}-1} \;\Big|\; \tau_r(x, v) > \frac{t}{r^{\mathrm{D}-1}}\Big)\Big\}.$$

One could also choose to consider instead

$$\Psi_r(t) = \nu_r\Big(\Big\{(x, v) \in \tilde{\Gamma}_r^+ \;\Big|\; \tau_r(x, v) > \frac{t}{r^{\mathrm{D}-1}}\Big)\Big\}.$$

However, the formula in Lemma 3.2 can be recast in the form

$$\int_0^\infty f(t)\Psi_r(r^{\mathrm{D}-1}t)\,dt = \frac{1 - |\mathbb{B}^{\mathrm{D}}|r^{\mathrm{D}}}{|\mathbb{B}^{\mathrm{D}-1}|r^{\mathrm{D}-1}} \int_0^\infty f'(t)\Phi_r(r^{\mathrm{D}-1}t)\,dt$$

for each $f \in C^1(\mathbb{R}_+)$ such that $f(0) = 0$, which means that

$$\Psi_r = -\tfrac{1-|\mathbb{B}^{\mathrm{D}}|r^{\mathrm{D}}}{|\mathbb{B}^{\mathrm{D}-1}|}\Phi_r' \quad \text{on } \mathbb{R}_+^*. \tag{10}$$

Hence it suffices to study Φ_r.

We begin with the following uniform bounds on Φ_r.

Theorem 3.3 (Bourgain–Golse–Wennberg [6], [18]). *For any space dimension* D *such that* $\mathrm{D} > 1$, *there exists two positive constants* $C'_{\mathrm{D}} > C_{\mathrm{D}}$ *such that*

$$\frac{C_{\mathrm{D}}}{t} \leq \Phi_r(t) \leq \frac{C'_{\mathrm{D}}}{t} \quad \textit{for each } t > 1 \textit{ and } r \in \big(0, \tfrac{1}{2}\big).$$

The proof of the upper estimate uses Fourier series in a way that is somewhat reminiscent of Siegel's proof [27] of Minkowski's convex body theorem – see also Theorem 9 in chapter 5 of [24].

The proof of the lower bound is very different in spirit: it is based on a precise counting of infinite open strips included in the billiard table Z_r, very similar to Bleher's analysis for the diffusion limit of the periodic Lorentz gas in [2]. Indeed, the free path length $\tau_r(x, v)$ for x in any such strip is bounded from below by the time $\tilde{\tau}_r(x, v)$ at which the trajectory $\{x + tv \mid t > 0\}$ exits the strip. Since $\tilde{\tau}_r(x, v)$ is explicitly known, its distribution is also explicit, and this provides the lower bound for Φ_r.

Since the function $t \mapsto 1/t$ does not belong to $L^1([1, +\infty))$, the lower estimate in Theorem 3.3 implies that the second notion of mean free path in (9) is

$$\int_{Y_r \times \mathbb{S}^{\mathrm{D}-1}} \tau_r(x, v)\,d\mu_r(x, v) = \int_0^\infty \Phi_r(r^{\mathrm{D}-1}t)\,dt = +\infty \quad \text{for each } r \in \big(0, \tfrac{1}{2}\big).$$

3.3. The distribution of free path lengths for $\mathrm{D} = 2$ as $r \to 0$. Numerical simulations in [18] suggest that the double inequality in Theorem 3.3 could be strengthened into some asymptotic equivalence as $r \to 0$. However, given the very different nature of the proofs for the upper and the lower bounds in Theorem 3.3, one cannot expect this asymptotic equivalence to be established by the same techniques as in [6].

The proof of Theorem 3.3 suggests that rational approximation plays an important role in the slow decay of the distribution of free path lengths. It is well-known that continued fractions provide a fast algorithm for finding the best rational approximants of any irrational number. For that reason, the Lorentz gas in the case $\mathrm{D} = 2$ can be analyzed in a quite detailed manner with continued fractions, as we shall see below. The same analysis in the case of dimension $\mathrm{D} > 2$ would require using simultaneous rational approximation, a much more difficult problem for which no satisfying analogue of the continued fraction algorithm seems to be available at the time of this writing.

For each $v \in \mathbb{S}^1$, define

$$\phi_r(t, v) = \frac{1}{|Y_r|} \left| \left\{ x \in Y_r \;\middle|\; \tau_r(x, v) > \frac{t}{r^{\mathrm{D}-1}} \right\} \right|.$$

Theorem 3.4 (Caglioti–Golse [7], [8]). *In the case of space dimension* $\mathrm{D} = 2$*,*

- *for each $t > 0$ and a.e. $v \in \mathbb{S}^1$, $\phi_r(t, v)$ converges in the sense of Cesàro as $r \to 0$: there exists $\phi(t) \in \mathbb{R}_+$ such that*

$$\lim_{\varepsilon \to 0} \frac{1}{\ln \frac{1}{\varepsilon}} \int_{\varepsilon}^{r^*} \phi_r(t, v) \frac{dr}{r} = \phi(t);$$

- *one has*

$$\phi(t) = \frac{1}{\pi^2 t} + O\Big(\frac{1}{t^2}\Big) \quad \textit{as } t \to +\infty.$$

