Hydrodynamic Limits

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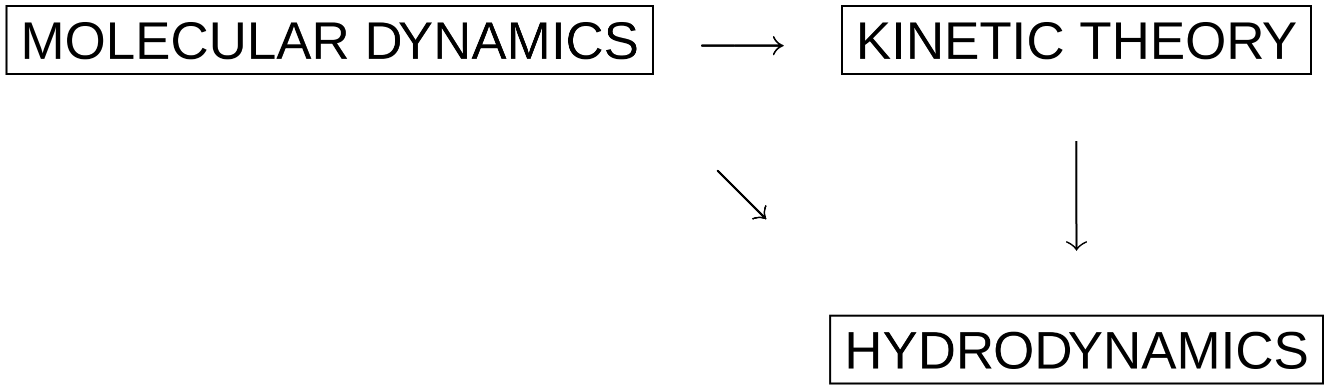
- In his 1866 paper on the kinetic theory of gases, Maxwell explained how the viscosity of a monatomic gas can be computed in terms of data at the molecular scale (scattering cross-section and diameter of the molecules) as well as macroscopic data (the pressure and temperature in the gas).

- Hilbert's 6th problem (1900):"[. . .] Boltzmann's work on the principles of mechanics suggests the problem of developing mathematically the limiting processes [. . .] which lead from the atomistic view to the laws of motion of *continua*"



- Derivation of the Boltzmann equation from molecular dynamics on short time intervals by O.E. Lanford (1975)

- "Formal" derivations of hydrodynamics from molecular dynamics by C.B. Morrey (1951)

- Rigorous results for stochastic models of molecular dynamics on short time intervals by S. Olla, S.R.S. Varadhan and H.T. Yau (1993)

In this talk, we discuss the derivation of the Navier-Stokes equations for incompressible flows from the Boltzmann equation

• Formal argument due to C. Bardos-F.G.-D. Levermore (CRAS 1988, and J. Stat. Phys. 1991)

• Case of global (in time) solutions of Navier-Stokes for small initial data done by C. Bardos-S. Ukai (Math. Models Methods Appl. Sci. 1991)

• Derivation based on a truncated Hilbert expansion sketched by A. DeMasi-R. Esposito-J. Lebowitz (Comm. Pure Appl. Math 1990)

•Case of initial data of arbitrary size: loss of regularity in finite time? for solutions to either the Boltzmann or the 3D Navier-Stokes equations ⇒ work with weak solutions

•Program (moment method as in the formal argument + compactness estimates) by C. Bardos-F.G.-D. Levemore (Comm. Pure Appl. Math. 1993)

•Various intermediate results in this program obtained by

P.-L. Lions-N. Masmoudi (Arch. Rational Mech. Anal. 2000)

F.G.-D. Levermore (Comm. Pure Appl. Math. 2002)

