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THE VLASOV–POISSON SYSTEM WITH STRONG MAGNETIC FIELD

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ABSTRACT. – This paper establishes various asymptotic limits of the Vlasov–Poisson equation with strong external magnetic field, some of which were announced in [14]. The so-called "guiding center approximation" is proved in the 2D case with a constant magnetic field orthogonal to the plane of motion, in various situations (noncollisional or weakly collisional). The 3D case is studied on the time scale of the motion along the lines of the magnetic field, much shorter than that of the guiding center motion. We discuss in particular the effect of nonconstant external magnetic fields. © Elsevier, Paris

1. Introduction

Consider a plasma consisting of light particles of mass m with individual electric charge q and of heavy particles of mass $m^* \gg m$ with individual electric charge $-q$. For simplicity, we assume that the heavy particles distribution is a uniform Maxwellian (even if collisions are taken into account, the effect on heavy particles of collisions with light particles is neglected). We call E the self-consistent electric field and $f \equiv f(t,x,v)$ the number density of the light particles. As usual, x is the position variable, v the velocity variable, t the time, and saying that f is the number density means that in an infinitesimal volume $\mathrm{d}x\,\mathrm{d}v$ of the phase space centered at (x,v), one can find, at time t, approximately $f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v$ particles. We assume in this paper that the characteristic speed of these particles is small compared to the speed of light c, so that the Maxwell equation for the electro-magnetic field reduces to the electrostatic approximation, i.e., E is governed by the Poisson equation [6]. However, we assume that some external magnetic field B is applied to this gas of particles, so that the Vlasov equation reads:

$$\partial_t f + v\cdot\nabla_x f + \frac{q}{m}\left(E + \frac{v}{c}\wedge B\right)\cdot\nabla_v f = 0, \tag{1.1}$$

while the Poisson equation is

$$E = -\nabla_x V, \qquad -\varepsilon_0\Delta_x V = q\int_{\mathbf{R}^D} f\,\mathrm{d}v - q\iint_{\mathbf{T}^D\times\mathbf{R}^D} f\,\mathrm{d}x\,\mathrm{d}v, \tag{1.2}$$

$$f(0,x,v) = f^{\mathrm{in}}(x,v), \tag{1.3}$$

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ε_0 denoting as usual the dielectric permittivity of the vacuum. For simplicity, we assume periodicity in the space variable: $(x, v) \in \mathbf{T}^D \times \mathbf{R}^D$. Here we set $\mathbf{T}^D = \mathbf{R}^D/\mathbf{Z}^D$, equipped with the measure $\mathrm{d}x$ identified with the restriction to $[0, 1[^D$ of the Lebesgue measure of $\mathbf{R}^D$.

The subject matter of this paper is the study of the Vlasov–Poisson system (1.1)–(1.3) in the limit as the intensity of the magnetic field $|B|$ tends to infinity. Studying the effect of strong magnetic fields on plasmas is of considerable importance for example in numerical simulations of tokamaks. An introduction to the modelling of plasmas in strong magnetic fields can be found in [16] and in [12]. A first picture of the effect of a strong external magnetic field in the Vlasov equation (1.1) can be seen from the following:

Heuristic argument. If E and B are constant fields, the motion of each individual charged particle in the electromagnetic field is given by:

$$x' = v, \qquad v' = \frac{qE}{m} + v \wedge \frac{qB}{cm}, \tag{1.4}$$

so that, after projecting v on the B direction and on the plane orthogonal to B, one sees that:

$$x_\parallel(t) = x_\parallel(0) + t v_\parallel(0) + \frac{t^2}{2}\frac{qE\cdot B}{m|B|}, \tag{1.5a}$$

$$x_\perp(t) = x_\perp(0) + ct\frac{E\wedge B}{|B|^2} + \mathrm{O}\left(\frac{mc}{q|B|}\right) + \mathrm{O}\left(\frac{c|E|}{|B|}\right) \tag{1.5b}$$

(where the subscript $\parallel$ denotes the projection on the B direction while the $\perp$ subscript designates that on the plane orthogonal to B). Hence one expects that, as the intensity of the magnetic field tends to infinity, particles should be advected:

- with acceleration $qE\cdot B/m|B|$ in the direction of B;
- with the macroscopic velocity $cE\wedge B/|B|^2$ (henceforth called the drift velocity) on the plane orthogonal to B.

In other words, particles move on helices with axis the direction of the magnetic field and radius the so-called Larmor radius. The motion of the axis, referred to as "guiding center" dynamics, is slow if measured in units of time defined by the reciprocal Larmor frequency (see below).

Also, since the drift velocity is macroscopic, one should expect that, to leading order, the limiting model of (1.1)–(1.3) for a strong external magnetic field B be kinetic in the direction of the magnetic field and macroscopic (i.e., hydrodynamic) on a slower time scale in the plane orthogonal to the magnetic field.

Scalings. Various time scales appear in the problem (1.1)–(1.3):

(a) $T_\mathrm{c} = mc/q|B|$, the reciprocal cyclotron frequency (cf. [16, §52]);

(b) $T_\mathrm{p} = mu/q[E]$, the reciprocal plasma frequency (cf. [16, §31]), where $[E]$ is the order of magnitude of the electric field, u being given by $\varepsilon_0[E]^2 = m[\rho]u^2$ where $[\rho]$ is the average macroscopic density;

(c) T_o, the macroscopic (observation) time scale. A first situation is the case where

$$T_\mathrm{o} \simeq T_\mathrm{p}, \qquad \frac{T_\mathrm{c}}{T_\mathrm{p}} = \varepsilon \ll 1. \tag{1.6}$$

In this case, the Vlasov–Poisson system can be put in dimensionless variables (which we denote with the same letters as the original variables with a slight abuse of notations):

(1.7a) $$\partial_t f_\varepsilon + v\cdot\nabla_x f_\varepsilon + E_\varepsilon\cdot\nabla_v f_\varepsilon + \frac{1}{\varepsilon} v\wedge B\cdot\nabla_v f_\varepsilon = 0,$$

(1.7b) $$E_\varepsilon = -\nabla_x V_\varepsilon,\quad -\Delta_x V_\varepsilon = \rho_\varepsilon - \overline{\rho}_\varepsilon,$$

(1.7c) $$f_\varepsilon(0,x,v) = f_\varepsilon^{\mathrm{in}}(x,v),$$

with the notations

(1.7d) $$\rho_\varepsilon = \int_{\mathbf{R}^3} f_\varepsilon\,\mathrm{d}v,\qquad \overline{\rho}_\varepsilon = \int_{\mathbf{T}^3} \rho_\varepsilon\,\mathrm{d}x,$$

the problem being posed for $(x,v)\in\mathbf{T}^3\times\mathbf{R}^3$ and $t\geqslant 0$. A detailed mathematical study of (1.7a–c) can be found in [10,11] mostly in the case of a constant magnetic field. Two cases of nonconstant magnetic fields will be considered in the present paper; they may give rise to some nontrivial geometric effects.

In the case of a constant magnetic field, the heuristic argument above indicates that in order to observe the drift velocity, one should consider exclusively the motion on the plane orthogonal to the magnetic field on a slower time scale than T_{p}, i.e., the 2D problem (1.1)–(1.3). This second situation corresponds to

(1.8) $$\varepsilon = \frac{T_{\mathrm{c}}}{T_{\mathrm{p}}} = \frac{T_{\mathrm{p}}}{T_{\mathrm{o}}} \ll 1.$$

Under this scaling assumption, the Vlasov equation can be recast in dimensionless variables, as follows:

(1.9) $$\varepsilon\partial_t f_\varepsilon + v\cdot\nabla_x f_\varepsilon + E_\varepsilon\cdot\nabla_v f_\varepsilon + \frac{1}{\varepsilon} v^\perp\cdot\nabla_v f_\varepsilon = 0,\quad t\in\mathbf{R}_+^*,\ (x,v)\in\mathbf{T}^2\times\mathbf{R}^2,$$

with $(v_1,v_2)^\perp = (-v_2,v_1)$. The Vlasov equation is supplemented with (1.7b,c) with the notations

(1.7d′) $$\rho_\varepsilon = \int_{\mathbf{R}^2} f_\varepsilon\,\mathrm{d}v,\qquad \overline{\rho}_\varepsilon = \int_{\mathbf{T}^2} \rho_\varepsilon\,\mathrm{d}x.$$

Finally, it may also be relevant to take into account collisions with the background gas of heavy particles the effect of which is to slow down the lighter particles. A very crude model for such collisions with a "thermal bath" is a Fokker–Planck linear operator

(1.10) $$Lf(v) = \sigma\Delta_v f + \nabla_v\cdot\big(b(v)f\big),$$

where $\sigma\geqslant 0$ is the diffusion constant and $b\equiv b(v)$ a friction term the form of which will be discussed later.

In the collisional case, two more time scales are involved:

(d) $T_{\mathrm{f}} = u/[b]$, the characteristic time scale of the friction effect, where $[b]$ is the average intensity of the vector fields b,

(e) $T_{\mathrm{d}} = u^2/\sigma$, the characteristic time scale of diffusion in the velocity space.

The two following conditions should be added to (1.8) in the collisional case;

(1.11) $$T_{\mathrm{o}} = \mathrm{O}(T_{\mathrm{d}}),\qquad \varepsilon \ll \frac{T_{\mathrm{f}}}{T_{\mathrm{o}}} = \beta^{-1} \ll 1.$$

Later, we shall give a more precise condition on β and relate it to ε. We can already say that observing the drift velocity is possible only if the friction on the background neutral particles is a weak effect occurring at high velocities only.