Obviously

$$\Phi_r(t) = \frac{1}{2\pi} \int_{\mathbb{S}^1} \phi_r(t, v) dv, \qquad \text{so that} \qquad \lim_{\varepsilon \to 0} \frac{1}{\ln \frac{1}{\varepsilon}} \int_{\varepsilon}^{r^*} \Phi_r(t) \frac{dr}{r} = \phi(t). \qquad (11)$$

The asymptotic expansion $\frac{1}{\pi^2 t} + O\big(\frac{1}{t^2}\big)$ has been identified for the first time in [7]. In fact, the result in [7] stated that both the lim sup and the lim inf of the Cesàro mean of Φ_r for the scaling invariant measure $\frac{dr}{r}$ as in (11) have that same asymptotic expansion. The a.e. pointwise (in v) convergence is new – see [8].

3.3.1. Method of proof. Before sketching the proof of the result above, let us recall some background on continued fractions.

The Gauss map is defined as

$$T : (0, 1) \setminus \mathbb{Q} \ni x \mapsto \frac{1}{x} - \left[\frac{1}{x}\right] \in (0, 1) \setminus \mathbb{Q};$$

it is an ergodic automorphism of $(0, 1) \setminus \mathbb{Q}$ with respect to the Gauss measure $dg(x) = \frac{1}{\ln 2}\frac{dx}{1+x}$ that is invariant under T.

Let $x \in (0, 1) \setminus \mathbb{Q}$; define the sequence of positive integers

$$a_k = \left[\frac{1}{T^{k-1}x}\right], \quad k \geq 1.$$

Then x is represented by the continued fraction

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} =: [0; a_1, a_2, a_3, \dots].$$

Define by induction the sequences of integers p_n and q_n by

$$\begin{aligned} p_{n+1} &= a_n p_n + p_{n-1}, \quad \text{for each } n \geq 1, \quad p_0 = 1,\ p_1 = 0, \\ q_{n+1} &= a_n q_n + q_{n-1}, \quad \text{for each } n \geq 1, \quad q_0 = 0,\ q_1 = 1, \end{aligned} \tag{12}$$

For each $n \geq 2$, the integers p_n and q_n are coprime, and the rational number $\frac{p_n}{q_n}$ is called the n-th convergent of x. The distance from x to its n-th convergent is measured by

$$d_n = (-1)^{n-1}(q_n x - p_n) > 0; \tag{13}$$

for each $n \geq 0$, one has

$$d_n = \prod_{k=0}^{n-1} T^k x. \tag{14}$$

(see for instance the third formula on p. 89 of [28]).

Step 1. *A three-term partition of* $\mathbb{T}^2$. A key idea in the proof of Theorem 3.4 is provided by the answer found by S. Blank and N. Krikorian [1] to the following question raised by R. Thom: *"What is the longest orbit of a linear flow with irrational slope on a flat torus with a disk removed?"*

Without loss of generality, assume that the linear flow is $x \mapsto x + tv$ with $v = (\cos\theta, \sin\theta)$ and $\theta \in (0, \frac{\pi}{4})$. The removed disk of radius r is then replaced with the vertical slit $S_r(v)$ of length $2r/\cos\theta$ as shown in Figure 2 (left). Blank and Krikorian found that the set of lengths of all orbits of the linear flow above on $\mathbb{T}^2 \setminus S_r(v)$ consists of exactly three positive values, $l_A(r, v) < l_B(r, v)$ and $l_C(r, v) = l_A(r, v) + l_B(r, v)$.

This suggests considering the three-term partition of $\mathbb{T}^2 \setminus S_r(v)$

$$\{Y_A(r, v), Y_B(r, v), Y_C(r, v)\}$$

defined as follows: $Y_A(r, v)$ (resp. $Y_B(r, v)$, $Y_C(r, v)$) is the union of all orbits of length $l_A(r, v)$ (resp. $l_B(r, v)$, $l_C(r, v)$). Set

$$S_A(r, v) = \{y \in S_r(v) \mid \text{ the orbit starting from } y \text{ is of type } A\}$$

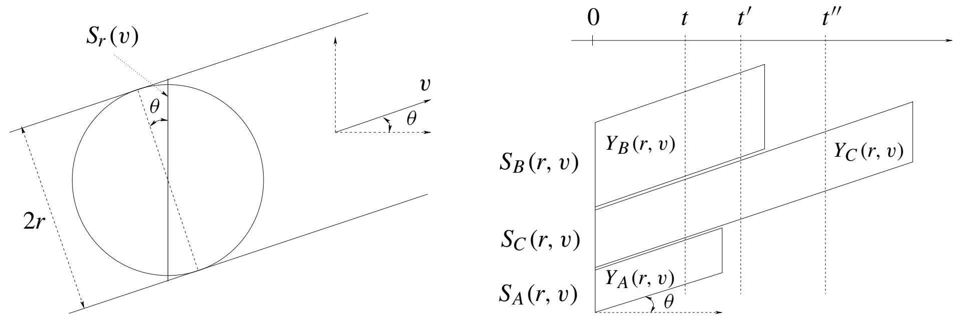

Figure 2. Left: The obstacle and the slit. Right: The 3-strip partition of the 2-torus. This figure gives a simple geometric interpretation of $\psi_r(rs, v)$ for $rs\cos\theta = t$, $rs\cos\theta = t'$ or $rs\cos\theta = t''$.