L. Saint-Raymond (Comm. PDEs 2002, Ann. Scient. ENS 2003)

The Navier-Stokes equations for incompressible flows

•Unknown: the velocity field $u \equiv u(t,x) \in \mathbf{R}^3$

•In the absence of external forces (electromagnetic force, gravity...) the velocity field u satisfies

$$\begin{aligned} \mathrm{div}_x\, u &= 0 \\ \partial_t u + (u \cdot \nabla_x) u + \nabla_x p &= \nu \Delta_x u \end{aligned}$$

where $\nu > 0$ is the kinematic viscosity

•NOTATION: $((u \cdot \nabla_x)u)^i := \sum_{j=1}^{3} u^j \frac{\partial u^i}{\partial x^j}$

•If u is a C^1 divergence-free vector field on $\mathbf{R}^3$, then

$$((u\cdot\nabla_x)u)^i=\sum_{j=1}^{3}u^j\frac{\partial u^i}{\partial x^j}=\sum_{j=1}^{3}\frac{\partial(u^iu^j)}{\partial x^j}=:(\mathrm{div}_x(u\otimes u))^i$$

Theorem. (Leray, Acta. Math. 1934) *For each $u^{in}\in L^2(\mathbf{R}^3;\mathbf{R}^3)$ such that $\mathrm{div}_x\, u^{in}=0$, there exists $u\in C(\mathbf{R}_+;L^2(\mathbf{R}^3;\mathbf{R}^3))$ that solves the Cauchy problem*

$$\partial_t u+\mathrm{div}_x(u\otimes u)+\nabla_x p=\nu\Delta_x u\,,\quad \mathrm{div}_x\, u=0$$
$$u\big|_{t=0}=u^{in}$$

in the sense of distributions on $\mathbf{R}_+^*\times\mathbf{R}^3$, and satisfies, for each $t>0$, the energy inequality

$$\tfrac12\int_{\mathbf{R}^3}|u(t,x)|^2dx+\nu\int_0^t\int_{\mathbf{R}^3}|\nabla_x u(s,x)|^2dxds\le\tfrac12\int_{\mathbf{R}^3}|u^{in}(x)|^2dx$$

The Boltzmann equation for a hard sphere gas

•Unknown: the number density $F \equiv F(t,x,v) \geq 0$ in the 1-particle phase space

•In the absence of external forces (electromagnetic force, gravity...) the number density F satisfies

$$\partial_t F + v \cdot \nabla_x F = \mathcal{C}(F)$$

where $\mathcal{C}(F)$ is the Boltzmann collision integral

•Collisions other than binary are neglected; besides, these collisions are viewed as instantaneous and purely local (molecular radius $\simeq$ 0)

