Hydrodynamic Limits for the Boltzmann Equation

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LECTURE 3

VELOCITY AVERAGING

PART 1:

VELOCITY AVERAGING

Fundamental formulas for the transport equation

• The solution to the Cauchy problem

$$\partial_t f + v\cdot\nabla_x f + a(t,x) f = S(t,x)\,, \quad f\Big|_{t=0} = f^{in}(x)\,, \quad t>0\,,\ x\in\mathbf{R}^D$$

with initial data $f^{in} \equiv f^{in}(x)$, source term S, amplification/absorption rate a, and unknown $f \equiv f(t,x)$ is given by

$$\begin{aligned} f(t,x) &= f^{in}(x-tv)\,\mathrm{exp}\left(-\int_0^t a(t-s,x-sv)ds\right) \\ &\quad + \int_0^t S(t-s,x-sv)\,\mathrm{exp}\left(-\int_0^s a(t-\sigma,x-\sigma v)d\sigma\right)ds \end{aligned}$$

• Method of characteristics: solve as a linear ODE in the variable t

$$\frac{d}{dt} f(t,z+tv) + a(t,z+tv) f(t,z+tv) = S(t,z+tv)$$

and set $z = x - tv$.

•<u>Stationary case:</u> for each $p > 0$, the solution to

$$pf + v \cdot \nabla_x f + a(x) f = S(x) \, , \quad x \in \mathbf{R}^D$$

where a is the amplification/absorption rate, S the source term, and with unknown $f \equiv f(x)$ is given by the formula

$$\boxed{f(x) = \int_0^{+\infty} S(x - tv) \exp\left(-pt - \int_0^t a(x - sv) ds\right) dt}$$

•<u>Proof:</u> Apply the Laplace transform to the evolution problem

$$\partial_t \phi + v \cdot \nabla_x \phi + a\phi = 0 \, , \quad \phi\Big|_{t=0} = S$$

with

$$f(x) = \int_0^{+\infty} e^{-pt} \phi(t, x) dt \Rightarrow \int_0^{+\infty} e^{-pt} \partial_t \phi(t, x) dt = pf(x) - S(x)$$

Velocity Averaging in L^2

•Setting: let m be a finite, positive Radon measure on $\mathbf{R}^D$ such that

(GC_0) $\qquad m(H) = 0$ for any hyperplane $H \ni 0$

Theorem. (G.-Perthame-Sentis, CRAS 1985) *Let $\mathcal{F}$ be a bounded subset of $L^2(\mathbf{R}^D_x \times \mathbf{R}^D_v; dx \otimes dm(v))$ such that*

$$\{v \cdot \nabla_x f \mid f \in \mathcal{F}\} \text{ is bounded in } L^2(\mathbf{R}^D_x \times \mathbf{R}^D_v; dx \otimes dm(v))$$

Then the set of velocity averages

$$\left\{ \int_{\mathbf{R}^D} f(x,v)dm(v) \,\middle|\, f \in \mathcal{F} \right\} \text{ is relatively compact in } L^2_{loc}(\mathbf{R}^D_x; dx)$$

•Earlier regularity remarks reported by Agoshkov (Dokl. AN 1984); general and systematic regularity results in G.-Lions-Perthame-Sentis (JFA 1988)

Velocity Averaging for Evolution Problems

•Set $z = (t, x) \in \mathbf{R} \times \mathbf{R}^D$, $w = (u, v) \in \mathbf{R} \times \mathbf{R}^D$ and

$$\mu = \delta_{u=1} \otimes m$$

•If $f(t, x, v) = F(t, x, u, v)\Big|_{u=1}$, then

$$w \cdot \nabla_z F \in L^2((\mathbf{R} \times \mathbf{R}^D) \times (\mathbf{R} \times \mathbf{R}^D); dtdx \otimes d\mu)$$

is equivalent to

$$\partial_t f + v \cdot \nabla_x f \in L^2(\mathbf{R} \times \mathbf{R}^D \times \mathbf{R}^D; dtdxdm(v))$$

•The homogeneous geometric condition (GC_0) on μ is equivalent to the following affine geometric condition on m:

(GC_a) $\qquad m(H) = 0$ for any affine hyperplane $H \subset \mathbf{R}^D$

Theorem. *Assume that m satisfies (GC_a). Let $\mathcal{F}$ be a bounded subset of $L^2(\mathbf{R}_x^D \times \mathbf{R}_v^D, dxdm(v))$ and assume that $\mathcal{G}$ is a bounded subset of $L^2(\mathbf{R}_+ \times \mathbf{R}_x^D \times \mathbf{R}_v^D, dtdxdm(v))$.*

For each $f^{in} \in \mathcal{F}$ and each $g \in \mathcal{G}$, let f be the solution of

$$\partial_t f + v \cdot \nabla_x f = g\,, \quad f\Big|_{t=0} = f^{in}$$

Then, the set of velocity averages

$$\left\{ \int_{\mathbf{R}^D} f(t,x,v)dm(v) \,\middle|\, f^{in} \in \mathcal{F} \text{ and } g \in \mathcal{G} \right\}$$

is relatively compact in $L^2_{loc}(\mathbf{R}_+ \times \mathbf{R}_x^D; dtdx)$

Proof of Velocity Averaging in L^2

•Rellich's compactness lemma: let $\mathcal{G}$ be a bounded subset of $L^2(\mathbf{R}^D)$. The set $\mathcal{G}$ is relatively compact in $L^2_{loc}(\mathbf{R}^D)$ iff

$$\int_{|\xi|>R} |\hat{g}(\xi)|^2 d\xi \to 0 \text{ as } R \to +\infty \text{ uniformly in } g \in \mathcal{G}$$

• Notation We denote by $\hat{g}$ the Fourier transform of g:

$$\hat{g}(\xi) = \int e^{-i\xi \cdot x} g(x) dx \text{ for each } g \in L^1 \cap L^2(\mathbf{R}^D)$$

•By Plancherel's theorem, the assumptions of the theorem are translated into

$$\{\hat{f} \mid f \in \mathcal{F}\} \text{ and } \{(v \cdot \xi)\hat{f} \mid f \in \mathcal{F}\} \text{ are bounded in } L^2(d\xi \otimes dm(v))$$

where $\hat{f}(\xi, v)$ is the Fourier transform of f in the x-variable:

$$\hat{f}(\xi, v) = \int e^{-i\xi \cdot x} f(x, v) dx$$

•Equivalently

$$\{\phi = (1 + iv \cdot \xi)\widehat{f} \mid f \in \mathcal{F}\} \text{ is bounded in } L^2(d\xi \otimes dm(v))$$

•Denote

$$\rho(x) = \int f(x,v)dm(v)\,, \quad \text{so that} \quad \widehat{\rho}(\xi) = \int \frac{\widehat{\phi}(\xi,v)dm(v)}{1 + i\xi \cdot v}$$

By Cauchy-Schwarz,

$$|\widehat{\rho}(\xi)|^2 \le \Lambda\left(|\xi|, \frac{\xi}{|\xi|}\right) \int |\widehat{g}(\xi,v)|^2 dm(v)$$

where

$$\Lambda(r,\omega) = \int \frac{dm(v)}{\sqrt{1 + r^2(v \cdot \omega)^2}}$$

•Since $m(\{v\cdot\omega=0\})=0$ for each unit vector ω,

$$\Lambda(r,\omega)\to 0 \text{ as } r\to+\infty\,,\qquad \text{pointwise in } \omega\in\mathbf{S}^{D-1}\,.$$

•Moreover, $\Lambda(r,\cdot)$ is continuous on the unit sphere, and $\Lambda(r,\omega)\downarrow 0$ as $r\to+\infty$; by Dini's theorem,

$$\Lambda(r,\omega)\to 0 \text{ as } r\to+\infty\,,\quad \text{uniformly in } \omega\in\mathbf{S}^{D-1}\,.$$

•Then

$$\int_{|\xi|>R}|\hat\rho(\xi)|^2d\xi\le\sup_{|\omega|=1}\Lambda(R,\omega)\iint|g(\xi,v)|^2d\xi dm(v)\to 0$$

$$\text{as } R\to+\infty \text{ uniformly in } g \text{ as } f \text{ runs through } \mathcal{F}$$

and conclude by Rellich's compactness lemma.