with analogous definitions for $S_B(r,v)$ and $S_C(r,v)$. Then $S_A(r,v)$, $S_B(r,v)$ and $S_C(r,v)$ are segments while $Y_A(r,v)$, $Y_B(r,v)$ and $Y_C(r,v)$ are (mod 1) parallelograms with one side being $S_A(r,v)$, $S_B(r,v)$ or $S_C(r,v)$ while the adjacent sides are of lengths $l_A(r,v)$, $l_B(r,v)$, or $l_C(r,v)$: see Figure 2 (right).

The orbit lengths $l_A(r,v)$, $l_B(r,v)$ and $l_C(r,v)$, and the lengths of the three segments $S_A(r,v)$, $S_B(r,v)$ and $S_C(r,v)$ are computed in terms of r and the continued fraction expansion of $\tan\theta$ as follows.

Set $\alpha = \tan\theta$, and denote by $\alpha = [0; a_1(\theta), a_2(\theta), a_3(\theta), \dots]$ the continued fraction expansion of $\alpha = \tan\theta$, also let $p_n(\alpha)/q_n(\alpha)$ be the n-th convergent of α as in (12). Finally, let $d_n(\alpha)$ be the sequence of errors as defined in (13).

Define

$$N(\alpha, r) = \min\left\{n \in \mathbb{N} \mid d_n(\alpha) \le 2r\sqrt{1+\alpha^2}\right\}; \tag{15}$$

and

$$k(\alpha, r) = -\left[\frac{2r\sqrt{1+\alpha^2} - d_{N(\alpha,r)-1}}{d_{N(\alpha,r)}}\right] \tag{16}$$

Then, the three-strip partition above is characterized by the formulas below:

$$\begin{aligned} l_A(r,v) &= q_{N(\alpha,r)}(\alpha)\sqrt{1+\alpha^2}, \\ l_B(r,v) &= \left(q_{N(\alpha,r)-1}(\alpha) + k(\alpha,r)q_{N(\alpha,r)}(\alpha)\right)\sqrt{1+\alpha^2}, \\ l_C(r,v) &= \left(q_{N(\alpha,r)-1}(\alpha) + (k(\alpha,r)+1)q_{N(\alpha,r)}(\alpha)\right)\sqrt{1+\alpha^2}, \end{aligned} \tag{17}$$

while

$$\begin{aligned} |S_A(\alpha,r)| &= 2r\sqrt{1+\alpha^2} - d_{N(\alpha,r)}(\alpha), \\ |S_B(\alpha,r)| &= 2r\sqrt{1+\alpha^2} - \left(d_{N(\alpha,r)-1}(\alpha) - k(\alpha,r)d_{N(\alpha,r)}(\alpha)\right), \\ |S_C(\alpha,r)| &= d_{N(\alpha,r)-1}(\alpha) - (k(\alpha,r)-1)d_{N(\alpha,r)}(\alpha) - 2r\sqrt{1+\alpha^2}. \end{aligned} \tag{18}$$

Step 2. *Computing* ϕ_r. Let $\lambda_r(x, v) = \inf\{t > 0 \mid x + tv \in S_r(v)\}$ for each $x \in \mathbb{T}^2 \setminus S_r(v)$; clearly

$$|\tau_r(x, v) - \lambda_r(x, v)| \leq r \quad \text{for each } x \in Y_r \setminus S_r(v). \tag{19}$$

Define

$$\psi_r(t, v) := \text{Prob}\{x \in \mathbb{T}^2 \setminus S_r(v) \mid \lambda_r(x, v) \geq t/r\},$$

where the probability is computed with respect to the uniform measure on Y_r. Because of (19), one has

$$\psi_r(t - r^2, v) - \pi r^2 \leq (1 - \pi r^2)\phi_r(t, v) \leq \psi_r(t + r^2, v) \quad \text{for each } t \geq r^2. \tag{20}$$

On the other hand, ψ_r can be computed explicitly with the help of the three-term partition above. It is found that

$$\begin{aligned}\psi_r(t, v) = \max\big(1 - 2t,\, 1 - \tfrac{1-\delta_N}{\delta_{N-1}}\mu_N - 2t\delta_N,\, 1 - \tfrac{(k-1)\delta_N+1}{\delta_{N-1}}\mu_N - \tfrac{\delta_N}{\delta_{N-2}}\mu_{N-1} \\ - (\delta_{N-1} - (k-1)\delta_N - 1)\big(2t - \tfrac{\mu_{N-1}}{\delta_{N-2}} - k\tfrac{\mu_N}{\delta_{N-1}}\big), 0\big).\end{aligned} \tag{21}$$

In the formula above, $N = N(\alpha, r)$ and $\delta_n = \frac{d_n(\alpha)}{2r\sqrt{1+\alpha^2}}$ while $\mu_n = d_{n-1}(\alpha)q_n(\alpha)$; also $k = k(\alpha, r) = -\big[-\big(\frac{\delta_{N-1}}{\delta_N} - \frac{1}{\delta_N}\big)\big]$. The direction is $v = \big(\frac{1}{\sqrt{1+\alpha^2}}, \frac{\alpha}{\sqrt{1+\alpha^2}}\big)$.