$\mathcal{C}$ is a bilinear operator acting only on the v variable in F

The Boltzmann collision integral

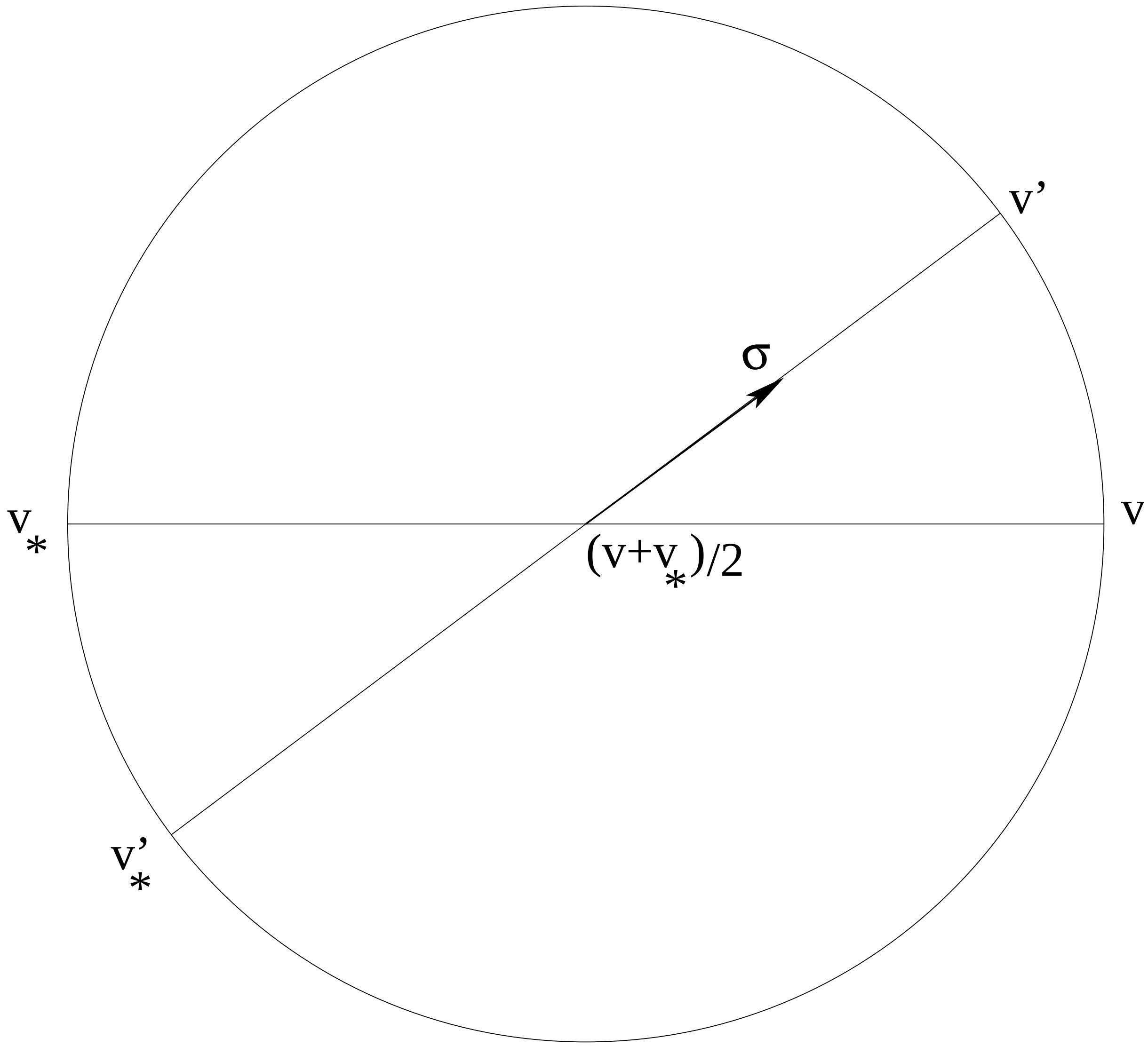
•For a hard sphere gas, the collision integral is

$$\mathcal{C}(F)(v) = \iint_{\mathbf{R}^3\times\mathbf{S}^2} (F(v')F(v'_*) - F(v)F(v_*))|v - v_*|dv_*d\sigma$$

where the velocities v' and v'_* are defined in terms of $v, v_* \in \mathbf{R}^3$ and $\sigma \in \mathbf{S}^2$ by

$$v' \equiv v'(v, v_*, \sigma) = \tfrac{1}{2}(v + v_*) + \tfrac{1}{2}|v - v_*|\sigma$$
$$v'_* \equiv v'_*(v, v_*, \sigma) = \tfrac{1}{2}(v + v_*) - \tfrac{1}{2}|v - v_*|\sigma$$