Weak compactness in L^1

•A sequence of functions f_n in $L^1(\mathbf{R}^N)$ converges weakly to f iff

$$\int_{\mathbf{R}^N} f_n(x)\phi(x)dx \to \int_{\mathbf{R}^N} f(x)\phi(x)dx\,, \quad \text{for all } \phi \in L^\infty(\mathbf{R}^N)$$

•A bounded subset of $L^1(\mathbf{R}^N)$ may not be weakly relatively compact:

a) there may be concentrations ($\|f_n\|_{L^1} = 1$ and $f_n \rightharpoonup \delta_0$ in the sense of Radon measures)

b) there maybe vanishing at infinity ($\|f_n\|_{L^1} = 1$ and $f\Big|_{|x|\leq R} \to 0$ in L^1 for each $R > 0$)

•<u>Exercise:</u> it may even happen that $\|f_n\|_{L^1} = 1$, that $f_n \rightharpoonup f \in L^1$ in the sense of Radon measures but NOT in the weak L^1 topology.

•Dunford-Pettis Theorem: a bounded subset $\mathcal{F} \subset L^1(\mathbf{R}^N)$ is relatively compact for the weak topology of L^1 iff

- $\mathcal{F}$ is uniformly integrable:

$$\int_A |f(z)|dz \to 0 \text{ as } |A| \to 0 \text{ UNIFORMLY IN } f \in \mathcal{F}$$

- $\mathcal{F}$ is tight:

$$\int_{|z|>R} |f(z)|dz \to 0 \text{ as } R \to +\infty \text{ UNIFORMLY IN } f \in \mathcal{F}$$

•Equivalently, $\mathcal{F}$ is uniformly integrable iff

$$\int_{|f(z)|>c} |f(z)|dz \to 0 \text{ as } c \to +\infty \text{ UNIFORMLY IN } f \in \mathcal{F}$$

•De La Vallée-Poussin Criterion: $\mathcal{F}$ is uniformly integrable iff there exists a function $H : \mathbf{R}_+ \to \mathbf{R}_+$ satisfying

$$\frac{H(r)}{r} \to +\infty \text{ as } r \to +\infty$$

and such that

$$\sup_{f\in\mathcal{F}} \int H(f(z))dz < +\infty$$

•Example: as a function H, one can choose $H(r) = r(\ln r)_+$; in the context of the kinetic theory of gases, an entropy bound implies the uniform integrability of the number densities.

Velocity Averaging in L^1-1

Theorem. *Let* $\mathcal{F} \subset L^1(\mathbf{R}^D \times \mathbf{R}^D; dxdv)$ *be weakly relatively compact and such that* $\{v \cdot \nabla_x f \mid f \in \mathcal{F}\}$ *is bounded in* L^1 *and uniformly integrable. Then the set*

$$\left\{ \int f(x,v)dv \,\middle|\, f \in \mathcal{F} \right\} \text{ is relatively compact in } L^1(\mathbf{R}^D)$$

Theorem. *Let* $\mathcal{F} \subset L^1([0,T] \times \mathbf{R}^D \times \mathbf{R}^D; dtdxdv)$ *be weakly relatively compact and such that* $\{\partial_t f + v \cdot \nabla_x f \mid f \in \mathcal{F}\}$ *is bounded in* L^1 *and uniformly integrable. Then the set*

$$\left\{ \int f(t,x,v)dv \,\middle|\, f \in \mathcal{F} \right\} \text{ is relatively compact in } L^1([0,T] \times \mathbf{R}^D)$$

- Both theorems were proved in G.-Lions-Perthame-Sentis (JFA 1988)

Proof: By Dunford-Pettis, $\mathcal{F}$ is tight, and therefore one can assume WLOG that all the functions in $\mathcal{F}$ are supported in $\{|x|+|v|<r\}$ modulo a small error in L^1 norm.

•Consider the resolvent of the transport operator: for $\lambda > 0$, we define $\boxed{R_\lambda = (\lambda I + v\cdot\nabla_x)^{-1}}$ by the formula

$$R_\lambda S(x,v) = \int_0^{+\infty} e^{-\lambda t} S(x-tv,v)dt$$

(i.e. $R_\lambda S$ is the solution $f \equiv f(x,v)$ of $\lambda f + v\cdot\nabla_x f = S$).

