Mathematical Models and Methods in Applied Sciences
Vol. 13, No. 5 (2003) 661–714

THE VLASOV–POISSON SYSTEM WITH STRONG MAGNETIC FIELD IN QUASINEUTRAL REGIME

F. GOLSE
Institut Universitaire de France & Département de Mathématiques et Applications, Ecole Normale Supérieure Paris, 45 rue d'Ulm, 75230 Paris cedex 05, France
golse@dma.ens.fr

L. SAINT-RAYMOND
Laboratoire d'Analyse Numérique, Université Paris VI, 175 rue du Chevaleret, 75013 Paris, France
saintray@ann.jussieu.fr

Received 28 October 2000
Revised 5 October 2002
Communicated by P. Degond

Consider the motion of a gas of electrons with a background of ions, subject to the self-consistent electric field and to a constant external magnetic field. As the Debye length and the Larmor radius vanish at the same rate, the asymptotic current density is governed by the 2D1/2 incompressible Euler equation. Establishing limit requires to overcome various difficulties: compactness with respect to the space variable, control of large velocities, oscillations in the time variable. Yet, for particular initial data, the simultaneous gyrokinetic and quasineutral approximation is completely justified.

Keywords: Vlasov–Poisson; quasineutral limit; Gyrokinetic approximation; relative entropy method; high frequency filtering.

AMS Subject Classification: 35Q99, 76X05

1. Introduction

The subject matter of this paper is the mathematical modeling of magnetized plasmas in regimes encountered for instance in tokamaks. In such regimes the plasma is subject to a strong axial magnetic field, inducing high frequencies in the number densities of charged species. In view of the need for numerical simulations in this context, establishing envelope equations that average out these high frequencies as the intensity $|B|$ of the magnetic field tends to infinity is therefore of considerable importance.

However, the features of the asymptotic regime so obtained strongly depend in particular upon the ordering of the following parameters[23]:

- $\rho_e = mc^2/e|B|$, the Larmor radius of the electrons (with mass m and charge $-e$)

- $\rho_i = Mc^2/Ze|B|$, the Larmor radius of the ions (with mass M and charge Ze)
- $\lambda_D = (\varepsilon_0 mc^2/n_e e^2)^{1/2}$, the Debye length where n_e denotes the average density of electrons
- L, the macroscopic (observation) length scale.

Here ε_0 and c denote as usual the dielectric permittivity of the vacuum and the speed of light.

1.1. *Scalings*

Near the axis of the tokamak, one can consider that

$$\rho_e = \lambda_D = 0\,, \quad \rho_i \ll L\,.$$

This implies that the plasma is quasineutral and that the density of electrons is given by the Boltzmann relation, i.e. $n_e(t,x) = n_0 \exp(e\phi(t,x))$ where ϕ is the electric potential and $-e$ the charge of the electron. The motion of ions is then obtained by the classical gyrokinetic approximation.

Closer to the "boundary", typically on a distance of many ion gyroradii, one can define a domain, called the "presheath", where

$$\rho_e \sim \lambda_D \ll L\,, \quad L \sim \rho_i\,.$$

There, the plasma is still quasineutral, but collisions between neutral and charged particles must be taken into account. In particular, a precise description of the motion of electrons is needed.

Still nearer to the "boundary", i.e. at the length scale of the Debye length, there is an electric sheath, i.e. a region where

$$L \sim \lambda_D \sim \rho_e\,, \quad \rho_i \gg L\,.$$

Quasineutrality is not verified there, and the interaction with the boundary (i.e. absorption by the divertors) rules the evolution.

Our purpose in this paper is to study the trajectory of an electron between two collisions in the presheath. In particular, one has to understand what becomes of the gyrokinetic approximation when gradient lengths are of the order of the Larmor radius. Below, we consider a gas of electrons with a background of ions with constant macroscopic density n_i so as to maintain global neutrality. Collisions are neglected. Denote by $f \equiv f(t,x,v)$ the number density of light particles (electrons). As usual, x is the position variable, v the velocity variable, t the time and f is the number density, which means that in any infinitesimal volume $dxdv$ of the phase space centered at (x,v), one can find at time t about $f(t,x,v)dxdv$ particles. A large magnetic field B is applied to this gas of particles; however we assume in this paper that the self-consistent magnetic field can be neglected, thereby reducing the Maxwell equations to their electrostatic approximation, meaning that E is governed by the Poisson equation.[9] These assumptions lead to the following variant of the Vlasov–Poisson equation:

$$\partial_t f + v \cdot \nabla_x f - \frac{e}{m}\left(E + \frac{v}{c} \wedge B\right) \nabla_v f = 0\,,$$

$$\nabla_x \wedge E = 0\,, \quad \varepsilon_0 \nabla_x \cdot E = -e \int_{\mathbb{R}^3} f dv + Zen_i\,,$$

$$f(0,x,v) = f^{in}(x,v)\,, \quad Ze \int n_i(x)\, dx - e \iint f^{in}(x,v)\, dxdv = 0\,.$$

(We recall at this point that Ze is the charge of the ions.)

In this paper, we consider a very simplified model where the magnetic field B is supposed to be homogeneous and stationary, i.e. constant. In particular, $B = |B|b$ where $b = (0,0,1)$. Define the dimensionless variables

$$\tilde{t} = \frac{ct}{L}\,, \quad \tilde{x} = \frac{x}{L}\,, \quad \tilde{v} = \frac{v}{c} \quad \text{and} \quad \tilde{f} = \frac{f}{Zn_i}\,.$$

The previous system can be recast in the form

$$\partial_{\tilde{t}} \tilde{f} + \tilde{v} \cdot \nabla_{\tilde{x}} \tilde{f} - \left(\tilde{E} + \frac{L}{\rho_e} \tilde{v} \wedge b\right) \nabla_{\tilde{v}} \tilde{f} = 0\,,$$

$$\nabla_{\tilde{x}} \wedge \tilde{E} = 0\,, \quad \nabla_{\tilde{x}} \cdot \tilde{E} = \frac{L^2}{\lambda_D^2}\left(1 - \int \tilde{f} d\tilde{v}\right),$$

$$\tilde{f}(0,\tilde{x},\tilde{v}) = \tilde{f}^{in}(\tilde{x},\tilde{v})\,, \quad \iint \tilde{f}^{in} d\tilde{x} d\tilde{v} = 1\,.$$

In the sequel we denote the dimensionless variables with the same letters as the original ones. For simplicity, we assume periodicity in the space variable: hence $(x,v) \in \mathbf{T}^3 \times \mathbb{R}^3$ where $\mathbf{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$, equipped with the measure dx identified with the restriction to $[0,1[^3$ of the Lebesgue measure of $\mathbb{R}^3$.

With the ordering of the presheath, we introduce a small parameter

$$\varepsilon = \frac{\lambda_D}{L} = \frac{\rho_e}{L} \ll 1\,.$$

Then, if we denote by V_ε the electric potential, one eventually arrives at

$$\begin{aligned} &\partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon - \frac{1}{\varepsilon}(\nabla_x V_\varepsilon + v \wedge b) \cdot \nabla_v f_\varepsilon = 0\,, \\ &-\varepsilon \Delta_x V_\varepsilon = \int f_\varepsilon\, dv - 1\,, \\ &f_\varepsilon(0,x,v) = f_\varepsilon^{in}(x,v)\,, \quad \iint f_\varepsilon^{in}(x,v)\, dxdv = 1\,. \end{aligned} \tag{1.1}$$

The gyrokinetic approximation has already been studied in many different regimes.[6,14,18,20,26] In all cases, it is assumed that the Larmor radius of the particles is smaller than the Debye length by one order of magnitude, which means that the electric field induces a weak coupling, and that the guiding-center approximation remains valid. In other words, the variations of the electric field along Larmor circles are assumed to be negligible in all the references above.

By contrast, in the case considered here, the coupling is strong and the collective oscillations are comparable to the magnetic ones; thus we expect a rather different asymptotic regime.

1.2. *Formal analysis*

The existence theory of global weak solutions of the Vlasov–Poisson system is due to Arsen'ev[1] and can be adapted to (1.1) without difficulty. Classical computations lead to the global conservation of mass and energy

$$\iint f_\varepsilon(t,x,v)\,dxdv \equiv \iint f_\varepsilon^{in}dxdv = 1\,,$$

$$\iint f_\varepsilon(t,x,v)|v|^2dxdv + \int |\nabla_x V_\varepsilon|^2(t,x)\,dx$$

$$\equiv \iint f_\varepsilon^{in}|v|^2dxdv + \int |\nabla_x V_\varepsilon^{in}|^2dx = 2\mathcal{E}_\varepsilon^{in} \tag{1.2}$$

while the maximum principle implies that

$$0 \le f_\varepsilon(t,x,v) \le \|f_\varepsilon^{in}\|_{L^\infty(\mathbf{T}^3\times\mathbb{R}^3)} \quad \text{a.e. on } \mathbb{R}^+\times\mathbf{T}^3\times\mathbb{R}^3\,. \tag{1.3}$$

In (1.2) and (1.3), $\mathcal{E}_\varepsilon^{in}$ and $\|f_\varepsilon^{in}\|_{L^\infty}$ depend on ε in a way to be made precise later. If both sequences are bounded, there exist $f \in L^\infty(\mathbb{R}^+\times\mathbf{T}^3\times\mathbb{R}^3)$ and $V \in L^\infty(\mathbb{R}^+, H^1(\mathbf{T}^3))$ such that, up to extraction of a subsequence,

$$f_\varepsilon \rightharpoonup f \text{ weakly}^* \text{ in } L^\infty(\mathbb{R}^+\times\mathbf{T}^3\times\mathbb{R}^3)$$

$$f_\varepsilon(1+|v|^2) \rightharpoonup f(1+|v|^2) \text{ weakly in } L^\infty(\mathbb{R}^+, L^1(\mathbf{T}^3\times\mathbb{R}^3))$$

$$\nabla_x V_\varepsilon \rightharpoonup \nabla_x V \text{ weakly in } L^\infty(\mathbb{R}^+, L^2(\mathbf{T}^3))$$

and taking limits in the Poisson equation

$$\int f_\varepsilon\,dv \rightharpoonup \int f dv = 1 \text{ weakly in } L^1_{\text{loc}}(\mathbb{R}_+\times\mathbf{T}^3)\,.$$

The conservation laws of mass and momentum are obtained by integrating the kinetic equation against 1 and v.

$$\partial_t \int f_\varepsilon\,dv + \nabla_x\cdot\int f_\varepsilon v dv = 0\,,$$

$$\partial_t \int f_\varepsilon v dv + \nabla_x\cdot\int f_\varepsilon v^{\otimes 2}dv + \frac{1}{\varepsilon}\nabla_x V_\varepsilon\int f_\varepsilon\,dv + \frac{1}{\varepsilon}\int f_\varepsilon v\wedge b dv = 0\,.$$

Using the Poisson equation $\int f_\varepsilon\,dv = 1 - \varepsilon\Delta_x V_\varepsilon$, and the identity

$$\nabla_x V\Delta_x V = \nabla_x\left(\nabla_x V\otimes\nabla_x V - \frac{1}{2}|\nabla_x V|^2\,\text{Id}\right)$$

we can put the previous system in the form

$$-\varepsilon\partial_t\Delta_x V_\varepsilon + \nabla_x\cdot\int f_\varepsilon v dv = 0\,, \tag{1.4}$$

$$\partial_t\int f_\varepsilon v dv + \nabla_x\cdot\int f_\varepsilon v^{\otimes 2}dv + \frac{1}{\varepsilon}\nabla_x V_\varepsilon - \nabla_x\left((\nabla_x V_\varepsilon)^{\otimes 2} - \frac{1}{2}|\nabla_x V_\varepsilon|^2\,\mathrm{Id}\right)$$
$$+\frac{1}{\varepsilon}\int f_\varepsilon v\wedge b dv = 0\,. \tag{1.5}$$

Let $J_\varepsilon = \int f_\varepsilon v dv$ and $J = \int f v dv$. Taking limits in (1.4) gives

$$\nabla_x\cdot J = 0 \tag{1.6}$$

while (1.5) leads to

$$\nabla_x V = -J\wedge b\,,\quad \partial_{x_3}V = 0\,. \tag{1.7}$$

Combining (1.4) and (1.5) and integrating in x_3 lead to

$$-\partial_t\int\Delta_x V_\varepsilon\,dx_3 + \nabla_x\int\left(\partial_t J_\varepsilon^\perp + \nabla_x\cdot\int f_\varepsilon v^\perp\otimes v dv - \nabla_x^\perp V_\varepsilon\Delta_x V_\varepsilon\right)dx_3 = 0\,, \tag{1.8}$$

where $u^\perp = u\wedge b = (u_2, -u_1, 0)$. Taking limits formally in (1.8) leads to

$$-\partial_t\int\Delta_x V dx_3 + \nabla_x\int\left(\partial_t J^\perp + \nabla_x\cdot\int f v^\perp\otimes v dv - \nabla_x^\perp V\Delta_x V\right)dx_3 = 0$$

which, combined with (1.7), gives

$$-2\partial_t\Delta_x V + \nabla_x'\otimes\nabla_x' : \int f v^\perp\otimes v dv dx_3 - \nabla_x^\perp V\cdot\nabla_x\Delta_x V = 0 \tag{1.9}$$

(denoting by ∇_x' the operator $(\partial_{x_1}, \partial_{x_2}, 0)$).

Multiplying then the kinetic equation in (1.1) by ε and assuming that $\nabla_x V_\varepsilon\cdot\nabla_v f_\varepsilon \to \nabla_x V\cdot\nabla_v f$, one has

$$(v - \nabla_x^\perp V)^\perp\cdot\nabla_v f = 0\,. \tag{1.10}$$

We multiply this relation successively by v_1v_2 and $v_1^2 - v_2^2$ and integrate by parts, which leads to

$$\int(v_2^2 - v_1^2)f dv = -\partial_{x_1}V\int v_2 f dv - \partial_{x_2}V\int v_1 f dv = (\partial_{x_1}V)^2 - (\partial_{x_2}V)^2\,,$$

$$\int v_1v_2 f dv = -\frac{1}{2}\partial_{x_1}V\int v_1 f dv + \frac{1}{2}\partial_{x_2}V\int v_2 f dv = -\partial_{x_1}V\partial_{x_2}V\,.$$

As $\partial_{x_3}V = 0$, straightforward computations give

$$\begin{aligned}\nabla_x'\otimes\nabla_x' : \int f v^\perp\otimes v dv &= (\partial_{x_1x_1} - \partial_{x_2x_2})\int v_1v_2 f dv + \partial_{x_1}\partial_{x_2}\int(v_2^2 - v_1^2)f dv\\ &= (\partial_{x_2x_2} - \partial_{x_1x_1})(\partial_{x_1}V\partial_{x_2}V) + \partial_{x_1}\partial_{x_2}[(\partial_{x_1}V)^2 - (\partial_{x_2}V)^2]\\ &= -\nabla_x\Delta_x V\cdot\nabla_x^\perp V\,.\end{aligned}$$

Inserting this relation in (1.9), we obtain

$$\partial_t \Delta_x V + \nabla_x^\perp V \cdot \nabla_x \Delta_x V = 0 \tag{1.11}$$

which is the vorticity formulation of the incompressible 2D Euler equation. Then there exists $\Pi \in L^\infty(\mathbb{R}^+, L^1(\mathbf{T}^3))$ such that

$$\partial_t \nabla_x^\perp V + (\nabla_x^\perp V \cdot \nabla_x) \nabla_x^\perp V + \nabla_x' \Pi = 0 \,. \tag{1.12}$$

The pressure Π is then defined as the Lagrange multiplier associated to the incompressibility constraint $\nabla_x \cdot \nabla_x^\perp V = 0$.

It remains to obtain the evolution equation for J_3. By (1.6) and (1.7),

$$\partial_{x_3} J_3 = 0 \,. \tag{1.13}$$

Integrating (1.5) in x_3 and taking limits formally gives

$$\partial_t J_3 + \nabla_x' \cdot \int f v v_3 dv - \Delta_x V \partial_{x_3} V = 0 \,.$$

After multiplying (1.10) by vv_3 and integrating by parts, we obtain

$$\int f v^\perp v_3 dv = -J_3 \nabla_x V \,,$$

so that

$$\partial_t J_3 + \nabla_x \cdot (J_3 \nabla_x^\perp V) = 0 \,. \tag{1.14}$$

The relations (1.12), (1.14) and (1.7) show that J satisfies the incompressible 2D1/2 Euler equation

$$\begin{aligned} &\partial_t J + (J \cdot \nabla_x) J + \nabla_x \Pi = 0 \,, \\ &\nabla_x J = 0 \,, \quad \partial_{x_3} J = 0 \,. \end{aligned} \tag{1.15}$$

In order to obtain a rigorous asymptotic result, we must first justify the process of taking limits in the nonlinear terms. Compactness with regard to the dependence on the time variable is one of the first difficulties, but we shall see in the sequel (Secs. 3 and 5) that in most cases oscillations in the time variable can be fully described. Compactness with regard to the dependence on the space variable is not obvious since the only control available is the energy bound. In this paper, for certain classes of initial data, we have been able to use the regularity given by the limiting system to establish whatever compactness in the space variable was needed for taking limits in the nonlinear terms. However, there cannot exist *a priori* estimates giving this type of regularity unconditionally on solutions of (1.1): this is close to a similar observation by DiPerna–Lions on the 3D Euler equations and will be explained in Sec. 6.

The second problem is to take limits in moments of second order in v, which relies on controlling large velocities. As with the difficulties described above, this is not a purely technical matter: in fact, similar instabilities are known to exist in the quasineutral regime and modify the expected limit. Under the present scaling assumption, one can expect that the magnetic perturbation should stabilize the system in the plane orthogonal to b, but double-humped instabilities can appear in the direction parallel to b. In order to avoid this problem, we consider initial data with special velocity profiles.

2. Main Results

In order to address separately the issues described above, we begin by studying the system in some special regimes.

2.1. *Near-linear regime*

The formal analysis shows that the non-oscillatory part of the system (described by the current density and the electric potential) should be governed by the 2D1/2 incompressible Euler equation. If the global energy $\mathcal{E}_\varepsilon^{in}$ is small, the nonlinear terms become negligible with respect to the linear ones, and the non-oscillatory part stays constant. This situation allows us to study separately the oscillatory behavior of the system.

Oscillations are created by the electrical coupling and the Larmor rotation. The next proposition shows how both effects are combined and generate oscillations with frequencies of order $1/\varepsilon$.

Proposition 2.1. *Let* (f_ε^{in}) *be a family of non-negative functions of* $L^1(\mathbf{T}^3 \times \mathbb{R}^3)$ *such that*

$$\iint f_\varepsilon^{in} dv dx = 1 \quad \text{and} \quad \mathcal{E}_\varepsilon^{in} \to 0 \text{ as } \varepsilon \to 0 \tag{2.1}$$

(*with* $\mathcal{E}_\varepsilon^{in}$ *defined in* (1.2)). *For every* $\varepsilon > 0$, *let* $(f_\varepsilon, V_\varepsilon)$ *be a solution of the scaled Vlasov–Poisson system* (1.1). *Define*

$$j_\varepsilon = \frac{1}{\sqrt{\mathcal{E}_\varepsilon^{in}}} \int f_\varepsilon v dv \,, \quad \phi_\varepsilon = \frac{1}{\sqrt{\mathcal{E}_\varepsilon^{in}}} (-\Delta_x)^{1/2} V_\varepsilon \,. \tag{2.2}$$

Then, the family $((j_\varepsilon, \phi_\varepsilon))_{\varepsilon>0}$ *is bounded in* $L^\infty(\mathbb{R}^+, L^1(\mathbf{T}^3) \times L^2(\mathbf{T}^3))$, *and for all* $s > 3/2$,

$$\mathcal{R}\left(\frac{t}{\varepsilon}\right)(j_\varepsilon, \phi_\varepsilon) - (j_\varepsilon^{in}, \phi_\varepsilon^{in}) \to 0 \quad \text{strongly in } C(\mathbb{R}^+, H^{-s}(\mathbf{T}^3)) \text{ as } \varepsilon \to 0 \,,$$

where $\mathcal{R}(t)$ *is the group of isometries generated by the linear operator*

$$R : (j, \phi) \mapsto (j \wedge b + \nabla_x(-\Delta_x)^{-1/2}\phi \,, \, (-\Delta_x)^{-1/2}\nabla_x \cdot j) \,.$$

2.2. *Well-prepared initial data*

Next, we establish rigorously the asymptotic equations for the non-oscillatory part of the system, in the case where no oscillation occurs. Assuming that the initial data are well-prepared, meaning that the initial density is such that $(\int f_\varepsilon^{in} v dv, (-\Delta_x)^{1/2} V_\varepsilon^{in})$ lies in the nullspace of the operator R defined in Proposition 2.1, we prove that no oscillation appears.

As mentioned in the introduction, we are not able to get *a priori* compactness with respect to space variables either on the electric field or on the current density; and consequently we cannot take limits in the quadratic terms. Then, in order to establish the asymptotic behavior of the system, we will use a stability property of the limiting equation. More precisely we will use the concept of dissipative solutions introduced by Lions[24] to prove the convergence of the incompressible 3D Navier–Stokes equations to the Euler equations, and already used to study the asymptotic behavior of kinetic equations by Golse,[5] Brenier[6] and more recently by Lions and Masmoudi.[25]

Definition 2.1. A dissipative solution of the 2D1/2 incompressible Euler equation (1.15) is a vector field $u \in L^\infty([0,T], L^2(\mathbf{T}^3)) \cap C^0([0,T], w - L^2(\mathbf{T}^3))$ satisfying $\nabla_x \cdot u = 0$ and $\partial_{x_3} u = 0$ in the sense of distributions, as well as $u(0,.) = u^{in}$ and such that

$$\int |w-u|^2(t,x)\,dx \le \int |w-u|^2(0,x)\,dx \exp\left(\int_0^t 2\|D(w)(\tau)\|_\infty ds\right) + 2\int_0^t \exp\left(\int_\tau^t 2\|D(w)(s)\|_\infty ds\right) \int E(w)(u-w)(\tau,x)dxd\tau \tag{2.3}$$

for each vector field $w \in C^\infty([0,T] \times \mathbf{T}^3)$ such that $\nabla_x \cdot w = 0$ and $\partial_{x_3} w = 0$, with $D(w) = \frac{1}{2}(\nabla_x w + (\nabla_x w)^{\mathrm{T}}) \in L^1([0,T], L^\infty(\mathbf{T}^3))$, and $E(w) = \partial_t w + (w \cdot \nabla_x) w \in L^1([0,T], L^2(\mathbf{T}^3))$.

Such solutions always exist; they are not weak solutions of (1.15) in conservative form, but coincide with smooth solutions as long as the latter exist.[24]

Proposition 2.2. *If there exists a solution* $u \in C([0,T], L^2(\mathbf{T}^3))$ *of* (1.15) *on* $[0,T] \times \mathbf{T}^3$ *such that* $D(u) = \frac{1}{2}(\nabla_x u + (\nabla_x u)^{\mathrm{T}}) \in L^1([0,T], L^\infty(\mathbf{T}^3))$ *and* $E(u) = \partial_t u + (u \cdot \nabla_x) u \in L^1([0,T], L^2(\mathbf{T}^3)))$, *then any dissipative solution of* (1.15) *is equal to* u *on* $[0,T] \times \mathbf{T}^3$.

In other words, proving that the current density associated to the solution of (1.1) converges to a dissipative solution of (1.15) entails a strong convergence result in the case where the limiting equation has a smooth solution. The strategy for obtaining dissipative solutions consists of modulating some conserved quantity of the initial system (1.1) by test functions and in establishing a stability inequality similar to (2.3).

A first case is that of well-prepared and almost monokinetic initial data, i.e.

$$f_\varepsilon^{in} \to \delta_{v=J^{in}} \quad \text{as } \varepsilon \to 0 \tag{2.4}$$

in a sense to be made precise later, for some divergence-free vector field J^{in} such that $\partial_{x_3} J^{in} = 0$. In this case, the quantity which is expected to satisfy a stability inequality is the modulated Hamiltonian

$$\frac{1}{2}\iint f_\varepsilon |v - J|^2 dvdx + \frac{1}{2}\int |\nabla_x V_\varepsilon + J \wedge b|^2\, dx\,.$$

One can then prove that, up to extraction of a subsequence,

$$f_\varepsilon \to \delta_{v=J} \quad \text{as } \varepsilon \to 0 \tag{2.5}$$

in some sense, where J is a dissipative solution of (1.15) with initial data J^{in}. More precisely, the following convergence result holds:

Theorem 2.1. *Let $T > 0$ and (f_ε^{in}) be a family of non-negative functions in $L^1(\mathbf{T}^3 \times \mathbb{R}^3)$ such that there exists $J^{in} \in L^2(\mathbf{T}^3)$ with $\nabla_x \cdot J^{in} = 0$ and $\partial_{x_3} J^{in} = 0$ satisfying*

$$\iint f_\varepsilon^{in} dvdx = 1\,, \quad \sup_{\varepsilon>0} \mathcal{E}_\varepsilon^{in} < +\infty\,,$$

$$\sup \left| \int f_\varepsilon^{in} dv - 1 \right| \to 0 \quad as\ \varepsilon \to 0\,, \tag{2.6}$$

$$\frac{1}{2}\iint |v - J^{in}|^2 f_\varepsilon^{in} dvdx + \frac{1}{2}\int |\nabla_x V_\varepsilon^{in} + J^{in} \wedge b|^2 dx \to 0 \quad as\ \varepsilon \to 0\,.$$

For every $\varepsilon > 0$, let $(f_\varepsilon, V_\varepsilon)$ be a solution of the scaled Vlasov–Poisson equation (1.1). *Then, up to extraction of a sequence $\varepsilon_n \to 0$, the current density $\int f_{\varepsilon_n} v dv$ and the scaled electric field $\nabla_x V_{\varepsilon_n}$ converge weakly in $L^1([0,T] \times \mathbf{T}^3) \times L^2([0,T] \times \mathbf{T}^3)$ to $(J, -J \wedge b)$ where $J \in C^0([0,T], w - L^2(\mathbf{T}^3))$ is a dissipative solution of the incompressible* 2D1/2 *Euler equation* (1.15).