•Usual notation: F_*, F' and F'_* designate resp. $F(v_*)$, $F(v')$ and $F(v'_*)$



v'
σ
v*
v
(v+v*)/2
v'*

Boltzmann's H Theorem

•Assume that $F \equiv F(v) > 0$ a.e. is rapidly decaying and such that $\ln F$ has polynomial growth at infinity. Then, the local entropy production rate

$$R(F) = -\int_{\mathbf{R}^3} \mathcal{C}(F) \ln F dv \geq 0$$

•The following conditions are equivalent:

$$R(F) = 0 \text{ a.e.} \quad \Leftrightarrow \quad \mathcal{C}(F) = 0 \text{ a.e.} \quad \Leftrightarrow F \text{ is a Maxwellian}$$

i.e. there exists $\rho, \theta > 0$ and $u \in \mathbf{R}^3$ such that

$$F(v) = \mathcal{M}_{\rho,u,\theta}(v) := \frac{\rho}{(2\pi\theta)^{3/2}} e^{-\frac{|v-u|^2}{2\theta}} \quad \text{a.e. in } v \in \mathbf{R}^3$$

Hydrodynamic limits of kinetic theory leading to incompressible flows consider solutions to the Boltzmann equation that are fluctuations of some uniform Maxwellian state.

•WLOG, we henceforth set this uniform equilibrium state to be

$$M = \mathcal{M}_{(1,0,1)} \qquad \text{(the centered, reduced Gaussian distribution)}$$

•The size of the number density fluctuations around the equilibrium state M will be measured in terms of the relative entropy defined as

$$H(F|M) = \iint_{\mathbf{R}^3\times\mathbf{R}^3} \left[F \ln\left(\frac{F}{M}\right) - F + M \right] dx dv \quad (\geq 0)$$

for each $F \in L^1(\mathbf{R}^3 \times \mathbf{R}^3)$

Renormalized solutions of $(\partial_t + v \cdot \nabla_x)F = \mathcal{C}(F)$

Theorem. (DiPerna-Lions, Ann. Math. 1990) *For each $F^{in} \geq 0$ a.e. such that $H(F^{in}|M) < +\infty$, there exists $F \in C(\mathbf{R}_+; L^1_{loc}(\mathbf{R}^3 \times \mathbf{R}^3))$ that solves the Cauchy problem*

$$(\partial_t + v \cdot \nabla_x) \ln(1+F) = \frac{\mathcal{C}(F)}{1+F}, \quad F\Big|_{t=0} = F^{in}$$

in the sense of distributions on $\mathbf{R}^*_+ \times \mathbf{R}^3 \times \mathbf{R}^3$, and satisfies, for each $t > 0$ the entropy inequality

$$H(F(t)|M) + \int_0^t \int_{\mathbf{R}^3} R(F)(s,x)dxds \leq H(F^{in}|M) .$$

The Navier-Stokes limit theorem

Theorem. *Let* u^{in} *be a divergence-free vector field in* $L^2(\mathbf{R}^3;\mathbf{R}^3)$*. For each* $\epsilon > 0$*, let* $F_\epsilon \equiv F_\epsilon(t,x,v)$ *be a renormalized solution to the Boltzmann equation with initial data*

$$F_\epsilon(0,x,v) = \mathcal{M}_{(1,\epsilon u^{in}(\epsilon x),1)}(v)$$

Then the family of vector fields $u_\epsilon \equiv u_\epsilon(t,x) \in \mathbf{R}^3$ *defined by*

$$u_\epsilon(t,x) = \frac{1}{\epsilon}\int_{\mathbf{R}^3} v F_\epsilon\left(\frac{t}{\epsilon^2},\frac{x}{\epsilon},v\right)dv$$

is relatively compact in $L^1_{loc}(\mathbf{R}_+ \times \mathbf{R}^3;\mathbf{R}^3)$ *and each of its limit points as* $\epsilon \to 0$ *is a Leray solution of the Navier-Stokes equations with initial data* u^{in} *and viscosity*

$$\nu = \tfrac{1}{5}\mathcal{D}^*(v\otimes v - \tfrac{1}{3}|v|^2 I)$$

where $\mathcal{D}^*$ *is the Legendre dual of the Dirichlet form of the collision integral.*

•The Dirichlet form of the linearized collision integral is given by

$$\mathcal{D}(\Phi) = \frac{1}{2}\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} |\Phi + \Phi_* - \Phi' - \Phi'_*|^2 |v - v_*| M M_* dv dv_* d\sigma$$

for $\Phi \in C_c(\mathbf{R}^3_v; M_3(\mathbf{R}))$ (with $|\cdot|$ denoting the Hilbert-Schmidt norm).