In which case, the 2D Vlasov equation reads:

$$\partial_t f_\varepsilon + \frac{1}{\varepsilon}(v \cdot \nabla_x f_\varepsilon + E_\varepsilon \cdot \nabla_v f_\varepsilon) + \frac{1}{\varepsilon^2} v^\perp \cdot \nabla_v f_\varepsilon = L_\varepsilon(f_\varepsilon), \quad t \in \mathbf{R}_+^*,\ (x, v) \in \mathbf{T}^2 \times \mathbf{R}^2, \tag{1.12}$$

with the notation

$$L_\varepsilon(f_\varepsilon) = \sigma_\varepsilon \Delta_v f_\varepsilon + \beta(\varepsilon) \nabla_v \cdot \big(b(v) f_\varepsilon\big). \tag{1.13}$$

2. Main results

We shall not dwell on the existence theory for all the models presented in Section 1. In the noncollisional case, the theory of global weak solutions of the Vlasov–Poisson system is due to Arsen'ev [1] and can be adapted without difficulty to (1.1)–(1.3) with a given, smooth magnetic field. As regards the existence theory, the 2D collisional model (1.12) is very close to the Fokker–Planck model considered by Degond in [7] and can be treated by essentially the same method.

2.A. The 2D results

This subsection is based on the scaling (1.8), except in the collisional case (i.e., for Theorem E below) which uses both (1.8) and (1.11).

THEOREM A. – *Let* $f^{\text{in}} \in L^\infty \cap L^1(\mathbf{T}^2 \times \mathbf{R}^2)$ *satisfy*

$$f^{\text{in}} \geqslant 0 \text{ a.e. and } \mathcal{E}\big(f^{\text{in}}\big) = \iint \tfrac{1}{2}|v|^2 f^{\text{in}}(x, v)\,\mathrm{d}x\,\mathrm{d}v + \int \tfrac{1}{2}\big|E^{\text{in}}(x)\big|^2\,\mathrm{d}x < +\infty. \tag{2.1}$$

Let $(f_\varepsilon)_{\varepsilon>0}$ *be a family of weak solutions of* (1.9), (1.7b,c). *Then, there exists*:

(a) *a subsequence of* $(f_\varepsilon)_{\varepsilon>0}$ *(still denoted by* (f_ε)*)*;

(b) $F \in L^\infty(\mathbf{R}_+; L^\infty \cap L^1(\mathbf{T}^2 \times \mathbf{R}_+))$ *such that*

$$f_\varepsilon \to F\big(t, x, |v|\big) \quad \text{in } L^\infty\big(\mathbf{R}_+ \times \mathbf{T}^2 \times \mathbf{R}^2\big) \text{ weak-* as } \varepsilon \to 0; \tag{2.2}$$

(c) *a defect measure* $\nu \in L^\infty(\mathbf{R}_+; \mathcal{M}_+(\mathbf{T}^2 \times S^1))$ [3] *such that, for any function* $\phi \in C^0(S^1)$

$$\int_{\mathbf{R}^2} \big[f_\varepsilon(t, x, v) - F\big(t, x, |v|\big)\big]\phi\left(\frac{v}{|v|}\right)|v|^2\,\mathrm{d}v \to \int_{S^1} \phi(\theta)\,\mathrm{d}\nu(\theta) \quad \text{as } \varepsilon \to 0 \tag{2.3}$$

in the sense of distributions. Moreover, the limiting density

$$\rho(t, x) = \int_{\mathbf{R}^2} F\big(t, x, |v|\big)\,\mathrm{d}v \tag{2.4}$$

satisfies: [4]

[3] We denote by $\mathcal{M}(X)$ the set of bounded measures on X and by $\mathcal{M}_+(X)$ its positive cone.

[4] $\partial_1 = \partial/\partial x_1$ and $\partial_2 = \partial/\partial x_2$.

$$\partial_t \rho + \nabla_x \cdot \big(\rho E^{\perp}\big) = \big(\partial_1^2 - \partial_2^2\big) \int\limits_{S^1} \theta_1\theta_2 \,\mathrm{d}\nu(\theta) + \partial_1\partial_2 \int\limits_{S^1} \big(\theta_2^2 - \theta_1^2\big)\,\mathrm{d}\nu(\theta), \tag{2.5}$$

$$E = -\nabla_x V, \qquad -\Delta_x V = \rho - \overline{\rho}, \tag{2.6}$$

$$\rho(0,x) = \int\limits_{\mathbf{R}^2} f^{\mathrm{in}}\,\mathrm{d}v, \qquad \overline{\rho} = \int\limits_{\mathbf{T}^2} \rho(0,x)\,\mathrm{d}x. \tag{2.7}$$

Notice that, without the right-hand side involving the defect measure, Eq. (2.5) is the vorticity formulation of the 2D incompressible Euler equation. Indeed, $\rho - \overline{\rho}$ is analogous to the vorticity field (which is scalar in 2D), $E^{\perp}$ is analogous to the velocity field while V is the corresponding stream function (up to a sign).

In various physical situations, the constraint that the f_ε should be uniformly bounded in L^∞ is not relevant (for example, it might be interesting to use the guiding center approximation in cases where the distribution f_ε is of the form

$$f_\varepsilon(t,x,v) = \rho_\varepsilon(t,x)\delta\big(v - u_\varepsilon(t,x)\big), \tag{2.8}$$

for some macroscopic density ρ_ε and bulk velocity $u_\varepsilon(t,x)$). While we have not been able to directly deal with measure solutions of the Vlasov equation, we can however treat the case of initial data converging to the form (2.8) – or more complicated variants of it – as $\varepsilon \to 0$. Specifically we have the:

THEOREM B. – *Let $f_\varepsilon^{\mathrm{in}}$ be a family of functions in $L^\infty \cap L^1(\mathbf{T}^2 \times \mathbf{R}^2)$ satisfying*

$$f_\varepsilon^{\mathrm{in}} \geqslant 0 \text{ a.e.}, \qquad \lim_{\varepsilon\to 0^+} \varepsilon \big\| f_\varepsilon^{\mathrm{in}} \big\|_{L^\infty_{x,v}} = 0 \text{ and } \sup_\varepsilon \big[\big\| f_\varepsilon^{\mathrm{in}} \big\|_{L^1_{x,v}} + \mathcal{E}\big(f_\varepsilon^{\mathrm{in}}\big)\big] < +\infty. \tag{2.9}$$

Let $(f_\varepsilon)_{\varepsilon>0}$ be a family of weak solutions of (1.9), (1.7b) *with initial data*

$$f_\varepsilon(0,x,v) = f_\varepsilon^{\mathrm{in}}(x,v). \tag{2.10}$$

Then, conclusions (a)–(c) *as well as* (2.2)–(2.7) *in Theorem* A *hold, with the only difference that $F \in L^\infty(\mathbf{R}_+; \mathcal{M}_+(\mathbf{T}^2 \times \mathbf{R}_+))$, that the convergence holds in $L^\infty(\mathbf{R}_+; \mathcal{M}_+(\mathbf{T}^2 \times \mathbf{R}_+))$ weak-*, and that the notation $\nabla_x \cdot (\rho E^{\perp})$ designates the second order distribution*:

$$\partial_1\partial_2\big(E_1^2 - E_2^2\big) + \big(\partial_1^2 - \partial_2^2\big)(E_1E_2).$$

Although Theorem B seems to be a harmless modification of Theorem A, one should keep in mind that it uses a highly nontrivial compactness argument which is useless in the proof of Theorem A, namely the key theorem in Delort's proof [8] of global existence of weak solutions to the 2D Euler in the case of vortex sheets. We recall Delort's theorem in Section 3 below and refer to [8] for its proof.

The appearance of a defect measure in the right-hand side of (2.5) is a definitely unpleasant feature of the guiding center approximation. It is fairly easy to construct sequences of stationary solutions of (1.9), (1.7b,c) with nonzero defect measures. In fact, more is true: it is likely that the part of the defect measure coming from velocities of order $1/\varepsilon$ and higher evolves according to the free dynamics corresponding to the electric field generated by particles slower that $1/\varepsilon^\alpha$, $0 \leqslant \alpha < 1$. In the next proposition, we substantiate this picture by studying the case of an initial distribution of particles with velocities of order $1/\varepsilon$.

PROPOSITION C. – *Let* $(f_\varepsilon^{\mathrm{in}})$ *be any family of nonnegative functions in* $C_c^\infty(\mathbf{T}^2 \times \mathbf{R}^2)$ *such that as* $\varepsilon \to 0$,

$$\iint_{\mathbf{T}^2\times\mathbf{R}^2} |v|^2 f_\varepsilon^{\mathrm{in}}\,\mathrm{d}x\,\mathrm{d}v \to 1, \qquad \|f_\varepsilon^{\mathrm{in}}\|_{L^\infty_{x,v}} = \mathrm{O}(\varepsilon^3). \tag{2.11}$$

Let $(f_\varepsilon)_{\varepsilon>0}$ *be the family of solutions of* (1.9), (1.7b) *with initial data given by* (2.11). *There does not exist a subsequence of* $(f_\varepsilon)_{\varepsilon>0}$ *for which the defect measure* ν *predicted by Theorem* A(c) *vanishes.*

Actually, in the previous example, the defect measure is always positive, but is also invariant under all transformations $(t,x,\theta) \mapsto (t,x,\mathcal{R}\theta)$ where $\mathcal{R}$ runs through the group of orthogonal transformations of $\mathbf{R}^2$. Therefore, both terms

$$\int_{S^1} \theta_1\theta_2\,\mathrm{d}\nu(\theta) \quad \text{and} \quad \int_{S^1} (\theta_1^2 - \theta_2^2)\,\mathrm{d}\nu(\theta)$$

vanish, as can be seen by a straightforward change of variables. Such rotation invariant defect measures do not affect Eq. (2.5) governing the limiting macroscopic density ρ. It is therefore a natural question to find criteria ensuring that the defect measure is rotation invariant. Theorem D below gives one such sufficient condition. Unfortunately, this condition cannot be directly verified on the initial data; however, the second part of Theorem D shows that this sufficient condition is not far from being verified for general initial data.