•One checks that

$$\begin{aligned}\|R_\lambda S\|_{L^p} &\leq \int_0^{+\infty} e^{-\lambda t}\|S(x-tv,v)\|_{L^p_{x,v}}dt\\ &= \|S\|_{L^p}\int_0^{+\infty} e^{-\lambda t}dt = \frac{\|S\|_{L^p}}{\lambda}\end{aligned}$$

•Let E be a Banach space, and $H \subset E$. To check that H is relatively compact in E, check that

$$\text{for each } \epsilon > 0\,,\ \text{there exists } K_\epsilon \subset\subset E \text{ s.t. } H \subset K_\epsilon + B(0,\epsilon)$$

•By assumption, $\mathcal{G} = \{g = f + v \cdot \nabla_x f \mid f \in \mathcal{F}\}$ is uniformly integrable; for each $c > 0$, decompose

$$f = f_c^< + f_c^> \,, \quad f_c^< = R_1\left(g\mathbf{1}_{|g|\le c}\right)\,,\ f_c^> = R_1\left(g\mathbf{1}_{|g|>c}\right)$$

•First

$$\rho_c^>(x) = \int_{|v|\le R} f_c^>(x,v)dv$$

satisfies

$$\|\rho_c^>\|_{L^1_x} \le \|f_c^>\|_{L^1_{x,v}} \le \left\|g\mathbf{1}_{|g|>c}\right\|_{L^1_{x,v}} \to 0 \text{ as } c \to +\infty \text{ uniformly in } g \in \mathcal{G}$$

•Then, for each $c > 0$, $g\mathbf{1}_{|g|\leq c}$ is bounded in $L^2_{x,v}$ and hence, by the L^2-Velocity Averaging theorem

$$\rho_c^<(x) = \int_{|v|\leq R} f_c^<(x,v)dv \text{ is relatively compact in } L^1(\mathbf{R}^D)$$

•<u>Conclusion:</u> therefore, for each $\epsilon > 0$, we have found a compact $K_\epsilon \subset L^1(\mathbf{R}^D)$ such that

$$\int f(x,v)dv = \rho_c^< + \rho_c^> \in K_\epsilon + B_{L^1_x}(0,\epsilon)$$

Velocity Averaging in L^1-2

•In fact, one can even drop the assumption of uniform integrability on derivatives (G.-Saint-Raymond, CRAS2002)

Theorem. *Let* $\mathcal{F} \subset L^1(\mathbf{R}^D \times \mathbf{R}^D; dxdv)$ *be weakly relatively compact and such that* $\{v \cdot \nabla_x f \,|\, f \in \mathcal{F}\}$ *is bounded in* L^1*. Then the set*

$$\left\{ \int f(x,v)dv \,\middle|\, f \in \mathcal{F} \right\} \text{ is relatively compact in } L^1(\mathbf{R}^D)$$

Theorem. *Let* $\mathcal{F} \subset L^1([0,T] \times \mathbf{R}^D \times \mathbf{R}^D; dtdxdv)$ *be weakly relatively compact and such that* $\{\partial_t f + v \cdot \nabla_x f \,|\, f \in \mathcal{F}\}$ *is bounded in* L^1*. Then the set*

$$\left\{ \int f(t,x,v)dv \,\middle|\, f \in \mathcal{F} \right\} \text{ is relatively compact in } L^1([0,T] \times \mathbf{R}^D)$$

•<u>Proof:</u> for each $\lambda > 0$, set $R_\lambda = (\lambda I + v \cdot \nabla_x)^{-1}$. We recall that

$$\|R_\lambda\|_{\mathcal{L}(L^1_{x,v})} \leq \frac{1}{\lambda}.$$

Write

$$f = R_\lambda(\lambda f + v \cdot \nabla_x f) = \lambda R_\lambda f + R_\lambda(v \cdot \nabla_x f)$$

so that

$$\int f dv = \lambda \int R_\lambda f dv + \int R_\lambda(v \cdot \nabla_x f) dv$$

Since $\{v \cdot \nabla_x f \mid f \in \mathcal{F}\}$ is bounded in $L^1_{x,v}$, the second term on the r.h.s. can be made arbitrarily small in $L^1_{x,v}$ for some $\lambda > 0$ large enough.

For such a λ, the first term on the r.h.s. is relatively compact in L^1_x by the previous L^1 Velocity Averaging theorem.