•The above theorem was proved by F.G. & L. Saint-Raymond — in the case of Maxwell molecules, see F.G.-L.S-R., Invent. Math. 2004.

•<u>REMARK:</u> the definition of u_ϵ consists in intertwining the evolution of the Boltzmann equation with the invariance group of the Navier-Stokes equations, i.e., for each $\lambda > 0$

if $u \equiv u(t,x)$ is a solution of the Navier-Stokes equations, then
$T_\lambda u :\equiv \lambda u(\lambda^2 t, \lambda x)$ is also a solution of the Navier-Stokes equations

Sketch of the proof

•Introduce the relative number density fluctuation g_ϵ:

$$g_\epsilon(t,x,v) = \frac{F_\epsilon\left(\frac{t}{\epsilon^2}, \frac{x}{\epsilon}, v\right) - M(v)}{\epsilon M(v)}, \quad \text{where } M(v) = \frac{1}{(2\pi)^{3/2}} e^{-\frac{|v|^2}{2}}$$

•In terms of g_ϵ, the Boltzmann equation becomes

$$\epsilon \partial_t g_\epsilon + v \cdot \nabla_x g_\epsilon + \frac{1}{\epsilon} \mathcal{L} g_\epsilon = \mathcal{Q}(g_\epsilon, g_\epsilon)$$

where the linearized collision operator $\mathcal{L}$ and $\mathcal{Q}$ are defined by

$$\mathcal{L} g = -M^{-1} D\mathcal{C}[M](Mg), \qquad \mathcal{Q}(g,g) = \tfrac{1}{2} M^{-1} D^2 \mathcal{C}[M](Mg, Mg)$$

Lemma. (Hilbert, Math. Ann. 1912) *The operator $\mathcal{L}$ is self-adjoint, Fredholm, unbounded on $L^2(\mathbf{R}^3; M dv)$ with* $\ker \mathcal{L} = \mathrm{span}\{1, v_1, v_2, v_3, |v|^2\}$

1. Asymptotic fluctuations

•Multiplying the Boltzmann equation by ϵ and letting $\epsilon \to 0$ suggests that

$$g_\epsilon \to g \quad \text{with } \mathcal{L}g = 0$$

By Hilbert's lemma, g is an infinitesimal Maxwellian, i.e. is of the form

$$g(t,x,v) = \rho(t,x) + u(t,x)\cdot v + \tfrac{1}{2}\theta(t,x)(|v|^2 - 3)$$

Notice that g is parametrized by its own moments, since

$$\rho = \langle g \rangle\,, \quad u = \langle vg \rangle\,, \quad \text{and } \theta = \langle (\tfrac{1}{3}|v|^2 - 1) g \rangle$$

•<u>NOTATION:</u>

$$\langle \phi \rangle = \int_{\mathbf{R}^3} \phi(v) M(v)\, dv$$

2. Local conservation laws

For each F rapidly decaying at infinity (in v), the collision integral satisfies

$$\int_{\mathbf{R}^3} \mathcal{C}(F)dv = \int_{\mathbf{R}^3} v_k \mathcal{C}(F)dv = 0\,, \quad k = 1,2,3$$

•The first relation entails the continuity equation

$$\epsilon\partial_t\langle g_\epsilon\rangle + \mathrm{div}_x\langle v g_\epsilon\rangle = 0\,, \quad \text{and thus } \mathrm{div}_x\langle v g\rangle = \mathrm{div}_x\, u = 0$$

which is the incompressibility condition in the Navier-Stokes equations.