THEOREM D. – *Let* $f^{\mathrm{in}} \in L^\infty \cap L^1(\mathbf{T}^2 \times \mathbf{R}^2)$ *satisfy* (2.1); *let* $(f_\varepsilon)_{\varepsilon>0}$ *be a family of weak solutions of* (1.9), (1.7b,c).

(a) *Assume that there exists* $\alpha > 2$ *such that*:

$$\iint |v|^\alpha f^{\mathrm{in}}\,\mathrm{d}x\,\mathrm{d}v < +\infty. \tag{2.12}$$

The defect measure ν *predicted by Theorem* A *is invariant under all transformations of the form* $(t,x,\theta) \mapsto (t,x,\mathcal{R}\theta)$ *where* $\mathcal{R}$ *runs through the group of orthogonal transformations of* $\mathbf{R}^2$ *if and only if, as* $\varepsilon \to 0$,

$$\varepsilon\nabla_x \cdot \int v|v|^2 f_\varepsilon\,\mathrm{d}v \to 0 \quad \textit{in } \mathcal{D}'(\mathbf{R}_+^* \times \mathbf{T}^2).$$

(*In particular,* ν *is rotation invariant if*

$$\int_0^T \iint |v|^3 f_\varepsilon\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v = \mathrm{o}\left(\frac{1}{\varepsilon}\right) \tag{2.13}$$

for all $T > 0$).

(b) *Assume that*

$$\iint |v|^3 f^{\mathrm{in}}\,\mathrm{d}x\,\mathrm{d}v < +\infty.$$

Then, for all $T > 0$,

$$(2.13') \qquad \int_0^T \iint |v|^3 f_\varepsilon \,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v = \mathrm{O}\left(\frac{\sqrt{|\log\varepsilon|}}{\varepsilon}\right).$$

Estimate (2.13′) shows that (2.13) does not fail by much if it does; more precisely it indicates that the possible loss of energy and effective appearance of a defect measure in the right-hand side of (2.5) depends on the behavior of the particles that have velocities of order $1/\varepsilon$.

Another situation where no defect measure appears in the limiting process is the collisional model (1.12).

THEOREM E. – *Let $f^{\mathrm{in}} \in L^\infty \cap L^1(\mathbf{T}^2 \times \mathbf{R}^2)$ satisfying* (2.1). *Assume that b is given by*

$$(2.14) \qquad b(v) = \eta\big(|v|^2\big)v, \quad \eta \in C^\infty(\mathbf{R}_+),$$

with

$$(2.15) \qquad 0 \leqslant \eta \leqslant 1, \quad \eta|_{[0,R]} = 0, \quad \eta|_{[R+1,+\infty[} = 1, \quad \|\eta'\|_{L^\infty} \leqslant 2.$$

and consider the Fokker–Planck Eq. (1.12) *with Fokker–Planck collision operator* (1.13) *such that*

$$(2.16) \qquad 0 \leqslant \sigma_\varepsilon = \mathrm{O}(1), \qquad \beta(\varepsilon) = \log|\log\varepsilon|$$

supplemented with the Poisson equation (1.7b) *and the initial condition* (1.7c)*. There exists a family $(f_\varepsilon)_{\varepsilon>0}$ of weak solutions of* (1.12), (1.7b,c) *for which points* (a)–(c) *as well as* (2.2)–(2.7) *in Theorem* A *hold, except that $F \in L^\infty(\mathbf{R}_+; \mathcal{M}_+(\mathbf{T}^2 \times \mathbf{R}_+))$ and the convergence holds in $L^\infty(\mathbf{R}_+; \mathcal{M}(\mathbf{T}^2 \times \mathbf{R}_+))$ weak-*. In addition, the defect measure $\nu = 0$ in* (2.3) *and* (2.5).

Theorems A, B and E were announced in [14]. Theorems A and B are proved in Section 3 while Section 4 contains the proof of Theorem E. The class of examples shown in Proposition C is discussed in Section 5. The proof of Theorem D is given in Section 6.

A result analogous to Theorems A, B or E but local in time and valid only for smooth solutions has been proved by Grenier [15] on the pressureless Euler–Poisson system, with a slightly different but equivalent scaling. Formally, Grenier's result corresponds to the situation studied in Theorems A, B and E but in the case where f_ε is of the form (2.8) with bulk velocity of the form $u_\varepsilon(t,x) = \varepsilon U_\varepsilon(t,x)$.

Recently, Brenier [4] proved that the bulk velocity fields of solutions of the gyrokinetic Vlasov–Poisson system converge to dissipative solutions of the 2D Euler equation (see [17, p. 153], where this notion is introduced). This result supersedes that in [15], for any smooth solution of the Euler equation is a dissipative solution. Since it is unknown whether dissipative solutions of the Euler equation are solutions in the sense of distributions, Brenier's result is disjoint from Theorems A, B or E above for all initial data such that the solution of the limiting 2D Euler equation is not smooth.

2.B. The 3D results

In this subsection, we give two elementary results which complete the picture proposed in [10, 11]. Both results are based on the scaling assumption (1.6).

Our first result concerns the case of a magnetic field of constant direction but variable strength. We shall use the following notations: first, as an extension of the 2D notation, $v^\perp = (-v_2, v_1, 0)$;

also

$$\langle\phi\rangle(x,r,v_3)=\frac{1}{2\pi}\int_{S^1}\phi(x,r\omega,v_3)\,\mathrm{d}\omega. \tag{2.16}$$

THEOREM F. – *Let* $f^{\mathrm{in}}\in L^\infty\cap L^1(\mathbf{T}^3\times\mathbf{R}^3)$ *satisfy* (2.1). *Let* $b\in C^1(\mathbf{T}^2)$ *such that* $b(x)\neq 0$ *for all* $x\in\mathbf{T}^2$ *and let* (f_ε) *be a family of weak solutions of*

$$\partial_t f_\varepsilon+v\cdot\nabla_x f_\varepsilon+E_\varepsilon\cdot\nabla_v f_\varepsilon+\frac{1}{\varepsilon}b(x_1,x_2)v^\perp\cdot\nabla_v f_\varepsilon=0,\quad t\in\mathbf{R}_+^*,\ (x,v)\in\mathbf{T}^3\times\mathbf{R}^3, \tag{2.17}$$

coupled to the Poisson equation (1.7b) *and with initial condition* (1.7c). *Then, the family* (f_ε) *is relatively compact in* $L^\infty(\mathbf{R}_+\times\mathbf{T}^3\times\mathbf{R}^3)$ *weak-* and any of its limit points as* $\varepsilon\to 0$ *is of the form*

$$f\equiv f\big(t,x,\sqrt{v_1^2+v_2^2},v_3\big),$$

where f *solves*

$$\partial_t f+v_3\partial_{x_3}f+E_3\partial_{v_3}f=0,\quad t,r>0,\ x\in\mathbf{T}^3,\ v_3\in\mathbf{R}; \tag{2.18a}$$

$$E=-\nabla_x V,\qquad -\Delta_x V=\rho-\overline{\rho}, \tag{2.18b}$$

$$f(0,x,r,v_3)=\big\langle f^{\mathrm{in}}\big\rangle(x,r,v_3),\quad t,r\geqslant 0,\ x\in\mathbf{T}^3,\ v_3\in\mathbf{R}. \tag{2.18c}$$

Our second and last 3D result concerns the case of a magnetic field of constant strength but variable direction. To be consistent with Maxwell's equation, the magnetic field B should also be divergence-free. However, there exist many divergence-free fields of constant length: pick any 2D divergence-free field $B_\perp\equiv B_\perp(x_1,x_2)\in L^\infty(\mathbf{T}^2)$ and let $B_3=\sqrt{4\|B_\perp\|_{L^\infty}^2-|B_\perp|^2}$: one easily check that the vector field $B=(B_\perp,B_3)$ has constant length $2|B_\perp|$ and is divergence free.