In particular, if the incompressible 2D1/2 *Euler equation* (1.15) *with initial data J^{in} has a strong solution on $[0,T]$ (i.e. when J^{in} is smooth), the whole family $(\int f_\varepsilon v dv, \nabla_x V_\varepsilon)$ satisfies in addition the strong convergences*

$$\begin{aligned} &\int f_\varepsilon\, v dv - \left(\int f_\varepsilon\, dv\right) J \to 0 && in\ L^1([0,T] \times \mathbf{T}^3) \\ &\nabla_x V_\varepsilon \to -J \wedge b && in\ L^2([0,T] \times \mathbf{T}^3) \end{aligned} \tag{2.7}$$

as $\varepsilon \to 0$.

This result can be extended to the case of velocity profiles more general than (2.4) and (2.5) provided that they satisfy some stability condition. Let h be a convex function defined on $\mathbb{R}^+$, such that $h'(x) \to -\infty$ as $x \to 0$ and $h(x)/x \to +\infty$ as $x \to +\infty$, more precisely

$$\forall\, p \geq 0\,, \quad \int |h'^{-1}(-r^2))| r^p dr < +\infty\,.$$

(Note that this class of functions is not empty since it contains the usual physical entropy $h : x \mapsto x \log x - x$.) Then it is easy to check that h is an entropy for the Vlasov–Poisson system, i.e. for each solution f of (1.1) with initial data f^{in},

$$\iint h(f(t,x,v))\, dxdv \leq \iint h(f^{in}(x,v))\, dxdv\,.$$

Define the thermodynamic equilibrium $M_{n,J,\theta} \in L^1(\mathbb{R}^3)$ as the minimizer of the entropy

$$\int h(M_{n,J,\theta})\,dv$$
$$= \min\left\{\int h(f)\,dv \middle| \int f dv = n,\ \int fv dv = J,\ \int f|v|^2 dv = \frac{1}{n}J^2 + 3n\theta\right\}.$$

Elementary techniques of the calculus of variations, coupled with Legendre's identity $(h')^{-1} = (h^*)'$ for the Legendre dual h^* of h and symmetry properties, imply that

$$M_{n,J,\theta}(v) = (h^*)'\left(\lambda - \nu\left|v - \frac{J}{n}\right|^2\right), \tag{2.8}$$

where λ and $\nu > 0$ are the Lagrange multipliers associated to the constraints depending only on n and θ. Such profiles are expected to be stable — indeed, as the entropy decreases, if the initial data is close to a local equilibrium at each point, the corresponding solution of (1.1) will be of small entropy and thus should remain close of a local equilibrium at each point. Thus, in the case where the initial data is well-pepared and has a velocity profile (2.8), the quantity expected to satisfy a stability inequality is the modulated free energy

$$\iint (h(f_\varepsilon) - h(M_{1,J,1}) - (f_\varepsilon - M_{1,J,1})h'(M_{1,J,1}))\,dvdx + \nu\int |\nabla_x V_\varepsilon + J\wedge b|^2 dx\,.$$

Actually we will establish that up to extraction of a subsequence,

$$f_\varepsilon \to M_{1,J,1} \quad \text{as } \varepsilon \to 0$$

in some sense, where J is a dissipative solution of (1.15). More precisely

Theorem 2.2. *Let $T > 0$ and (f_ε^{in}) be a family of non-negative functions of $L^1(\mathbf{T}^3 \times \mathbb{R}^3)$ such that there exists $J^{in} \in L^2(\mathbf{T}^3)$ with $\nabla_x \cdot J^{in} = 0$ and $\partial_{x_3} J^{in} = 0$ satisfying*

$$\iint f_\varepsilon^{in} dvdx = 1\,, \quad \sup_{\varepsilon>0} \mathcal{E}_\varepsilon^{in} < +\infty\,,$$
$$\sup\left|\int f_\varepsilon^{in} dv - 1\right| \to 0 \quad as\ \varepsilon \to 0\,,$$

$$\iint (h(f_\varepsilon^{in}) - h(M_{1,J^{in},1}) - (f_\varepsilon^{in} - M_{1,J^{in},1})h'(M_{1,J^{in},1}))dvdx$$
$$+\nu\int |\nabla_x V_\varepsilon^{in} + J^{in}\wedge b|^2 dx \to 0 \quad as\ \varepsilon \to 0\,. \tag{2.9}$$

For each $\varepsilon > 0$, let $(f_\varepsilon, V_\varepsilon)$ be a solution of the scaled Vlasov–Poisson equation (1.1). *Then, up to extraction of a sequence $\varepsilon_n \to 0$, the current density $\int f_{\varepsilon_n} v dv$ and the* *scaled electric field* $\nabla_x V_{\varepsilon_n}$ *converge weakly in* $L^1([0,T] \times \mathbf{T}^3)$ *and* $L^2([0,T] \times \mathbf{T}^3)$ *respectively to* J *and* $-J \wedge b$ *where* $J \in C^0([0,T], w - L^2(\mathbf{T}^3))$ *is a dissipative solution of the* 2D1/2 *incompressible Euler equation* (1.15).

In particular, if the incompressible 2D1/2 *Euler equation* (1.15) *with initial data* J^{in} *has a strong solution on* $[0,T]$ (*i.e. when* J^{in} *is smooth*), *the whole family* $(\int f_\varepsilon v dv, \nabla_x V_\varepsilon)$ *satisfy in addition the strong convergences* (2.7) *as* $\varepsilon \to 0$.

2.3. *General initial data*

It remains to consider the general case where both the oscillating and the non-oscillating parts contribute to the limiting model. Following Babin, Mahalov and Nicolaenko,[3] we prove that the non-oscillating part (corresponding to the weak limits of the current density and the electrical field) is generically governed by the incompressible 2D1/2 Euler equation, while the oscillating part is governed by a linear system of equations whose coefficients depend on the non-oscillating part.

The crucial point in this asymptotic analysis is to see that the limiting equations for the non-oscillating part are decoupled — indeed there is no resonance of the oscillating part in the equation governing the evolution of the non-oscillating part. Such a result comes from a precise study of the oscillating frequencies and is established only generically; as the oscillating frequencies depend on the size of the periodic box, we state a convergence result which holds only for almost all periodic boxes.[3]

The second restriction of our result is the regularity of the initial data. As we need precise estimates on the non-oscillating part to describe the oscillating part, we will restrict our attention to regular initial data providing strong solutions of the limiting system.

Finally, for the sake of simplicity, we state and prove our result in the case of almost monokinetic initial data.

Theorem 2.3. *Denote by* $Q_{a_1,a_2,a_3} = (\mathbb{R}/a_1\mathbb{Z}) \times (\mathbb{R}/a_2\mathbb{Z}) \times (\mathbb{R}/a_3\mathbb{Z})$. *There exists a set* $\mathcal{A} \subset (\mathbb{R}_*^+)^3$ *of Lebesgue measure zero such that for all* $(a_1,a_2,a_3) \in (\mathbb{R}_*^+)^3 \setminus \mathcal{A}$, *the asymptotic behavior of the current density and electric field on* Q_{a_1,a_2,a_3} *can be completely described.*

Let (J^{in}, Φ^{in}) *be a function of* $C^r(Q_{a_1,a_2,a_3})$ *with* $r > 13/2$ *and* $(a_1,a_2,a_3) \in (\mathbb{R}_*^+)^3 \setminus \mathcal{A}$. *Denote by* $\bar{J}$ *the* (*unique*) *smooth solution of the incompressible* 2D1/2 *Euler equation* (1.15) *on* $\mathbb{R}^+ \times Q_{a_1,a_2,a_3}$ *with initial data*

$$\left(\frac{1}{2}\nabla_x'(-\Delta_x')^{-1/2}\int (\Phi^{in}\wedge b)'dx_3 + \frac{1}{2}P'\int (J^{in})'dx_3\,,\ \int J_3^{in}dx_3\right)$$

with the notations $\nabla_x' = (\partial_{x_1}, \partial_{x_2})$, $\Delta_x' = \partial^2_{x_1x_1} + \partial^2_{x_2x_2}$ *and* P' *for the* 2D *Leray projection. Let* (f_ε^{in}) *be a family of non-negative functions of* $L^1(Q_{a_1,a_2,a_3} \times \mathbb{R}^3)$ *such that*

$$\iint f_\varepsilon^{in} dv dx = 1\,, \quad \sup_{\varepsilon>0} \mathcal{E}_\varepsilon^{in} < +\infty\,,$$

$$\sup \left| \int f_\varepsilon^{in} dv - 1 \right| \to 0 \quad as\ \varepsilon \to 0\,,$$

$$\frac{1}{2} \iint \left| v - J^{in} \right|^2 f_\varepsilon^{in} dv dx + \frac{1}{2} \int \left| \nabla_x V_\varepsilon^{in} - \nabla_x (-\Delta_x)^{-1/2} \Phi^{in} \right|^2 dx \to 0 \quad as\ \varepsilon \to 0\,. \tag{2.10}$$

For each $\varepsilon > 0$, *let* $(f_\varepsilon, V_\varepsilon)$ *be a solution of the scaled Vlasov–Poisson equation* (1.1) *with initial data* f_ε^{in}. *Then, the whole family* $(\int f_\varepsilon v dv, \nabla_x V_\varepsilon)_\varepsilon$ *converges weakly to* $(\bar{J}, -\bar{J} \wedge b)$ *in* $H^{-1}_{\text{loc}}(\mathbb{R}^+, L^1 \times L^2(Q_{a_1,a_2,a_3}))$.

More precisely, defining the group of isometries $s \mapsto \mathcal{R}(s) = e^{sR}$ *as in Proposition* 2.1, *the family* $\mathcal{R}(t/\varepsilon)(\int f_\varepsilon v dv, (-\Delta_x)^{1/2} V_\varepsilon)$ *converges strongly in* $L^\infty_{\text{loc}}(\mathbb{R}^+, W^{-1,3/2} \times L^2(Q_{a_1,a_2,a_3}))$ *to a function* $\Psi = \bar{\Psi} + \Psi_{\text{osc}}$ *where* $\bar{\Psi} = (\bar{J}, -(-\Delta_x)^{-1/2} \nabla_x \cdot (\bar{J} \wedge b))$ *is the projection of* Ψ *on the kernel of* R, *and where* Ψ_{osc} *is governed by a linear system of equations whose coefficients depend on* $\bar{J}$.

A natural question is then to determine the asymptotic behavior of the system when (a_1, a_2, a_3) belongs to $\mathcal{A}$. In the framework of 2D rotating fluids (which corresponds to the scaling of the 2D gyrokinetic approximation with fixed Debye length), recent improvements of Schochet's trick[8] show that the weak convergence actually holds for all $(a_1, a_2, a_3) \in (\mathbb{R}_+^*)^3$. The method consists of a precise characterization of the resonant frequencies. It is not easy to extend this result to our case because the singular perturbation is given by a pseudo-differential operator of order 0 that is much more complicated than a rotation.

Extending the weak convergence result stated in Theorem 2.3 to all periodic boxes and to all stable velocity profiles would achieve partially the program of deriving mathematically the gyrokinetic limit in quasineutral regime. The remaining open questions are two well-known mathematical problems, i.e. the global existence of weak solutions for the 2D1/2 incompressible Euler equation (1.15) and the control of instabilities generated by electrical interactions in the quasineutral regime.

3. Near-Linear Regime: Study of the Fast Time Oscillations

We start with a precise description of the oscillatory behavior of the system. The framework of this study is given in Proposition 2.1: there is no particular assumption on the initial data, on the velocity profile, and on the associated macroscopic quantities. We just assume that the initial energy is very small, so that the evolution is governed essentially by the linear part of the system.

Proof of Proposition 2.1. The first step consists of establishing the *a priori* bounds on the families $(j_\varepsilon)_{\varepsilon>0}$ and $(\phi_\varepsilon)_{\varepsilon>0}$ defined by (2.2). From (1.2) we deduce that

$$\|\nabla_x V_\varepsilon\|^2_{L^2(\mathbf{T}^3)} \le 2\mathcal{E}_\varepsilon^{in}\,,$$

$$\left\|\int f_\varepsilon v dv\right\|_{L^1(\mathbf{T}^3)}^2 \leq \left(\iint f_\varepsilon\, dxdv\right)\left(\iint f_\varepsilon |v|^2 dxdv\right) \leq 2\mathcal{E}_\varepsilon^{in}\,.$$

By the definition of j_ε and ϕ_ε and since $(-\Delta_x)^{-1/2}\nabla_x$ is a bounded operator on L^2,

$$\|j_\varepsilon\|_{L^1(\mathbf{T}^3)} \leq \sqrt{2}\,, \quad \|\phi_\varepsilon\|_{L^2(\mathbf{T}^3)} \leq \sqrt{2}\,. \tag{3.1}$$

Next we derive the equations governing the evolution of j_ε and ϕ_ε. Rewriting (1.4) and (1.5) in terms of j_ε and ϕ_ε leads to

$$\begin{aligned} &\varepsilon\partial_t\phi_\varepsilon + (-\Delta_x)^{-1/2}\nabla_x\cdot j_\varepsilon = 0\,,\\ &\partial_t j_\varepsilon + \frac{1}{\varepsilon}\nabla_x(-\Delta_x)^{-1/2}\phi_\varepsilon + \frac{1}{\varepsilon}j_\varepsilon\wedge b = S_\varepsilon\,, \end{aligned} \tag{3.2}$$

where S_ε is defined by

$$S_\varepsilon = \nabla_x\cdot\left(-\frac{1}{\sqrt{\mathcal{E}_\varepsilon^{in}}}\int f_\varepsilon v^{\otimes 2}dv + \frac{1}{\sqrt{\mathcal{E}_\varepsilon^{in}}}\left(\nabla_x V_\varepsilon\otimes\nabla_x V_\varepsilon - \frac{1}{2}|\nabla_x V_\varepsilon|^2\,\mathrm{Id}\right)\right)$$

and therefore satisfies

$$\|S_\varepsilon\|_{L^\infty(\mathbb{R}^+,W^{-1,1}(\mathbf{T}^3))} \leq c\sqrt{\mathcal{E}_\varepsilon^{in}}\,. \tag{3.3}$$

Equipped with these preliminary results, we can now describe the asymptotic behavior of j_ε and ϕ_ε as $\varepsilon\to 0$. Define the operator

$$R : (j,\phi) \mapsto (\nabla_x(-\Delta_x)^{-1/2}\phi + j\wedge b,\ (-\Delta_x)^{-1/2}\nabla_x\cdot j)$$

and recast (3.2) in the form

$$\partial_t(j_\varepsilon,\phi_\varepsilon) + \frac{1}{\varepsilon}R(j_\varepsilon,\phi_\varepsilon) = (S_\varepsilon, 0)\,. \tag{3.4}$$

As R is a bounded skew-adjoint operator on $L^2(\mathbf{T}^3)$, it generates a unitary group on $L^2(\mathbf{T}^3)$ denoted $\mathcal{R}$. In particular,

$$\partial_t\mathcal{R}\left(-\frac{t}{\varepsilon}\right)(j_\varepsilon^{in},\phi_\varepsilon^{in}) + \frac{1}{\varepsilon}R\mathcal{R}\left(-\frac{t}{\varepsilon}\right)(j_\varepsilon^{in},\phi_\varepsilon^{in}) = (0,0)\,. \tag{3.5}$$

Then, in order to establish that $(j_\varepsilon,\phi_\varepsilon)$ and $\mathcal{R}(-\frac{t}{\varepsilon})(j_\varepsilon^{in},\phi_\varepsilon^{in})$ are asymptotically close to one another as $\varepsilon\to 0$, we need a stability result in $L^2(\mathbf{T}^3)$. Define $\Lambda = (-\Delta_x)^{-1/2}$. By Sobolev embedding and (3.3)

$$\|\Lambda^{5/2}S_\varepsilon\|_{L^\infty(\mathbb{R}^+,L^2(\mathbf{T}^3))} \leq C\sqrt{\mathcal{E}_\varepsilon^{in}}\,.$$

Assumption (2.1) implies then that $(\Lambda^{5/2}S_\varepsilon)$ converges strongly to 0 in $L^\infty(\mathbb{R}^+,L^2(\mathbf{T}^3))$ as $\varepsilon\to 0$. Using (3.4) and (3.5) with the commutation properties $\partial_t\Lambda = \Lambda\partial_t$ and $R\Lambda = \Lambda R$ gives

$$\Lambda^{5/2}\left(\partial_t + \frac{1}{\varepsilon}R\right)\Lambda^{5/2}\left((j_\varepsilon,\phi_\varepsilon) - \mathcal{R}\left(-\frac{t}{\varepsilon}\right)(j_\varepsilon^{in},\phi_\varepsilon^{in})\right) = (\Lambda^5 S_\varepsilon, 0)$$

from which we deduce that

$$\left\|(j_\varepsilon,\phi_\varepsilon)-\mathcal{R}\left(-\frac{t}{\varepsilon}\right)(j_\varepsilon^{in},\phi_\varepsilon^{in})\right\|_{H^{-5/2}}^2 \le \left\|\Lambda^{5/2}\left((j_\varepsilon,\phi_\varepsilon)-\mathcal{R}\left(-\frac{t}{\varepsilon}\right)(j_\varepsilon^{in},\phi_\varepsilon^{in})\right)\right\|_{L^2(\pi^3)}^2$$
$$\le \int_0^t \|\Lambda^{5/2}S_\varepsilon(s,x)\|_{L^2(\mathbf{T}^3)}^2 ds$$

converges to 0 in $C(\mathbb{R}^+)$. By (3.1) and Sobolev embeddings, as $\mathcal{R}$ is unitary on $L^2(\mathbf{T}^3)$,

$$\left\|(j_\varepsilon,\phi_\varepsilon)-\mathcal{R}\left(-\frac{t}{\varepsilon}\right)(j_\varepsilon^{in},\phi_\varepsilon^{in})\right\|_{H^{-3/2}(\mathbf{T}^3)} \le 2\|(j_\varepsilon^{in},\phi_\varepsilon^{in})\|_{H^{-3/2}(\mathbf{T}^3)} \le 2\sqrt{2}\,.$$

A standard interpolation argument concludes the proof. □

4. Well-Prepared Initial Data: Convergence of the Non-Oscillating Part

The previous result shows that the linear part of the system does not create oscillations as long as the current density $J_\varepsilon = \int f_\varepsilon v dv$ and Φ_ε defined in terms of the electric potential by $\Phi_\varepsilon = (-\Delta_x)^{1/2}V_\varepsilon$ satisfy

$$\nabla_x\cdot J_\varepsilon = 0\,,\quad \nabla_x(-\Delta_x)^{-1/2}\Phi_\varepsilon + J_\varepsilon\wedge b = 0\,, \tag{4.1}$$

i.e. as long as $(J_\varepsilon,\Phi_\varepsilon)$ belongs to the kernel of R.

In order to study the asymptotic behavior of the non-oscillating part of the system, we begin by eliminating the oscillations. We assume that the initial data are well-prepared, meaning that they satisfy both conditions (4.1). For such initial data, we see that no oscillation occurs, which implies that the conditions (4.1) are in some sense stable. The corresponding solutions of (1.1) as well as their time derivatives are uniformly bounded, and one can prove the desired convergence by using the stability of the limiting system (1.15).

4.1. *A preliminary computation*

Before prooving Theorems 2.1 and 2.2, we restate the fundamental stability result for the Vlasov–Poisson system (1.1), which is on the modulated Hamiltonian.

Lemma 4.1. *For each scalar field $\Phi\in C^\infty([0,T]\times\mathbf{T}^3)$ and each vector field $\bar J\in C^\infty([0,T]\times\mathbf{T}^3)$, the following identity holds*:

$$\frac{d}{dt}\left(\frac12\iint|v-\bar J|^2 f_\varepsilon(t,x,v)\,dvdx + \frac12\int|\nabla_x V_\varepsilon-\nabla_x(-\Delta_x)^{-1/2}\Phi|^2(t,x)\,dx\right)$$
$$= -\int D(\bar J):\left(\int(v-\bar J)^{\otimes2}f_\varepsilon\,dv-(\nabla_x V_\varepsilon-\nabla_x(-\Delta_x)^{-1/2}\Phi)^{\otimes2}\right)(t,x)\,dx$$
$$-\frac12\int(\nabla_x\bar J)|\nabla_x V_\varepsilon-\nabla_x(-\Delta_x)^{-1/2}\Phi|^2(t,x)\,dx$$

$$- \int \left(\partial_t \bar{J} + (\bar{J} \cdot \nabla_x) \bar{J} + \frac{\bar{J} \wedge b}{\varepsilon} + \frac{\nabla_x (-\Delta_x)^{-1/2} \Phi}{\varepsilon} \right) \cdot \int (v - \bar{J}) f_\varepsilon(t, x, v) \, dv dx$$

$$- \int \left(\partial_t \nabla_x (-\Delta_x)^{-1/2} \Phi - \bar{J} (-\Delta_x)^{1/2} \Phi - \frac{\bar{J}}{\varepsilon} \right) \cdot (\nabla_x V_\varepsilon - \nabla_x (-\Delta_x)^{-1/2} \Phi)(t, x) \, dx \,, \tag{4.2}$$

where $D(\bar{J}) = \frac{1}{2}(\nabla_x \bar{J} + (\nabla_x \bar{J})^{\mathrm{T}})$.

