Hydrodynamic Limits for the Boltzmann Equation

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LECTURE 5

THE NAVIER-STOKES LIMIT: SETUP AND A PRIORI ESTIMATES

The incompressible Navier-Stokes scaling

•Consider the dimensionless Boltzmann equation in the incompressible Navier-Stokes scaling, i.e. with $\boxed{\mathrm{St} = \pi \mathrm{Kn} = \epsilon \ll 1}$:

$$\epsilon \partial_t F_\epsilon + v \cdot \nabla_x F_\epsilon = \frac{1}{\epsilon} \mathcal{B}(F_\epsilon, F_\epsilon)$$

•Start with an initial data that is a perturbation of some uniform Maxwellian (say, the centered reduced Gaussian $M = M_{1,0,1}$) with Mach number $\boxed{\mathrm{Ma} = O(\epsilon)}$:

$$F_\epsilon^{in} = M_{1,0,1} + \epsilon f_\epsilon^{in}$$

•Example 1: pick $u^{in} \in L^2(\mathbf{R}^3)$ a divergence-free vector field; then the distribution function

$$F_\epsilon^{in}(x,v) = M_{1,\epsilon u^{in}(x),1}(v)$$

is of the type above.

•Example 2: If in addition $\theta^{in} \in L^2 \cap L^\infty(\mathbf{R}^3)$, the distribution function

$$F_\epsilon^{in}(x,v) = M_{1-\epsilon\theta^{in}(x),\frac{\epsilon u^{in}(x)}{1-\epsilon\theta^{in}(x)},\frac{1}{1-\epsilon\theta^{in}(x)}}(v)$$

is also of the type above. (Pick $0 < \epsilon < \frac{1}{\|\theta^{in}\|_{L^\infty}}$, then $1 - \epsilon\theta^{in} > 0$ a.e.).

• Problem : to prove that

$$\frac{1}{\epsilon}\int_{\mathbf{R}^3} vF_\epsilon(t,x,v)dv \to u(t,x) \text{ as } \epsilon \to 0$$

where u solves the incompressible Navier-Stokes equations

$$\partial_t u + u \cdot \nabla_x u + \nabla_x p = \nu \Delta_x u\,, \quad \operatorname{div}_x u = 0$$
$$u\Big|_{t=0} = u^{in}$$

The viscosity ν is given by the same formula as in the Chapman-Enskog expansion.

Renormalized solutions relatively to M

•The DiPerna-Lions theory of renormalized solutions considered initial data vanishing at infinity. In the context of the Navier-Stokes limit, we shall need solutions that approach a uniform Maxwellian state at infinity.

Definition. *A renormalized solution relatively to M of the scaled Boltzmann equation is a nonnegative $F \in C(\mathbf{R}_+; L^1_{loc}(\mathbf{R}^3 \times \mathbf{R}^3))$ such that $H(F_\epsilon(t)|M) < +\infty$ for all $t \geq 0$ and $\Gamma'\left(\frac{F_\epsilon}{M}\right)\mathcal{B}(F_\epsilon, F_\epsilon) \in L^1_{loc}(dtdxdv)$, as well as*

$$M(\epsilon\partial_t + v \cdot \nabla_x)\Gamma\left(\frac{F_\epsilon}{M}\right) = \frac{1}{\epsilon}\Gamma'\left(\frac{F_\epsilon}{M}\right)\mathcal{B}(F_\epsilon, F_\epsilon)$$

in the sense of distributions on $\mathbf{R}_+^* \times \mathbf{R}^3 \times \mathbf{R}^3$, for all $\Gamma \in C^1(\mathbf{R}_+)$ such that $\Gamma(0) = 0$ and $|\Gamma'(Z)| \leq \frac{C}{\sqrt{1+Z}}$.

•In a later paper (CPDEs 1994), P.-L. Lions studied the existence of renormalized solutions to the Boltzmann equation with various limiting conditions at infinity. His results imply the following

Theorem. *Let $F^{in} \geq 0$ a.e. satisfy $H(F_\epsilon|M) < +\infty$. Then there exists a renormalized solution relatively to M of the scaled Boltzmann equation such that $F_\epsilon\Big|_{t=0} = F^{in}$. Moreover, this solution satisfies*

- *the continuity equation (local conservation of mass), and*
- *the DiPerna-Lions relative entropy inequality*

A priori estimates

• The only a priori estimate satisfied by renormalized solutions to the Boltzmann equation is the DiPerna-Lions entropy inequality:

$$H(F_\epsilon|M)(t) + \frac{1}{\epsilon^2}\int_0^t \int_{\mathbf{R}^3} \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} d(F_\epsilon)|(v-v_*)\cdot\omega|\,dv\,dv_*\,d\omega\,dx\,ds \leq H(F_\epsilon^{in}|M)$$