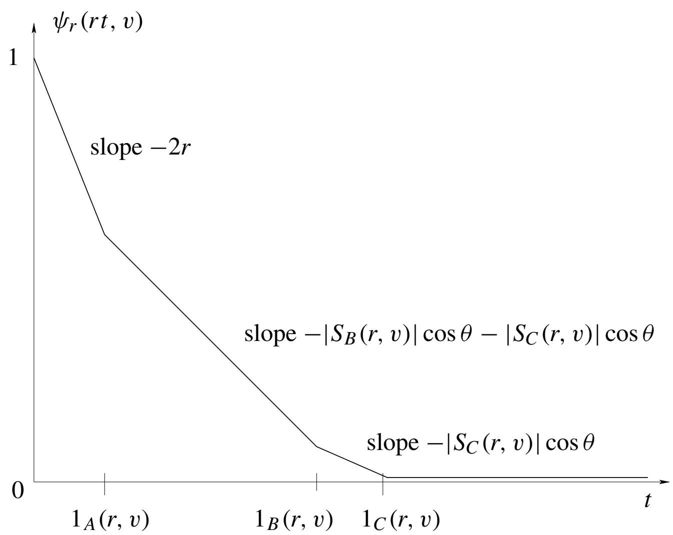

Figure 3. Graph of $\psi_r(rt, v)$.

Step 3. *Using the ergodicity of the Gauss map.* Birkhoff's ergodic theorem says that, for each $h \in L^1\big((0,1), \frac{dx}{1+x}\big)$, one has

$$\frac{1}{N}\sum_{j=0}^{N-1} h(T^m\alpha) \to \frac{1}{\ln 2}\int_0^1 \frac{h(z)dz}{1+z} \quad \text{a.e. in } \alpha \in (0,1) \text{ as } N \to +\infty. \tag{22}$$

Together with formula (14) and the definition of $N(\alpha, r)$, the convergence in (22) for $h = \ln$ implies that

$$N(\alpha, r) \sim \tfrac{12\ln 2}{\pi^2}|\ln r| \quad \text{as } r \to 0, \text{ for a.e. } \alpha \in (0,1). \tag{23}$$

Define

$$\Delta_j(\alpha, x) := -\ln \delta_{N(\alpha,e^{-x})-j}(\alpha) = -\ln d_{N(\alpha,e^{-x})-j}(\alpha) - x + \ln(2\sqrt{1+\alpha^2}) \tag{24}$$

for each $j \geq 0$, $\alpha \in (0,1) \setminus \mathbb{Q}$ and $x > \ln 2$. A further application of Birkhoff's theorem (22) leads to

Lemma 3.5. *Let f be a bounded continuous function on $\mathbb{R}^{m+1}$. Then, for each $x_* \geq \ln 2$, one has*

$$\frac{1}{\ln(1/r)}\int_{x^*}^{\ln(1/r)} f(\Delta_0(\alpha, x), \dots, \Delta_m(\alpha, x))dx \to \int_0^1 \frac{F(\theta)d\theta}{1+\theta} \quad \textit{a.e. in } \alpha \in (0,1)$$

as $r \to 0$, where

$$F(\theta) = \int_0^{|\ln(T^m\theta)|} f(Y_m(y,\theta))\,dy.$$

In the formula above, $Y_m(y,\theta)$ denotes

$$Y_m(y,\theta) = (y, y + \ln T^m\theta, y + \ln T^m\theta + \ln T^{m-1}\theta, \dots, y + \ln T^m\theta + \dots + \ln T\theta).$$

Step 4. *The small scatterer limit for the Cesàro mean of Φ_r.* We seek to apply the lemma above to compute

$$\frac{1}{\ln(1/\varepsilon)}\int_\varepsilon^{1/2} \psi_r(t,v)\,\tfrac{dr}{r} \quad \text{in the limit as } \varepsilon \to 0.$$

Unfortunately, ψ_r given by (21) is not a function of any fixed, finite number of ratios of the form $\frac{\delta_n}{\delta_{n-1}}$, but also involves a few μ_ns – in the original variables, ψ_r explicitly depends on the $q_n(\alpha)$s which involve the complete string of all the $T^j\alpha$s for $j = 0, 1, \dots, n-1$, not only the last one.