Proof. Because of the global energy conservation (1.2),

$$\frac{d}{dt} \left(\frac{1}{2} \iint |v - \bar{J}|^2 f_\varepsilon \, dv dx + \frac{1}{2} \int |\nabla_x V_\varepsilon - \nabla_x (-\Delta_x)^{-1/2} \Phi|^2 \, dx \right)$$

$$= \frac{d}{dt} \Bigg(\iint \left(\frac{1}{2} \bar{J}^2 - v \cdot \bar{J} \right) f_\varepsilon \, dv dx$$

$$+ \int \left(\frac{1}{2} (\nabla_x (-\Delta_x)^{-1/2} \Phi)^2 - \nabla_x V_\varepsilon \cdot \nabla_x (-\Delta_x)^{-1/2} \Phi \right) dx \Bigg).$$

As f_ε is a solution of the scaled Vlasov equation, integrating by parts gives

$$\frac{d}{dt} \left(\frac{1}{2} \iint |v - \bar{J}|^2 f_\varepsilon \, dv dx + \frac{1}{2} \int |\nabla_x V_\varepsilon - \nabla_x (-\Delta_x)^{-1/2} \Phi|^2 \, dx \right)$$

$$= \frac{1}{2} \iint f_\varepsilon \left(\partial_t + v \nabla_x - \frac{1}{\varepsilon} \nabla_x V_\varepsilon \cdot \nabla_v - \frac{1}{\varepsilon} v \wedge b \cdot \nabla_v \right) (\bar{J}^2 - 2 v \cdot \bar{J}) \, dv dx$$

$$+ \int (\nabla_x (-\Delta_x)^{-1/2} \Phi - \nabla_x V_\varepsilon) \cdot \partial_t \nabla_x (-\Delta_x)^{-1/2} \Phi dx$$

$$+ \int (-\Delta_x)^{-1/2} \Phi \, \partial_t \Delta_x V_\varepsilon \, dx \,.$$

Using the local conservation of mass (1.4) gives

$$\frac{d}{dt} \left(\frac{1}{2} \iint |v - \bar{J}|^2 f_\varepsilon \, dv dx + \frac{1}{2} \int |\nabla_x V_\varepsilon - \nabla_x (-\Delta_x)^{-1/2} \Phi|^2 \, dx \right)$$

$$= \iint f_\varepsilon (\bar{J} - v) \left(\partial_t + v \nabla_x \right) \bar{J} dv dx + \frac{1}{\varepsilon} \iint f_\varepsilon (\nabla_x V_\varepsilon + v \wedge b) \cdot \bar{J} dv dx$$

$$+ \int (\nabla_x (-\Delta_x)^{-1/2} \Phi - \nabla_x V_\varepsilon) \cdot \partial_t \nabla_x (-\Delta_x)^{-1/2} \Phi dx$$

$$+ \frac{1}{\varepsilon} \int (-\Delta_x)^{-1/2} \Phi \nabla_x \cdot \int f_\varepsilon v dv dx \,.$$

Decomposing $v = \bar{J} + v - \bar{J}$ leads to

$$\frac{d}{dt}\left(\frac{1}{2}\iint |v-\bar{J}|^2 f_\varepsilon\, dvdx + \frac{1}{2}\int |\nabla_x V_\varepsilon - \nabla_x(-\Delta_x)^{-1/2}\Phi|^2\, dx\right)$$
$$= \iint f_\varepsilon(\bar{J}-v)\cdot\left(\partial_t\bar{J} + (\bar{J}\cdot\nabla_x)\bar{J} + \frac{1}{\varepsilon}\bar{J}\wedge b + \frac{1}{\varepsilon}\nabla_x(-\Delta_x)^{-1/2}\Phi\right) dvdx$$
$$- \iint D(\bar{J}) : f_\varepsilon(v-\bar{J})^{\otimes 2} dvdx + \frac{1}{\varepsilon}\iint f_\varepsilon \nabla_x V_\varepsilon\cdot \bar{J} dvdx$$
$$+ \int (\nabla_x(-\Delta_x)^{-1/2}\Phi - \nabla_x V_\varepsilon)\cdot \partial_t\nabla_x(-\Delta_x)^{-1/2}\Phi dx$$
$$- \frac{1}{\varepsilon}\iint \bar{J}\cdot\nabla_x(-\Delta_x)^{-1/2}\Phi dvdx\,.$$

By the Poisson equation $\int f_\varepsilon\, dv = 1-\varepsilon\Delta_x V_\varepsilon$,

$$\frac{d}{dt}\left(\frac{1}{2}\iint |v-\bar{J}|^2 f_\varepsilon\, dvdx + \frac{1}{2}\int |\nabla_x V_\varepsilon - \nabla_x(-\Delta_x)^{-1/2}\Phi|^2\, dx\right)$$
$$= \iint f_\varepsilon(\bar{J}-v)\left(\partial_t\bar{J} + (\bar{J}\cdot\nabla_x)\bar{J} + \frac{1}{\varepsilon}\bar{J}\wedge b + \frac{1}{\varepsilon}\nabla_x(-\Delta_x)^{-1/2}\Phi\right) dvdx$$
$$- \iint D(\bar{J}) : f_\varepsilon(v-\bar{J})^{\otimes 2} dvdx$$
$$+ \int (\nabla_x(-\Delta_x)^{-1/2}\Phi - \nabla_x V_\varepsilon)\cdot\left(\partial_t\nabla_x(-\Delta_x)^{-1/2}\Phi dx - \frac{1}{\varepsilon}\bar{J} + \bar{J}\Delta_x V_\varepsilon\right).$$

Decomposing $\nabla_x V_\varepsilon = \nabla_x(-\Delta_x)^{-1/2}\Phi + \nabla_x V_\varepsilon - \nabla_x(-\Delta_x)^{-1/2}\Phi$ and using the identity $\nabla_x V\Delta_x V = \nabla_x(\nabla_x V\otimes\nabla_x V - \frac{1}{2}|\nabla_x V|^2\,\mathrm{Id})$ eventually leads to

$$\frac{d}{dt}\left(\frac{1}{2}\iint |v-\bar{J}|^2 f_\varepsilon\, dvdx + \frac{1}{2}\int |\nabla_x V_\varepsilon - \nabla_x(-\Delta_x)^{-1/2}\Phi|^2\, dx\right)$$
$$= \iint f_\varepsilon(\bar{J}-v)\left(\partial_t\bar{J} + (\bar{J}\cdot\nabla_x)\bar{J} + \frac{1}{\varepsilon}\bar{J}\wedge b + \frac{1}{\varepsilon}\nabla_x(-\Delta_x)^{-1/2}\Phi\right) dvdx$$
$$- \iint D(\bar{J}) : f_\varepsilon(v-\bar{J})^{\otimes 2} dvdx$$
$$+ \int (\nabla_x(-\Delta_x)^{-1/2}\Phi - \nabla_x V_\varepsilon)\cdot\left(\partial_t\nabla_x(-\Delta_x)^{-1/2}\Phi dx - \frac{1}{\varepsilon}\bar{J} - \bar{J}(-\Delta_x)^{1/2}\Phi\right)$$
$$+ \int D(\bar{J}) : \left((\nabla_x(-\Delta_x)^{-1/2}\Phi - \nabla_x V_\varepsilon)^{\otimes 2} - \frac{1}{2}|\nabla_x(-\Delta_x)^{-1/2}\Phi - \nabla_x V_\varepsilon|^2\,\mathrm{Id}\right) dx\,.$$

Remarking that $\mathrm{tr}(D(\bar{J})) = \nabla_x\cdot\bar{J}$ completes the proof. □

Lemma 4.1 gives a stability equality very similar to the inequality (2.3) that defines the notion of dissipative solution for the 2D1/2 incompressible Euler equation. Indeed, with the acceleration term denoted

$$E_\varepsilon(\bar J,\phi) = \Bigg(\partial_t \bar J + \bar J\cdot\nabla_x \bar J + \frac{\bar J\wedge b}{\varepsilon} + \frac{\nabla_x(-\Delta_x)^{-1/2}\Phi}{\varepsilon}\,,$$

$$\partial_t\nabla_x(-\Delta_x)^{-1/2}\Phi - \bar J(-\Delta_x)^{1/2}\Phi - \frac{\bar J}{\varepsilon}\Bigg),$$

we deduce from (4.2) that

$$\begin{aligned}
&\frac{d}{dt}\left(\frac12\iint |v-\bar J|^2 f_\varepsilon(t,x,v)\,dvdx + \frac12\int |\nabla_x V_\varepsilon - \nabla_x(-\Delta_x)^{-1/2}\Phi|^2(t,x)\,dx\right)\\
&\quad\le \|D(\bar J)\|_{L^\infty}\left(\iint |v-\bar J|^2 f_\varepsilon(t,x,v)dvdx + \int|\nabla_x V_\varepsilon - \nabla_x(-\Delta_x)^{-1/2}\Phi|^2(t,x)\,dx\right)\\
&\qquad - \int E_\varepsilon(\bar J,\phi)\left(\int (v-\bar J)f_\varepsilon\,dv,\ (\nabla_x V_\varepsilon - \nabla_x(-\Delta_x)^{-1/2}\Phi)\right)(t,x)\,dx\,.
\end{aligned}$$

The acceleration operator $E_\varepsilon(\bar J,\Phi)$ so defined involves a linear part of order $1/\varepsilon$ (which is exactly the oscillation operator studied in the previous section), and nonlinear terms of order 1. If we assume that $(\bar J,\Phi)$ satisfies the conditions (4.1), only the non-oscillating part remains.

Corollary 4.1. *For each scalar field* $\Phi\in C^\infty([0,T]\times\mathbf{T}^3)$ *and each vector field* $\bar J\in C^\infty([0,T]\times\mathbf{T}^3)$ *satisfying* (4.1), *the following identity holds*:

$$\begin{aligned}
&\frac{d}{dt}\left(\frac12\iint |v-\bar J|^2 f_\varepsilon(t,x,v)\,dvdx + \frac12\int |\nabla_x V_\varepsilon - \nabla_x(-\Delta_x)^{-1/2}\Phi|^2(t,x)\,dx\right)\\
&\quad = -\int D(\bar J) : \left(\int (v-\bar J)^{\otimes 2} f_\varepsilon\,dv - (\nabla_x V_\varepsilon - \nabla_x(-\Delta_x)^{-1/2}\Phi)^{\otimes 2}\right)(t,x)\,dx\\
&\qquad - \int \big(\partial_t\bar J + (\bar J\nabla_x)\bar J\big)\int (v-\bar J)f_\varepsilon(t,x,v)\,dvdx\\
&\qquad - \int (\partial_t\nabla_x(-\Delta_x)^{-1/2}\Phi - \bar J(-\Delta_x)^{1/2}\Phi)(\nabla_x V_\varepsilon - \nabla_x(-\Delta_x)^{-1/2}\Phi)(t,x)\,dx\,,
\end{aligned}\tag{4.3}$$

where $D(\bar J)$ *denotes the symmetrized gradient of* $\bar J$.

Notice that requiring that $(\bar J,\Phi)$ satisfies conditions (4.1) is equivalent to assuming that

$$\nabla_x\cdot\bar J = 0\,,\quad \partial_{x_3}\bar J = 0\,. \tag{4.4}$$

Then Φ is uniquely defined by $\nabla_x(-\Delta_x)^{-1/2}\Phi = -\bar J\wedge b$. It is easily seen that

$$\begin{aligned}
\partial_t\nabla_x(-\Delta_x)^{-1/2}\Phi - \bar J(-\Delta_x)^{1/2}\Phi &= -\partial_t\bar J\wedge b - \bar J\nabla_x\cdot(\bar J\wedge b)\\
&= -\partial_t\bar J\wedge b - \bar J(\partial_{x_1}\bar J_2 - \partial_{x_2}\bar J_1)\,.
\end{aligned}$$

Compute each component of the second term using (4.4)

$$\bar{J}_1(\partial_{x_1}\bar{J}_2 - \partial_{x_2}\bar{J}_1) = \bar{J}_1\partial_{x_1}\bar{J}_2 + \bar{J}_2\partial_{x_2}\bar{J}_2 - \partial_{x_2}\frac{\bar{J}_1^2 + \bar{J}_2^2}{2}$$

$$\bar{J}_2(\partial_{x_1}\bar{J}_2 - \partial_{x_2}\bar{J}_1) = -\bar{J}_1\partial_{x_1}\bar{J}_1 - \bar{J}_2\partial_{x_2}\bar{J}_1 + \partial_{x_1}\frac{\bar{J}_1^2 + \bar{J}_2^2}{2}\,.$$

We deduce that

$$\partial_t\nabla_x(-\Delta_x)^{-1/2}\Phi - \bar{J}(-\Delta_x)^{1/2}\Phi = -\left(\partial_t\bar{J} + (\bar{J}\cdot\nabla_x)\bar{J} - \nabla_x\frac{\bar{J}_1^2 + \bar{J}_2^2}{2}\right)\wedge b - \bar{J}_3 b\nabla_x\cdot(\bar{J}\wedge b)\,.$$

Then (4.3) can be rewritten in terms of $\bar{J}$.

Corollary 4.2. *For each vector field $\bar{J} \in C^\infty([0,T]\times\mathbf{T}^3)$ satisfying* (4.4), *the following identity holds*:

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\iint |v-\bar{J}|^2 f_\varepsilon(t,x,v)\,dvdx + \frac{1}{2}\int |\nabla_x V_\varepsilon + \bar{J}\wedge b|^2(t,x)\,dx\right)\\
&\quad = -\int D(\bar{J}) : \left(\int (v-\bar{J})^{\otimes 2} f_\varepsilon\,dv - (\nabla_x V_\varepsilon + \bar{J}\wedge b)^{\otimes 2}\right)(t,x)\,dx\\
&\qquad - \int E(\bar{J})\cdot\left(\int (v-\bar{J})f_\varepsilon\,dv + (\nabla_x V_\varepsilon + \bar{J}\wedge b)\wedge b\right)(t,x)\,dx\\
&\qquad + \int \bar{J}_3\nabla_x\cdot(\bar{J}\wedge b)\partial_{x_3}V_\varepsilon(t,x)\,dx\,,
\end{aligned} \tag{4.5}$$

where $D(\bar{J}) = \frac{1}{2}(\nabla_x\bar{J} + (\nabla_x\bar{J})^{\mathrm{T}})$ and $E(\bar{J}) = \partial_t\bar{J} + (\bar{J}\cdot\nabla_x)\bar{J}$.

4.2. *Convergence proof in the case of monokinetic profiles*

In the case of monokinetic profiles, the assumption (2.6) on the initial data implies that there exists $J^{in} \in L^2(\mathbf{T}^3)$ with $\nabla_x\cdot J^{in} = 0$ and $\partial_{x_3}J^{in} = 0$ satisfying

$$\frac{1}{2}\iint |v - J^{in}|^2 f_\varepsilon^{in} dvdx + \frac{1}{2}\int |\nabla_x V_\varepsilon^{in} + J^{in}\wedge b|^2 dx \to 0 \quad \text{as } \varepsilon\to 0\,.$$

Then, because of (4.5), we expect that

$$\frac{1}{2}\iint |v-J|^2 f_\varepsilon(t,x,v)\,dvdx + \frac{1}{2}\int |\nabla_x V_\varepsilon + J\wedge b|^2(t,x)\,dx \to 0 \quad \text{as } \varepsilon\to 0\,,$$

where J is the solution of the 2D1/2 incompressible Euler equation (1.15) with initial data J^{in}. Indeed for such a vector field, conditions (4.4) are satisfied and $E(J) = -\nabla_x\Pi$.

Nevertheless, since (1.15) does not have a unique strong solution for general initial data $J^{in} \in L^2(\mathbf{T}^3)$ with $\nabla_x\cdot J^{in} = 0$ and $\partial_{x_3}J^{in} = 0$, we can establish only a weak form of this convergence result.

Proof of Theorem 2.1. From the global conservations (1.2) and the bounds on the initial conditions (2.6), we infer the existence of $f \in L^\infty(\mathbb{R}^+, \mathcal{M}^+(\mathbf{T}^3 \times \mathbb{R}^3))$ and $V \in L^\infty(\mathbb{R}^+, H^1(\mathbf{T}^3))$ such that, up to extraction of a subsequence,

$$f_\varepsilon(1+|v|^2) \rightharpoonup f(1+|v|^2) \quad \text{weakly-}^* \text{ in } L^\infty(\mathbb{R}^+, \mathcal{M}(\mathbf{T}^3 \times \mathbb{R}^3))$$

and

$$\nabla_x V_\varepsilon \rightharpoonup \nabla_x V \quad \text{weakly-}^* \text{ in } L^\infty(\mathbb{R}^+, L^2(\mathbf{T}^3))\,.$$

Taking limits in the Poisson equation leads to $\int f dv = 1$, while the local conservations (1.4) and (1.5) give, in the limit as $\varepsilon \to 0$

$$\nabla_x \cdot \int f v dv = 0 \quad \text{and} \quad \nabla_x V = -\int f v dv \wedge b \tag{4.6}$$

in the sense of distributions.

Let $\bar{J} \in C^\infty([0,T] \times \mathbf{T}^3)$ be any vector field satisfying (4.4). From (4.5) we deduce the Gronwall type inequality

$$\begin{aligned}
&\frac{1}{2}\iint |v-\bar{J}|^2 f_\varepsilon(t,x,v)\, dvdx + \frac{1}{2}\int |\nabla_x V_\varepsilon + \bar{J}\wedge b|^2(t,x)\, dx \\
&\quad \le \left(\frac{1}{2}\iint |v-\bar{J}^{in}|^2 f_\varepsilon^{in} dvdx + \frac{1}{2}\int |\nabla_x V_\varepsilon^{in} + \bar{J}^{in}\wedge b|^2\, dx\right) \\
&\qquad\qquad \times \int_0^t \exp(\|D(\bar{J})(s)\|_{L^\infty(\mathbf{T}^3)})\, ds \\
&\quad\quad - \int_0^t \int E(\bar{J})\cdot\left(\int (v-\bar{J}) f_\varepsilon dv + (\nabla_x V_\varepsilon + \bar{J}\wedge b)\wedge b\right)(\tau,x) dx \\
&\qquad\qquad \times \int_\tau^t \exp(\|D(\bar{J})(s)\|_{L^\infty(\mathbf{T}^3)})\, ds d\tau \\
&\quad\quad + \int_0^t \int \bar{J}_3 \nabla_x\cdot(\bar{J}\wedge b)\partial_{x_3} V_\varepsilon(\tau,x) dx \int_\tau^t \exp(\|D(\bar{J})(s)\|_{L^\infty(\mathbf{T}^3)})\, ds d\tau\,,
\end{aligned} \tag{4.7}$$

where $\bar{J}^{in}$ is the initial value of $\bar{J}$.

Denote by $n_\varepsilon = \int f_\varepsilon\, dv$, $J_\varepsilon = \int f_\varepsilon v dv$ and $J = \int f v dv$. By the Cauchy–Schwarz inequality and the positivity of f_ε,

$$\frac{|J_\varepsilon - n_\varepsilon \bar{J}|^2}{n_\varepsilon} \le \frac{\left(\int f_\varepsilon (v-\bar{J})\, dv\right)^2}{\int f_\varepsilon\, dv} \le \int f_\varepsilon |v-\bar{J}|^2\, dv\,.$$

Then, by (4.7),

$$\frac{1}{2}\int \left(\frac{|J_\varepsilon - n_\varepsilon \bar{J}|^2}{n_\varepsilon} + |\nabla_x V_\varepsilon + \bar{J}\wedge b|^2\right)(t,x)\, dx$$

$$\leq \left(\frac{1}{2}\iint |v-\bar{J}^{in}|^2 f_\varepsilon^{in} dvdx + \frac{1}{2}\int |\nabla_x V_\varepsilon^{in} + \bar{J}^{in}\wedge b|^2 dx\right)$$
$$\times \int_0^t \exp(\|D(\bar{J})(s)\|_{L^\infty(\mathbf{T}^3)})\, ds$$
$$- \int_0^t \int E(\bar{J})\cdot((J_\varepsilon - n_\varepsilon \bar{J}) + (\nabla_x V_\varepsilon + \bar{J}\wedge b)\wedge b)(\tau,x)\, dx$$
$$\times \int_\tau^t \exp(\|D(\bar{J})(s)\|_{L^\infty(\mathbf{T}^3)})\, dsd\tau$$
$$+ \int_0^t \int \bar{J}_3 \nabla_x\cdot(\bar{J}\wedge b)\partial_{x_3} V_\varepsilon(\tau,x)\, dx \int_\tau^t \exp(\|D(\bar{J})\|_{L^\infty(\mathbf{T}^3)})\, dsd\tau\,. \tag{4.8}$$

As the functional $(n, J) \mapsto \int n^{-1}|J - n\bar{J}|^2 dx$ is convex and lsc. (lower semi-continuous) with respect to the weak convergence of measures, the convergences $n_\varepsilon \to 1$ and $J_\varepsilon \to J$ in the sense of measures imply that J belongs to $L^\infty([0,T], L^2(\mathbf{T}^3))$ and that

$$\int |J-\bar{J}|^2\, dx \leq \liminf_{\varepsilon\to 0} \int \frac{|J_\varepsilon - n_\varepsilon \bar{J}|^2}{n_\varepsilon}\, dx\,.$$

In order to take limits in (4.8), it remains to study the term coming from the initial data.

$$\frac{1}{2}\iint |v-\bar{J}^{in}|^2 f_\varepsilon^{in} dvdx + \frac{1}{2}\int |\nabla_x V_\varepsilon^{in} + \bar{J}^{in}\wedge b|^2\, dx$$
$$= \frac{1}{2}\iint |v-J^{in}|^2 f_\varepsilon^{in}\, dvdx + \frac{1}{2}\int |\nabla_x V_\varepsilon^{in} + J^{in}\wedge b|^2\, dx$$
$$+ \frac{1}{2}\iint |J^{in}-\bar{J}^{in}|^2 f_\varepsilon^{in}\, dvdx + \frac{1}{2}\int |(J^{in}-\bar{J}^{in})\wedge b|^2\, dx$$
$$+ \iint (J^{in}-\bar{J}^{in})\cdot(v-J^{in}) f_\varepsilon^{in}\, dvdx$$
$$- \int (J^{in}\wedge b - \bar{J}^{in}\wedge b)\cdot(\nabla_x V_\varepsilon^{in} + J^{in}\wedge b)\, dx\,.$$

The assumption on the initial data (2.6) imply that

$$\frac{1}{2}\iint |v-\bar{J}^{in}|^2 f_\varepsilon^{in}\, dvdx + \frac{1}{2}\int |\nabla_x V_\varepsilon^{in} + \bar{J}^{in}\wedge b|^2\, dx$$
$$\to \frac{1}{2}\int |J^{in}-\bar{J}^{in}|^2\, dx + \frac{1}{2}\int |(J^{in}-\bar{J}^{in})\wedge b|^2\, dx \quad \text{as } \varepsilon\to 0\,.$$

Taking limits in (4.8) and using the relations (4.6) leads to

$$\frac{1}{2}\int \left(|J-\bar{J}|^2 + |\nabla_x V + \bar{J}\wedge b|^2\right)(t,x)\, dx$$

$$
\begin{aligned}
&\leq \int |J^{in} - \bar{J}^{in}|^2 \, dx \int_0^t \exp(\|D(\bar{J})(s)\|_{L^\infty(\mathbf{T}^3)}) \, ds \\
&\quad - \int_0^t \int E(\bar{J}) \cdot \big((J - \bar{J}) + (\nabla_x V + \bar{J} \wedge b) \wedge b\big) (\tau, x) \, dx \\
&\qquad\qquad \times \int_\tau^t \exp(\|D(\bar{J})(s)\|_{L^\infty(\mathbf{T}^3)}) \, ds d\tau \, . \qquad (4.9)
\end{aligned}
$$

Extending (4.9) by a density argument to all vector fields $\bar{J} \in C([0,T], L^2(\mathbf{T}^3))$ satisfying $\nabla_x \bar{J} = 0$, $\partial_{x_3} \bar{J} = 0$, $D(\bar{J}) \in L^1([0,T], L^\infty(\mathbf{T}^3))$, $E(\bar{J}) \in L^1([0,T], L^2(\mathbf{T}^3))$ shows that J is a dissipative solution of the 2D1/2 incompressible Euler equation (1.15). □

4.3. *Convergence proof in the case of local thermodynamic equilibria*

In the case of velocity profiles defined as minimizers of an entropy with given temperature, the modulated Hamiltonian expected to converge to 0 must be replaced by the modulated free energy

$$
\begin{aligned}
&\iint (h(f_\varepsilon) - h(M_{1,J,1}) - (f_\varepsilon - M_{1,J,1}) h'(M_{1,J,1})) \, dv dx \\
&\quad + \nu \int |\nabla_x V_\varepsilon + J \wedge b|^2 dx \to 0 \quad \text{as } \varepsilon \to 0 \, ,
\end{aligned}
$$

where J is the solution of (1.15) with appropriate initial data and $\nu > 0$ is the Lagrange multiplier defined by the relation (2.8)

$$
M_{1,0,1}(v) = (h^*)'(\lambda - \nu |v|^2)
$$

and the constraints

$$
\int M_{1,0,1}(v) \, dv = 1 \, , \qquad \int |v|^2 M_{1,0,1}(v) \, dv = 3 \, .
$$

Note that this implies the convergence of the current density $\int f_\varepsilon v dv$ by the following inequality:

Lemma 4.2. *For each non-negative function f with finite relative entropy*

$$
\frac{|\int f(v - \bar{J}) \, dv|^2}{\int f dv} \leq \frac{1}{\nu} \int (h(f) - h(M_{1,\bar{J},1}) - (f - M_{1,\bar{J},1}) h'(M_{1,\bar{J},1})) \, dv \, .
$$

Proof. Let M_f be the minimizer of

$$
\int h(g) \, dv
$$

with the same moments as f

$$\int M_f dv = \int f dv = n\,, \quad \int M_f v dv = \int f v dv = J\,, \quad \text{and}$$

$$\int M_f \left|v - \frac{J}{n}\right|^2 dv = \int f \left|v - \frac{J}{n}\right|^2 dv = 3n\theta\,.$$

By definition

$$\int (h(M_f) - h(M_{1,\bar{J},1}) - (M_f - M_{1,\bar{J},1})h'(M_{1,\bar{J},1}))\, dv$$
$$\leq \int (h(f) - h(M_{1,\bar{J},1}) - (f - M_{1,\bar{J},1})h'(M_{1,\bar{J},1}))\, dv\,. \tag{4.10}$$

We deduce from (2.8) that

$$M_f = h'^{-1}\left(\lambda(n,\theta) - \nu(n,\theta)\left|v - \frac{J}{n}\right|^2\right)$$

which implies in particular that $\int h(M_f)dv$ depends on n and θ only. Thus

$$\int (h(M_f) - h(M_{1,\bar{J},1}) - (M_f - M_{1,\bar{J},1})h'(M_{1,\bar{J},1}))\, dv$$
$$= \int (h(M_f) - h(M_{1,\bar{J},1}) + (\nu|v - \bar{J}|^2 - \lambda)(M_f - M_{1,\bar{J},1}))\, dv$$
$$= \nu n \left|\frac{J}{n} - \bar{J}\right|^2 + \bar{H}(n,\theta) \tag{4.11}$$

where $\bar{H}$ is non-negative. Combining (4.10) and (4.11) gives the expected inequality. □

As above, we establish the convergence result on the free-energy in the weak sense only because a unique strong solution for general initial data does not exist for the limiting equation. The key to the proof is the identity

$$\frac{d}{dt}\left(\iint (h(f_\varepsilon) - h(M_{1,\bar{J},1}) - (f_\varepsilon - M_{1,\bar{J},1})h'(M_{1,\bar{J},1}))\, dv dx + \nu \int |\nabla_x V_\varepsilon + \bar{J} \wedge b|^2 dx\right)$$
$$= \frac{d}{dt}\left(\iint (f_\varepsilon - M_{1,\bar{J},1})(-\lambda + \nu|v - \bar{J}|^2)\, dv dx + \nu \int |\nabla_x V_\varepsilon + \bar{J} \wedge b|^2 dx\right)$$
$$= \nu \frac{d}{dt}\left(\iint f_\varepsilon |v - \bar{J}|^2 dv dx + \int |\nabla_x V_\varepsilon + \bar{J} \wedge b|^2\, dx\right).$$

Thus, by Corollary 4.2, for each vector field $\bar{J} \in C^\infty([0,T] \times \mathbf{T}^3)$ satisfying conditions (4.4)

$$\frac{d}{dt}\left(\iint(h(f_\varepsilon) - h(M_{1,\bar{J},1}) - (f_\varepsilon - M_{1,\bar{J},1})h'(M_{1,\bar{J},1}))\,dvdx + \nu\int|\nabla_x V_\varepsilon + \bar{J}\wedge b|^2\,dx\right)$$
$$= -2\nu\int D(\bar{J}) : \left(\int(v-\bar{J})^{\otimes 2} f_\varepsilon\,dv - (\nabla_x V_\varepsilon + \bar{J}\wedge b)^{\otimes 2}\right)(t,x)\,dx$$
$$- 2\nu\int E(\bar{J})\cdot\left(\int(v-\bar{J})f_\varepsilon\,dv + (\nabla_x V_\varepsilon + \bar{J}\wedge b)\wedge b\right)(t,x)\,dx$$
$$+ 2\nu\int \bar{J}_3\nabla_x\cdot(\bar{J}\wedge b)\,\partial_{x_3}V_\varepsilon(t,x)\,dx\,. \tag{4.12}$$

The main difficulty consists of estimating the flux term

$$-2\nu\int D(\bar{J}) : \left(\int(v-\bar{J})^{\otimes 2} f_\varepsilon\,dv - (\nabla_x V_\varepsilon + \bar{J}\wedge b)^{\otimes 2}\right)(t,x)\,dx$$

in terms of the modulated free energy, in order to conclude by a Gronwall type inequality as before.