• <u>Notation</u>:

$$H(f|g) = \iint_{\mathbf{R}^3\times\mathbf{R}^3}\left(f\ln\left(\frac{f}{g}\right) - f + g\right)dx\,dv \qquad \text{(relative entropy)}$$

$$d(f) = \tfrac{1}{4}(f'f'_* - ff_*)\ln\left(\frac{f'f'_*}{ff_*}\right) \qquad \text{(dissipation integrand)}$$

•Introduce the relative number density, and the relative number density fluctuation:

$$G_\epsilon = \frac{F_\epsilon}{M}, \qquad g_\epsilon = \frac{F_\epsilon - M}{\epsilon M}$$

•Pointwise inequalities: one easily checks that

$$(\sqrt{G_\epsilon} - 1)^2 \leq C(G_\epsilon \ln G_\epsilon - G_\epsilon + 1)$$
$$\left(\sqrt{G'_\epsilon G'_{\epsilon *}} - \sqrt{G_\epsilon G_{\epsilon *}}\right)^2 \leq \frac{1}{4}(G'_\epsilon G'_{\epsilon *} - G_\epsilon G_{\epsilon *}) \ln \left(\frac{G'_\epsilon G'_{\epsilon *}}{G_\epsilon G_{\epsilon *}}\right)$$
$$= d(G_\epsilon)$$

•Notice that $Z \ln Z - Z + 1 \sim \frac{1}{2}(Z-1)^2$ near $Z = 1$.

•Express that the initial data is a perturbation of the uniform Maxwellian M with Mach number Ma $= O(\epsilon)$:

$$H(F_\epsilon^{in}) \leq C^{in}\epsilon^2$$

•With the DiPerna-Lions entropy inequality, and the pointwise inequalities above, one gets the following uniform in ϵ bounds

$$\iint_{\mathbf{R}^3\times\mathbf{R}^3} (\sqrt{G_\epsilon} - 1)^2 M dv dx \leq C\epsilon^2$$

$$\int_0^{+\infty}\int_{\mathbf{R}^3}\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2}\left(\sqrt{G'_\epsilon G'_{\epsilon *}} - \sqrt{G_\epsilon G_{\epsilon *}}\right)^2 d\mu dx dt \leq C\epsilon^4$$

where μ is the collision measure:

$$d\mu(v, v_*, \omega) = |(v - v_*)\cdot\omega| d\omega M_* dv_* M dv$$

References

- C. Bardos, F. G., D. Levermore: CPAM 1993 (Stokes limit+stationary incompressible Navier-Stokes, assuming local conservation of momentum + nonlinear compactness estimate)

- P.-L. Lions, N. Masmoudi: ARMA 2000 (evolution Navier-Stokes under the same assumptions)

- C.B.-F.G.-D.L.: ARMA 2000 + F.G.-D.L.: CPAM 2002 (local conservation of momentum and energy PROVED in the hydrodynamic limit, for the acoustic and Stokes limits)

- L. Saint-Raymond (CPDEs 2002 + Ann. Sci. ENS 2003): complete derivation of incompressible Navier-Stokes from BGK

- F.G.+L.S.-R.: (Invent. Math. 2004) complete derivation of incompressible Navier-Stokes from Boltzmann for cutoff Maxwell molecules

- L.S.-R. (Bull. Sci. Math. 2002 + ARMA 2003): complete derivation of dissipative solutions to incompressible Euler from BGK and Boltzmann equations

- N. M.+L.S.-R. (CPAM 2003) Stokes limit for the boundary value problem

The BGL Program (CPAM 1993)

•Let $F_\epsilon^{in} \geq 0$ be any sequence of measurable functions satisfying the entropy bound $H(F_\epsilon^{in}|M) \leq C^{in}\epsilon^2$, and let F_ϵ be a renormalized solution relative to M of the scaled Boltzmann equation

$$\epsilon\partial_t F_\epsilon + v \cdot \nabla_x F_\epsilon = \frac{1}{\epsilon}\mathcal{B}(F_\epsilon, F_\epsilon)\,, \quad F_\epsilon\Big|_{t=0} = F_\epsilon^{in}$$

•Let $g_\epsilon \equiv g_\epsilon(x,v)$ be such that $G_\epsilon := 1 + \epsilon g_\epsilon \geq 0$ a.e.. We say that $g_\epsilon \to g$ entropically at rate ϵ as $\epsilon \to 0$ iff

$$g_\epsilon \rightharpoonup g \text{ in } L^1_{loc}(M\,dv\,dx)\,, \text{ and } \frac{1}{\epsilon^2}H(MG_\epsilon|M) \to \tfrac{1}{2}\iint g^2 M\,dv\,dx$$