Next observe that $\frac{\mu_N}{\delta_{N-1}} \leq 1$, and hence $t \geq 1$ implies that

$$\begin{aligned}\psi_r(t,v) &= \max\big(1 - \tfrac{1-\delta_N}{\delta_{N-1}}\mu_N - 2t\delta_N, 1 - \tfrac{(k-1)\delta_N+1}{\delta_{N-1}}\mu_N - \tfrac{\delta_N}{\delta_{N-2}}\mu_{N-1}\\ &\qquad - (\delta_{N-1} - (k-1)\delta_N - 1)\big(2t - \tfrac{\mu_{N-1}}{\delta_{N-2}} - k\tfrac{\mu_N}{\delta_{N-1}}\big), 0\big)\\ &= \max\big(1 - \tfrac{1-\delta_N}{\delta_{N-1}}\mu_N - 2t\delta_N,\\ &\qquad (\delta_{N-1} - (k-1)\delta_N - 1)\big(\tfrac{\mu_{N-1}}{\delta_{N-2}} + (k+1)\tfrac{\mu_N}{\delta_{N-1}} - 2t\big), 0\big).\end{aligned} \tag{25}$$

In this last equality we have used formula (8) in chapter 1 of [21].

On the other hand, $\delta_{N-1} - (k-1)\delta_N - 1 \leq \delta_N$ so that

$$
\begin{aligned}
0 &\leq \psi_r(t,v) - (1 - \tfrac{1-\delta_N}{\delta_{N-1}}\mu_N - 2t\delta_N)_+ \\
&\leq \tfrac{\delta_N}{\delta_{N-1}}\mu_N \mathbf{1}_{2t \leq \frac{\mu_{N-1}}{\delta_{N-2}} + (k+1)\frac{\mu_N}{\delta_{N-1}}} \leq \tfrac{\delta_N}{\delta_{N-1}}\mu_N \mathbf{1}_{2t \leq (k+2)\frac{\mu_N}{\delta_{N-1}}} \leq \tfrac{1}{k}\mathbf{1}_{k+2\geq t}.
\end{aligned}
$$

Finally $\frac{\delta_N}{\delta_{N-1}}\mu_N \leq \frac{1}{k}$ and $1 - \mu_N \leq \frac{2}{k}$ (see Lemma 4.1 in [7]) so that

$$
\begin{aligned}
\left|\left(1 - \tfrac{1-\delta_N}{\delta_{N-1}}\mu_N - 2t\delta_N\right)_+ - \left(1 - \tfrac{1}{\delta_{N-1}} - 2t\delta_N\right)_+\right| &\leq \left(\tfrac{\delta_N}{\delta_{N-1}}\mu_N + \tfrac{1-\mu_N}{\delta_{N-1}}\right)\mathbf{1}_{k+2\geq t} \\
&\leq \tfrac{3}{k}\mathbf{1}_{k+2\geq t}
\end{aligned}
$$

and $\psi_r(t,v)$ can be replaced with $(1 - \frac{1}{\delta_{N-1}} - 2t\delta_N)_+$ modulo an error term controlled by $\frac{3}{k}\mathbf{1}_{k+2\geq t}$. Applying Lemma 3.5 to $f(\Delta_0, \Delta_1) = (1 - e^{\Delta_1} - 2te^{-\Delta_0})_+$ leads to the asymptotic behavior in the second part of Theorem 3.4.

The proof of the a.e. in v convergence uses Steps 1–3 above, in a way that is somehow more involved: see [8] for more details.

3.3.2. Later improvements. Theorem 3.4 was later strengthened by F. Boca and A. Zaharescu [4], in two different ways. First, they were able to remove the need for Cesàro averaging in the convergence statement of (11). Also, they obtained a (semi-)explicit formula for ϕ. Here is their result:

Theorem 3.6 (Boca–Zaharescu [4]). *In the case of space dimension* $\mathrm{D} = 2$*, one has*

$$\Phi_r(t) \to \phi(t) \quad \text{as } r \to 0 \text{ for each } t > 0$$

where

$$
\begin{aligned}
\phi(t) &= 1 - 2t + \tfrac{12}{\pi^2}t^2, && t \in \left(0, \tfrac{1}{2}\right], \\
\phi(t) &= \tfrac{6}{\pi^2}\int_0^{2t-1} a(x,t)\,dx + \tfrac{6}{\pi^2}\int_{2t-1}^1 b(x,t)\,dx, && t \in \left(\tfrac{1}{2}, 1\right], \\
\phi(t) &= \tfrac{6}{\pi^2}\int_0^1 a(x,t)\,dx, && t \in (1, +\infty),
\end{aligned}
$$

with the functions a and b given by

$$
\begin{aligned}
a(x,t) &= \tfrac{(1-x)^2}{x}\left(2\ln\tfrac{2t-x}{2(t-x)} - \tfrac{2t}{x}\ln\tfrac{(2t-x)^2}{4t(t-x)}\right), \\
b(x,t) &= \tfrac{1-2t}{x}\ln\tfrac{1}{2t-x} + \tfrac{(2t-x)(x+1-2t)}{x} + \tfrac{(1-x)^2}{x}\left(2\ln\tfrac{2t-x}{1-x} - \tfrac{2t}{x}\ln\tfrac{(2t-x)}{2t(1-x)}\right).
\end{aligned}
$$

The formulas above for Φ were first conjectured by P. Dahlqvist in [11], by an argument involving Farey fractions, which however remained incomplete since it ultimately relied on the equidistribution of a certain geometrical quantity, which remained to be proved.

The proof by Boca and Zaharescu is essentially based on two ideas: a) using the same 3-strip partition as in [7], in the language of Farey instead of continuous fractions, and b) computing certain sums indexed by lattice points with coprime coordinates by replacing them with integrals while controlling the resulting error terms.