Proposition 4.3. *Let $T > 0$ and (f_ε^{in}) be a family of non-negative functions of $L^1(\mathbf{T}^3\times\mathbb{R}^3)$ such that*

$$\iint f_\varepsilon^{in}\,dvdx = 1\,,\quad \sup_{\varepsilon>0}\mathcal{E}_\varepsilon^{in} < +\infty\,.$$

For each $\varepsilon > 0$, let $(f_\varepsilon, V_\varepsilon)$ be a solution of the scaled Vlasov–Poisson equation (1.1). *Then, there exists a non-negative constant C such that for each divergence-free vector field $\bar{J}\in C^\infty([0,T]\times\mathbf{T}^3)$ and all $t\in[0,T]$,*

$$\left|\int_0^t\iint D(\bar{J}) : (v-\bar{J})^{\otimes 2} f_\varepsilon\,dvdxdt\right|$$
$$\le C\int_0^t\|D(\bar{J})\|_{L^\infty(\mathbf{T}^3)}\iint(h(f_\varepsilon) - h(M_{1,\bar{J},1}) - (f_\varepsilon - M_{1,\bar{J},1})h'(M_{1,\bar{J},1}))\,dvdxds$$
$$+ C\int_0^t\|D(\bar{J})\|_{L^\infty(\mathbf{T}^3)}\int|\nabla_x V_\varepsilon + \bar{J}\wedge b|^2\,dxds + \eta_\varepsilon(t)\,, \tag{4.13}$$

where η_ε converges to 0 in $L^\infty([0,T])$ as $\varepsilon\to 0$.

Proposition 4.3 is based on the identity

$$\iint D(\bar{J}) : (v-J)^{\otimes 2} f_\varepsilon\,dxdv = \iint(\partial_{x_1}\bar{J}_2 + \partial_{x_2}\bar{J}_1)(v_1-\bar{J}_1)(v_2-\bar{J}_2)f_\varepsilon\,dxdv$$
$$+ \iint\partial_{x_1}\bar{J}_1[(v_1-\bar{J}_1)^2 - (v_2-\bar{J}_2)^2]f_\varepsilon\,dxdv \tag{4.14}$$

which holds because $\bar{J}$ is divergence-free. Then, as $(\nabla_x V_\varepsilon + v \wedge b) \cdot \nabla_v f_\varepsilon = O(\varepsilon)$ in some appropriate sense, we expect that

$$\iint D(\bar{J}) : (v - J)^{\otimes 2} f_\varepsilon \, dx dv \sim - \int (\partial_{x_1} \bar{J}_2 + \partial_{x_2} \bar{J}_1)(\partial_2 V_\varepsilon - \bar{J}_1)(\partial_1 V_\varepsilon + \bar{J}_2) \, dx$$
$$+ \int \partial_{x_1} \bar{J}_1 [(\partial_2 V_\varepsilon - \bar{J}_1)^2 - (\partial_1 V_\varepsilon + \bar{J}_2)^2] \, dx \, .$$

In order to establish such a claim, we have to integrate by parts both terms on the right-hand side of (4.14), after truncating large values of the density.

Lemma 4.3. *There exists a function $g \in C^1(\mathbb{R}^+)$ with*

$$\int \left(\frac{|g'(r^2)|^2}{|g(r^2)|} r^6 + g(r^2) r^4 \right) dr < +\infty$$

such that the inequality

$$f(v)(1 + |v - \bar{J}|^2) \mathbf{1}_{f(v) \geq 2g(\frac{1}{2}|v - \bar{J}|^2)} \leq \frac{4}{\nu} (h(f) - h(M_{1,\bar{J},1}) - (f - M_{1,\bar{J},1}) h'(M_{1,\bar{J},1}))$$

holds for each $\bar{J}$ and for each function f for which the right hand side is defined.

Proof. Define the convex function H by

$$H(x) = \frac{1}{M_{1,\bar{J},1}} (h(M_{1,\bar{J},1}(1 + x)) - h(M_{1,\bar{J},1}) - x M_{1,\bar{J},1} h'(M_{1,\bar{J},1})) \, .$$

A direct computation shows that

$$H^*(y) = \frac{1}{M_{1,\bar{J},1}} (h^*(y + h'(M_{1,\bar{J},1})) + h(M_{1,\bar{J},1}) - M_{1,\bar{J},1}(y + h'(M_{1,\bar{J},1}))) \, ,$$

then, by Young's inequality,

$$\begin{aligned} \frac{\nu}{2}(f - M_{1,\bar{J},1})(1 + |v - \bar{J}|^2) &\leq M_{1,\bar{J},1} \left(H\left(\frac{f - M_{1,\bar{J},1}}{M_{1,\bar{J},1}} \right) + H^*\left(\frac{\nu}{2}(1 + |v - \bar{J}|^2) \right) \right) \\ &\leq (h(f) - h(M_{1,\bar{J},1}) - (f - M_{1,\bar{J},1}) h'(M_{1,\bar{J},1})) \\ &\quad + h^*\left(\lambda + \frac{\nu}{2} - \frac{\nu}{2}|v - \bar{J}|^2 \right) + h(M_{1,\bar{J},1}) \\ &\quad + M_{1,\bar{J},1}\left(\frac{\nu}{2}|v - \bar{J}|^2 - \frac{\nu}{2} - \lambda \right). \end{aligned} \tag{4.15}$$

Define then g by

$$g(r) = \frac{2}{\nu}\left(h^*\left(\lambda + \frac{\nu}{2} - \nu r \right) + h({h'}^{-1}(\lambda - 2\nu r)) + {h'}^{-1}(\lambda - 2\nu r)(2\nu r - \lambda) \right). \tag{4.16}$$

Then (4.15) can be recast as

$$f(1 + |v - \bar{J}|^2) \leq \frac{2}{\nu}(h(f) - h(M_{1,\bar{J},1}) - (f - M_{1,\bar{J},1}) h'(M_{1,\bar{J},1})) + g\left(\frac{1}{2}|v - \bar{J}|^2 \right)$$

from which we deduce that

$$f(v)(1+|v-\bar{J}|^2)\mathbf{1}_{f(v)\geq 2g(\frac{1}{2}|v-\bar{J}|^2)} \leq \frac{4}{\nu}(h(f)-h(M_{1,\bar{J},1})-(f-M_{1,\bar{J},1})h'(M_{1,\bar{J},1}))\,.$$

Moreover, the assumptions made on h guarantee that g defined by (4.16) verifies the expected regularity and integrability conditions. Indeed

$$g(r) = \frac{2}{\nu}h^*\left(\lambda+\frac{\nu}{2}-\nu r\right) - \frac{2}{\nu}h^*\left(\lambda-2\nu r\right) \underset{r\to\infty}{\sim} 2\int_{\infty}^{r}(h')^{-1}\left(\lambda+\frac{\nu}{2}-\nu s\right)ds,$$

$$g'(r) = -2(h')^{-1}\left(\lambda+\frac{\nu}{2}-\nu r\right) + 4(h')^{-1}(\lambda-2\nu r) \underset{r\to\infty}{\sim} -2(h')^{-1}\left(\lambda+\frac{\nu}{2}-\nu r\right)$$

and these, together with the assumption that

$$\forall\, p\geq 0\,,\quad \int |{h'}^{-1}(-r^2))|r^p\,dr < +\infty\,,$$

leads to the expected bound. □

Equipped with this preliminary result, we can now prove Proposition 4.3.

Proof of Proposition 4.3. Introduce a smooth truncation $\gamma \in C^\infty(\mathbb{R}^+,[0,1])$ such that

$$\gamma \equiv \begin{cases} 1 & \text{on } [0,2]\,, \\ 0 & \text{on } [3,+\infty[\,. \end{cases}$$

Denote by f_ε a solution of (1.1) and by $\tilde{f}_\varepsilon = f_\varepsilon\gamma(\frac{f_\varepsilon}{g})$ where g and all its derivatives are always taken at point $|v-\bar{J}|^2/2$. Then,

$$\varepsilon\partial_t\tilde{f}_\varepsilon + \varepsilon v\cdot\nabla_x\tilde{f}_\varepsilon - (\nabla_x V_\varepsilon + v\wedge b)\cdot\nabla_v\tilde{f}_\varepsilon = \frac{f_\varepsilon^2}{g^2}\gamma'\left(\frac{f_\varepsilon}{g}\right)g'(\nabla_x V_\varepsilon + \bar{J}\wedge b)\cdot(v-\bar{J})$$
$$+\frac{f_\varepsilon^2}{g^2}\gamma'\left(\frac{f_\varepsilon}{g}\right)g'(\varepsilon\partial_t\bar{J}+\varepsilon v\cdot\nabla_x\bar{J})\cdot(v-\bar{J})\,. \tag{4.17}$$

Integrating (4.17) against $(v_1-\bar{J}_1)(v_2-\bar{J}_2)$ leads to

$$\int \tilde{f}_\varepsilon\left[(v_2-\bar{J}_2)(\partial_1 V_\varepsilon+v_2)+(v_1-\bar{J}_1)(\partial_2 V_\varepsilon - v_1)\right]dv$$
$$= -\varepsilon\partial_t\int\tilde{f}_\varepsilon(v_1-\bar{J}_1)(v_2-\bar{J}_2)\,dv - \varepsilon\nabla_x\cdot\int\tilde{f}_\varepsilon(v_1-\bar{J}_1)(v_2-\bar{J}_2)\,vdv$$
$$-\,\varepsilon\int\tilde{f}_\varepsilon(\partial_t\bar{J}_1(v_2-\bar{J}_2)+\partial_t\bar{J}_2(v_1-\bar{J}_1))\,dv$$
$$-\,\varepsilon\int\tilde{f}_\varepsilon(v\cdot\nabla_x\bar{J}_1(v_2-\bar{J}_2)+v\cdot\nabla_x\bar{J}_2(v_1-\bar{J}_1))\,dv$$

$$+\varepsilon\int \frac{f_\varepsilon^2}{g^2}\gamma'\left(\frac{f_\varepsilon}{g}\right) g'(\partial_t \bar J + v\cdot\nabla_x \bar J)\cdot(v-\bar J)(v_1-\bar J_1)(v_2-\bar J_2)\,dv$$

$$+\int \frac{f_\varepsilon^2}{g^2}\gamma'\left(\frac{f_\varepsilon}{g}\right) g'(\nabla_x V_\varepsilon + \bar J\wedge b)(v-\bar J)(v_1-\bar J_1)(v_2-\bar J_2)\,dv\,,$$

while integrating against $(v_1-\bar J_1)^2-(v_2-\bar J_2)^2$ gives

$$\begin{aligned}
&2\int \tilde f_\varepsilon[(v_1-\bar J_1)(\partial_1 V_\varepsilon + v_2)-(v_2-\bar J_2)(\partial_2 V_\varepsilon - v_1)]\,dv\\
&\quad= -\varepsilon\partial_t\int \tilde f_\varepsilon[(v_1-\bar J_1)^2-(v_2-\bar J_2)^2]\,dv - \varepsilon\nabla_x\cdot\int \tilde f_\varepsilon[(v_1-\bar J_1)^2-(v_2-\bar J_2)^2]\,vdv\\
&\qquad - 2\varepsilon\int \tilde f_\varepsilon(\partial_t\bar J_1(v_1-\bar J_1)-\partial_t\bar J_2(v_2-\bar J_2))\,dv\\
&\qquad - 2\varepsilon\int \tilde f_\varepsilon(v\cdot\nabla_x\bar J_1(v_1-\bar J_1)-v\cdot\nabla_x\bar J_2(v_2-\bar J_2))\,dv\\
&\qquad +\varepsilon\int \frac{f_\varepsilon^2}{g^2}\gamma'\left(\frac{f_\varepsilon}{g}\right) g'(\partial_t\bar J + v\nabla_x\bar J)\cdot(v-\bar J)[(v_1-\bar J_1)^2-(v_2-\bar J_2)^2]\,dv\\
&\qquad +\int \frac{f_\varepsilon^2}{g^2}\gamma'\left(\frac{f_\varepsilon}{g}\right) g'(\nabla_x V_\varepsilon + \bar J\wedge b)\cdot(v-\bar J)[(v_1-\bar J_1)^2-(v_2-\bar J_2)^2]\,dv\,.
\end{aligned}$$

Decomposing $f_\varepsilon = \tilde f_\varepsilon + f_\varepsilon(1-\gamma(\frac{f_\varepsilon}{g}))$ leads to

$$\begin{aligned}
&\int f_\varepsilon[(v_2-\bar J_2)^2-(v_1-\bar J_1)^2]\,dv\\
&\quad = \int f_\varepsilon\left(1-\gamma\left(\frac{f_\varepsilon}{g}\right)\right)[(v_2-\bar J_2)^2-(v_1-\bar J_1)^2]\,dv\\
&\qquad + \int f_\varepsilon\gamma\left(\frac{f_\varepsilon}{g}\right)[(v_2-\bar J_2)(-\partial_1 V_\varepsilon-\bar J_2)-(v_1-\bar J_1)(\partial_2 V_\varepsilon-\bar J_1)]\,dv\\
&\qquad + \int f_\varepsilon\gamma\left(\frac{f_\varepsilon}{g}\right)[(v_2-\bar J_2)(v_2+\partial_1 V_\varepsilon)-(v_1-\bar J_1)(v_1-\partial_2 V_\varepsilon)]\,dv\\
&\quad = \int f_\varepsilon\left(1-\gamma\left(\frac{f_\varepsilon}{g}\right)\right)[(v_2-\bar J_2)^2-(v_1-\bar J_1)^2]\,dv\\
&\qquad + \int \tilde f_\varepsilon[(v_2-\bar J_2)(-\partial_1 V_\varepsilon-\bar J_2)-(v_1-\bar J_1)(\partial_2 V_\varepsilon-\bar J_1)]\,dv\\
&\qquad -\varepsilon\partial_t\int \tilde f_\varepsilon(v_1-\bar J_1)(v_2-\bar J_2)\,dv - \varepsilon\nabla_x\int \tilde f_\varepsilon(v_1-\bar J_1)(v_2-\bar J_2)\,vdv\\
&\qquad -\varepsilon\int \tilde f_\varepsilon(\partial_t\bar J_1(v_2-\bar J_2)+\partial_t\bar J_2(v_1-\bar J_1))\,dv
\end{aligned}$$

$$
\begin{aligned}
&- \varepsilon \int \tilde{f}_\varepsilon (v \cdot \nabla_x \bar{J}_1 (v_2 - \bar{J}_2) + v \cdot \nabla_x \bar{J}_2 (v_1 - \bar{J}_1))\, dv \\
&+ \varepsilon \int \frac{f_\varepsilon^2}{g^2} \gamma' \left(\frac{f_\varepsilon}{g} \right) g' (\partial_t \bar{J} + v \cdot \nabla_x \bar{J}) \cdot (v - \bar{J})(v_1 - \bar{J}_1)(v_2 - \bar{J}_2)\, dv \\
&+ \int \frac{f_\varepsilon^2}{g^2} \gamma' \left(\frac{f_\varepsilon}{g} \right) g' (\nabla_x V_\varepsilon + \bar{J} \wedge b) \cdot (v - \bar{J})(v_1 - \bar{J}_1)(v_2 - \bar{J}_2)\, dv \\
&= I_1 + I_2 + \partial_t I_3 + \nabla_x I_4 + I_5 + I_6
\end{aligned} \tag{4.18}
$$

and

$$
\begin{aligned}
&4 \int f_\varepsilon (v_1 - \bar{J}_1)(v_2 - \bar{J}_2)\, dv \\
&\quad = 4 \int f_\varepsilon \left(1 - \gamma \left(\frac{f_\varepsilon}{g} \right) \right) (v_1 - \bar{J}_1)(v_2 - \bar{J}_2)\, dv \\
&\qquad + 2 \int \tilde{f}_\varepsilon [-(v_1 - \bar{J}_1)(\partial_1 V_\varepsilon + \bar{J}_2) + (v_2 - \bar{J}_2)(\partial_2 V_\varepsilon - \bar{J}_1)]\, dv \\
&\qquad - \varepsilon \partial_t \int \tilde{f}_\varepsilon [(v_1 - \bar{J}_1)^2 - (v_2 - \bar{J}_2)^2]\, dv - \varepsilon \nabla_x \int \tilde{f}_\varepsilon [(v_1 - \bar{J}_1)^2 - (v_2 - \bar{J}_2)^2]\, v dv \\
&\qquad - 2\varepsilon \int \tilde{f}_\varepsilon (\partial_t \bar{J}_1 (v_1 - \bar{J}_1) - \partial_t \bar{J}_2 (v_2 - \bar{J}_2))\, dv \\
&\qquad - 2\varepsilon \int \tilde{f}_\varepsilon (v \cdot \nabla_x \bar{J}_1 (v_1 - \bar{J}_1) - v \cdot \nabla_x \bar{J}_2 (v_2 - \bar{J}_2))\, dv \\
&\qquad + \varepsilon \int \frac{f_\varepsilon^2}{g^2} \gamma' \left(\frac{f_\varepsilon}{g} \right) g' (\partial_t \bar{J} + v \nabla_x \bar{J}) \cdot (v - \bar{J}) [(v_1 - \bar{J}_1)^2 - (v_2 - \bar{J}_2)^2]\, dv \\
&\qquad + \int \frac{f_\varepsilon^2}{g^2} \gamma' \left(\frac{f_\varepsilon}{g} \right) g' (\nabla_x V_\varepsilon + \bar{J} \wedge b) \cdot (v - \bar{J}) [(v_1 - \bar{J}_1)^2 - (v_2 - \bar{J}_2)^2]\, dv \\
&\quad = I_1' + I_2' + \partial_t I_3' + \nabla_x I_4' + I_5' + I_6' \,.
\end{aligned} \tag{4.19}
$$

It remains to estimate each term on the right-hand sides of (4.18) and (4.19). By Lemma 4.3,

$$
|I_1| + |I_1'| \leq C \int (h(f_\varepsilon) - h(M_{1,\bar{J},1}) - (f_\varepsilon - M_{1,\bar{J},1}) h'(M_{1,\bar{J},1}))\, dv \,. \tag{4.20}
$$

By the Cauchy–Schwarz inequality, as $\int \tilde{f}_\varepsilon dv \leq 2 \int g dv \leq C$,

$$
|I_2| + |I_2'| \leq C |\nabla_x V_\varepsilon + \bar{J} \wedge b|^2 + C \frac{|\int \tilde{f}_\varepsilon (v - \bar{J})\, dv|^2}{1 + \int \tilde{f}_\varepsilon dv} \,.
$$

Decomposing $f_\varepsilon = \tilde{f}_\varepsilon + f_\varepsilon (1 - \gamma(\frac{f_\varepsilon}{g}))$ leads to

$$
\frac{|\int \tilde{f}_\varepsilon (v - \bar{J})\, dv|^2}{1 + \int \tilde{f}_\varepsilon dv} \leq \frac{|\int \tilde{f}_\varepsilon (v - \bar{J})\, dv|^2}{1 + \int f_\varepsilon\, dv} + \frac{|\int \tilde{f}_\varepsilon (v - \bar{J})\, dv|^2 \int f_\varepsilon (1 - \gamma(\frac{f_\varepsilon}{g}))\, dv}{(1 + \int \tilde{f}_\varepsilon dv)(1 + \int f_\varepsilon\, dv)}
$$

$$\leq 2\frac{|\int f_\varepsilon(v-\bar{J})\,dv|^2}{1+\int f_\varepsilon\,dv} + 2\frac{|\int f_\varepsilon(1-\gamma(\frac{f_\varepsilon}{g}))|v-\bar{J}|dv|^2}{1+\int f_\varepsilon\,dv}$$

$$+4\left|\int g|v-\bar{J}|dv\right|^2 \int f_\varepsilon\left(1-\gamma\left(\frac{f_\varepsilon}{g}\right)\right)dv$$

$$\leq 2\frac{|\int f_\varepsilon(v-\bar{J})\,dv|^2}{1+\int f_\varepsilon\,dv} + 2\int f_\varepsilon\left(1-\gamma\left(\frac{f_\varepsilon}{g}\right)\right)|v-\bar{J}|^2\,dv$$

$$+4\left|\int g|v-\bar{J}|dv\right|^2 \int f_\varepsilon\left(1-\gamma\left(\frac{f_\varepsilon}{g}\right)\right)dv\,.$$

By Lemmas 4.3 and 4.2

$$\frac{|\int \tilde{f}_\varepsilon(v-\bar{J})\,dv|^2}{1+\int \tilde{f}_\varepsilon dv} \leq C\int (h(f_\varepsilon)-h(M_{1,\bar{J},1})-(f_\varepsilon-M_{1,\bar{J},1})h'(M_{1,\bar{J},1}))\,dv\,.$$

Thus,

$$(|I_2|+|I_2'|) \leq C\int (h(f_\varepsilon)-h(M_{1,\bar{J},1})-(f_\varepsilon-M_{1,\bar{J},1})h'(M_{1,\bar{J},1}))\,dv$$
$$+C|\nabla_x V_\varepsilon+\bar{J}\wedge b|^2\,. \tag{4.21}$$

In the same way,

$$|I_6|+|I_6'| \leq C|\nabla_x V_\varepsilon+\bar{J}\wedge b|^2 + C\left|\int \frac{f_\varepsilon^2}{g^2}\gamma'\left(\frac{f_\varepsilon}{g}\right)|g'|\,|v-\bar{J}|^3\,dv\right|^2$$

with

$$\left|\int \frac{f_\varepsilon^2}{g^2}\gamma'\left(\frac{f_\varepsilon}{g}\right)|g'|\,|v-\bar{J}|^3\,dv\right|^2$$
$$\leq C\left(\int g\gamma'\left(\frac{f_\varepsilon}{g}\right)|v-\bar{J}|^2\,dv\right)\left(\int \frac{|g'|^2}{g}|v-\bar{J}|^4\,dv\right)$$
$$\leq C\int (h(f_\varepsilon)-h(M_{1,\bar{J},1})-(f_\varepsilon-M_{1,\bar{J},1})h'(M_{1,\bar{J},1}))\,dv$$

by Lemma 4.3. Then,

$$(|I_6|+|I_6'|) \leq C\int (h(f_\varepsilon)-h(M_{1,\bar{J},1})-(f_\varepsilon-M_{1,\bar{J},1})h'(M_{1,\bar{J},1}))\,dv$$
$$+C|\nabla_x V_\varepsilon+\bar{J}\wedge b|^2\,. \tag{4.22}$$

From the trivial bound $\tilde{f}_\varepsilon \leq 2g$ combined with the estimates on g stated in Lemma 4.3, we deduce that I_3, I_3', I_4, I_4', I_5 and I_5' converge to 0 in $L^\infty([0,T]\times\mathbf{T}^3)$. Then, as $\bar{J}\in C^\infty([0,T]\times\mathbf{T}^3)$,

$$\int (\partial_{x_1}\bar{J}_2+\partial_{x_2}\bar{J}_1)I_5'\,dxds \to 0 \quad \text{in } L^\infty([0,T])\,,$$

$$\int (\partial_{x_1}\bar{J}_1) I_5 \, dxds \to 0 \quad \text{in } L^\infty([0,T])\,, \tag{4.23}$$

$$\int (\partial_{x_1}\bar{J}_2 + \partial_{x_2}\bar{J}_1)\nabla_x I_4' \, dx \to 0 \quad \text{in } L^\infty([0,T])$$

$$\int (\partial_{x_1}\bar{J}_1)\nabla_x I_4 \, dx \to 0 \quad \text{in } L^\infty([0,T]) \tag{4.24}$$

and

$$\int_0^t \int (\partial_{x_1}\bar{J}_2 + \partial_{x_2}\bar{J}_1)\partial_t I_3' \, dxds \to 0 \quad \text{in } L^\infty([0,T])\,,$$

$$\int_0^t \int (\partial_{x_1}\bar{J}_1)\partial_t I_3 \, dxds \to 0 \quad \text{in } L^\infty([0,T])\,. \tag{4.25}$$

Combining estimates (4.20)–(4.25) with identities (4.14), (4.18) and (4.19) gives the expected bound (4.13) on the flux term. □

Combining (4.12) with the estimates in Lemma 4.2 and in Proposition 4.3 gives the expected convergence.