Define $R(B(x),\theta)$ the rotation of an angle θ around the oriented axis of direction $B(x)$. Define then:

$$g_\varepsilon(t,x,w)=f_\varepsilon\big(t,x,R(x,-t/\varepsilon)w\big). \tag{2.19}$$

To simplify notations, we shall also denote, for all $\phi\equiv\phi(x,w)$ and all vector field V on $\mathbf{T}_x^3\times\mathbf{R}_w^3$

$$\nabla_V\phi=V\cdot\nabla_x\phi. \tag{2.20}$$

THEOREM G. – *Let* $f^{\mathrm{in}}\in L^\infty\cap L^1(\mathbf{T}^3\times\mathbf{R}^3)$ *satisfy* (2.1). *Let* $B\in C^1(\mathbf{T}^3)$ *satisfy* $\nabla_x\cdot B=0$ *and* $|B|\equiv 1$; *and let* (f_ε) *be a family of weak solutions of* (1.7a–c). *Then, the family* (g_ε) *is relatively compact in* $L^\infty(\mathbf{R}_+\times\mathbf{T}^3\times\mathbf{R}^3)$ *and any of its limit points* g, *as* $\varepsilon\to 0$, *satisfies*:

$$\begin{aligned}
&\partial_t g+(w\cdot B)B\cdot\nabla_x g+(E\cdot B)B\cdot\nabla_w g\\
&\quad=\tfrac{1}{2}w\wedge\big[3(w\cdot B)(B\wedge\nabla_B B)-B\wedge\nabla_w B-\nabla_{B\wedge w}B\big]\cdot\nabla_w g,\\
&E=\nabla_x\Delta_x^{-1}\bigg(\int_{\mathbf{R}^3}g\,\mathrm{d}w-\iint_{\mathbf{T}^3\times\mathbf{R}^3}f^{\mathrm{in}}\,\mathrm{d}x\,\mathrm{d}v\bigg),\\
&g(0,x,w)=f^{\mathrm{in}}(x,w),\quad (x,w)\in\mathbf{T}^3\times\mathbf{R}^3.
\end{aligned}$$

Theorems F and G are proved in Section 8.

3. Proofs of Theorems A and B

Throughout this paper, we shall need the following elementary interpolation result, which we record in the form of a lemma.

LEMMA 3.1. – *Let $f \equiv f(x,v)$ be an a.e. nonnegative measurable function on $\mathbf{T}^d \times \mathbf{R}^d$. Then, for all $0 \leqslant k \leqslant m$,*

$$\left\| \int f|v|^k \, \mathrm{d}v \right\|_{L^{(m+d)/(d+k)}} \leqslant C(d,k) \|f\|_{L^\infty}^{\frac{m-k}{m+d}} \left(\iint f|v|^m \, \mathrm{d}v \, \mathrm{d}x \right)^{\frac{d+k}{m+d}}, \tag{3.1}$$

where $C(d,k)$ is a positive constant depending only on the dimension d and on k.

Proof. – One has, for a.e. $x \in \mathbf{T}^d$

$$\int f|v|^k \, \mathrm{d}v = \int_{|v| \leqslant R} f|v|^k \, \mathrm{d}v + \int_{|v|>R} f|v|^k \, \mathrm{d}v$$

$$\leqslant \|f\|_{L^\infty} \frac{|S^{d-1}|}{k+d} R^{k+d} + \frac{1}{R^{m-k}} \int f|v|^m \, \mathrm{d}v. \tag{3.2}$$

Choose $R = (\int f|v|^m \, \mathrm{d}v / \|f\|_{L^\infty})^{1/(m+d)}$; (3.2) gives

$$\int f|v|^k \, \mathrm{d}v \leqslant \left(1 + \frac{|S^{d-1}|}{k+d}\right) \|f\|_{L^\infty}^{\frac{m-k}{m+d}} \left(\int f|v|^m \, \mathrm{d}v \right)^{\frac{d+k}{m+d}}; \tag{3.3}$$

raising each side of (3.3) to the $\frac{m+d}{k+d}$-th power and integrating in x gives the announced result with

$$C(d,k) = \left(1 + \frac{|S^{d-1}|}{k+d}\right). \qquad \square$$

Let $(f^{\mathrm{in}}) \in L^\infty \cap L^1(\mathbf{T}^2 \times \mathbf{R}^2)$ satisfy

$$f^{\mathrm{in}} \geqslant 0 \text{ a.e.} \quad \text{and} \quad \mathcal{E}\big(f^{\mathrm{in}}\big) < +\infty. \tag{3.4}$$

For any $\varepsilon > 0$, there exists a weak solution f_ε to the Cauchy problem (1.9), (1.7b,c), which satisfies (1.9), (1.7b,c) in the sense of distributions as well as

$$f_\varepsilon \geqslant 0 \text{ a.e.}, \quad \forall t > 0 \iint f_\varepsilon(t,x,v) \, \mathrm{d}x \, \mathrm{d}v = \iint f^{\mathrm{in}} \, \mathrm{d}x \, \mathrm{d}v, \quad \|f_\varepsilon\|_{L^\infty} = \|f^{\mathrm{in}}\|_{L^\infty}, \tag{3.5}$$

and the energy inequality:

$$\mathcal{E}\big(f_\varepsilon(t,\cdot,\cdot)\big) \leqslant \mathcal{E}\big(f^{\mathrm{in}}\big) \quad \text{for all } t \geqslant 0. \tag{3.6}$$

In particular, (3.5) implies that $\overline{\rho}_\varepsilon(t) = \overline{\rho}_\varepsilon(0)$ for all $t \geqslant 0$. All the statements above can be proved easily by the same methods as in [1,9].

The first step in the proof of Theorems A and B is to cast the local conservation laws in a form that is convenient to take limits as $\varepsilon \to 0$. This is done in:

LEMMA 3.2. – *Let* $(f_\varepsilon^{\mathrm{in}})$ *be a family of functions in* $L^\infty \cap L^1(\mathbf{T}^2 \times \mathbf{R}^2)$ *satisfying* (3.4) *for all* $\varepsilon > 0$. *Then*

$$\begin{aligned}&\partial_t \rho_\varepsilon + \nabla_x \cdot \big(\rho_\varepsilon E_\varepsilon^\perp\big)\\&\quad = \big(\partial_1^2 - \partial_2^2\big) \int_{\mathbf{R}^2} v_1 v_2 f_\varepsilon \,\mathrm{d}v + \partial_1\partial_2 \int_{\mathbf{R}^2} \big(v_2^2 - v_1^2\big) f_\varepsilon \,\mathrm{d}v + \varepsilon \partial_t \nabla_x \cdot \int_{\mathbf{R}^2} v^\perp f_\varepsilon \,\mathrm{d}v.\end{aligned} \tag{3.7}$$

Proof. – For each $\varepsilon > 0$, f_ε solves (1.9) in the sense of distributions and belongs to $L^\infty(\mathbf{R}_+; L^1(\mathbf{T}^2 \times \mathbf{R}^2; \mathrm{d}x(1+|v|^2)\,\mathrm{d}v))$ by the energy inequality (3.6). By the same token, $E_\varepsilon \in L^\infty(\mathbf{R}_+; L^2(\mathbf{T}^2))$. Next we test (1.9) on functions of the form $\phi_0(t,x)\chi_R(|v|)$ and $\phi_0(t,x)\chi_R(|v|)v$ with $\phi_0 \in C_c^\infty(\mathbf{R}_+ \times \mathbf{T}^2)$ and $\chi_R \in C^\infty(\mathbf{R}_+)$ such that $\chi_R \equiv 1$ on $[0,R]$, $\chi_R \equiv 0$ on $[2R,+\infty[$, $0 \leqslant \chi_R \leqslant 1$ and $\|\chi_R\|_{L^\infty} \leqslant 2/R$. Letting $R \to +\infty$, one gets, by dominated convergence, the relations

$$\partial_t \rho_\varepsilon + \frac{1}{\varepsilon}\nabla_x \cdot \int v f_\varepsilon \,\mathrm{d}v = 0, \tag{3.8}$$

$$\partial_t \int v f_\varepsilon \,\mathrm{d}v + \frac{1}{\varepsilon}\nabla_x \cdot \int v \otimes v f_\varepsilon \,\mathrm{d}v - \frac{1}{\varepsilon}\rho_\varepsilon E_\varepsilon - \frac{1}{\varepsilon^2}\int v^\perp f_\varepsilon \,\mathrm{d}v = 0 \tag{3.9}$$

which hold in the distribution sense on $\mathbf{R}_+^* \times \mathbf{T}^2$ and are respectively the continuity equation and the momentum equation. Applying the rotation $v \mapsto v^\perp$ to (3.9) after multiplying it by ε, and eliminating $\frac{1}{\varepsilon}\int v f_\varepsilon \,\mathrm{d}v$ between the resulting equation and (3.8) leads to (3.7). □

The following formula will be fundamental in the proof of Theorem B: the vector field $\rho_\varepsilon E_\varepsilon^\perp$ can be recast as

$$(\rho_\varepsilon - \overline{\rho}_\varepsilon)E_\varepsilon^\perp = \big(\tfrac{1}{2}\partial_2\big(E_{1\varepsilon}^2 - E_{2\varepsilon}^2\big) - \partial_1(E_{1\varepsilon}E_{2\varepsilon});\ \tfrac{1}{2}\partial_1\big(E_{1\varepsilon}^2 - E_{2\varepsilon}^2\big) + \partial_2(E_{1\varepsilon}E_{2\varepsilon})\big), \tag{3.10}$$

by using the formulas

$$\nabla_x \cdot E_\varepsilon = \rho_\varepsilon - \overline{\rho}_\varepsilon, \qquad \nabla_x \cdot E_\varepsilon^\perp = 0. \tag{3.11}$$

(The second equality above holds because E_ε is the gradient of the electrostatic potential).

The second step is to establish the asymptotic form of the number density f_ε as $\varepsilon \to 0$.