However, being based on averaging in x and v, their proof fails to provide a.e. pointwise convergence in v, unlike the proof of Theorem 3.4, based on Birkhoff's ergodic theorem for the Gauss map, which requires instead averaging in r, thereby proving only convergence in Cesàro's sense.

3.4. The entropy of the billiard map as $r \to 0$. The semi-explicit formula for Φ in Theorem 3.6 has at least one important application besides the problem of justifying the Lorentz equation (1). Define the *billiard map* in the case of the Lorentz gas to be

$$\mathcal{B}_r : \tilde{\Gamma}_r^+ \to \tilde{\Gamma}_r^+, \quad (x, v) \mapsto \mathcal{B}_r(x, v) = (x + \tau_r(x, v)v, \mathcal{R}[n_{x+\tau_r(x,v)v}]v); \tag{26}$$

one easily checks that the measure ν_r is invariant under the map $\mathcal{B}_r$. Denote by $h(\mathcal{B}_r)$ the Kolmogorov–Sinai entropy of the billiard map $\mathcal{B}_r$ with respect to the measure ν_r. A consequence of Theorem 3.6 and of formula (10)[4] is the following asymptotic formula for the entropy of the billiard map in dimension $\mathrm{D} = 2$ and in the small obstacle limit:

$$h(\mathcal{B}_r) = 2\ln\frac{1}{r} + 2 + C + o(1) \quad \text{as } r \to 0.$$

Here the constant C is defined as

$$C = \lim_{r\to 0}\left(\int_{\tilde{\Gamma}_r^+} \ln \tau_r(x, v)\, d\nu_r(x, v) - \ln \int_{\tilde{\Gamma}_r^+} \tau_r(x, v)\, d\nu_r(x, v)\right) = \frac{9\zeta(3)}{4\zeta(2)} - 3\ln 2$$

while ζ is Riemann's zeta function.

In 1991, N. Chernov had proved that, in dimension D, the entropy of the billiard map satisfies

$$h(\mathcal{B}_r) = \mathrm{D}(\mathrm{D}-1)\ln\frac{1}{r} + O(1) \quad \text{as } r \to 0;$$

see [9] and the references therein. That the $O(1)$ error term should actually converge as $r \to 0$ had been conjectured earlier by B. Friedman, Y. Oono and I. Kubo on the basis of numerical simulations; the correct value of the limit was then proposed by Dahlqvist in [11] before Boca–Zaharescu's proof in [4].

4. The Boltzmann-Grad limit: a negative result

In this section, we return to the formulation of the Boltzmann-Grad limit for the periodic Lorentz gas in terms of a homogenization problem for the transport equation, as in Section 2.

[4]In [4], Boca and Zaharescu do not use formula (10); instead they derive the formula for the distribution Ψ_r by using again the 3-term partition and the approximation of sums over coprime lattice points as in the proof of Theorem 3.6.

Going back to the free transport equation (8), we set

$$\vec{C} := \varepsilon\mathbb{Z}^{\mathrm{D}}, \quad r_\varepsilon = \varepsilon^{\frac{\mathrm{D}}{\mathrm{D}-1}}, \quad \Omega_\varepsilon := Z_{r_\varepsilon}[\varepsilon\mathbb{Z}^{\mathrm{D}}] \quad \text{and} \quad f_\varepsilon(t,x,v) := f_{r_\varepsilon}(t,x,v;\varepsilon\mathbb{Z}^{\mathrm{D}}),$$

where $\varepsilon \in (0, 2^{-\mathrm{D}})$. Hence f_ε satisfies

$$\begin{aligned} &\partial_t f_\varepsilon + v\cdot\nabla_x f_\varepsilon = 0, && x\in\Omega_\varepsilon,\ v\in\mathbb{S}^{\mathrm{D}-1},\\ &f_\varepsilon(t,x,v) = f_\varepsilon(t,x,\mathcal{R}[n_x]v), && x\in\partial\Omega_\varepsilon,\ v\in\mathbb{S}^{\mathrm{D}-1}. \end{aligned} \tag{27}$$

For simplicity, we shall assume that ε is of the form $\varepsilon = \frac{1}{n}$ for $n > 2^{\mathrm{D}}$, and that the initial data $f^{\mathrm{in}} \equiv f^{\mathrm{in}}(x,v)$ is continuous and periodic in x with period 1 in each coordinate direction. In other words, $f^{\mathrm{in}} \in C(\mathbb{T}^{\mathrm{D}}\times\mathbb{S}^{\mathrm{D}-1})$.