Proof of Theorem 2.2. The global conservations (1.2) and the bounds on the initial conditions (2.9) imply the existence of $f \in L^\infty(\mathbb{R}^+, L^1 \cap L^\infty(\mathbf{T}^3 \times \mathbb{R}^3))$ and of $V \in L^\infty(\mathbb{R}^+, H^1(\mathbf{T}^3))$ such that, up to extraction of a subsequence,

$$f_\varepsilon(1+|v|^2) \rightharpoonup f(1+|v|^2) \quad \text{weakly-}* \text{ in } L^\infty(\mathbb{R}^+, L^1(\mathbf{T}^3\times\mathbb{R}^3))\,,$$

$$\nabla_x V_\varepsilon \rightharpoonup \nabla_x V \quad \text{weakly-}* \text{ in } L^\infty(\mathbb{R}^+, L^2(\mathbf{T}^3))\,.$$

Taking limits in the Poisson equation shows that $\int f dv = 1$, while the local conservations laws (1.4) and (1.5) give asymptotically

$$\nabla_x \cdot \int f v dv = 0 \quad \text{and} \quad \nabla_x V = -\int f v dv \wedge b$$

in the sense of distributions.

Let $\bar{J} \in C^\infty([0,T]\times\mathbf{T}^3)$ be any vector field verifying the conditions (4.4). From (4.12) and (4.3) we deduce the existence of a non-negative constant C such that, for all $t \in [0,T]$

$$\iint M_{1,\bar{J},1} H\left(\frac{f_\varepsilon - M_{1,\bar{J},1}}{M_{1,\bar{J},1}}\right) dvdx + \nu \int |\nabla_x V_\varepsilon + \bar{J}\wedge b|^2 \, dx$$

$$\le \iint M_{1,\bar{J}^{in},1} H\left(\frac{f_\varepsilon^{in} - M_{1,\bar{J}^{in},1}}{M_{1,\bar{J}^{in},1}}\right) dvdx + \nu \int |\nabla_x V_\varepsilon + \bar{J}^{in}\wedge b|^2 \, dx$$

$$+ C\int_0^t \|D(\bar{J})\|_{L^\infty(\mathbf{T}^3)} \left(\iint M_{1,\bar{J},1} H\left(\frac{f_\varepsilon - M_{1,\bar{J},1}}{M_{1,\bar{J},1}}\right) dvdx + \int |\nabla_x V_\varepsilon + \bar{J}\wedge b|^2 \, dx\right) ds$$

$$-2\nu \int_0^t \int E(\bar{J}) \cdot \left(\int (v - \bar{J}) f_\varepsilon \, dv + (\nabla_x V_\varepsilon + \bar{J} \wedge b) \wedge b \right) (s, x) \, dx ds$$

$$+ 2\nu \int_0^t \int \bar{J}_3 \nabla_x (\bar{J} \wedge b) \partial_{x_3} V_\varepsilon (s, x) \, dx ds + \eta_\varepsilon (t) \,,$$

where $\eta_\varepsilon \to 0$ in $L^\infty([0,T])$ for all $T > 0$. Below, we denote $n_\varepsilon = \int f_\varepsilon \, dv$, $J_\varepsilon = \int f_\varepsilon v dv$ and $J = \int f v dv$. Gronwall's inequality and Lemma 4.2 imply that

$$\begin{aligned} &\nu \int \frac{|J_\varepsilon - n_\varepsilon \bar{J}|^2}{n_\varepsilon} \, dx + \nu \int |\nabla_x V_\varepsilon + \bar{J} \wedge b|^2 \, dx \\ &\quad \le \left(\iint M_{1,\bar{J}^{in},1} H \left(\frac{f_\varepsilon^{in} - M_{1,\bar{J}^{in},1}}{M_{1,\bar{J}^{in},1}} \right) dv dx + \nu \int |\nabla_x V_\varepsilon^{in} + \bar{J}^{in} \wedge b|^2 \, dx + \eta_\varepsilon \right) \\ &\qquad\qquad \times \exp \left(C \int_0^t \|D(\bar{J})\|_{L^\infty_x} \, ds \right) \\ &\quad -2\nu \int_0^t \int E(\bar{J}) \cdot \left(\int (v - \bar{J}) f_\varepsilon \, dv + (\nabla_x V_\varepsilon + \bar{J} \wedge b) \wedge b \right) (s, x) \\ &\qquad\qquad \times \exp \left(C \int_s^t \|D(\bar{J})(s)\|_{L^\infty_x} \, d\tau \right) dx ds \,. \end{aligned} \tag{4.26}$$

A direct computation shows that

$$\begin{aligned} \iint M_{1,\bar{J}^{in},1} H \left(\frac{f_\varepsilon^{in} - M_{1,\bar{J}^{in},1}}{M_{1,\bar{J}^{in},1}} \right) dv dx &= \iint M_{1,J^{in},1} H \left(\frac{f_\varepsilon^{in} - M_{1,J^{in},1}}{M_{1,J^{in},1}} \right) dv dx \\ &\quad + \nu \iint f_\varepsilon^{in} |J^{in} - \bar{J}^{in}|^2 \, dv dx \,. \end{aligned}$$

Then, taking limits in (4.26) as $\varepsilon \to 0$ leads to

$$\begin{aligned} &\nu \left(\int |J - \bar{J}|^2 \, dx + |\nabla_x V + \bar{J} \wedge b|^2 \, dx \right) \\ &\quad \le \left(2\nu \int |J^{in} - \bar{J}^{in}|^2 \, dx \right) \exp \left(C \int_0^t \|D(\bar{J})(s)\|_{L^\infty_x} \, ds \right) \\ &\quad - 2\nu \int_0^t \int E(\bar{J}) \left((J - \bar{J}) + (\nabla_x V \wedge b - \bar{J}') \right) (s, x) \\ &\quad \times \exp \left(C \int_s^t (s) \|D(\bar{J})(s)\|_{L^\infty_x} \, d\tau \right) dx ds \end{aligned}$$

by the same convexity argument as in the proof of Theorem 2.1. Extending the stability inequality so obtained to all divergence-free vector fields $\bar{J} \in C([0,T], L^2(\mathbf{T}^3))$ satisfying $\nabla_x \cdot \bar{J} = 0$, $\partial_{x_3} \bar{J} = 0$, $D(\bar{J}) \in L^\infty([0,T], L^\infty(\mathbf{T}^3))$, $E(\bar{J}) \in L^1([0,T], L^2(\mathbf{T}^3))$, concludes the proof. □

5. General Initial Data: Study of the Coupling Between Oscillating Terms

If $(\bar{J}, \Phi)$ does not satisfy conditions (4.1), we must consider the general stability inequality given by Lemma 4.1

$$\frac{d}{dt}\left(\frac{1}{2}\iint |v-\bar{J}|^2 f_\varepsilon(t,x,v)\,dvdx + \frac{1}{2}\int |\nabla_x V_\varepsilon - \nabla_x(-\Delta_x)^{-1/2}\Phi|^2(t,x)\,dx\right)$$
$$= -\int D(\bar{J}) : \left(\int (v-\bar{J})^{\otimes 2} f_\varepsilon\,dv - (\nabla_x V_\varepsilon - \nabla_x(-\Delta_x)^{-1/2}\Phi)^{\otimes 2}\right)(t,x)\,dx$$
$$-\frac{1}{2}\int (\nabla_x\cdot\bar{J})|\nabla_x V_\varepsilon - \nabla_x(-\Delta_x)^{-1/2}\Phi|^2(t,x)\,dx$$
$$-\int E_\varepsilon(\bar{J},\Phi)\cdot\left(\int (v-\bar{J})f_\varepsilon\,dv,\ (\nabla_x V_\varepsilon - \nabla_x(-\Delta_x)^{-1/2}\Phi)\right)(t,x)\,dx\,,$$

where the acceleration operator $E_\varepsilon(\bar{J},\Phi)$ is defined by

$$E_\varepsilon(\bar{J},\Phi) = \left(\partial_t\bar{J} + \bar{J}\nabla_x\bar{J} + \frac{\bar{J}\wedge b}{\varepsilon} + \frac{\nabla_x(-\Delta_x)^{-1/2}\Phi}{\varepsilon}\,,\right.$$
$$\left.\partial_t\nabla_x(-\Delta_x)^{-1/2}\Phi - \bar{J}(-\Delta_x)^{1/2}\Phi - \frac{\bar{J}}{\varepsilon}\right).$$

By analogy with the previous results, we expect that the current $J_\varepsilon = \int f_\varepsilon\, vdv$ and the electric potential V_ε behave respectively as $\bar{J}_\varepsilon$ and $(-\Delta_x)^{-1/2}\Phi_\varepsilon$ where $(\bar{J}_\varepsilon,\Phi_\varepsilon)$ denotes the solution of $E_\varepsilon(\bar{J},\Phi)=0$.

As E_ε depends crucially on ε, solutions $(\bar{J}_\varepsilon,\Phi_\varepsilon)$ of $E_\varepsilon(\bar{J},\Phi)=0$ also depend on ε. Then, in order to describe the asymptotic behavior of the family $((\bar{J}_\varepsilon,\Phi_\varepsilon))_\varepsilon$ (which is the problem we want to study), we need to have a good notion of solution for the equation $E_\varepsilon(\bar{J},\Phi)=0$. In particular, we will require that

- the life span T_ε of these solutions are bounded from below: $\inf_{\varepsilon>0} T_\varepsilon > 0$;
- the solutions satisfy uniform bounds (implying that the family of solutions indexed by ε is compact in some appropriate function space).

Indeed we will restrict our attention to smooth initial data (with smoothness to be made precise later). For such initial data, the unique solution of $E_\varepsilon(\bar{J},\Phi)=0$ can be decomposed as the sum of a non-oscillating part approximately governed by the 2D1/2 incompressible Euler equation (1.15), and of terms oscillating at high-frequency. Such a solution exists as long as does the strong solution of (1.15).

5.1. *Construction of approximate solutions for $E_\varepsilon(\bar{J},\Phi)=0$*

Proposition 2.1 shows that the linear part of the equation $E_\varepsilon(\bar{J},\Phi)=0$ generates a group of isometries $\mathcal{R}(t)=\exp(tR)$. Proceeding as in Schochet,[27] we remove the fast temporal oscillations in the solution by considering instead of $(\bar{J}_\varepsilon, \Phi_\varepsilon)$ the new unknown

$$\Psi_\varepsilon = \mathcal{R}\left(\frac{t}{\varepsilon}\right)(\bar{J}_\varepsilon, \Phi_\varepsilon)\,. \tag{5.1}$$

Thus Ψ_ε is expected to have uniformly bounded time derivatives and to converge strongly, up to extraction of a subsequence, in some function space. In order to establish such a convergence result, we first have to determine the structure of the equation governing the evolution of Ψ_ε and then study its asymptotic behavior as $\varepsilon \to 0$.

Lemma 5.4. (Van der Pol transform) *Let $(\bar{J}_\varepsilon, \Phi_\varepsilon)$ be a solution of the equation $E_\varepsilon(\bar{J}, \phi) = 0$. Define Ψ_ε by* (5.1), *then Ψ_ε satisfies*

$$\partial_t \Psi_\varepsilon + Q\left(\frac{t}{\varepsilon}, \Psi_\varepsilon, \Psi_\varepsilon\right) = 0\,, \tag{5.2}$$

where the non-autonomous bilinear operator $(a, b) \mapsto Q(t, a, b)$ is defined by its Fourier coefficients

$$\forall\, k \in \mathbb{Z}^3\,, \quad \mathcal{F}_k Q(t, a, b) = \sum_{l+m=k} \sum_{\eta \in [[1,4]]^3} \exp(it\omega_\eta(k, l, m)) s_\eta(k, l, m)[\mathcal{F}_l a, \mathcal{F}_m b]\,. \tag{5.3}$$

The phase $\omega_\eta(k, l, m)$ is given by

$$\omega_\eta(k, l, m) = \lambda_{\eta_1}(k) - \lambda_{\eta_2}(l) - \lambda_{\eta_3}(m)\,,$$

where $(i\lambda_j(k))_{j\in[[1,4]]}$ are the eigenvalues of the symbol $\mathcal{F}_k R$ of R. The bilinear map $(\alpha, \beta) \mapsto s_\eta(k, l, m)[\alpha, \beta]$ — defined in (5.10) *below — satisfies the estimate*

$$\|s_\eta(l + m, l, m)\| \leq C(|l| + |m|)$$

for some non-negative constant C and $|k|^2 = \frac{k_1^2}{a_1^2} + \frac{k_2^2}{a_2^2} + \frac{k_3^2}{a_3^2}$.

Proof. The equation $E_\varepsilon(\bar{J}, \Phi) = 0$ is equivalent to the following system:

$$\partial_t \bar{J} + (\bar{J} \cdot \nabla_x)\bar{J} + \frac{\bar{J} \wedge b}{\varepsilon} + \frac{\nabla_x(-\Delta_x)^{-1/2}\Phi}{\varepsilon} = 0\,,$$

$$\partial_t \Phi + (-\Delta_x)^{-1/2}\nabla_x \cdot (\bar{J}(-\Delta_x)^{1/2}\Phi) + \frac{(-\Delta_x)^{-1/2}\nabla_x \cdot \bar{J}}{\varepsilon} = 0\,,$$

which can be rewritten

$$\partial_t(\bar{J}, \Phi) + \frac{1}{\varepsilon} R(\bar{J}, \Phi) + ((\bar{J} \cdot \nabla_x)\bar{J},\, (-\Delta_x)^{-1/2}\nabla_x \cdot (\bar{J}(-\Delta_x)^{1/2}\Phi)) = 0\,.$$

Conjugating by $\mathcal{R}(\frac{t}{\varepsilon})$ leads to

$$\partial_t \mathcal{R}\left(\frac{t}{\varepsilon}\right)(\bar{J}, \Phi) + \mathcal{R}\left(\frac{t}{\varepsilon}\right)((\bar{J} \cdot \nabla_x)\bar{J},\, (-\Delta_x)^{-1/2}\nabla_x \cdot (\bar{J}(-\Delta_x)^{1/2}\Phi)) = 0\,.$$

Then $\Psi_\varepsilon = \mathcal{R}(\frac{t}{\varepsilon})(\bar{J}_\varepsilon, \Phi_\varepsilon)$ satisfies

$$\partial_t \Psi_\varepsilon + \mathcal{R}\left(\frac{t}{\varepsilon}\right)\mathcal{B}\left[\mathcal{R}\left(-\frac{t}{\varepsilon}\right)\Psi_\varepsilon,\, \mathcal{R}\left(-\frac{t}{\varepsilon}\right)\Psi_\varepsilon\right] = 0\,, \tag{5.4}$$

where $\mathcal{B}$ is the symmetric bilinear operator defined by

$$\mathcal{B}[\Psi,\Psi] = ((\Psi'\cdot\nabla_x)\Psi',\, (-\Delta_x)^{-1/2}\nabla_x\cdot(\Psi'(-\Delta_x)^{1/2}\Psi_4)) \tag{5.5}$$

with the notation Ψ' for the 3D vector (Ψ_1, Ψ_2, Ψ_3).

Notice that $\mathcal{R}$ is translation invariant, i.e.

$$\mathcal{F}_k(\mathcal{R}(t)a) = \mathcal{R}_k(t)\mathcal{F}_k a = \exp(\mathcal{F}_k R t)\mathcal{F}_k a$$

we get on the Fourier side

$$\begin{aligned}
\partial_t \mathcal{F}_k \Psi_\varepsilon &= -\mathcal{F}_k\left(\mathcal{R}\left(\frac{t}{\varepsilon}\right)\mathcal{B}\left[\mathcal{R}\left(-\frac{t}{\varepsilon}\right)\Psi_\varepsilon,\, \mathcal{R}\left(-\frac{t}{\varepsilon}\right)\Psi_\varepsilon\right]\right)\\
&= -\mathcal{R}_k\left(\frac{t}{\varepsilon}\right)\mathcal{F}_k\left(\mathcal{B}\left[\mathcal{R}\left(-\frac{t}{\varepsilon}\right)\Psi_\varepsilon, \mathcal{R}\left(-\frac{t}{\varepsilon}\right)\Psi_\varepsilon\right]\right)\\
&= -\mathcal{R}_k\left(\frac{t}{\varepsilon}\right)\sum_{l+m=k} B_{l,m}\left[\mathcal{F}_l\left(\mathcal{R}\left(-\frac{t}{\varepsilon}\right)\Psi_\varepsilon\right),\, \mathcal{F}_m\left(\mathcal{R}\left(-\frac{t}{\varepsilon}\right)\Psi_\varepsilon\right)\right]
\end{aligned}$$

from which we deduce

$$\partial_t \mathcal{F}_k \Psi_\varepsilon + \mathcal{R}_k\left(\frac{t}{\varepsilon}\right)\sum_{l+m=k} B_{l,m}\left[\mathcal{R}_l\left(-\frac{t}{\varepsilon}\right)\mathcal{F}_l\Psi_\varepsilon, \mathcal{R}_m\left(-\frac{t}{\varepsilon}\right)\mathcal{F}_m\Psi_\varepsilon\right] = 0\,, \tag{5.6}$$

where $B_{l,m}$ is defined by

$$\begin{aligned}
B_{l,m}[\Psi_l\Psi_m] = \frac{1}{2}\Bigg(& i\langle\Psi_l, m\rangle\Psi'_m + i\langle\Psi_m, l\rangle\Psi'_l,\\
&+ i|m|\Psi_{m,4}\left\langle\frac{m+l}{|m+l|},\, \Psi_l\right\rangle + i|l|\Psi_{l,4}\left\langle\frac{m+l}{|m+l|},\, \Psi_m\right\rangle\Bigg)
\end{aligned} \tag{5.7}$$

where $\langle\Psi, k\rangle = \Psi_1\frac{k_1}{a_1} + \Psi_2\frac{k_2}{a_2} + \Psi_3\frac{k_3}{a_3}$.

In order to obtain the required form for the equation governing Ψ_ε, it remains to describe precisely the symbol of $\mathcal{R}$. For each $k \in \mathbb{Z}^3$, the skew-symmetric matrix $R_k = \mathcal{F}_k R$ can be reduced to the diagonal form on $\mathbb{C}$ with purely imaginary eigenvalues

$$\begin{aligned}
i\lambda_1(k) &= i\sqrt{1+k^*}\,, \quad i\lambda_2(k) = -i\sqrt{1+k^*}\,,\\
i\lambda_3(k) &= i\sqrt{1-k^*}\,, \quad i\lambda_4(k) = -i\sqrt{1-k^*}\,,
\end{aligned}$$

where

$$k^* = \left(\frac{k_1^2}{a_1^2} + \frac{k_2^2}{a_2^2}\right)^{1/2}\left(\frac{k_1^2}{a_1^2} + \frac{k_2^2}{a_2^2} + \frac{k_3^2}{a_3^2}\right)^{-1/2}\,. \tag{5.8}$$

The canonical basis of $\mathbb{C}^4$ is denoted by $(V_j)_{j\in[[1,4]]}$ and $A_j = V_j \otimes V_j^{\mathrm{T}}$. Let P_k be a unitary transition matrix such that

$$R_k = P_k^* \left(\sum_{j=1}^{4} i\lambda_j(k) A_j \right) P_k \,.$$

Then $\mathcal{R}_k(t)$ is also reducible to diagonal form on $\mathbb{C}$ with eigenvalues $(\exp(it\lambda_j(k)))_{j\in[[1,4]]}$

$$\mathcal{R}_k(t) = P_k^* \left(\sum_{j=1}^{4} \exp(it\lambda_j(k)) A_j \right) P_k \,.$$

Replacing in (5.6) leads to

$$\partial_t \mathcal{F}_k \Psi_\varepsilon + \sum_{l+m=k} \sum_{\eta\in[[1,4]]^3} \exp(it\omega_\eta(k,l,m)) s_\eta(k,l,m)[\mathcal{F}_l\Psi_\varepsilon,\, \mathcal{F}_m\Psi_\varepsilon] = 0\,,$$

where

$$\omega_\eta(k,l,m) = \lambda_{\eta_1}(k) - \lambda_{\eta_2}(l) - \lambda_{\eta_3}(m) \tag{5.9}$$

and

$$s_\eta(k,l,m)[\alpha,\beta] = (P_k^* A_{\eta_1} P_k) B_{l,m}[P_l^* A_{\eta_2} P_l \alpha, P_m^* A_{\eta_3} P_m \beta] \,. \tag{5.10}$$

From (5.7) we infer that $\|B_{l,m}\| \le C(|l| + |m|)$ for some positive constant C. Combining both results gives the expected bound on $s_\eta(k,l,m)$. □

The formal limit Ψ of Ψ_ε as $\varepsilon \to 0$ is obtained by formal time averaging and is supposed to solve

$$\partial_t \Psi + Q_\infty(\Psi,\Psi) = 0\,, \tag{5.11}$$

where the non-autonomous bilinear operator $(a,b) \mapsto Q_\infty(a,b)$ is defined by its Fourier coefficients

$$\forall\, k \in \mathbb{Z}^3\,, \quad \mathcal{F}_k Q_\infty(a,b) = \sum_{\substack{l+m=k\\ \omega_\eta(k,l,m)=0}} s_\eta(k,l,m)[\mathcal{F}_l a, \mathcal{F}_m b] \,. \tag{5.12}$$

In order to make precise the structure of the limiting Eq. (5.11), we need an accurate description of the resonances, i.e. of the set $\{(l,m,\eta) | \omega_\eta(l+m,l,m) = 0\}$.

Lemma 5.5. (Structure of the limiting equation) *For all $\eta \in [[1,4]]^3$ and all $k,l,m \in \mathbb{Z}^3$, define $\omega_\eta(k,l,m)$ by (5.9). Then there exists a set $\mathcal{A} \subset (\mathbb{R}_*^+)^3$ of Lebesgue measure zero such that for all $(a_1,a_2,a_3) \in (\mathbb{R}_*^+)^3 \setminus \mathcal{A}$,*

- $\omega_\eta(l+m,l,m) = 0$ *implies that* $l_3 = 0$ *or* $m_3 = 0$ *or* $l_3 + m_3 = 0$,
- $\omega_\eta(l+m,l,m) = 0$ *with* $l_3, m_3 \neq 0 \Leftrightarrow \eta_1 \in \{3,4\}$ *and* $\lambda_{\eta_2}(l) + \lambda_{\eta_3}(m) = 0$,
- $\omega_\eta(l+m,l,m) = 0$ *with* $l_3 + m_3, m_3 \neq 0 \Leftrightarrow \eta_2 \in \{3,4\}$ *and* $\lambda_{\eta_1}(l+m) = \lambda_{\eta_3}(m)$,
- $\omega_\eta(l+m,l,m) = 0$ *with* $l_3 + m_3, l_3 \neq 0 \Leftrightarrow \eta_3 \in \{3,4\}$ *and* $\lambda_{\eta_1}(l+m) = \lambda_{\eta_2}(l)$.

In particular, for such periodic boxes, any solution Ψ *of* (5.11) *can be decomposed in*

$$\Psi = \bar{\Psi} + \Psi_{\mathrm{osc}}\,,$$

where $E(\bar{\Psi}) = 0$ (*which means that* $\bar{\Psi}'$ *satisfies the incompressible* 2D1/2 *Euler Eq.* (1.15) *while* $\nabla_x(-\Delta_x)^{-1/2}\bar{\Psi}_4 + \bar{\Psi}' \wedge b = 0$), *and* Ψ_{osc} *is governed by a linear system of equations whose coefficients depend on* $\bar{\Psi}$.