LEMMA 3.3. – *Let* $(f_\varepsilon^{\mathrm{in}})$ *be a family of functions in* $L^\infty \cap L^1(\mathbf{T}^2 \times \mathbf{R}^2)$ *satisfying* (2.9), *and let* (f_ε) *be a family of weak solutions of* (1.9), (1.7b) *with initial data* (2.10). *Then* (f_ε) *is relatively compact in* $L^\infty(\mathbf{R}_+; \mathcal{M}(\mathbf{T}^2 \times \mathbf{R}^2))$ *weak-* and any of its limit point is invariant under all transformations of the form*

$$(t,x,v) \mapsto (t,x,\mathcal{R}v), \tag{3.12}$$

where $\mathcal{R}$ *runs through the group of orthogonal transforms of* $\mathbf{R}^2$.

In other words, any weak-* limit point of (f_ε) is radial in the velocity variable.

Proof. – Multiplying (1.9) by ε^2 leads to

$$v^\perp \cdot \nabla_v f_\varepsilon = -\partial_t\big(\varepsilon^2 f_\varepsilon\big) - \nabla_x \cdot (\varepsilon v f_\varepsilon) - \nabla_v(\varepsilon E_\varepsilon f_\varepsilon). \tag{3.13}$$

By the energy inequality (3.6), the family (f_ε) is bounded in $L^\infty(\mathbf{R}_+; L^1(\mathrm{d}x(1+|v|^2)\,\mathrm{d}v))$ so that the first two terms in the right-hand side of (3.13) converge to zero in the distribution sense. The Maximum Principle (3.5) and the L^∞ estimate in (2.9) imply that, as $\varepsilon \to 0$, $\varepsilon f_\varepsilon \to 0$ in

$L^\infty(\mathbf{R}_+\times\mathbf{T}^2\times\mathbf{R}^2)$ while the family (E_ε) is uniformly bounded in $L^\infty(\mathbf{R}_+;L^2(\mathbf{T}^2))$ by the energy estimate (3.6). Therefore (3.13) implies that

$$v^\perp\cdot\nabla_v f_\varepsilon\to 0\quad\text{in }\mathcal{D}'\big(\mathbf{R}_+^*\times\mathbf{T}^2\times\mathbf{R}^2\big).\tag{3.14}$$

Next, the family (f_ε) is bounded in $L^\infty(\mathbf{R}_+;L^1(\mathrm{d}x(1+|v|^2)\,\mathrm{d}v)$ and is therefore relatively compact in $L^\infty(\mathbf{R}_+;\mathcal{M}(\mathbf{T}^2\times\mathbf{R}^2))$ weak-*; let $f\in L^\infty(\mathbf{R}_+;\mathcal{M}(\mathbf{T}^2\times\mathbf{R}^2))$ be any of its limit point. It follows from (3.14) that

$$v^\perp\cdot\nabla_v f=0.\tag{3.15}$$

Since the operator $v^\perp\cdot\nabla_v$ generates the group of transformations (3.12), any element of the nullspace of this operator must be invariant under this group, which establishes our claim. □

A last but important preparation is the following lemma, which controls the oscillations of the macroscopic density in terms of the time variable only.

LEMMA 3.4. – *Let* $(f_\varepsilon^{\mathrm{in}})$ *be a family of functions in* $L^\infty\cap L^1(\mathbf{T}^2\times\mathbf{R}^2)$ *satisfying* (2.9)*, and let* (f_ε) *be a family of weak solutions of* (1.9), (1.7b) *with initial data* (2.10)*. Then, the associated family* (ρ_ε) *is bounded in* $C^{1/2}(\mathbf{R}_+;W^{-2,1}(\mathbf{T}^2))$.

Proof. – Define

$$\pi_\varepsilon=\rho_\varepsilon-\varepsilon\nabla_x\cdot\int v^\perp f_\varepsilon\,\mathrm{d}v;\tag{3.16}$$

by Lemma 3.2,

$$\partial_t\pi_\varepsilon=-\nabla_x\cdot\big(\rho_\varepsilon E_\varepsilon^\perp\big)+\big(\partial_1^2-\partial_2^2\big)\int_{\mathbf{R}^2}v_1v_2f_\varepsilon\,\mathrm{d}v+\partial_1\partial_2\int_{\mathbf{R}^2}\big(v_2^2-v_1^2\big)f_\varepsilon\,\mathrm{d}v.\tag{3.17}$$

By the energy estimate (3.6), the family (E_ε) is bounded in $L^\infty(\mathbf{R}_+;L^2(\mathbf{T}^2))$; hence, using formula (3.10) shows that there exists $C>0$ such that, for all $\varepsilon>0$,

$$\big\|\nabla_x\cdot\big(\rho_\varepsilon E_\varepsilon^\perp\big)\big\|_{L^\infty(\mathbf{R}_+;W^{-2,1}(\mathbf{T}^2))}\leqslant C\|E_\varepsilon\|^2_{L^\infty(\mathbf{R}_+;L^2(\mathbf{T}^2))}\leqslant C\sup_\varepsilon\mathcal{E}\big(f_\varepsilon^{\mathrm{in}}\big).\tag{3.18}$$

Applying again the energy estimate (3.6) to the last two terms in the right-hand side of (3.17) shows that

$$\partial_t\pi_\varepsilon\text{ is bounded in }L^\infty\big(\mathbf{R}_+;W^{-2,1}\big(\mathbf{T}^2\big)\big).\tag{3.19}$$

The formulas (3.9), (3.10) shows that there exists $C>0$ such that, for all $\varepsilon>0$ and all $0\leqslant t\leqslant t'$

$$\begin{aligned}\varepsilon&\left\|\int v^\perp f_\varepsilon(t',x,v)\,\mathrm{d}v-\int v^\perp f_\varepsilon(t,x,v)\,\mathrm{d}v\right\|_{W^{-1,1}(\mathbf{T}^2)}\\&\leqslant C(t'-t)\left[\|E_\varepsilon\|^2_{L^\infty(\mathbf{R}_+;L^2(\mathbf{T}^2))}+\left(1+\frac{1}{\varepsilon}\right)\left\|\int(1+|v|^2)f_\varepsilon\,\mathrm{d}v\right\|_{L^\infty(\mathbf{R}_+;L^1(\mathbf{T}^2))}\right]\\&\leqslant C\sup_{\varepsilon>0}\big(\tfrac12\big\|f_\varepsilon^{\mathrm{in}}\big\|_{L^1}+\mathcal{E}\big(f_\varepsilon^{\mathrm{in}}\big)\big)(t'-t)\left(2+\frac{2}{\varepsilon}\right).\end{aligned}\tag{3.20}$$

Set

$$A=\sup_{\varepsilon>0}\big(\tfrac12\big\|f_\varepsilon^{\mathrm{in}}\big\|_{L^1}+\mathcal{E}\big(f_\varepsilon^{\mathrm{in}}\big)\big).$$

Here we assume that $0<\varepsilon<1$; indeed, it is only the limit as $\varepsilon\to 0$ which is of interest to us. If $t'-t>\varepsilon^2$, we estimate

$$\begin{aligned}
&\varepsilon\left\| \int v^{\perp} f_\varepsilon(t',x,v)\,\mathrm{d}v - \int v^{\perp} f_\varepsilon(t,x,v)\,\mathrm{d}v\right\|_{W^{-1,1}(\mathbf{T}^2)}\\
&\quad\leqslant \varepsilon\left\| \int v^{\perp} f_\varepsilon(t',x,v)\,\mathrm{d}v\right\|_{L^1(\mathbf{T}^2)} + \varepsilon\left\| \int v^{\perp} f_\varepsilon(t,x,v)\,\mathrm{d}v\right\|_{L^1(\mathbf{T}^2)}\\
&\quad\leqslant 2A\varepsilon \leqslant 2A\sqrt{t'-t}; \qquad (3.21)
\end{aligned}$$

if on the other hand $t'-t<\varepsilon^2$, one has, by (3.20)

$$\begin{aligned}
&\varepsilon\left\| \int v^{\perp} f_\varepsilon(t'x,v)\,\mathrm{d}v - \int v^{\perp} f_\varepsilon(t,x,v)\,\mathrm{d}v\right\|_{W^{-1,1}(\mathbf{T}^2)}\\
&\quad\leqslant CA(t'-t)\left(2+\frac{2}{\sqrt{t'-t}}\right) \leqslant 4CA\sqrt{t'-t}. \qquad (3.22)
\end{aligned}$$

Combining (3.21) and (3.22) shows that, for all $\varepsilon\in]0,1[$ and $t'\geqslant t\geqslant 0$ we have:

$$\varepsilon\left\| \int v^{\perp} f_\varepsilon(t'x,v)\,\mathrm{d}v - \int v^{\perp} f_\varepsilon(t,x,v)\,\mathrm{d}v\right\|_{W^{-1,1}(\mathbf{T}^2)} \leqslant (2+4C)A\sqrt{t'-t} \qquad (3.23)$$

which, coupled to (3.19) and the decomposition (3.16) establishes our claim. □

Equipped with the lemmas above, we can now proceed to prove Theorems A and B.