With the choice of $\varepsilon = \frac{1}{n}$, the solution f_ε of (27) with initial data

$$f_\varepsilon(0,x,v) = f^{\mathrm{in}}(x,v), \quad x\in\Omega_\varepsilon,\ v\in\mathbb{S}^{\mathrm{D}-1}, \tag{28}$$

is also periodic in the variable x with period 1 in each coordinate direction; if one extends f_ε by 0 inside the obstacles and abuse the notation f_ε to designate this extension, one sees that

$$f_\varepsilon \in L^\infty(\mathbb{R}_+\times\mathbb{T}^{\mathrm{D}}\times\mathbb{S}^{\mathrm{D}-1}) \quad \text{with } \|f_\varepsilon\|_{L^\infty} = \|f^{\mathrm{in}}\|_{L^\infty}.$$

By the Banach–Alaoglu theorem, the sequence f_ε (for $\varepsilon = \frac{1}{n}$ with $n > 2^{\mathrm{D}}$) is relatively compact in $L^\infty(\mathbb{R}_+\times\mathbb{T}^{\mathrm{D}}\times\mathbb{S}^{\mathrm{D}-1})$ for the weak-* topology. It is therefore natural to investigate the limit points of f_ε as $\varepsilon\to 0$ – this being exactly the Boltzmann-Grad limit of the periodic Lorentz gas viewed as a homogenization problem for the transport equation.

We begin with the following negative result:

Theorem 4.1 (Golse [17]). *There exists initial data $f^{\mathrm{in}} \in L^\infty(\mathbb{T}^{\mathrm{D}}\times\mathbb{S}^{\mathrm{D}-1})$ such that no subsequence of f_ε converges in $L^\infty(\mathbb{R}_+\times\mathbb{T}^{\mathrm{D}}\times\mathbb{S}^{\mathrm{D}-1})$ weak-* to the solution of the Lorentz kinetic equation* (1).

In fact, the result in [17] is stronger: it excludes the possibility that any subsequence of f_ε converges in $L^\infty(\mathbb{R}_+\times\mathbb{T}^{\mathrm{D}}\times\mathbb{S}^{\mathrm{D}-1})$ weak-* to the solution of any linear Boltzmann equation of the form

$$(\partial_t + v\cdot\nabla_x)f(t,x,v) = \sigma\int_{\mathbb{S}^{\mathrm{D}-1}} k(v,v')(f(t,x,v') - f(t,x,v))\,dv' \tag{29}$$

with $\sigma > 0$ and $k \in C(\mathbb{S}^{\mathrm{D}-1}\times\mathbb{S}^{\mathrm{D}-1})$ such that

$$k(v,w) = k(w,v) > 0, \quad \int_{\mathbb{S}^{\mathrm{D}-1}} k(v,w)\,dv = 1.$$

The proof is based on the fact that the operator

$$f \mapsto v\cdot\nabla_x f + \sigma\int_{\mathbb{S}^{\mathrm{D}-1}} k(v,v')(f(t,x,v) - f(t,x,v'))\,dv'$$

with domain

$$\{f \equiv f(x,v) \in L^2(\mathbb{T}^{\mathrm{D}} \times \mathbb{S}^{\mathrm{D}-1}) \mid v \cdot \nabla_x f \in L^2(\mathbb{T}^{\mathrm{D}} \times \mathbb{S}^{\mathrm{D}-1})\}$$

is Fredholm with nullspace the set of constant functions. Hence there exists $c > 0$ such that

$$\begin{aligned} &\left\| f(t,\cdot,\cdot) - \tfrac{1}{|\mathbb{S}^{\mathrm{D}-1}|} \iint_{\mathbb{T}^{\mathrm{D}} \times \mathbb{S}^{\mathrm{D}-1}} f(t,y,w)\,dydw \right\|_{L^2(\mathbb{T}^{\mathrm{D}} \times \mathbb{S}^{\mathrm{D}-1})} \\ &\quad \le C \| f|_{t=0} \|_{L^2(\mathbb{T}^{\mathrm{D}} \times \mathbb{S}^{\mathrm{D}-1})} e^{-ct} \end{aligned} \tag{30}$$

for each solution of (29). On the other hand, if $f^{\mathrm{in}} \ge 0$ a.e., the solution f_ε of (27) satisfies

$$f_\varepsilon(t,x,v) \ge f^{\mathrm{in}}(x - tv, v) \mathbf{1}_{t \le \varepsilon \tau_{\varepsilon^{1/(\mathrm{D}-1)}}(x,v)}$$

– the right-hand side being the solution of the same transport equation as in (27) but with absorbing boundary condition

$$f_\varepsilon(t,x,v) = 0 \quad \text{for } x \in \partial\Omega_\varepsilon \text{ and } v \cdot n_x > 0.$$

Hence, if f is any weak-* limit point of f_ε in $L^\infty(\mathbb{R}_+ \times \mathbb{T}^{\mathrm{D}} \times \mathbb{S}^{\mathrm{D}-1})$ as $\varepsilon \to 0$, it must satisfy

$$f(t,x,v) \ge \frac{C_{\mathrm{D}}}{t} f^{\mathrm{in}}(x - tv, v)$$

by Theorem 3.3. This is incompatible with (30) as can be seen by taking $f^{\mathrm{in}}(x,v) \equiv \rho(x)$ with $\|\rho\|_{L^2(\mathbb{T}^{\mathrm{D}})} = 1$ while $\|\rho\|_{L^1(\mathbb{T}^{\mathrm{D}})} = o(1)$.