Proof. The results concerning the resonances, i.e. the solutions of the dispersion equation

$$\omega_\eta(l+m, l, m) = 0$$

come from algebraic properties of the functions $k \mapsto \lambda_j(k)$ defined by (5.8): the main argument is the small divisor estimate stated in Appendix B. Define

$$q(l,m) = \Pi_{\eta \in [[1,4]]^3}\, \omega_\eta(l+m, l, m)\,.$$

By (5.9), $q(l,m)$ is a polynomial with respect to the variables $\lambda_j(k)$ (for $j \in [[1,4]]$ and $k \in \{l, m, l+m\}$). Considerations of symmetry ensure that it is in fact a polynomial in the $\sigma_j(k)$'s ($j \in [[1,4]], k \in \{l, m, l+m\}$) where $(\sigma_j)_{j\in[[1,4]]}$ are the elementary symmetrical functions in the $(\lambda_j)_{j\in[[1,4]]}$. Computing these elementary symmetrical functions shows that $q(l,m)$ is a polynomial with respect to $(l^*)^2$, $(m^*)^2$ and $((l+m)^*)^2$. Then there exist an integer N and a polynomial P such that

$$q(l,m) = |l|^{-N}|m|^{-N}|l+m|^{-N} P\left(\frac{l_1}{a_1}, \frac{l_2}{a_2}, \frac{l_3}{a_3}, \frac{m_1}{a_1}, \frac{m_2}{a_2}, \frac{m_3}{a_3}\right).$$

By Proposition B.1, there exists a set $\mathcal{A} \subset (\mathbb{R}_*^+)^3$ of Lebesgue measure zero and $\Omega \subset \mathbb{Z}^6$ such that for all $(a_1, a_2, a_3) \in (\mathbb{R}_+^*)^3 \setminus \mathcal{A}$ there exists (C, s) such that

$$\begin{aligned} &\forall\,(l,m) \in \Omega\,, \quad q(l,m) \equiv 0\,,\\ &\forall\,(l,m) \in \mathbb{Z}^6 \setminus \Omega\,, \quad |q(l,m)|^{-1} \le C(1+|l|)^s(1+|m|)^s\,. \end{aligned} \tag{5.13}$$

In particular, as $|\omega_\eta(l+m,l,m)| \le 3\sqrt{2}$ for all $\eta \in [[1,4]]$ and all $l, m \in \mathbb{Z}^3$, this implies

$$\forall\,(l,m) \in \mathbb{Z}^6 \setminus \Omega\,, \quad \forall\,\eta \in [[1,4]]\,, \quad |\omega_\eta(l+m,l,m)|^{-1} \le C(1+|l|)^s(1+|m|)^s\,.$$

In order to establish the characterization of the resonances, we have to describe the set Ω. Consider $(l,m) \in \mathbb{Z}^6$. If l_3, m_3 and $l_3 + m_3$ are not equal to zero, l^*, m^* and $(l+m)^*$ tend to 0 as $a_3 \to 0$. Then, for each η, $\omega_\eta(l+m,l,m)$ converges to an odd number as $a_3 \to 0$. Thus $q(l,m) \not\equiv 0$ and $(l,m) \notin \Omega$. Conversely, if l_3, m_3 or l_3+m_3 is equal to zero, it is easy to check that $q(l,m) \equiv 0$. Thus,

$$\Omega = \{(l,m) \in \mathbb{Z}^6 | l_3 = 0,\ m_3 = 0, \text{ or } l_3 + m_3 = 0\}\,. \tag{5.14}$$

Combining (5.13) and (5.14) provides the first assertion in Lemma 5.5, i.e. a necessary condition for a resonance to appear. To complete our description, we must determine exactly which (l,m,η) with $(l,m) \in \Omega$ and $\eta \in [[1,4]]$ satisfy the dispersion relation. If $l_3 = m_3 = l_3 + m_3 = 0$, we easily obtain the various (η_1, η_2, η_3) leading to $\omega_\eta(l+m,l,m) = 0$. Then, by a symmetry argument, it is enough to consider the case with $l_3 \neq 0$ and $l_3 + m_3 = 0$.

- If $\eta_1 = 3$ or $\eta_1 = 4$, then $\lambda_{\eta_1}(l+m) = 0$ and the result is clear.
- If $\eta_1 = 1$ or $\eta_1 = 2$, $\lambda_{\eta_1}(l+m) = \pm\sqrt{2}$. Define

$$q'(l,m) = \Pi_{(\eta_2,\eta_3)\in[[1,4]]^2}(\sqrt{2} - \lambda_{\eta_2}(l) - \lambda_{\eta_3}(m)).$$

The same arguments as before imply the existence of $\mathcal{A}'$ of Lebesgue measure zero such that $\forall\,(a_1,a_2,a_3) \in (\mathbb{R}_+^*)^3 \setminus \mathcal{A}'$, there exists (C,s) such that

$$\begin{gathered}\forall\,(l,m) \in \mathbb{Z}^6 \setminus \Omega', \quad \forall\,(\eta_2,\eta_3),\\ |\sqrt{2} - \lambda_{\eta_2}(l) - \lambda_{\eta_3}(m)|^{-1} \leq C(1+|l|)^s(1+|m|)^s \\ \Omega' = \{(l,m) \in \mathbb{Z}^6 | l_3 = 0 \text{ or } m_3 = 0\}.\end{gathered} \tag{5.15}$$

Upon replacing $\mathcal{A}$ by $\mathcal{A} \cup \mathcal{A}'$, also of Lebesgue measure zero, we cannot have $\omega_\eta(k,l,m) = 0$ with $l_3, m_3 \neq 0$ if $\eta_1 = 1$ or $\eta_1 = 2$.

This completes the characterization of resonances. Notice that, as a consequence of (5.13)–(5.15), we obtain the existence of a set $\mathcal{A}$ of Lebesgue measure zero such that $\forall\,(a_1,a_2,a_3) \in (\mathbb{R}_+^*)^3 \setminus \mathcal{A}$ there exists (C,s) such that

$$\begin{aligned}\forall\,(l,m,\eta), \quad & \omega_\eta(l+m,l,m) = 0 \\ & \text{or } |\omega_\eta(l+m,l,m)|^{-1} \leq C(1+|l|)^s(1+|m|)^s.\end{aligned} \tag{5.16}$$

We next turn to the second part of Lemma 5.5, showing how the results above on resonances allow to decouple the equations governing $\bar\Psi = \mathcal{P}(\Psi)$ where $\mathcal{P}$ denotes the projection on the kernel of R:

$$\mathcal{F}_k\bar\Psi = \begin{cases} P_k^*(A_3+A_4)P_k\mathcal{F}_k\Psi & \text{if } k_3 = 0, \\ 0 & \text{otherwise}. \end{cases} \tag{5.17}$$

By (5.11), (5.12) and (5.10),

$$\partial_t \mathcal{F}_k\Psi + \sum_{\substack{l+m=k \\ \omega_\eta(k,l,m)=0}} (P_k^{-1}A_{\eta_1}P_k)B_{l,m}[P_l^{-1}A_{\eta_2}P_l\mathcal{F}_l\Psi,\, P_m^{-1}A_{\eta_3}P_m\mathcal{F}_m\Psi] = 0.$$

Then, if $k_3 = 0$,

$$\partial_t \mathcal{F}_k\bar\Psi + \sum_{\substack{l+m=k,\eta_1\in\{3,4\} \\ \omega_\eta(k,l,m)=0}} (P_k^{-1}A_{\eta_1}P_k)B_{l,m}[P_l^{-1}A_{\eta_2}P_l\mathcal{F}_l\Psi,\, P_m^{-1}A_{\eta_3}P_m\mathcal{F}_m\Psi] = 0.$$

The previous results on resonances show that if $\eta_1 \in \{3,4\}$, $l_3 + m_3 = 0$,

$$\omega_\eta(l+m,l,m) = 0 \Leftrightarrow \lambda_{\eta_2}(l) + \lambda_{\eta_3}(m) = 0.$$

Considerations of symmetry as in Proposition C.1 show that

$$(P_k^{-1}(A_3 + A_4)P_k) \sum_{\substack{l+m=k \\ \lambda_{\eta_2}(l)+\lambda_{\eta_3}(m)=0}} B_{l,m}[P_l^{-1}A_{\eta_2}P_l\mathcal{F}_l\Psi,\, P_m^{-1}A_{\eta_3}P_m\mathcal{F}_m\Psi]$$

$$= (P_k^{-1}(A_3 + A_4)P_k) \sum_{\substack{l+m=k \\ \eta_2,\eta_3\in\{3,4\}}} B_{l,m}[P_l^{-1}A_{\eta_2}P_l\mathcal{F}_l\Psi,\, P_m^{-1}A_{\eta_3}P_m\mathcal{F}_m\Psi]\,.$$

Finally the equation for $\mathcal{F}_k\bar{\Psi}$ with $k_3 = 0$ is rewritten

$$\partial_t\mathcal{F}_k\bar{\Psi} + (P_k^{-1}(A_3 + A_4)P_k) \sum_{l+m=k} B_{l,m}[\mathcal{F}_l\bar{\Psi},\, \mathcal{F}_m\bar{\Psi}] = 0$$

meaning that

$$\partial_t\bar{\Psi} + \mathcal{P}(B(\bar{\Psi},\, \bar{\Psi})) = 0 \quad \text{with the constraint } \bar{\Psi} = \mathcal{P}(\bar{\Psi}) \tag{5.18}$$

or equivalently that $(\bar{\Psi}',\, \bar{\Psi}_4)$ verifies (4.1) and that $\bar{\Psi}'$ satisfies the 2D1/2 incompressible Euler equation (1.15). It remains to determine the equations governing $\Psi_{\text{osc}} = \Psi - \bar{\Psi}$.

- If $k_3 = 0$,

$$\partial_t\mathcal{F}_k\Psi_{\text{osc}} + \sum_{\substack{l+m=k,\ \eta_1\in\{1,2\} \\ \omega_\eta(k,l,m)=0}} (P_k^{-1}A_{\eta_1}P_k)B_{l,m}[P_l^{-1}A_{\eta_2}P_l\mathcal{F}_l\Psi,\, P_m^{-1}A_{\eta_3}P_m\mathcal{F}_m\Psi] = 0\,.$$

  The previous results on resonances imply that

$$\begin{aligned} &l_3 + m_3 = 0\,, \quad \omega_\eta(l+m, l, m) = 0\,, \quad \eta_1 \in \{1,2\} \\ &\Leftrightarrow\ l_3 = m_3 = 0\,, \quad \eta_1 = \eta_2 \in \{1,2\}\,, \quad \eta_3 \in \{3,4\} \\ &\quad\text{or } l_3 = m_3 = 0\,, \quad \eta_1 = \eta_3 \in \{1,2\}\,, \quad \eta_2 \in \{3,4\}\,. \end{aligned}$$

  By symmetry,

$$\partial_t\mathcal{F}_k\Psi_{\text{osc}} + 2\sum_{l+m=k,\eta_1\in\{1,2\}} (P_k^{-1}A_{\eta_1}P_k)B_{l,m}[\mathcal{F}_l\bar{\Psi}, P_m^{-1}A_{\eta_1}P_m\mathcal{F}_m\Psi] = 0\,. \tag{5.19}$$

- If $k_3 \neq 0$,

$$\partial_t\mathcal{F}_k\Psi_{\text{osc}} + \sum_{\substack{l+m=k \\ \omega_\eta(k,l,m)=0}} (P_k^{-1}A_{\eta_1}P_k)B_{l,m}[P_l^{-1}A_{\eta_2}P_l\mathcal{F}_l\Psi,\, P_m^{-1}A_{\eta_3}P_m\mathcal{F}_m\Psi] = 0\,.$$

  The previous results on resonances imply that

$$\begin{aligned} &l_3 + m_3 \neq 0\,, \quad \omega_\eta(l+m, l, m) = 0\,, \\ &\Leftrightarrow\ l_3 = 0\,, \quad \eta_1 = \eta_3\,, \quad \eta_2 \in \{3,4\}\,, \quad m^* = (l+m)^* \\ &\quad\text{or } m_3 = 0\,, \quad \eta_1 = \eta_2\,, \quad \eta_3 \in \{3,4\}\,, \quad l^* = (l+m)^*\,. \end{aligned}$$

Then, by symmetry,

$$\partial_t \mathcal{F}_k \Psi_{\rm osc} + 2 \sum_{\substack{l+m=k,\eta_1\in\{3,4\}\\ m^*=(l+m)^*}} (P_k^{-1} A_{\eta_1} P_k) B_{l,m}[\mathcal{F}_l \bar\Psi, P_m^{-1} A_{\eta_1} P_m \mathcal{F}_m \Psi] = 0\,. \tag{5.20}$$

Combining both results proves that $\Psi_{\rm osc}$ is governed by a linear system of equations whose coefficients depend on $\bar\Psi$. □

In order to estimate the error made in replacing Ψ_ε by the solution Ψ of the limiting Eq. (5.11), we will need some regularity estimates on Ψ. The projection $\bar\Psi$ of Ψ on the kernel of R is smooth, at least locally in time, since $\bar\Psi$ satisfies the 2D1/2 incompressible Euler equation. It remains to establish that $\Psi_{\rm osc}$ can also be controlled in the Sobolev norms.

Lemma 5.6. (Regularity estimates) *Consider a periodic box Q_{a_1,a_2,a_3} with $(a_1,a_2,a_3)\in(\mathbb{R}_*^+)^3\setminus\mathcal{A}$. Let Ψ be a solution of* (5.11) *on* $[0,T]$ *with smooth initial data. Denote by $\bar\Psi$ its projection on* $\mathrm{Ker}(R)$ *and by $\Psi_{\rm osc}=\Psi-\bar\Psi$. Then, for all $s\in\mathbb{N}$, there exists some non-negative constant C such that* $\forall\, t\in[0,T]$

$$\|\Psi_{\rm osc}(t)\|_{H^s(Q_{a_1,a_2,a_3})} \le \|\Psi_{\rm osc}^{in}\|_{H^s(Q_{a_1,a_2,a_3})} \times \exp\left(C\int_0^t \|\bar\Psi(\tau)\|_{H^{s+7/2}(Q_{a_1,a_2,a_3})}\,d\tau\right). \tag{5.21}$$

Proof. We will obtain the propagation of regularity on $\Psi_{\rm osc}$ as an easy property of $\mathcal{B}$, after rewriting the limiting equation in a convenient form.

Indeed we consider the following decomposition of $\Psi_{\rm osc}$:

$$\Psi_{\rm osc} = \sum_{n\in\mathbb{N}} \psi_n\,,$$

where

$$\mathcal{F}\psi_0 = \sum_{k^*=1} (P_k^{-1}(A_1+A_2)P_k\mathcal{F}_k\Psi_{\rm osc})\,,$$

$$\mathcal{F}\psi_n = \sum_{k^*=\alpha_n} (P_k^{-1}(A_3+A_4)P_k\mathcal{F}_k\Psi_{\rm osc})\,,\quad \forall\, n\in\mathbb{N}^*$$

and α_n goes over the set $\{k^*\neq 1/k\in\mathbb{Z}^3\}$ (which is of course a countable set). Checking that

$$\forall\, i,j\in\mathbb{N}\,,\quad i\neq j\,,\quad \{k\in\mathbb{Z}^3/k^*=\alpha_i\}\cap\{k\in\mathbb{Z}^3/k^*=\alpha_j\}=\emptyset\,,$$

we obtain

$$\|\Psi_{\rm osc}\|^2_{H^s(Q_{a_1,a_2,a_3})} = \sum_{n\in\mathbb{N}} \|\psi_n\|^2_{H^s(Q_{a_1,a_2,a_3})}\,. \tag{5.22}$$

Equations (5.19)–(5.20) of the limiting system can be rewritten as:

$$\forall\, n\in\mathbb{N}\,,\quad \partial_t\psi_n + 2\mathcal{B}(\bar\Psi,\,\psi_n) = 0\,. \tag{5.23}$$

In view of (5.22) and (5.23), Lemma 5.6 will be established if we prove that the transport operator $(\partial_t + 2\mathcal{B}(\bar{\Psi}, .))$ propagates the Sobolev regularity. Let ψ be a solution of

$$\partial_t \psi + 2\mathcal{B}(\bar{\Psi}, \psi) = 0\,. \tag{5.24}$$

Differentiating (5.24) leads to

$$\forall\, s \in \mathbb{N}^3\,,\ \partial_t D^s \psi + 2\mathcal{B}(\bar{\Psi}, D^s\psi) + 2 \sum_{\sigma \neq 0, \sigma_i \leq s_i} \prod_{i=1}^{3} \begin{pmatrix} s_i \\ \sigma_i \end{pmatrix} \mathcal{B}(D^\sigma \bar{\Psi}, D^{s-\sigma}\psi) = 0\,.$$

Then, using the definition of $\mathcal{B}$, we get

$$\begin{aligned} &\partial_t D^s \psi' + (\bar{\Psi}' \cdot \nabla_x)\, D^s \psi' \\ &\quad = -(\psi' \cdot \nabla_x)\, D^s \bar{\Psi}' - 2 \sum_{\sigma \neq 0, \sigma_i \leq s_i} \prod_{i=1}^{3} \begin{pmatrix} s_i \\ \sigma_i \end{pmatrix} \mathcal{B}'(D^\sigma \bar{\Psi}, D^{s-\sigma}\psi)\,, \\ &\partial_t D^s (-\Delta_x)^{1/2} \psi_4 + \nabla_x (\bar{\Psi}' D^s (-\Delta_x)^{1/2} \psi_4) \\ &\quad = -\nabla_x (\psi' D^s (-\Delta_x)^{1/2} \bar{\Psi}_4 - 2 \sum_{\sigma \neq 0, \sigma_i \leq s_i} \prod_{i=1}^{3} \begin{pmatrix} s_i \\ \sigma_i \end{pmatrix} (-\Delta_x)^{1/2} \mathcal{B}_4 (D^\sigma \bar{\Psi}, D^{s-\sigma}\psi)\,. \end{aligned}$$

As $\nabla_x \bar{\Psi}' = 0$, this implies

$$\frac{1}{2}\frac{d}{dt} \|\psi'\|^2_{H^s(Q_{a_1,a_2,a_3})} \leq C \|\bar{\Psi}'\|_{H^{s+1+3/2}(Q_{a_1,a_2,a_3})} \|\psi'\|^2_{H^s(Q_{a_1,a_2,a_3})}$$

and

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt} \|(-\Delta_x)^{1/2}\psi_4\|^2_{H^s(Q_{a_1,a_2,a_3})} \\ &\quad \leq C \|\bar{\Psi}\|_{H^{s+2+3/2}(Q_{a_1,a_2,a_3})} (\|\psi'\|^2_{H^s(Q_{a_1,a_2,a_3})} + \|(-\Delta_x)^{1/2}\psi_4\|^2_{H^s(Q_{a_1,a_2,a_3})})\,. \end{aligned}$$

We conclude by Gronwall's lemma that

$$\|\psi(t)\|_{H^s(Q_{a_1,a_2,a_3})} \leq \|\psi^{in}\|_{H^s(Q_{a_1,a_2,a_3})} \exp\left(C \int_0^t \|\bar{\Psi}(\tau)\|_{H^{s+7/2}(Q_{a_1,a_2,a_3})} \| d\tau \right). \tag{5.25}$$

Applying estimate (5.25) to each ψ_n and using (5.22) gives the expected regularity on Ψ_{osc}. □

5.2. *Error estimates*

In the sequel, we assume that (a_1, a_2, a_3) is chosen in $(\mathbb{R}_*^+)^3 \backslash \mathcal{A}$, where $\mathcal{A}$ is the set of Lebesgue measure zero in Lemma 5.5. A consequence of this lemma is the global existence of a unique solution for (5.11) and (5.12) with initial data $(J^{in}, \Phi^{in}) \in C^r(Q_{a_1,a_2,a_3})$ $(r > 1)$. Indeed the theory of the 2D1/2 incompressible Euler equation (1.15) guarantees the global existence of a unique solution $\bar{\Psi}$ for

(5.18)–(5.5) with initial data $\mathcal{P}(J^{in}, \Phi^{in}) \in C^r(Q_{a_1,a_2,a_3})$, while the theory of linear differential equations gives the existence and the uniqueness of $\Psi_{\rm osc}$ satisfying (5.19) and (5.20) with initial data $(\mathrm{Id} - \mathcal{P})(J^{in}, \Phi^{in})$ as long as $\bar\Psi$ is defined.

For all $\varepsilon > 0$, let Ψ_ε be a solution of (5.2) and (5.3) with initial data (J^{in}, Φ^{in}). We expect that the asymptotic behavior of Ψ_ε as $\varepsilon \to 0$ is described by the solution Ψ of the formal limiting Eqs. (5.11), (5.12). Yet we cannot prove directly that

$$\partial_t \Psi + Q\left(\frac{t}{\varepsilon}, \Psi, \Psi\right) \rightharpoonup 0$$

in a weak sense to be made precise. Then, using a standard method in singular perturbation problems,[27] we introduce a small quantity $\varepsilon y_\varepsilon$ such that

$$\partial_t(\Psi + \varepsilon y^\varepsilon) + Q\left(\frac{t}{\varepsilon}, (\Psi + \varepsilon y_\varepsilon), (\Psi + \varepsilon y_\varepsilon)\right) \to 0\,,$$

where the convergence here holds strongly in some appropriate sense. Of course, $\Psi + \varepsilon y_\varepsilon$ has the same asymptotic behavior as Ψ.

Lemma 5.7. *Let* $(J^{in}, \Phi^{in}) \in C^{r_0}(Q_{a_1,a_2,a_3})$ *with* $r_0 > 13/2$. *Denote by* $\Psi \in L^\infty([0,T], H^r)$ *the solution of* (5.11) *and* (5.12) *with initial data* (J^{in}, Φ^{in}); *define* y_ε *by its Fourier coefficients*

$$\forall\, k \in \mathbb{Z}^3, \quad \mathcal{F}_k y_\varepsilon = - \sum_{\substack{l+m=k,\ |l|+|m|\le|\log\varepsilon|,\\ \eta\in[[1,4]]^3,\ \omega_\eta(k,l,m)\neq 0}} \frac{\exp\left(\frac{it}{\varepsilon}\omega_\eta(k,l,m)\right)}{i\omega_\eta(k,l,m)} s_\eta(k,l,m)[\mathcal{F}_l\Psi, \mathcal{F}_m\Psi]\,. \tag{5.26}$$

Then,

- *there exists a non-negative constant* C *such that, for each* $r \le r_0 - 7/2$

$$\|y_\varepsilon\|_{L^\infty([0,T],H^r)} \le C|\log\varepsilon|^{2s+1}\,,$$

where s *depends only on* $(a_1, a_2, a_3) \in (\mathbb{R}_*^+)^3 \setminus \mathcal{A}$.
- *there exists* $\delta \in C(\mathbb{R}^+)$ *with* $\delta(0) = 0$ *such that*

$$\left\|\partial_t(\Psi + \varepsilon y_\varepsilon) + Q\left(\frac{t}{\varepsilon}, (\Psi + \varepsilon y_\varepsilon), (\Psi + \varepsilon y_\varepsilon)\right)\right\|_{L^\infty([0,T],H^2)} \le \delta(\varepsilon)\,.$$

Proof. The classical theory of the 2D1/2 incompressible Euler equation (1.15) gives the following *a priori* bounds for the solution $\bar\Psi$ of (5.18)–(5.5) with initial data $\mathcal{P}(J^{in}, \Phi^{in})$

$$\|\bar\Psi\|_{L^\infty([0,T],C^r)} + \|\partial_t\bar\Psi\|_{L^\infty([0,T],C^{r-1})} \le C_T\,,$$

where C_T depends on T and on $\|(J^{in}, \Phi^{in})\|_{C^r}$. By Lemma 5.6, $\Psi_{\rm osc}$ solves a linear system that preserves all Sobolev norms. It is then easily checked that, for each $r \le r_0 - 7/2$

$$\|\Psi_{\rm osc}\|_{L^\infty([0,T],H^r)} + \|\partial_t\Psi_{\rm osc}\|_{L^\infty([0,T],H^{r-1})} \le C_T\,.$$

Then, for all $T > 0$, there exists $C_T > 0$ such that

$$\|\Psi\|_{L^\infty([0,T],H^r)} + \|\partial_t \Psi\|_{L^\infty([0,T],H^{r-1})} \le C_T\,. \tag{5.27}$$

By (5.16), there exist non-negative constants (C, s) such that

$$\forall\,(l,m,\eta),\quad \omega_\eta(l+m,l,m) \neq 0 \Rightarrow |\omega_\eta(l+m,l,m)|^{-1} \le C(1+|l|)^s(1+|m|)^s\,.$$

By Lemma 5.4,

$$|s_\eta(l+m,l,m)| \le C(|l|+|m|)\,.$$

Combining these last two estimates leads to

$$\begin{aligned}|\mathcal{F}_k y_\varepsilon| &\le C \sum_{l+m=k,|l|+|m|\le|\log\varepsilon|} (1+|l|)^s(1+|m|)^s(|l|+|m|)|\mathcal{F}_l\Psi|\,|\mathcal{F}_m\Psi| \\ &\le C|\log\varepsilon|^{2s+1} \sum_{l+m=k} |\mathcal{F}_l\Psi|\,|\mathcal{F}_m\Psi|\,.\end{aligned}$$

Then

$$(1+|k|^2)^{r/2}|\mathcal{F}_k y_\varepsilon| \le C|\log\varepsilon|^{2s+1} \sum_{l+m=k} (1+|l|^2)^{r/2}|\mathcal{F}_l\Psi|(1+|m|^2)^{r/2}|\mathcal{F}_m\Psi|$$

which, together with (5.27) gives the expected bound on $\|y_\varepsilon\|_{L^\infty([0,T],H^r)}$.