Proof of Theorem A. – Consider a subsequence of (f_ε), still denoted by (f_ε) for simplicity, converging to f in $L^\infty(\mathbf{R}_+;\mathcal{M}(\mathbf{T}^2\times\mathbf{R}^2))$ weak-* as in Lemma 3.3 above. By the energy inequality (3.6), the sequence $(|v|^2 f_\varepsilon)$ is bounded in $L^\infty(\mathbf{R}_+;\mathcal{M}(\mathbf{T}^2\times\mathbf{R}^2))$. Thus the sequence

$$\mu_\varepsilon = \int_0^\infty r^2 f_\varepsilon(t,x,r\theta)r\,\mathrm{d}r \qquad (3.24)$$

of push-forwards of f_ε under the map $(t,x,v)\mapsto(t,x,v/|v|)$ is bounded in $L^\infty(\mathbf{R}_+;\mathcal{M}(\mathbf{T}^2\times S^1))$. Hence, there exists a subsequence of (f_ε) denoted by $(f_{\varepsilon'})$ such that $\mu_{\varepsilon'}$ converges to μ in $L^\infty(\mathbf{R}_+;\mathcal{M}(\mathbf{T}^2\times S^1))$ weak-*. We next define the defect measure associated to the subsequence $(f_{\varepsilon'})$ by:

$$\langle\nu;\psi\rangle = \int_{\mathbf{R}_+\times\mathbf{T}^2\times S^1} \psi(t,x,\theta)\,\mathrm{d}\mu(t,x,\theta) - \int_{\mathbf{R}_+\times\mathbf{T}^2\times\mathbf{R}^2} \psi\left(t,x,\frac{v}{|v|^2}\right)|v|^2\,\mathrm{d}f(t,x,v) \qquad (3.25)$$

for every $\psi\in C_c^0(\mathbf{R}_+\times\mathbf{T}^2\times S^1)$.

Let $R>0$, $\chi\in C_c^0(\mathbf{R}_+)$ be such that $\chi|_{[0,1]}\equiv 1$, $\chi|_{[2,+\infty[}\equiv 0$ and $0\leqslant\chi\leqslant 1$; define χ_R by $\chi_R(v)=\chi(|v|/R)$. For every nonnegative function $\psi\in C_c^0(\mathbf{R}_+\times\mathbf{T}^2\times S^1)$, and all $\varepsilon'>0$,

$$\left\langle |v|^2 f_\varepsilon;\psi\big(t,x,v/|v|\big)\chi_R(v)\right\rangle \leqslant \left\langle |v|^2 f_\varepsilon;\psi\big(t,x,v/|v|\big)\right\rangle = \langle\mu_\varepsilon;\psi\rangle;$$

taking limits as $\varepsilon'\to 0$ gives

$$\left\langle |v|^2 f;\psi(t,x,v/|v|)\chi_R(v)\right\rangle \leqslant \langle\mu;\phi\rangle. \qquad (3.26)$$

Letting $R \to +\infty$ in (3.26) proves that ν is a positive measure.

Specializing formula (3.7) to the subsequence $(f_{\varepsilon'})$ and letting $\varepsilon' \to 0$ shows that the right-hand side of (3.7) converges to

$$\begin{aligned}&\left(\partial_1^2-\partial_2^2\right)\int_{\mathbf{R}^2} v_1 v_2 f+\left(\partial_1^2-\partial_2^2\right)\int_{S^1}\theta_1\theta_2\,\mathrm{d}\nu(\theta)+\partial_1\partial_2\int_{\mathbf{R}^2}\left(v_2^2-v_1^2\right)f\\ &\quad+\partial_1\partial_2\int_{S^1}\left(\theta_2^2-\theta_1^2\right)\mathrm{d}\nu(\theta)\end{aligned}\tag{3.27}$$

in the sense of distributions. By Lemma 3.3, f is radial in the velocity variable; therefore

$$\int_{\mathbf{R}^2} v_1 v_2 f=\int_{\mathbf{R}^2}\left(v_2^2-v_1^2\right)f=0.\tag{3.28}$$

It remains to find the limit of $\rho_{\varepsilon'}E_{\varepsilon'}$ as $\varepsilon' \to 0$. By the energy inequality (3.6), (E_ε) is bounded in $L^\infty(\mathbf{R}_+; L^2(\mathbf{T}^2))$. By the Maximum Principle (3.5), the energy inequality (3.6) and Lemma 3.1 with $k=0$ and $m=2$, we obtain:

$$\|\rho_\varepsilon\|_{L^\infty(\mathbf{R}_+;L^2(\mathbf{T}^2))}\leqslant\sqrt{2}C(2,0)\left\|f^{\mathrm{in}}\right\|_{L^\infty}^{1/2}\mathcal{E}\left(f^{\mathrm{in}}\right)^{1/2}.\tag{3.29}$$

Since $E_\varepsilon=\nabla_x\Delta_x^{-1}(\rho_\varepsilon-\overline{\rho}_\varepsilon)$, we conclude that (E_ε) is bounded in $L^\infty(\mathbf{R}_+; H^1(\mathbf{T}^2))$. Let $\psi \in C_c^\infty(\mathbf{R}_+\times\mathbf{T}^2)$; by Lemma 3.4, $(\psi\rho_\varepsilon)$ is bounded in, say, $C^{1/2}(\mathbf{R}_+; H^{-4}(\mathbf{T}^2))$ (by the Sobolev embedding) as well as $L^\infty(\mathbf{R}_+; L^2(\mathbf{T}^2))$ by (3.29); thus it is bounded in $C^{1/16}(\mathbf{R}_+; H^{-1/2}(T^2))$ by a standard interpolation argument. Since $(\psi\rho_\varepsilon)$ has support included in the (compact) support of ψ, one sees that $(\psi\rho_\varepsilon)$ is relatively compact in $L^\infty(\mathbf{R}_+; H^{-1}(\mathbf{T}^2))$, so that

$$\psi\rho_{\varepsilon'}E_{\varepsilon'}\to\psi\rho E\quad\text{in } L^\infty\left(\mathbf{R}_+;\mathcal{M}\left(\mathbf{T}^2\right)\right)\text{ weak-*},\tag{3.30}$$

where

$$\rho(t,x)=\int_{\mathbf{R}_v^2} f(t,x,v)\,\mathrm{d}v,\qquad E=\nabla_x\Delta_x^{-1}\rho.\tag{3.31}$$

The convergences (3.27) and (3.30), together with formula (3.31) establish Theorem A. □

In the proof of Theorem B, we use the following compactness argument due to Delort; we recall that it is the key argument in the proof of global existence of weak solutions to the 2D incompressible Euler equation in the case of vortex sheets: see [8].

THEOREM (see Delort [8], Theorem 1.2.1). – *Let $T>0$ and $(\omega_\varepsilon)_{0<\varepsilon<1}$ be a family of functions in $L^\infty([-T,T], C^\infty(\mathbf{T}^2))$ which can be decomposed as $\omega_\varepsilon=\omega_\varepsilon'+\omega_\varepsilon''$ and satisfies the following assumptions*:

(a) *the family (ω_ε) is equicontinuous in $[-T,T]$ with values in $\mathcal{D}'(\mathbf{T}^2)$) and such that $\int_{\mathbf{T}^2}\omega(t,x)\,\mathrm{d}x=0$;*
(b) *the family (ω_ε') is bounded in $L^\infty([-T,T], L^1\cap H^{-1}(\mathbf{T}^2))$ and, for each $0<\varepsilon<1$, $\omega_\varepsilon\geqslant 0$;*
(c) *the family (ω_ε'') is bounded in $L^\infty([-T,T], L^1\cap L^p(\mathbf{T}^2))$ for some $p>1$;*
(d) *setting $v_\varepsilon=\nabla_x^\perp\Delta_x^{-1}\omega_\varepsilon$, the family (v_ε) converges to v as $\varepsilon\to 0^+$ in the sense of distributions on $]-T,T[\times\mathbf{T}^2$.*

Then $v \in L^\infty([-T,T]; L^2(\mathbf{T}^2))$ *and*

$$v_{1\varepsilon}^2 - v_{2\varepsilon}^2 \to v_1^2 - v_2^2 \quad \textit{and} \quad v_{1\varepsilon} v_{2\varepsilon} \to v_1 v_2$$

in the sense of distributions on $]-T,T[\times \mathbf{T}^2$.

Proof of Theorem B. – The part of the proof of Theorem A leading to the existence of the defect measure ν applies verbatim in the present case. The only difference lies in the convergence of the nonlinear terms $\nabla_x \cdot (\rho_\varepsilon E_\varepsilon^\perp)$ in (3.7). This is where Delort's result is needed. We first extend ρ_ε and E_ε respectively by $\int f_\varepsilon^{\mathrm{in}}\,\mathrm{d}v$ and $E_\varepsilon^{\mathrm{in}}$ for $t \leqslant 0$ and abuse the notation ρ_ε and E_ε for the resulting extensions. We must regularize the families (E_ε) and (ρ_ε) in the x-variable in order to comply with the first assumption in Delort's theorem. For all $\varepsilon \in]0,1[$ there exists $\delta(\varepsilon) > 0$ such that

$$\left\| \mathrm{e}^{\delta(\varepsilon)\Delta_x} E_\varepsilon - E_\varepsilon \right\|_{L^\infty([-T,T];L^2(\mathbf{T}^2))} \leqslant \varepsilon. \tag{3.32}$$

Set

$$v_\varepsilon = \mathrm{e}^{\delta(\varepsilon)\Delta_x} E_\varepsilon. \tag{3.33}$$

Then the family (ω_ε) defined by $\omega_\varepsilon = \nabla^\perp \cdot v_\varepsilon$ can, for all ε, be decomposed as

$$\omega_\varepsilon = \omega_\varepsilon' + \omega_\varepsilon'', \tag{3.34}$$

with

$$\omega_\varepsilon' = \mathrm{e}^{\delta(\varepsilon)\Delta_x} \rho_\varepsilon \tag{3.35}$$

and

$$\omega_\varepsilon'' = -\overline{\rho}_\varepsilon = \iint_{\mathbf{T}^2 \times \mathbf{R}^2} f_\varepsilon^{\mathrm{in}}\,\mathrm{d}x\,\mathrm{d}v. \tag{3.36}$$