Theorem. *Assume that*

$$\frac{F_\epsilon^{in}(x,v) - M(v)}{\epsilon M(v)} \to u^{in}(x) \cdot v$$

entropically at rate ϵ. Then the family of bulk velocity fluctuations

$$\frac{1}{\epsilon} \int_{\mathbf{R}^3} v F_\epsilon dv$$

is relatively compact in $w - L^1_{loc}(dtdx)$ and each of its limit points as $\epsilon \to 0$ is a Leray solution of

$$\partial_t u + \operatorname{div}_x(u \otimes u) + \nabla_x p = \nu \Delta_x u\,, \quad \operatorname{div}_x u = 0\,, \quad u\Big|_{t=0} = u^{in}$$

with viscosity given by the formula

$$\nu = \tfrac{1}{10} \int A : \widehat{A} M dv\,, \quad \text{where } \widehat{A} = \mathcal{L}^{-1} A$$

Method of proof

•Renormalization: pick $\gamma \in C^\infty(\mathbf{R}_+)$ a nonincreasing function such that

$$\gamma\Big|_{[0,3/2]} \equiv 1\,, \quad \gamma\Big|_{[2,+\infty)} \equiv 0\,; \qquad \text{set } \widehat{\gamma}(z) = \frac{d}{dz}((z-1)\gamma(z))$$

•The Boltzmann equation is renormalized (relatively to M) as follows:

$$\partial_t(g_\epsilon\gamma_\epsilon) + \frac{1}{\epsilon} v \cdot \nabla_x(g_\epsilon\gamma_\epsilon) = \frac{1}{\epsilon^3}\widehat{\gamma}_\epsilon \mathcal{Q}(G_\epsilon, G_\epsilon)$$

where $\gamma_\epsilon := \gamma(G_\epsilon)$, $\widehat{\gamma}_\epsilon = \widehat{\gamma}(G_\epsilon)$ and $\mathcal{Q}(G,G) = M^{-1}\mathcal{B}(MG, MG)$

• Continuity equation Renormalized solutions of the Boltzmann equation satisfy the local conservation of mass:

$$\epsilon\partial_t\langle g_\epsilon\rangle + \mathrm{div}_x\langle vg_\epsilon\rangle = 0$$

• The entropy bound and Young's inequality imply that

$$(1+|v|^2)g_\epsilon \text{ is relatively compact in } w-L^1_{loc}(dtdx; L^1(Mdv))$$

Modulo extraction of a subsequence

$$g_\epsilon \rightharpoonup g \text{ in } L^1_{loc}(dtdx; L^1(Mdv))$$

and hence $\langle vg_\epsilon\rangle \rightharpoonup \langle vg\rangle =: u$ in $L^1_{loc}(dtdx)$; passing to the limit in the continuity equation leads to the incompressibility condition

$$\mathrm{div}_x\, u = 0$$

•High velocity truncation: pick $K > 6$ and set $K_\epsilon = K|\ln \epsilon|$; for each function $\xi \equiv \xi(v)$, define $\xi_{K_\epsilon}(v) = \xi(v)\mathbf{1}_{|v|^2 \leq K_\epsilon}$

•Multiply both sides of the scaled, renormalized Boltzmann equation by each component of v_{K_ϵ}:

$$\partial_t \langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle + \mathrm{div}_x \mathbf{F}_\epsilon(A) + \nabla_x \frac{1}{\epsilon} \langle \tfrac{1}{3} |v|^2_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle = \mathbf{D}_\epsilon(v)$$

where

$$\mathbf{F}_\epsilon(A) = \tfrac{1}{\epsilon} \langle A_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle \,, \quad \mathbf{D}_\epsilon(v) = \frac{1}{\epsilon^3} \langle\!\langle v_{K_\epsilon} \widehat{\gamma}_\epsilon (G'_\epsilon G'_{\epsilon *} - G_\epsilon G_{\epsilon *}) \rangle\!\rangle$$

•Notation: with $d\mu = |(v - v_*) \cdot \omega| M dv M_* dv_* d\omega$ (collision measure)

$$\langle \phi \rangle = \int_{\mathbf{R}^3} \phi(v) M dv \,, \quad \langle\!\langle \psi \rangle\!\rangle = \iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} \psi(v, v_*, \omega) d\mu$$

•The plan is to prove that, modulo extraction of a subsequence

$$\langle v_{K_\epsilon} g_\epsilon \gamma_\epsilon \rangle \rightharpoonup \langle vg \rangle =: u \qquad \text{in } L^1_{loc}(\mathbf{R}_+ \times \mathbf{R}^3)$$
$$\mathbf{D}_\epsilon(v) \to 0 \qquad \text{in } L^1_{loc}(\mathbf{R}_+ \times \mathbf{R}^3) \text{ and}$$
$$P\left(\mathrm{div}_x \, \mathbf{F}_\epsilon(A)\right) \rightharpoonup P \, \mathrm{div}_x(u^{\otimes 2}) - \nu \boldsymbol{\Delta}_x u \qquad \text{in } L^1_{loc}(dt, W^{-s,1}_{x,loc})$$

for $s > 1$ as $\epsilon \to 0$, where P is the Leray projection (i.e. the L^2 orthogonal projection on divergence-free vector fields).