In the next step we check that $\Psi + \varepsilon y_\varepsilon$ approximately verifies both (5.2) and (5.3). By (5.11), (5.12) and (5.26),

$$\begin{aligned}\partial_t \mathcal{F}_k(\Psi + \varepsilon y_\varepsilon) = &- \sum_{\substack{l+m=k,\\ \omega_\eta(k,l,m)=0}} s_\eta(k,l,m)[\mathcal{F}_l\Psi,\,\mathcal{F}_m\Psi] \\ &- \sum_{\substack{l+m=k,|l|+|m|\le|\log\varepsilon|,\\ \omega_\eta(k,l,m)\neq 0}} \exp\left(\frac{it}{\varepsilon}\omega_\eta(k,l,m)\right) s_\eta(k,l,m)[\mathcal{F}_l\Psi,\,\mathcal{F}_m\Psi] \\ &- \varepsilon \sum_{\substack{l+m=k,|l|+|m|\le|\log\varepsilon|,\\ \omega_\eta(k,l,m)\neq 0}} \frac{\exp\left(\frac{it}{\varepsilon}\omega_\eta(k,l,m)\right)}{i\omega_\eta(k,l,m)} \partial_t s_\eta(k,l,m)[\mathcal{F}_l\Psi,\,\mathcal{F}_m\Psi]\end{aligned}$$

from which we deduce that

$$\begin{aligned}&\partial_t \mathcal{F}_k(\Psi + \varepsilon y_\varepsilon) + \mathcal{F}_k Q\left(\frac{t}{\varepsilon},\,\Psi + \varepsilon y_\varepsilon,\,\Psi + \varepsilon y_\varepsilon\right) \\ &\quad= \sum_{\substack{l+m=k,|l|+|m|>|\log\varepsilon|\\ \omega_\eta(k,l,m)\neq 0}} \exp\left(\frac{it}{\varepsilon}\omega_\eta(k,l,m)\right) s_\eta(k,l,m)[\mathcal{F}_l\Psi,\,\mathcal{F}_m\Psi] \\ &\qquad+ \varepsilon\mathcal{F}_k Q\left(\frac{t}{\varepsilon},\,y_\varepsilon,\,2\Psi + \varepsilon y_\varepsilon\right) \\ &\qquad- \varepsilon \sum_{\substack{l+m=k,|l|+|m|\le|\log\varepsilon|,\\ \omega_\eta(k,l,m)\neq 0}} \frac{\exp\left(\frac{it}{\varepsilon}\omega_\eta(k,l,m)\right)}{i\omega_\eta(k,l,m)} \partial_t s_\eta(k,l,m)[\mathcal{F}_l\Psi,\,\mathcal{F}_m\Psi]\,.\end{aligned}$$

The estimates on $s_\eta(l+m,l,m)$ and $\omega_\eta(l+m,l,m)$ then give

$$
\begin{aligned}
(1+|k|^2)&\left|\partial_t\mathcal{F}_k(\Psi+\varepsilon y_\varepsilon)+\mathcal{F}_kQ\left(\frac{t}{\varepsilon},\,\Psi+\varepsilon y_\varepsilon,\,\Psi+\varepsilon y_\varepsilon\right)\right|\\
&\le C\sum_{l+m=k,|l|+|m|>|\log\varepsilon|}(|l|+|m|)^{-(r-3)}(1+|l|^2)^{r/2}|\mathcal{F}_l\Psi|(1+|m|^2)^{r/2}|\mathcal{F}_m\Psi|\\
&\quad+C\varepsilon\sum_{l+m=k}(1+|l|+|m|)^3|\mathcal{F}_l y_\varepsilon|\,|\mathcal{F}_m(2\Psi+\varepsilon y_\varepsilon)|\\
&\quad+C\varepsilon\sum_{\substack{l+m=k,|l|+|m|\le|\log\varepsilon|,\\ \omega_\eta(k,l,m)\neq 0}}(1+|l|)^s(1+|m|)^s(1+|l|+|m|)^3|\partial_t\mathcal{F}_l\Psi|\,|\mathcal{F}_m\Psi|
\end{aligned}
$$

which can be rewritten

$$
\begin{aligned}
&\left\|\partial_t\mathcal{F}_k(\Psi+\varepsilon y_\varepsilon)+\mathcal{F}_kQ\left(\frac{t}{\varepsilon},\,\Psi+\varepsilon y_\varepsilon,\,\Psi+\varepsilon y_\varepsilon\right)\right\|_{H^2}\\
&\quad\le C|\log\varepsilon|^{-(r-3)}\|\Psi\|_{H^r}^2+C\varepsilon\|y_\varepsilon\|_{H^3}\|2\Psi+\varepsilon y_\varepsilon\|_{H^3}\\
&\qquad+C\varepsilon|\log\varepsilon|^{2s+3}\|\partial_t\Psi\|_{L^2}\|\Psi\|_{L^2}\,.
\end{aligned}
$$

Using the *a priori* estimates on Ψ and y_ε leads to the expected result. □

As an immediate consequence of Lemma 5.7, we get the convergence as $\varepsilon\to 0$ of the family (Ψ_ε) to the solution Ψ of the limiting Eq. (5.11) and (5.12).

Corollary 5.3. *Consider $(a_1,a_2,a_3)\in(\mathbb{R}_*^+)^3\setminus\mathcal{A}$ where $\mathcal{A}$ is the set of Lebesgue measure zero defined in Lemma* 5.5. *Let $(J^{in},\Phi^{in})\in C^{r_0}(Q_{a_1,a_2,a_3})$ with $r_0>13/2$. Assume that, for all $\varepsilon>0$, there exists a solution $\Psi_\varepsilon\in L^\infty_{\rm loc}(\mathbb{R}^+,L^\infty((Q_{a_1,a_2,a_3}))$ of* (5.2) *and* (5.3) *with initial data (J^{in},Φ^{in}). Denote by $\Psi\in L^\infty_{\rm loc}(\mathbb{R}^+,H^r(Q_{a_1,a_2,a_3}))$, $r>3$, the solution of* (5.11) *and* (5.12) *with initial data (J^{in},Φ^{in}). Then $(\Psi_\varepsilon)_\varepsilon$ converges to Ψ strongly in $L^\infty_{\rm loc}(\mathbb{R}^+,L^2(Q_{a_1,a_2,a_3}))$ as $\varepsilon\to 0$.*

Proof. Define y_ε by (5.26). By Lemma 5.7 and Sobolev embeddings,

$$\partial_t(\Psi_\varepsilon-\Psi-\varepsilon y_\varepsilon)+Q\left(\frac{t}{\varepsilon},\,\Psi_\varepsilon+\Psi+\varepsilon y_\varepsilon,\,\Psi_\varepsilon-\Psi-\varepsilon y_\varepsilon\right)=r_\varepsilon\to 0$$

in $L^\infty_{\rm loc}(\mathbb{R}^+,L^\infty)$. Rewriting Q in terms of $\mathcal{B}$ defined by (5.5) leads to

$$
\begin{aligned}
&\partial_t(\Psi_\varepsilon-\Psi-\varepsilon y_\varepsilon)\\
&\quad+\mathcal{R}\left(\frac{t}{\varepsilon}\right)\mathcal{B}\left[\mathcal{R}\left(-\frac{t}{\varepsilon}\right)(\Psi_\varepsilon+\Psi+\varepsilon y_\varepsilon),\,\mathcal{R}\left(-\frac{t}{\varepsilon}\right)(\Psi_\varepsilon-\Psi-\varepsilon y_\varepsilon)\right]=r_\varepsilon
\end{aligned}
$$

from which we deduce that

$$\frac{d}{dt}\|\Psi_\varepsilon-\Psi-\varepsilon y_\varepsilon\|_{L^\infty}\le\|r_\varepsilon\|_{L^\infty}+C\|\Psi_\varepsilon-\Psi-\varepsilon y_\varepsilon\|_{L^\infty}\|\Psi+\varepsilon y_\varepsilon\|_{W^{1,\infty}}\,.$$

By Sobolev embeddings and the *a priori* estimates in Lemma 5.7, there exists C_T such that

$$\forall\, t \in [0,T]\,, \quad \|\Psi\|_{W^{1,\infty}} + \varepsilon^{1/2}\|y_\varepsilon\|_{W^{1,\infty}} \le C_T\,.$$

Then, by Gronwall's lemma and the initial bound $\|\Psi_\varepsilon^{in} - \Psi^{in} - \varepsilon y_\varepsilon^{in}\|_{L^\infty} \le C\varepsilon$,

$$\Psi_\varepsilon - \Psi - \varepsilon y_\varepsilon \to 0 \quad \text{strongly in } L^\infty_{\text{loc}}(\mathbb{R}^+, L^\infty(Q_{a_1,a_2,a_3}))\,,$$

which is the expected convergence. □

5.3. *Convergence proof for general initial data with monokinetic profiles*

The expected asymptotic behavior of $\int f_\varepsilon v dv$ and $\nabla_x V_\varepsilon$ is obtained by combining the stability results on the Vlasov–Poisson equation (1.1) stated in Lemma 4.1 with the asymptotic results on the equation $E_\varepsilon(\bar{J}_\varepsilon, \Phi_\varepsilon) = 0$ coming from Lemmas 5.4 and 5.7.

Proof of Theorem 2.3. By a density argument, the stability inequality given by Lemma 4.1 can be extended to all $(\bar{J}, \Phi) \in L^\infty_{\text{loc}}(\mathbb{R}^+, L^\infty(Q_{a_1,a_2,a_3}))$ such that $D(\bar{J})$ and $E_\varepsilon(\bar{J}, \phi)$ belong to $L^\infty_{\text{loc}}(\mathbb{R}^+, L^\infty(Q_{a_1,a_2,a_3}))$. Consider $(a_1, a_2, a_3) \in (\mathbb{R}^+_*)^3 \backslash \mathcal{A}$ where $\mathcal{A}$ is the set of Lebesgue measure zero defined in Lemma 5.5, and define

$$(\bar{J}_\varepsilon,\, \Phi_\varepsilon) = \mathcal{R}\left(-\frac{t}{\varepsilon}\right)(\Psi + \varepsilon y_\varepsilon)\,, \tag{5.28}$$

where Ψ is the unique global solution of (5.11) and (5.12) with initial data (J^{in}, Φ^{in}) and y_ε is defined in terms of Ψ by (5.26). Then,

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\iint |v - \bar{J}_\varepsilon|^2 f_\varepsilon(t,x,v)\, dvdx + \frac{1}{2}\int |\nabla_x V_\varepsilon - \nabla_x(-\Delta_x)^{-1/2}\Phi_\varepsilon|^2(t,x)\, dx\right)\\
&\quad = -\int D(\bar{J}_\varepsilon) : \left(\int (v - \bar{J}_\varepsilon)^{\otimes 2} f_\varepsilon\, dv - (\nabla_x V_\varepsilon - \nabla_x(-\Delta_x)^{-1/2}\Phi_\varepsilon)^{\otimes 2}\right)(t,x)\, dx\\
&\qquad - \frac{1}{2}\int (\nabla_x \cdot \bar{J}_\varepsilon)|\nabla_x V_\varepsilon - \nabla_x(-\Delta_x)^{-1/2}\Phi_\varepsilon|^2(t,x)\, dx\\
&\qquad - \int E_\varepsilon(\bar{J}_\varepsilon,\, \Phi_\varepsilon) \cdot \left(\int (v - \bar{J}_\varepsilon) f_\varepsilon\, dv,\, (\nabla_x V_\varepsilon - \nabla_x(-\Delta_x)^{-1/2}\Phi_\varepsilon)\right)(t,x)\, dx
\end{aligned}$$

from which we deduce that

$$\begin{aligned}
&\iint |v - \bar{J}_\varepsilon|^2 f_\varepsilon(t,x,v)\, dvdx + \int |\nabla_x V_\varepsilon - \nabla_x(-\Delta_x)^{-1/2}\Phi_\varepsilon|^2(t,x)\, dx\\
&\quad \le \left(\iint |v - \bar{J}_\varepsilon^{in}|^2 f_\varepsilon^{in}\, dvdx + \int |\nabla_x V_\varepsilon^{in} - \nabla_x(-\Delta_x)^{-1/2}\Phi_\varepsilon^{in}|^2\, dx\right) e^{4\int_0^t \|D(\bar{J}_\varepsilon)\|_{L^\infty}\, ds}
\end{aligned}$$

$$- \int_0^t \int E_\varepsilon(\bar{J}_\varepsilon, \Phi_\varepsilon) \cdot \left(\int (v - \bar{J}_\varepsilon) f_\varepsilon \, dv, \, \nabla_x V_\varepsilon - \nabla_x(-\Delta_x)^{-1/2} \Phi_\varepsilon \right) (s, x) \times e^{4 \int_s^t \|D(\bar{J}_\varepsilon)\|_{L^\infty} \, d\tau} \, dx ds \,. \tag{5.29}$$

By Lemma 5.7 and Sobolev embeddings, as $\mathcal{R}$ is a group of isometries,

$$\forall\, t \leq T \,, \quad \|(\bar{J}_\varepsilon, \, \Phi_\varepsilon)(t)\|_{W^{1,\infty}} \leq \|(\Psi + \varepsilon y_\varepsilon)(t)\|_{W^{1,\infty}} \leq \|(\Psi + \varepsilon y_\varepsilon)(t)\|_{H^3} \leq C_T \,,$$

$$\|(\bar{J}_\varepsilon^{in} - J^{in}, \, \Phi_\varepsilon^{in} - \Phi^{in})\|_{L^\infty} \leq \|(\bar{J}_\varepsilon^{in} - J^{in}, \, \Phi_\varepsilon^{in} - \Phi^{in})\|_{H^2} \leq \|\varepsilon y_\varepsilon^{in}\|_{H^2} \leq C\varepsilon^{1/2} \,,$$

and

$$\begin{aligned} E_\varepsilon(\bar{J}_\varepsilon, \, \Phi_\varepsilon) &= \partial_t(\bar{J}_\varepsilon, \, \Phi_\varepsilon) + \frac{1}{\varepsilon} R(\bar{J}_\varepsilon, \, \Phi_\varepsilon) + \mathcal{B}[(\bar{J}_\varepsilon, \, \Phi_\varepsilon), (\bar{J}_\varepsilon, \, \Phi_\varepsilon)] \\ &= \mathcal{R}\left(-\frac{t}{\varepsilon}\right) \left(\partial_t(\Psi + \varepsilon y_\varepsilon) + Q\left(\frac{t}{\varepsilon}, \, \Psi + \varepsilon y_\varepsilon, \, \Psi + \varepsilon y_\varepsilon\right)\right) \\ &= \mathcal{R}\left(-\frac{t}{\varepsilon}\right) r_\varepsilon \end{aligned}$$

converges to 0 in $L^\infty_{\text{loc}}(\mathbb{R}^+, H^2(Q_{a_1,a_2,a_3}))$. From (1.2) and the bounds on the initial data (2.10), we deduce that

$$\iint (1 + |v|) f_\varepsilon \, dv dx + \int |\nabla_x V_\varepsilon| \, dx \leq C \,.$$

Then both terms in the right of (5.29) converges to 0 in $L^\infty_{\text{loc}}(\mathbb{R}^+)$. Thus

$$\iint |v - \bar{J}_\varepsilon|^2 f_\varepsilon(t, x, v) \, dv dx + \int |\nabla_x V_\varepsilon - \nabla_x(-\Delta_x)^{-1/2} \Phi_\varepsilon|^2(t, x) \, dx \to 0 \quad \text{in } L^\infty_{\text{loc}}(\mathbb{R}^+) \,.$$

By (5.28) and Lemma 5.5, this is equivalent to the convergence stated in Theorem 2.3. □

6. Lack of *a priori* Compactness

All the results above lead to strong convergence of $(\int f_\varepsilon \, v dv, \nabla_x V_\varepsilon)_\varepsilon$ as $\varepsilon \to 0$ and are based on restrictive assumptions on the sequence of initial data (smoothness and convergence in a very strong sense). Whether it is possible to relax these assumptions and obtain convergence in some weaker sense is a natural question. As always, the difficulty lies in passing to the limit in nonlinear terms. As we explained in Sec. 5, time oscillations can be handled provided that the current $\int f_\varepsilon \, v dv$ and the electric potential V_ε verify some kind of compactness property with respect to space variables. One might therefore attempt to prove that (1.1) propagates some low regularity in the x variable. This section is aimed at showing why this strategy is likely to fail.

6.1. *Non-propagation of low Besov regularity by* (1.15)

This part elaborates on remarks originally made by DiPerna–Lions[13] (see pp. 7 and 8), (with additional details to be found in Lions' monograph,[24] pp. 150–152). These remarks show that the 3D Euler flow does not propagate $W^{1,p}$ estimates in the variable x for each $1 < p < \infty$. Here we prove a slightly more general result, with $W^{1,p}$ replaced by the Besov space $B_s^{p,\infty}$ for arbitrarily small values of $s > 0$. While the analogous statement with $B_s^{p,\infty}$ replaced by $W^{s,p}$ is most likely true — and in any case mentioned by DiPerna–Lions[13] — its proof would be somewhat more technical than the argument for $B_s^{p,\infty}$ presented below. On the other hand, establishing that a sequence of functions is bounded in $B_s^{p,\infty}$ is as natural a way of proving that this sequence is relatively compact in L^p_{loc} or as proving that it is bounded in $W^{s',p}$ for some $s' \in (0, s)$. Thus, we have chosen to show that the 3D Euler flow does not propagate $B_s^{p,\infty}$ regularity for small $s > 0$ as an indication of the difficulty in proving compactness in the space variables for families of solutions of the 3D incompressible Euler equation.

We first recall the definition of $B_s^{p,\infty}$ (see Stein,[28] pp. 150–159 where $B_s^{p,q}$ is denoted by $\Lambda_s^{p,q}$).

Definition 6.2. Let $p \in (1, +\infty)$ and $s \in (0, 1)$; the Besov $B_s^{p,\infty}$-norm is defined, for each smooth function with compact support, by

$$\|f\|_{B_s^{p,\infty}} = \|f\|_{L^p} + \sup_{z\neq 0} \frac{1}{|z|^s} \left(\int |f(x+z) - f(x)|^p \right)^{1/p} .$$

The space $B_s^{p,\infty}$ consists of all L^p functions such that $\|f\|_{B_s^{p,\infty}}$ is finite.

The next lemma contains some essential preparation for the non-propagation result on 3D Euler.

Lemma 6.8. *For all $p \geq 1$ and all $s \in (0, 1/p)$, there exist a sequence $(W_n)_n$ bounded in $B_s^{p,\infty}(\mathbf{T}^2)$, and a sequence $(V_n)_n$ of C^∞ functions bounded in $B_s^{p,\infty}(\mathbf{T})$ such that the sequence U_n defined by*

$$U_n : (x_1, x_2) \in \mathbf{T}^2 \mapsto U_n(x_1, x_2) = W_n(x_1 + V_n(x_2), x_2)$$

satisfies $\|U_n\|_{B_s^{p,\infty}(\mathbf{T}^2)} \to +\infty$ as $n \to +\infty$.

Proof. Pick $W(x_1, x_2) = W^1(x_1)W^2(x_2)$ with $W^1(x_1) = \sin(\pi x_1)$ and $W^2(x_2) = x_2^\beta$ with $\beta = s - \frac{1}{p}$. (In this discussion, $\mathbf{T}^2$ is identified with $[0, 1)^2$.) Pick then $\alpha = s/(\frac{1}{p} - s)$ and for each $n \geq 10$, define

$$V_n = \text{the indicator function of } \bigcup_{k=1}^{n} \left[\frac{2k-1}{n^{\alpha+1}}, \frac{2k}{n^{\alpha+1}} \right].$$

First, observe that, for each $z \in (0,1)$

$$\begin{aligned}
&\int_0^1 |W^2(x+z) - W^2(x)|^p \, dx \\
&\quad \le \int_0^{1-z} |(x+z)^\beta - x^\beta|^p \, dx + \int_0^z |y^\beta - (y-z+1)^\beta|^p \, dx \\
&\quad \le z^{\beta p+1} \int_0^{z^{-1}-1} |(u+1)^\beta - u^\beta|^p \, dx + \int_0^z y^{\beta p} \, dx \\
&\quad \le C z^{\beta p+1} \qquad (6.30)
\end{aligned}$$

which shows that $W^2 \in B_s^{p,\infty}(\mathbf{T})$ since $s = \beta + \frac{1}{p}$.

Next we observe that, since V_n takes its values in $\{0,1\}$,

$$\begin{aligned}
\int_0^1 |V_n(x+z) - V_n(x)|^p \, dx &\le \int_0^1 |V_n(x+z) - V_n(x)| \, dx \\
&\le 2 \int_0^1 |V_n(x)| \, dx = \frac{2}{n^\alpha}\,; \qquad (6.31)
\end{aligned}$$

further, if $0 < z < n^{-\alpha-1}$, then

$$|V_n(x+z) - V_n(x)| = \begin{cases} 1 & \text{if } x \bigcup_{l=1}^{2n} \left[\frac{l}{n^{\alpha+1}} - z,\, \frac{l}{n^{\alpha+1}} \right]\,; \\ 0 & \text{otherwise}\,. \end{cases}$$

Hence, for $0 < z < n^{-\alpha-1}$,

$$\int_0^1 |V_n(x+z) - V_n(x)|^p \, dx \le 2nz \le 2z^{-\frac{1}{\alpha+1}} z\,. \qquad (6.32)$$

By using (6.32) if $0 < z < n^{-\alpha-1}$ and (6.31) if $n^{-\alpha-1} < z < 1$, in both cases one arrives at the inequality

$$\int_0^1 |V_n(x+z) - V_n(x)|^p \, dx \le 2z^{\frac{\alpha}{\alpha+1}}$$

which implies that V_n is bounded in $B_s^{p,\infty}(\mathbf{T})$ since $s = \frac{\alpha}{p(\alpha+1)}$.

On the other hand, one has

$$\begin{aligned}
&|W(x_1 + V_n(x_2+z), x_2) - W(x_1 + V_n(x_2), x_2)| \\
&\quad = \left| \cos\left(\pi \frac{2x_1 + V_n(x_2+z) + V_n(x_2)}{2} \right) \sin\left(\pi \frac{V_n(x_2+z) - V_n(x_2)}{2} \right) \right| W^2(x_2) \\
&\quad \ge \left| \cos\left(\pi \frac{2x_1 + V_n(x_2+z) + V_n(x_2)}{2} \right) \right| |V_n(x_2+z) - V_n(x_2)| W^2(x_2)\,.
\end{aligned}$$

Pick $z_n = n^{-\alpha-1}$, then

$$\int_{\mathbf{T}^2} |W(x_1 + V_n(x_2 + z_n), x_2) - W(x_1 + V_n(x_2), x_2)|^p \, dx_1 \, dx_2$$

$$\geq \int_0^{2n^{-\alpha}} W^2(x_2) \int_0^1 \left| \cos\left(\pi \frac{2x_1 + V_n(x_2 + z) + V_n(x_2)}{2}\right)\right| \, dx_1 \, dx_2$$

$$= \int_0^1 |\cos(\pi y_1)| \, dy_1 \int_0^{2n^{-\alpha}} W^2(x_2) \, dx_2 = 2 \int_0^{2n^{-\alpha}} x_2^{\beta p} \, dx_2$$

$$= \frac{2}{\beta p + 1}(2n)^{-\alpha(\beta p + 1)} = \frac{2^{1-\alpha ps}}{ps} z_n^{\frac{\alpha ps}{\alpha+1}}$$

which shows that

$$\frac{1}{z_n^s}\left(\int_{\mathbf{T}^2} |W(x_1 + V_n(x_2 + z_n), x_2) - W(x_1 + V_n(x_2), x_2)|^p \, dx_1 \, dx_2\right)^{1/p}$$

$$\geq \frac{2^{\frac{1}{p} - \alpha s}}{(ps)^{1/p}} z_n^{-\frac{s}{\alpha+1}} \to +\infty \tag{6.33}$$

as $n \to +\infty$. Besides

$$\int_{\mathbf{T}^2} |W(x_1 + V_n(x_2 + z_n), x_2 + z_n) - W(x_1 + V_n(x_2 + z_n), x_2)|^p \, dx_1 \, dx_2$$

$$= \int_{\mathbf{T}} |W^2(x_2 + z_n) - W^2(x_2)|^p \int_{\mathbf{T}} |W_1(x_1 + V_n(x_2 + z_n)|^p \, dx_1 \, dx_2$$

$$\leq \int_{\mathbf{T}} |W^2(x_2 + z_n) - W^2(x_2)|^p \, dx_2 \leq C z_n^{ps}$$

by (6.30). This estimate and (6.33) eventually imply that

$$\frac{1}{z_n^s}\left(\int_{\mathbf{T}^2} |W(x_1 + V_n(x_2 + z_n), x_2 + z_n) - W(x_1 + V_n(x_2), x_2)|^p \, dx_1 \, dx_2\right)^{1/p} \to +\infty$$

as $n \to +\infty$, showing that U_n is not bounded in $B_s^{p,\infty}$. □

The implications of Lemma 6.8 on the 3D incompressible Euler equations are summarized in the next proposition — a variant of the remark due to DiPerna–Lions.[13]

Proposition 6.4. *Let $p \geq 1$, $s \in (0, 1/p)$ and $T > 0$. There exists a sequence of smooth divergence-free vector fields $(J_n^{in})_n$ that is uniformly bounded in $B_s^{p,\infty}(\mathbf{T}^2, \mathbb{R}^3)$ and such that the sequence (J_n) of solutions of the incompressible* 2D1/2 *Euler equations* (1.15) *with initial data (J_n^{in}) satisfies*

$$\|J_n(T, \cdot)\|_{B_s^{p,\infty}(\mathbf{T}^2, \mathbb{R}^3)} \to +\infty \quad \textit{as } n \to +\infty\,.$$

Since solutions of the 2D1/2 Euler equations (1.15) are also solutions of the 3D incompressible Euler equations, Proposition 6.4 implies the non-propagation of $B^{p,\infty}_s$ estimates in x for $s > 0$ arbitrarily small.