By Lemma 3.4, the families (ω_ε') and (ω_ε'') satisfy assumption (a) in Delort's theorem; by (3.5) and (3.6), the family (ρ_ε) satisfies assumption (b) and so does (ω_ε'), by the positivity of the heat semigroup. By (2.9), the family (ω_ε'') satisfies assumption (c). Finally, modulo extraction of a subsequence, the family (E_ε) converges to E in $L^\infty([-T,T]; L^2(\mathbf{T}^2))$ weak-* as $\varepsilon \to 0$; thus the family (v_ε) converges to $E^\perp$ in the sense of distributions as $\varepsilon \to 0$ and satisfies assumption (d). Therefore,

$$v_{1\varepsilon}^2 - v_{2\varepsilon}^2 \to E_2^2 - E_1^2 \quad \text{and} \quad v_{1\varepsilon} v_{2\varepsilon} \to -E_1 E_2 \tag{3.37}$$

in the sense of distributions on $]-T,T[\times \mathbf{T}^2$ as $\varepsilon \to 0$. By (3.32), one also has

$$E_{1\varepsilon}^2 - E_{2\varepsilon}^2 \to E_2^2 - E_1^2 \quad \text{and} \quad E_{1\varepsilon} E_{2\varepsilon} \to -E_1 E_2 \tag{3.38}$$

in the sense of distributions on $]-T,T[\times \mathbf{T}^2$ as $\varepsilon \to 0$. Using the obvious formula

$$\nabla_x \cdot \left(\rho_\varepsilon E_\varepsilon^\perp\right) = \nabla_x \cdot \left[(\rho_\varepsilon - \overline{\rho}_\varepsilon) E_\varepsilon^\perp\right] \tag{3.39}$$

together with (3.10), (3.37) shows that, after extracting a subsequence if necessary, we have:

$$\nabla_x \cdot \left(\rho_\varepsilon E_\varepsilon^\perp\right) \to \nabla_x \cdot \left(\rho E^\perp\right), \tag{3.40}$$

in the sense of distributions on $]-T,T[\times \mathbf{T}^2$ as $\varepsilon \to 0$. This completes the proof of Theorem B. □

4. Proof of Theorem E

We first address briefly the question of global existence of a weak solution of the collisional model (1.12), (1.7b,c) for fixed $\varepsilon > 0$. The method is essentially the same as in [7]: the only difference between (1.12) and the Vlasov–Fokker–Planck equation treated in [7] is that [7] deals with the case without external magnetic field and where the friction term $b \equiv 0$ in the Fokker–Planck operator (1.13). Also, [7] focuses on smooth solutions.

Here we first regularize and truncate the initial data for (1.12). Let $\psi \in C_c^\infty(\mathbf{R}^2)$ such that $\psi(x) = 1$ for $|x| \leqslant 1$, $\psi(x) = 0$ for $|x| \geqslant 2$ and $0 \leqslant \psi \leqslant 1$. For all $\delta > 0$, let $\psi_\delta(v) = \psi(\delta v)$. In the problem (1.12), (1.7b), we replace the initial data (1.7c) by

$$f_\delta^{\mathrm{in}} = \psi_\delta \mathrm{e}^{\delta \Delta_x} f^{\mathrm{in}}. \tag{4.1}$$

By the trivial amplification of Degond's results in [7] recalled above, (1.12), (1.7b), (4.1) has a unique global smooth solution f_ε^δ.

These smooth solutions satisfy the following estimates:

$$\begin{gathered}\iint f_\varepsilon^\delta(t,x,v)\,\mathrm{d}x\,\mathrm{d}v = \iint f_\delta^{\mathrm{in}}\,\mathrm{d}x\,\mathrm{d}v = m_\delta,\\ 0 \leqslant f_\varepsilon^\delta(t,x,v) \leqslant \|f^{\mathrm{in}}\|_{L^\infty} \mathrm{e}^{t\beta(\varepsilon)},\\ \mathcal{E}\big(f_\varepsilon^\delta(t,\cdot,\cdot)\big) + \beta(\varepsilon)\int_0^t \iint \eta\big(|v|^2\big)|v|^2 f_\varepsilon^\delta(s,x,v)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}v \leqslant \mathcal{E}\big(f^{\mathrm{in}}\big) + 2m_\delta\sigma_\varepsilon t.\end{gathered} \tag{4.2}$$

We then remove the regularization and the truncations of the initial data and pass to the limit after extracting subsequences in (1.12) keeping ε fixed, based on the a priori estimates (4.2) only.

The only nontrivial term is the nonlinear one, i.e., $f_\varepsilon^\delta E_\varepsilon^\delta$. As in the proof of Theorem A, we use the L^∞ estimate and the energy inequality in (4.2), together with Lemma 3.1 with $k = 0$ and $m = 2$ to show that the family (E_ε^δ) is bounded in $L^\infty_{\mathrm{loc}}(\mathbf{R}_+; H^1(\mathbf{T}^2))$ for $\varepsilon > 0$ fixed, as $\delta \to 0$. On the other hand, the continuity equation

$$\partial_t \rho_\varepsilon^\delta + \nabla_x \cdot \int v f_\varepsilon^\delta\,\mathrm{d}v = 0 \tag{4.3}$$

implies that the family $(\rho_\varepsilon^\delta)$ is bounded in $W^{1,\infty}_{\mathrm{loc}}(\mathbf{R}_+; W^{-1,1}(\mathbf{T}^2))$ for $\varepsilon > 0$ fixed, as $\delta \to 0$, and also in $W^{1,\infty}_{\mathrm{loc}}(\mathbf{R}_+; H^{-3}(\mathbf{T}^2))$ by Sobolev embedding and duality. Thus, the family (E_ε^δ) is bounded in $W^{1,\infty}_{\mathrm{loc}}(\mathbf{R}_+; H^{-2}(\mathbf{R}^2))$ for $\varepsilon > 0$ fixed, as $\delta \to 0$. It is therefore relatively compact in $L^\infty_{\mathrm{loc}}(\mathbf{R}_+; L^2(\mathbf{T}^2))$. This shows that, if $f_\varepsilon^\delta \to f_\varepsilon$ and $E_\varepsilon^\delta \to E_\varepsilon$ in the sense of distributions on $\mathbf{R}_+ \times \mathbf{T}^2 \times \mathbf{R}^2$ as $\delta \to 0$ while $\varepsilon > 0$ is kept fixed, a situation to which the general case reduces after extraction of subsequences, then $f_\varepsilon^\delta E_\varepsilon^\delta \to f_\varepsilon E_\varepsilon$ in the sense of distributions on $\mathbf{R}_+ \times \mathbf{T}^2 \times \mathbf{R}^2$.

By this procedure, we have constructed weak solutions of (1.12), (1.7b,c) which satisfy

$$\iint f_\varepsilon(t,x,v)\,\mathrm{d}x\,\mathrm{d}v = \iint f^{\mathrm{in}}\,\mathrm{d}x\,\mathrm{d}v = m, \tag{4.4}$$

$$0 \leqslant f_\varepsilon(t,x,v) \leqslant \|f^{\mathrm{in}}\|_{L^\infty} \mathrm{e}^{t\beta(\varepsilon)}, \tag{4.5}$$

$$\mathcal{E}\big(f_\varepsilon(t,\cdot,\cdot)\big) + \beta(\varepsilon)\int_0^t \iint \eta\big(|v|^2\big)|v|^2 f_\varepsilon(s,x,v)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}v \leqslant \mathcal{E}\big(f^{\mathrm{in}}\big) + 2m\sigma_\varepsilon t, \tag{4.6}$$

since the estimates (4.2) are obviously uniform in δ.

The proof of Theorem E follows then the same lines as that of Theorem B. Notice, that one does not have a uniform bound on $\|f_\varepsilon\|_{L^\infty}$, which explains why the proof of Theorem B (and not simply that of Theorem A) is needed.

The Maximum Principle applied to the Fokker–Planck equation (1.12), together with condition (2.16) on $\beta(\varepsilon)$, shows that, for all $T > 0$,

$$\|f_\varepsilon\|_{L^\infty([0,T]\times\mathbf{T}^2\times\mathbf{R}^2)} \leqslant \|f^{\mathrm{in}}\|_{L^\infty(\mathbf{T}^2\times\mathbf{R}^2)}|\log\varepsilon|^T = \mathrm{o}\left(\frac{1}{\varepsilon}\right). \tag{4.7}$$

Hence $\varepsilon f_\varepsilon \to 0$ in $L^\infty([0,T]\times\mathbf{T}^2\times\mathbf{R}^2)$, which is crucial in the proof of Lemma 3.3 (more specifically in that of (3.14)). In any case, Theorem B applies to this case for all $T > 0$.