The case of a Lorentz gas with purely absorbing obstacles is much simpler and yet not without interest. Let $g_\varepsilon \equiv g_\varepsilon(t,x,v)$ be the solution of

$$\begin{aligned} \partial_t g_\varepsilon + v \cdot \nabla_x g_\varepsilon &= 0, && x \in \Omega_\varepsilon,\ v \in \mathbb{S}^{\mathrm{D}-1},\ t > 0, \\ g_\varepsilon(t,x,v) &= 0, && x \in \partial\Omega_\varepsilon,\ v \cdot n_x > 0, \\ g_\varepsilon(0,x,v) &= f^{\mathrm{in}}(x,v), && x \in \Omega_\varepsilon,\ v \in \mathbb{S}^{\mathrm{D}-1}. \end{aligned} \tag{31}$$

In the 2-dimensional case, Theorem 3.4 provides a complete description of the limit:

Theorem 4.2 (Caglioti–Golse [7], [8]). *For each $f^{\mathrm{in}} \in L^\infty(\mathbb{T}^2 \times \mathbb{S}^1)$,*

$$\frac{1}{\ln \frac{1}{\eta}} \int_\eta^{1/2} g_\varepsilon \frac{d\varepsilon}{\varepsilon} \to g$$

weakly- in $L^\infty(\mathbb{T}^2 \times \mathbb{S}^1)$ and pointwise in $t \ge 0$ as $\varepsilon \to 0$, with g given by*

$$g(t,x,v) = f^{\mathrm{in}}(x - tv, v)\phi(t),$$

where ϕ is the small scatterer limit of the distribution of free path lengths, whose explicit expression is provided by Theorem 3.6.

In other words, g is the solution of

$$\begin{aligned} \partial_t g + v \cdot \nabla_x g &= \frac{\phi'(t)}{\phi(t)} g, \quad x \in \mathbb{R}^2,\ v \in \mathbb{S}^1,\ t > 0, \\ g_\varepsilon|_{t=0} &= f^{\text{in}}. \end{aligned} \tag{32}$$

Notice that $\frac{\phi'(t)}{\phi(t)} < 0$, so that the term on the right-hand side of (32) indeed models the loss of particles impinging on the obstacles.

This result can be viewed as a homogenization problem for the free transport equation in a domain with holes. The analogous problem for the diffusion (Laplace) equation has been analyzed in detail: see for instance [20], [10]. It describes the steady, D-dimensional motion of particles on Brownian trajectories in a periodic array of circular holes with radius $\varepsilon^{\frac{\mathrm{D}}{\mathrm{D}-2}}$ centered at the points of the cubic lattice $\varepsilon\mathbb{Z}^{\mathrm{D}}$, each particle falling into a hole being permanently removed. Notice the different critical size of the obstacles – $\varepsilon^{\frac{\mathrm{D}}{\mathrm{D}-2}}$ in the diffusion case, instead of $\varepsilon^{\frac{\mathrm{D}}{\mathrm{D}-1}}$ in the transport case – which comes from the fact that the diffusion and free transport operators are of order 2 and 1 respectively, thereby leading to different scalings. More importantly, in the case of the diffusion problem, the loss of particles falling into the holes is described in this limit with a constant absorption coefficient. Indeed, successive increments in Brownian trajectories are independent random variables, so that the periodic structure of the array of holes is somehow ignored by the particles. On the contrary, in the case of the free transport problem (31), the trajectories are straight lines, which introduces correlations between the obstacles. Intuitively, particles which have not encountered any obstacle over some interval of time $[0, T]$ move in a direction that is well approximated by a rational direction – with increasing quality of approximation as T increases. Such particles are therefore much less likely to encounter obstacles after time T, and this agrees with the fact that the absorption rate $\frac{\phi'(t)}{\phi(t)}$ vanishes as $t \to +\infty$.

5. Conclusion

The methods presented above explain why the Lorentz kinetic equation (1) fails to describe the Lorentz gas in the Boltzmann-Grad limit, when the obstacles are centered at the vertices of the cubic lattice $\mathbb{Z}^{\mathrm{D}}$. The ergodic theory of continued fractions provides additional insight on this example of periodic Lorentz gas in the case $\mathrm{D} = 2$, especially on the asymptotic distribution of free path lengths in the small obstacle limit.

Obviously, it would be desirable to obtain as much information in higher dimensions, particularly for the physically relevant case $\mathrm{D} = 3$. This could be difficult, as it might require accurate estimates on simultaneous rational approximation.

Otherwise, it would be useful to have analogues of the results above for 2-dimensional lattices other than $\mathbb{Z}^2$. Specifically, one would like to know whether

Theorems 3.3, 3.4 and 3.6 can be extended or adapted to the case of arbitrary 2-dimensional lattices. If so, it would be particularly interesting to find the intrinsic meaning of the constants $\frac{1}{\pi^2}$ and $\frac{9\zeta(3)}{4\zeta(2)} - 3\ln 2$ that appear in Theorem 3.4 and in Section 3.4.

Finally, the problem of finding an equation describing the Boltzmann-Grad limit of the periodic Lorentz gas – even in the simplest 2-dimensional case and for the cubic lattice $\mathbb{Z}^2$ – remains open. So far, we have no clue as to the structure of such an equation, should it exist: we only know that it cannot be a linear Boltzmann equation of the type (29).

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