•The second relation together with entropy production controls entails

$$\partial_t\langle v g_\epsilon\rangle + \mathrm{div}_x\left(\langle v g_\epsilon\rangle \otimes \langle v g_\epsilon\rangle\right) - \nu\Delta_x\langle v g_\epsilon\rangle \to 0 \text{ modulo gradients}$$

which gives the Navier-Stokes motion equation in the limit as $\epsilon \to 0$.

3. Compactness arguments

•The DiPerna-Lions entropy inequality gives *a priori* bounds on the number density fluctuations that are uniform in ϵ; therefore

$$(1+|v|^2)g_\epsilon \text{ is relatively compact in weak-}L^1_{loc}(\mathbf{R}_+;L^1(\mathbf{R}^3\times\mathbf{R}^3))$$

•Modulo extracting subsequences, for each $\phi = O(|v|^2)$ at infinity

$$\phi g_\epsilon \to \phi g \text{ weakly in } L^1_{loc}(\mathbf{R}_+;L^1(\mathbf{R}^3\times\mathbf{R}^3))$$

and this justifies passing to the limit in expressions that are linear in g_ϵ.

•It remains to pass to the limit in the nonlinear term, i.e. to justify that

$$\langle vg_\epsilon\rangle \otimes \langle vg_\epsilon\rangle \to \langle vg\rangle \otimes \langle vg\rangle \text{ as } \epsilon \to 0$$

and this requires a.e. pointwise, instead of weak convergence.

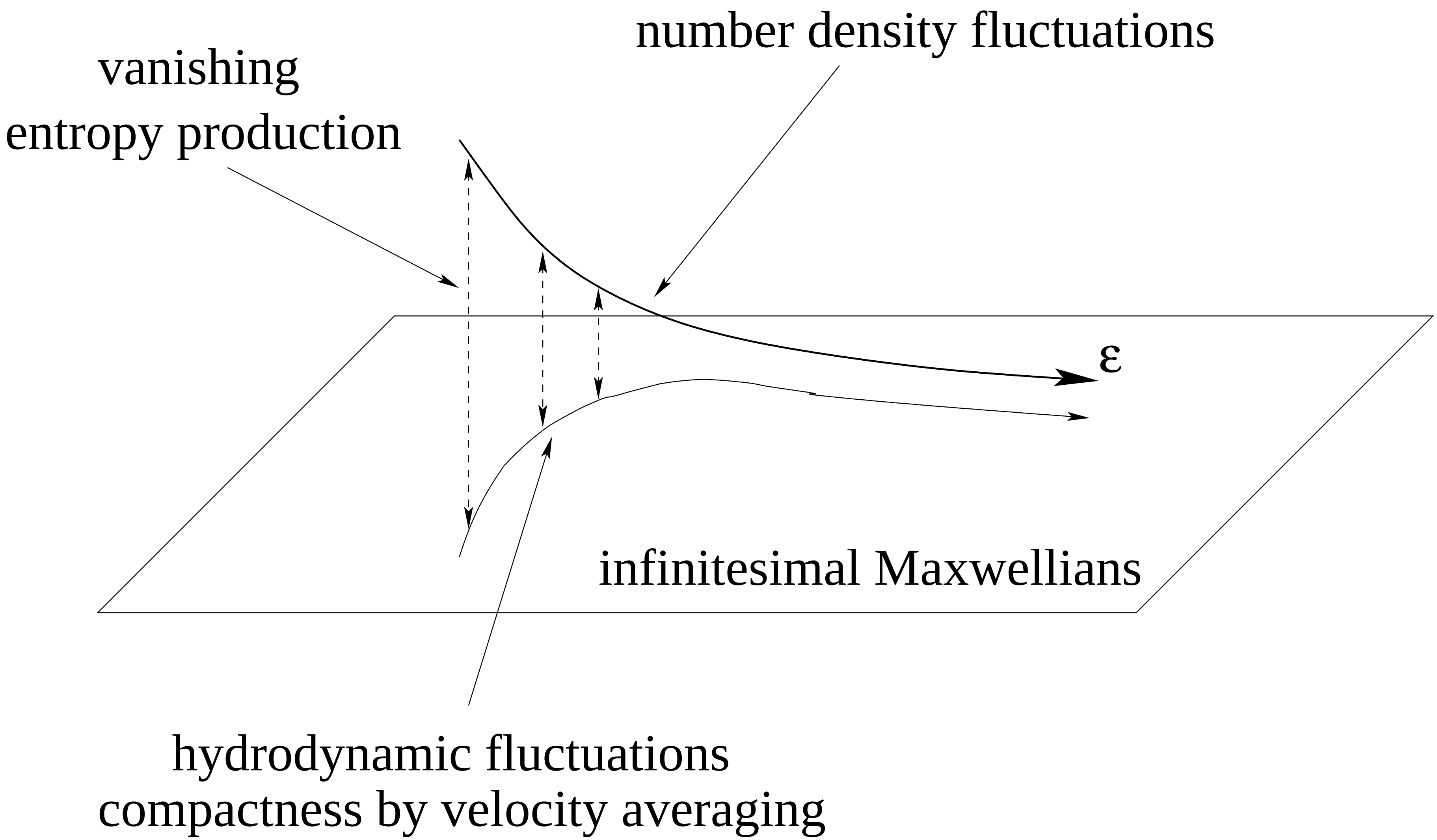
•This is done by using a “velocity averaging" lemma, a typical example of which (in a time-independent situation) is as follows:

Lemma. (F.G.-L. Saint-Raymond, CRAS 2002) *Let $f_n \equiv f(x,v)$ be a bounded sequence in $L^1(\mathbf{R}^D_x; L^p(\mathbf{R}^D_v))$ for some $p > 1$ such that the sequence $v \cdot \nabla_x f_n$ is bounded in $L^1(\mathbf{R}^D \times \mathbf{R}^D)$. Then*

•*the sequence f_n is weakly relatively compact in $L^1_{loc}(\mathbf{R}^D \times \mathbf{R}^D)$; and*

•*for each $\phi \in C_c(\mathbf{R}^D)$, the sequence of moments*

$$\int_{\mathbf{R}^D} f_n(x,v)\phi(v)dv \text{ is strongly relatively compact in } L^1_{loc}(\mathbf{R}^D)$$


number density fluctuations
vanishing
entropy production
ε
infinitesimal Maxwellians
hydrodynamic fluctuations
compactness by velocity averaging

REMARKS ON VELOCITY AVERAGING:

•L^2-variant proved with Fourier techniques (small divisors involving the symbol of $v \cdot \nabla_x$) by F.G.-B. Perthame-R. Sentis (CRAS 1985)

•L^2-based Sobolev regularity of moments by F.G. - P.-L. Lions - B.P. - R.S. (J. Funct. Anal. 1988)

•$L^1_x(L^p_v)$ case: in physical space (instead of Fourier space), one sees that the group generated by $v \cdot \nabla_x$ exchanges x- and v- regularity for $t \neq 0$

$$e^{tv\cdot\nabla_x}\phi(x,v) = \phi(x+tv,v)$$

$\Rightarrow$ dispersion estimates “à la Strichartz”; conclude by interpolation using $t > 0$ as parameter.

Other limits

●From the Boltzmann equation to the Euler equations for compressible flows: analogous to an infinite relaxation system (as in Bouchut's talk)

a) for smooth solutions, before onset of shock waves: see Nishida (Comm. Math. Phys. 1978), and Caflisch (Comm. Pure and Appl. Math. 1980)

b) acoustic limit, under sub-optimal scaling assumptions, done by F.G. - D. Levermore (Comm. Pure Appl. Math. 2002)

c) small BV solutions in the 1D case, "à la Glimm/Bressan"? major open problem, partial results obtained by T.P. Liu, H.S. Yu & T. Yang