Proof of Proposition 6.4. Let $T > 0$ be a fixed positive time. Let $W \in B^{p,\infty}_s(\mathbf{T}^2)$ and the sequence $(V_n)_n$ bounded in $B^{p,\infty}_s(\mathbf{T})$ be chosen as in Lemma 6.8. Then consider

$$J_n^{in}(x) = \left(\frac{1}{T}V_n(x_2),\, 0,\, W(x_1, x_2)\right). \tag{6.34}$$

Clearly

$$\nabla_x \cdot J_n^{in} = 0$$

and

$$\|J_n^{in}\|_{B^{p,\infty}_s(\mathbf{T}^2,\mathbb{R}^3)} \le \frac{1}{T}\|V_n\|_{B^{p,\infty}_s(\mathbf{T})} + \|W\|_{B^{p,\infty}_s(\mathbf{T}^2)} \le C\,.$$

A simple computation shows that the solution of the 2D1/2 incompressible Euler equation (1.15) with initial data J_n^{in} is

$$J_n(t,x) = \left(\frac{1}{T}V_n(x_2),\, 0,\, W\left(x_1 + \frac{t}{T}V_n(x_2), x_2\right)\right).$$

In particular,

$$J_{n,3}(T,x) = U_n(x_1, x_2)\,.$$

with U_n defined as in Lemma 6.8 by

$$U_n : (x_1, x_2) \in \mathbf{T}^2 \mapsto U_n(x_1, x_2) = W(x_1 + V_n(x_2), x_2)\,.$$

By Lemma 6.8, $\|\, J_{n,3}(T,\cdot)\,\|_{B^{p,\infty}_s(\mathbf{T}^2)} \to +\infty$ as $n \to +\infty$ and so does $\|J_n(T,\cdot)\|_{B^{p,\infty}_s(\mathbf{T}^2,\mathbb{R}^3)}$. □

6.2. *Nonpropagation of weak regularity by* (1.1)

The Vlasov–Poisson system (1.1) shares with the 2D1/2 incompressible Euler equations the property stated in Proposition 6.4, i.e. the fact that $B^{p,\infty}_s$ regularity is not propagated for low values of s. Consider indeed particle distributions of the form

$$f_{n,\varepsilon}(t,x,v) = (1 + \varepsilon\nabla_x(J_n(t,x) \wedge b))\delta(v - J(_nt,x))\,,$$

where J_n solves (1.15) with initial data J_n^{in} as in (6.34). Since J_n represents a shear flow, $\nabla_x \Pi = 0$ in (1.15) and $f_{n,\varepsilon}$ is a measure-valued solution of (1.1) with

$$f_{n,\varepsilon}^{in}(x,v) = (1 + \varepsilon\nabla_x(J_n^{in}(x)) \wedge b))\delta(v - J_n^{in}(x))\,.$$

By Proposition 6.4 the families

$$\int f_{n,\varepsilon}(T,x,v)\,vdv = J_n(T,x)(1+\varepsilon\nabla_x(J_n(T,x)\wedge b))) \quad \text{and}$$

$$\nabla_x V_{n,\varepsilon}(T,x) = -J_n(T,x)\wedge b$$

are not uniformly bounded in $B_s^{p,\infty}(\mathbf{T}_x^2)$ as n runs through $\mathbb{N}^*$ and ε runs through $(0;1)$.

However, this class of examples is not satisfactory because it is based upon dealing with monokinetic distributions. The usual methods to study propagation of regularity in the Vlasov–Poisson system consist of estimating simultaneously derivatives in the x and v variables. Alternative strategies, using for instance compactness by velocity averaging, require that the family of distributions under consideration be at least equi-integrable in the v variable.

Actually we can prove a stronger result, which claims that weak regularity estimates cannot be propagated by (1.1).

Proposition 6.5. *Let $p\geq 1$, $s\in(0,1/p)$ $T>0$ and $K>0$. Consider the set*

$$\mathcal{W} = \left\{ f^{in} \quad \textit{smooth in } v \,\middle|\, \mathcal{E}^{in} + \left\| \int f^{in}\,vdv \right\|_{B_s^{p,\infty}(\mathbf{T}^3)} + \|\nabla_x V^{in}\|_{B_s^{p,\infty}(\mathbf{T}^3)} \leq K \right\}.$$

Then

$$\sup_{\substack{0\leq t\leq T\\ \varepsilon>0,\, f^{in}\in\mathcal{W}}} \left(\left\| \int f_\varepsilon(t)\,vdv \right\|_{B_s^{p,\infty}(\mathbf{T}^3)} + \|\nabla_x V_\varepsilon\|_{B_s^{p,\infty}(\mathbf{T}^3)} \right) = +\infty\,,$$

where f_ε denotes any solution of (1.1) *with initial data f^{in}.*

Proof. Assume that for each $T>0$ and $K>0$, there exists $C_{T,K}>0$ such that any solution of (1.1) verifying

$$\left\| \int f_\varepsilon^{in}\,vdv \right\|_{B_s^{p,\infty}(\mathbf{T}^3)} + \|\nabla_x V_\varepsilon^{in}\|_{B_s^{p,\infty}(\mathbf{T}^3)} \leq K$$

satisfies the uniform estimate

$$\sup_{t\in[0,T]} \left(\left\| \int f_\varepsilon(t)\,vdv \right\|_{B_s^{p,\infty}(\mathbf{T}^3)} + \|\nabla_x V_\varepsilon(t)\|_{B_s^{p,\infty}(\mathbf{T}^3)} \right) \leq C_{T,K}\,. \tag{6.35}$$

Let $f_{n,\varepsilon}$ be a solution of (1.1) with initial data

$$f_{n,\varepsilon}^{in}(x,v) = (1+\varepsilon\nabla_x\cdot(J_n^{in}(x)\wedge b))M(v-J_n^{in}(x))\,,$$

where M denotes the centered, reduced Gaussian distribution and J_n^{in} is a sequence of smooth vector fields satisfying the assumptions in Proposition 6.4 — for instance, pick J_n^{in} to be the initial data in (6.34).

By Theorem 2.2, for each $t \in [0,T]$, $\int f_{n,\varepsilon}\, vdv$ and $\nabla_x V_\varepsilon$ converge to J_n and $-J_n \wedge b$ respectively as $\varepsilon \to 0$, where J_n denotes the solution of (1.15) with initial data J_n^{in}. By Proposition 6.4,

$$\lim_{n\to\infty} \|J_n(T,\cdot)\|_{B_s^{p,\infty}(\mathbf{T}^3)} = +\infty\,;$$

but this is in contradiction with (6.35). Hence the initial assumption is false. □

As we insisted at the beginning of this section, this result indicates that taking limits in the nonlinear terms requires at least a new idea to obtain compactness properties.

Appendix A. Reduction of the Oscillation Operator R to Diagonal Form

Consider R the oscillation operator defined in Proposition 2.1 and denote by $\mathcal{R}$ the unitary group generated by R. Describing the asymptotic behavior of $\mathcal{R}(\frac{t}{\varepsilon})\Psi$ for $\Psi \in L^2(\mathbf{T}^3, \mathbb{R}^4)$ requires a rather good understanding of the structure of R and in particular a precise description of its kernel.

Proposition A.1. *Consider the bounded skew-adjoint operator defined on* $L^2(\mathbf{T}^3, \mathbb{R}^4)$ *by*

$$R : (j,\phi) \mapsto (j\wedge b + \nabla_x(-\Delta_x)^{-1/2}\phi,\ (-\Delta_x)^{-1/2}\nabla_x\cdot j)\,.$$

The orthogonal projections on eigenspaces of R are pseudo-differential operators of order 0. *In particular, the projection on the nullspace of R is defined by*

$$\mathcal{P} : (j,\phi) \mapsto \left(\frac{1}{2}\nabla_x{}'(-\Delta_x')^{-1/2}\int \phi\, dx_3 \wedge b + \frac{1}{2}P'\int j'\, dx_3, \int j_3\, dx_3\,,\right.$$
$$\left.\frac{1}{2}\int \phi\, dx_3 + \frac{1}{2}(-\Delta_x')^{-1/2}\nabla_x'\cdot\left(\int (j\wedge b)'\, dx_3\right)\right)$$

with the notations $\nabla_x' = (\partial_{x_1}, \partial_{x_2})$, $\Delta_x' = \partial^2_{x_1x_1} + \partial^2_{x_2x_2}$ and where P' is the $L^2_{x_1x_2}$-orthogonal projection on divergence-free vector fields depending on the two variables (x_1, x_2).

Proof. Denote by R_k ($k \in \mathbb{Z}^3$) the symbol of R

$$\forall\, k \in \mathbb{Z}^3\,, \quad R_k = \begin{pmatrix} 0 & 1 & 0 & i\kappa_1 \\ -1 & 0 & 0 & i\kappa_2 \\ 0 & 0 & 0 & i\kappa_3 \\ i\kappa_1 & i\kappa_2 & i\kappa_3 & 0 \end{pmatrix},$$

$$\kappa_\alpha = \frac{k_\alpha}{a_\alpha}\left[\left(\frac{k_1}{a_1}\right)^2 + \left(\frac{k_2}{a_2}\right)^2 + \left(\frac{k_3}{a_3}\right)^2\right]^{-1/2}\,.$$

We recall the notations

$$|k|^2 = \left(\frac{k_1}{a_1}\right)^2 + \left(\frac{k_2}{a_2}\right)^2 + \left(\frac{k_3}{a_3}\right)^2 , \quad k^* = \sqrt{1-\kappa_3^2} .$$

A straightforward computation shows that

$$R_k = P_k^* D_k P_k$$

with

$$D_k = \begin{pmatrix} i(1+k^*)^{1/2} & 0 & 0 & 0 \\ 0 & -i(1+k^*)^{1/2} & 0 & 0 \\ 0 & 0 & i(1-k^*)^{1/2} & 0 \\ 0 & 0 & 0 & -i(1-k^*)^{1/2} \end{pmatrix},$$

$$P_k = \frac{1}{2k^*} \begin{pmatrix} -i\kappa_2 + \sqrt{1+k^*}\kappa_1 & i\kappa_1 + \sqrt{1+k^*}\kappa_2 & k^*(1+k^*)^{-1/2}\kappa_3 & k^* \\ -i\kappa_2 - \sqrt{1+k^*}\kappa_1 & i\kappa_1 - \sqrt{1+k^*}\kappa_2 & -k^*(1+k^*)^{-1/2}\kappa_3 & k^* \\ -i\kappa_2 + \sqrt{1-k^*}\kappa_1 & i\kappa_1 + \sqrt{1-k^*}\kappa_2 & k^*(1-k^*)^{-1/2}\kappa_3 & k^* \\ -i\kappa_2 - \sqrt{1-k^*}\kappa_1 & i\kappa_1 - \sqrt{1-k^*}\kappa_2 & -k^*(1-k^*)^{-1/2}\kappa_3 & k^* \end{pmatrix},$$

with the convention $(1-k^*)^{-1/2}\kappa_3 = \sqrt{2}$ whenever $k_3 = 0$. Then P_k and $P_k^{-1} = P_k^*$ are order zero pseudo-differential operators that satisfy the uniform bound

$$\forall\, k \in \mathbb{Z}^3 , \quad \|P_k\|_{L^\infty} \|P_k^{-1}\|_{L^\infty} \le 2 .$$

It remains to obtain a precise description of the nullspace of R. It is easily seen that

$$\lambda_\eta(k) = 0 \Leftrightarrow \eta \in \{3,4\} \quad \text{and} \quad k_3 = 0 .$$

Then $\mathcal{P}$ is given by its symbol

$$\forall\, k \in \mathbb{Z}^3 , \quad \mathcal{P}_k = \begin{cases} P_k^{-1}(A_3 + A_4)P_k & \text{if } k_3 = 0 \\ 0 & \text{if } k_3 \neq 0 . \end{cases}$$

with the notations of Sec. 5. For k such that $k_3 = 0$, we compute

$$\mathcal{P}_k = \frac{1}{4} \begin{pmatrix} 0 & 0 & i\kappa_2 & i\kappa_2 \\ 0 & 0 & -i\kappa_1 & -i\kappa_1 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i\kappa_2 & i\kappa_1 & \sqrt{2} & 1 \\ -i\kappa_2 & i\kappa_1 & -\sqrt{2} & 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} \kappa_2^2 & -\kappa_1\kappa_2 & 0 & i\kappa_2 \\ -\kappa_1\kappa_2 & \kappa_1^2 & 0 & -i\kappa_1 \\ 0 & 0 & 2 & 0 \\ -i\kappa_2 & i\kappa_1 & 0 & 1 \end{pmatrix}$$

which is the expected symbol for the projection $\mathcal{P}$ on the nullspace of R. □

Appendix B. Small Divisor Estimate

In order to describe the coupling of the various oscillating components by the nonlinear terms, a basic tool is the study of the resonances, i.e. of solutions $(l, m, \eta) \in (\mathbb{Z}^3)^2 \times [[1,4]]^3$ of the dispersion equation

$$\omega_\eta(l+m, l, m) = \lambda_{\eta_1}(l+m) - \lambda_{\eta_2}(l) - \lambda_{\eta_3}(m) = 0\,,$$

where $(i\lambda_j(k))_{j\in[[1,4]]}$ denote the eigenvalues of R_k for all $k \in \mathbb{Z}^3$. These eigenvalues are the roots of a polynomial with polynomial coefficients in the variables k_1/a_1, k_2/a_2 and k_3/a_3, implying the following small divisor estimate.[16]

Proposition B.1. *Consider $P(l,m)$, a polynomial in a_3^{-1} with coefficients that are polynomials in l, m. Then, there exist $\mathcal{A} \subset \mathbb{R}_*^+$ of Lebesgue measure zero and $\Omega \subset \mathbb{Z}^6$ such that*

$$\forall\,(l,m) \in \Omega\,, \quad P(l,m) \equiv 0$$

$$\forall\, a_3 \in \mathbb{R}_+^* \setminus \mathcal{A},\ \exists\,(C,s) \text{ s.t. } \forall\,(l,m) \in \mathbb{Z}^6 \setminus \Omega,\ |P(l,m)|^{-1} \le C(1+|l|)^s(1+|m|)^s\,.$$

Appendix C. Symmetry Properties of the Bilinear Operator $\mathcal{B}$

In order to determine the structure of the limiting equation, we use the symmetry properties of the bilinear operator $\mathcal{B}$ summarized below.

Proposition C.1. *For all $l, m \in \mathbb{Z}^3$, define $B_{l,m}$ by*

$$B_{l,m}[\Psi_l\Psi_m] = \frac{1}{2}\left(i\langle\Psi_l, m\rangle\Psi'_m + i\langle\Psi_m, l\rangle\Psi'_l,\ -i\left\langle\frac{m+l}{|m+l|},\ |m|\Psi_{m,4}\Psi_l + |l|\Psi_{l,4}\Psi_m\right\rangle\right)$$

with the notation $\langle\Psi, k\rangle = \Psi_1\frac{k_1}{a_1} + \Psi_2\frac{k_2}{a_2} + \Psi_3\frac{k_3}{a_3}$. For each $k \in \mathbb{Z}^3$, define R_k by

$$R_k\Psi_k = \left(\Psi_{k,2} + i\frac{k_1}{a_1|k|}\Psi_{k,4},\ -\Psi_{k,1} + i\frac{k_2}{a_2|k|}\Psi_{k,4},\ i\frac{k_3}{a_3|k|}\Psi_{k,4},\ i\langle\Psi,k\rangle\right).$$

Denote by $i\lambda_j(k)$ its eigenvalues ordered as in (5.8) *and by P_k the transition matrix*

$$R_k = P_k^{-1}\left(\sum_{j=1}^4 i\lambda_j(k)A_j\right)P_k\,.$$

Then, for each Ψ and each $k \in \mathbb{Z}^2 \times \{0\}$,

$$(P_k^{-1}(A_3+A_4)P_k)\sum_{\substack{l+m=k\\ \lambda_{\eta_2}(l)+\lambda_{\eta_3}(m)=0}} B_{l,m}[(P_l^{-1}A_{\eta_2}P_l\mathcal{F}_l\Psi),(P_m^{-1}A_{\eta_3}P_m\mathcal{F}_m\Psi)]$$

$$= (P_k^{-1}(A_3+A_4)P_k)\sum_{\substack{l+m=k\\ \eta_2,\eta_3\in\{3,4\}}} B_{l,m}[(P_l^{-1}A_{\eta_2}P_l\mathcal{F}_l\Psi),(P_m^{-1}A_{\eta_3}P_m\mathcal{F}_m\Psi)]\,.$$

Proof. For each $k \in \mathbb{Z}^2 \times \{0\}$, the set of (l, m, η_2, η_3)'s satisfying $l + m = k$ and $\lambda_{\eta_2}(l) + \lambda_{\eta_3}(m) = 0$ is decomposed as follows

- $\{l_3 = m_3 = 0$ and $\eta_2, \eta_3 \in \{3,4\}\}$,
- $\{l_3 = m_3 = 0$ and $\{\eta_2, \eta_3\} = \{1,2\}\}$,
- $\{l_3 = -m_3 \neq 0$ and $\{\eta_2, \eta_3\} = \{1,2\}\}$,
- $\{l_3 = -m_3 \neq 0$ and $\{\eta_2, \eta_3\} = \{3,4\}\}$.

Then, in order to prove Proposition C.1, it is enough to see that in the last three cases

$$(P_k^{-1}(A_3 + A_4)P_k)B_{l,m}[P_l^{-1}V_{\eta_2}, P_m^{-1}V_{\eta_3}] = 0\,.$$

In these three cases, $\lambda = -\lambda_{\eta_2}(l) = \lambda_{\eta_3}(m) \neq 0$ and $l_3 = -m_3$, so that in particular $l^* = m^*$ and $|l| = |m|$. Hence, with the notations as in the proof of Proposition A.1,

$$\begin{aligned}\langle P_m^{-1}V_{\eta_3},\, l\rangle &= \frac{iM_2 + \lambda M_1}{2m^*}L_1|l| + \frac{-iM_1 + \lambda M_2}{2m^*}L_2|l| + \frac{M_3}{2\lambda}L_3|l| \\ &= -\langle P_l^{-1}V_{\eta_2},\, m\rangle\,.\end{aligned}$$

and

$$\langle P_m^{-1}V_{\eta_3},\, m\rangle = \lambda\frac{m^*|m|}{2} + \frac{M_3^2}{2\lambda}|m| = -\langle P_l^{-1}V_{\eta_2},\, l\rangle$$

so that

$$B_{l,m}[P_l^{-1}V_{\eta_2}, P_m^{-1}V_{\eta_3}] = i\frac{\langle P_l^{-1}V_{\eta_2}, m\rangle}{2m^*}\begin{pmatrix} i(M_2 - L_2) + \lambda(M_1 + L_1) \\ -i(M_1 - L_1) + \lambda(M_2 + L_2) \\ 0 \\ 0 \end{pmatrix},$$

multiplying by $P_k^{-1}(A_3 + A_4)P_k = \mathcal{P}_k$, we obtain

$$(P_k^{-1}(A_3 + A_4)P_k)B_{l,m}[P_l^{-1}V_{\eta_2}, P_m^{-1}V_{\eta_3}] = 0$$

which is the result expected. □

References

1. A. A. Arsen'ev, *Existence in the large of a weak solution of Vlasov's system of equations*, *Z. Vyčisl. Mat. Mat. Fiz.* **15** (1975) 136–147.
2. K. Asano and S. Ukai, *On the Vlasov–Poisson limit of the Vlasov–Maxwell equation. Pattern and waves*, *Qualitative analysis of nonlinear differential equations*, *Stud. Math. Appl.* **18** (1986) 369–383.
3. A. Babin, A. Mahalov and B. Nicolaenko, *Global splitting, integrability and regularity of 3D Euler and Navier–Stokes equations for uniformly rotating fluids*, *Euro. J. Mech.* **15** (1996) 291–300.
4. A. Babin, A. Mahalov and B. Nicolaenko, *Regularity and integrability of 3D Euler and Navier–Stokes equations for rotating fluids*, *Asympt. Anal.* **15** (1997) 103–150.
5. F. Bouchut, F. Golse and M. Pulvirenti, *Kinetic Equations and Asymptotic Theory*, *Series in Applied Math. No. 4* (Gauthier-Villars, 2000).
6. Y. Brenier, *Convergence of the Vlasov–Poisson system to the incompressible Euler equations*, *Comm. P.D.E.* **25** (2000) 737–754.

7. Y. Brenier, *A remark on the low Rossby number theory*, Internal Report of the Newton Institute (1996).
8. J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier, *Fluids with anisotropic viscosity*, *Special issue for R. Temam's 60th birthday*, *Math. Model. Numer. Anal.* **34** (2000) 315–335.
9. P. Degond, *Local existence of solutions of the Vlasov–Maxwell equations and convergence to the Vlasov–Poisson equations for infinite light velocity*, *Math. Methods Appl. Sci.* **8** (1986) 533–558.
10. J.-M. Delort, *Existence de nappes de tourbillon en dimension deux*, *J. Amer. Math. Soc.* **4** (1991), 553–586.
11. B. Desjardins and E. Grenier, *Derivation of quasi-geostrophic potential vorticity equations*, *Adv. Differential Equations* **3** (1998) 715–752.
12. R. DiPerna, *Convergence of the viscosity method for isentropic gas dynamics*, *Comm. Math. Phys.* **91** (1983) 1–30.
13. R. DiPerna and P.-L. Lions, *Equations différentielles ordinaires et équations de transport avec des coefficients irréguliers*, *Séminaire sur les Equations aux Dérivées Partielles*, *1988–1989*, *Exp. No. XIV*, Ecole Polytech., Palaiseau, 1989.
14. E. Frénod and E. Sonnendrücker, *Homogenization of the Vlasov equation and the Vlasov–Poisson system with a strong external magnetic field*, *Asympt. Anal.* **18** (1998) 193–213.
15. E. Frénod and E. Sonnendrücker, *Long time behavior of the two-dimensional Vlasov equation with a strong external magnetic field*, *Math. Models Methods Appl. Sci.* **10** (2000) 539–553.
16. I. Gallagher, *Applications of Schochet's methods to parabolic equations*, *J. Math. Pures Appl.* **77** (1998) 989–1054.
17. I. Gallagher, *Asymptotic of the solutions of hyperbolic equations with a skew-symmetric perturbation*, *J. Differential Equations* **150** (1998) 363–384.
18. F. Golse and L. Saint-Raymond, *The guiding center approximation for the Vlasov–Poisson System*, *J. Math. Pures Appl.* **78** (1999) 791–817.
19. H. Grad, *The guiding center plasma*, *Proc. Symp. Appl. Math.* **XVIII** (1967) 162–248.
20. E. Grenier, *Pseudo-differential energy estimates of singular perturbations*, *Comm. Pure Appl. Math.* **50** (1997) 821–865.
21. E. Grenier, *Oscillations in quasineutral plasmas*, *Comm. P.D.E.* **21** (1996) 363–394.
22. E. Grenier, *On the stability of boundary layers of incompressible Euler equations*, *J. Differential Equations* **164** (2000) 180–222.
23. E. M. Lifshitz and L. P. Pitaevski, *Course of Theoretical Physics*, *Vol. 10* (Pergamon Press, N.Y. 1981).
24. P.-L. Lions, *Mathematical Topics in Fluid Mechanics. Vol. 1*: *Incompressible Models*, *Oxford Lecture Series in Mathematics and its Applications* (Clarendon Press, 1996).
25. P.-L. Lions and N. Masmoudi, *From Boltzmann equations to the Stokes and Euler equations II*, *Arch. Rat. Mech. Anal.* **158** (2001) 195–211.
26. L. Saint-Raymond, *The gyrokinetic approximation for the Vlasov–Poisson system*, *Math. Models Methods Appl. Sci.* **10** (2000) 1305–1332.
27. S. Schochet, *Fast singular limits of hyperbolic PDEs*, *J. Differential Equations* **114** (1994) 476–512.
28. E. Stein, *Singular Integrals and Differentiability Properties of Functions* (Princeton Univ. Press, 1970).
29. S. Ukai and T. Okabe, *On classical solutions in the large in time of two-dimensional Vlasov's equation*, *Osaka J. Math.* **15** (1978) 245–261.