The only remaining task is to prove that the defect measure ν predicted by Theorem B vanishes. By (4.6), again for some fixed $T > 0$, we have:

$$\int_0^T \int \int_{|v|\geqslant R} |v|^2 f_\varepsilon(s,x,v)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}v \leqslant \frac{\mathcal{E}(f^{\mathrm{in}}) + 2m\sigma_\varepsilon T}{\beta(\varepsilon)} \to 0 \tag{4.8}$$

as $\varepsilon \to 0$. We keep the notations of the part of the proof of Theorem A before formula (3.25). Let $\chi \in C_c^0(\mathbf{R}_+)$ be such that $0 \leqslant \chi \leqslant 1$, $\chi \equiv 1$ on $[0,R]$ and $\chi \equiv 0$ on $[2R,+\infty[$, then

$$\mu_\varepsilon - \int_0^\infty \chi(r) r^2 f_\varepsilon(t,x,r\theta) r\,\mathrm{d}r \leqslant \int_R^\infty r^2 f_\varepsilon(t,x,r\theta) r\,\mathrm{d}r \to 0 \tag{4.9}$$

in $L^1([0,T]; L^1(\mathbf{T}^2\times S^1))$. Restricting (4.9) to subsequences $(f_{\varepsilon'})$ and $(\mu_{\varepsilon'})$ as in the proof of Theorem A, gives, in the limit as $\varepsilon' \to 0$

$$\mu = \int_{\mathbf{R}_+} \chi\big(|v|\big)|v|^2 f(t,x,v)|v|\,\mathrm{d}|v|. \tag{4.10}$$

By (4.8), f is supported in $\mathbf{R}_+\times\mathbf{T}^2\times\overline{B}(0,R)$; thus (4.10) implies that

$$\mu = \int_{\mathbf{R}_+} |v|^2 f(t,x,v)|v|\,\mathrm{d}|v|. \tag{4.11}$$

By the rotation invariance of f (see Lemma 3.3) and the definition (3.25) of the defect measure ν, (4.11) implies $\nu = 0$. This concludes the proof. □

5. Proof of Proposition C

To begin with, for each $\varepsilon > 0$, $f_\varepsilon^{\mathrm{in}} \in C_c^\infty(\mathbf{T}^2\times\mathbf{R}^2)$. Therefore, the problem (1.9), (1.7b,c) has a unique classical solution f_ε on $\mathbf{R}_+\times\mathbf{T}^2\times\mathbf{R}^2$, as can be seen from a trivial modification of the arguments in [20] (adapted to treat the case of a constant external magnetic field). In particular, the energy inequality (3.6) becomes the equality:

$$\mathcal{E}\big(f_\varepsilon(t,\cdot,\cdot)\big) = \mathcal{E}\big(f_\varepsilon^{\mathrm{in}}\big), \quad \forall t \geqslant 0,\ \varepsilon > 0. \tag{5.1}$$

This remark is essential in the sequel.

LEMMA 5.1. – *Under the assumptions of Proposition C, there exists* $C > C' > 0$ *such that*

$$C' \leqslant \mathcal{E}\left(f_\varepsilon^{\mathrm{in}}\right) \leqslant C\left(1+\varepsilon^3\right), \qquad \|\rho_\varepsilon\|_{L_t^\infty(L_x^2)} \leqslant C\varepsilon^{3/2}, \qquad \|E_\varepsilon\|_{L_t^\infty(L_x^2)} \leqslant C\varepsilon^{3/2}. \tag{5.2}$$

Proof. – By the interpolation inequality in Lemma 3.1 and the inequalities (2.11), one has the estimate $\|\rho_\varepsilon|_{t=0}\|_{L_x^2} = \mathrm{O}(\varepsilon^{3/2})$. Hence

$$\left\|E_\varepsilon|_{t=0}\right\|_{L_x^2} = \left\|\nabla_x \Delta_x^{-1}(\rho_\varepsilon|_{t=0} - \overline{\rho}_\varepsilon)\right\|_{L_x^2} = \mathrm{O}\left(\varepsilon^{3/2}\right). \tag{5.3}$$

With the first statement in (2.11), this gives the estimate on the total energy at time $t=0$ in (5.2).

The next step is to propagate the various estimates to $t \geqslant 0$. The Maximum Principle (3.5) shows that

$$\|f_\varepsilon\|_{L^\infty_{t,x,v}} = \mathrm{O}\left(\varepsilon^3\right). \tag{5.4}$$

Using the interpolation inequality in Lemma 3.1, the estimate (5.4), the energy conservation (5.1) and the first estimate in (5.2) shows that

$$\|\rho_\varepsilon\|_{L_t^\infty(L_x^2)} \leqslant Cst\, \|f_\varepsilon\|^{1/2}_{L^\infty_{t,x,v}} \left(\iint\limits_{\mathbf{T}^2\times\mathbf{R}^2} |v|^2 f_\varepsilon \,\mathrm{d}x\,\mathrm{d}v \right)^{1/2} = \mathrm{O}\left(\varepsilon^{3/2}\right). \tag{5.5}$$

This proves the second estimate in (5.2); proceeding as in (5.3) gives the last estimate in (5.2). □

With these estimates, it is easy to prove that the defect measure associated to any subsequence can not vanish. Indeed, the energy conservation (5.1) and the estimates (5.2) show that for all $\varepsilon > 0$

$$\iint\limits_{\mathbf{T}^2\times\mathbf{R}^2} |v|^2 f_\varepsilon(t,x,v)\,\mathrm{d}x\,\mathrm{d}v = 2\mathcal{E}\left(f_\varepsilon^{\mathrm{in}}\right) - \int\limits_{\mathbf{T}^2} \left|E_\varepsilon(t,x)\right|^2 \mathrm{d}x \geqslant C' - \mathrm{O}\left(\varepsilon^3\right). \tag{5.6}$$

Let $(f_{\varepsilon'})$ be any subsequence of (f_ε) such that $\mu_{\varepsilon'}$ (defined in (3.24)) converges in $L^\infty(\mathbf{R}_+; \mathcal{M}(\mathbf{T}^2 \times S^1))$ weak-* to some limit μ; by (5.6)

$$\int\limits_{\mathbf{T}^2\times S^1} \mathrm{d}\mu(t,x,\theta) = \lim_{\varepsilon'\to 0} \iint\limits_{\mathbf{T}^2\times\mathbf{R}^2} |v|^2 f_\varepsilon(t,x,v)\,\mathrm{d}x\,\mathrm{d}v \geqslant C'. \tag{5.7}$$

On the other hand, (5.4) shows that the weak-* limit of any subsequence of (f_ε) is $f \equiv 0$. Therefore, the definition (3.25) shows that $\nu \neq 0$. □

6. Proof of Theorem D

Proof of (a). – Multiplying (1.9) by $\varepsilon^2|v|^2$ leads to:

$$v^\perp \cdot \nabla_v\left(|v|^2 f_\varepsilon\right) = -\partial_t\left(\varepsilon^2|v|^2 f_\varepsilon\right) - \nabla_x \cdot \left(\varepsilon v|v|^2 f_\varepsilon\right) - \nabla_v \cdot \left(\varepsilon E_\varepsilon |v|^2 f_\varepsilon\right) + 2\varepsilon(E_\varepsilon \cdot v) f_\varepsilon. \tag{6.1}$$

It is convenient to use polar coordinates in the velocity space: set $v = (r\cos\theta, r\sin\theta)$; in these coordinates, as noticed in the proof of Lemma 3.3, $v^\perp \cdot \nabla_v = \partial_\theta$.

Going back to the proof of Lemma 3.3, we see that

$$\partial_\theta \nu(t,x,\theta) = \lim_{\varepsilon'\to 0} \int_0^{+\infty} \partial_\theta f_{\varepsilon'}\big(t,x,r(\cos\theta,\sin\theta)\big) r^3\,\mathrm{d}r$$

$$= \lim_{\varepsilon\to 0} \int_0^{+\infty} v^\perp \cdot \nabla_v\big(|v|^2 f_{\varepsilon'}(t,x,v)\big)|v|\,\mathrm{d}|v| \tag{6.2}$$

for subsequences $(f_{\varepsilon'})$ as considered in the proof of Theorem A before formula (3.25).

By the energy inequality (3.6), the first term in the right-hand side of (6.1) converges to 0:

$$\varepsilon \int_0^{+\infty} |v|^2 f_{\varepsilon'}\big(t,x,|v|(\cos\theta,\sin\theta)\big)|v|\,\mathrm{d}|v| \to 0 \quad \text{in } L^\infty\big(\mathbf{R}_+; L^1\big(\mathbf{T}^2_x \times S^1_\theta\big)\big). \tag{6.3}$$

By the L^∞ bound (3.5), the energy inequality (3.6) and Lemma 3.1,

$$\rho_\varepsilon = \int f_\varepsilon\,\mathrm{d}v \text{ is uniformly bounded in } L^\infty\big(\mathbf{R}_+, L^2\big(\mathbf{T}^2\big)\big) \tag{6.4}$$

and

$$\int v f_\varepsilon\,\mathrm{d}v \text{ is uniformly bounded in } L^\infty\big(\mathbf{R}_+, L^{4/3}\big(\mathbf{T}^2\big)\big). \tag{6.5}$$

By (6.4), the energy inequality (3.6) and Sobolev embedding

$$\forall p \in [2,+\infty[\,, \quad E_\varepsilon \text{ is uniformly bounded in } L^\infty\big(\mathbf{R}_+, L^p\big(\mathbf{T}^2\big)\big). \tag{6.6}$$

Setting $p = 4$, we see that:

$$\varepsilon E_\varepsilon \cdot \int_0^\infty |v|(\cos\theta,\sin\theta) f_\varepsilon |v|\,\mathrm{d}|v| \to 0 \quad \text{in } L^\infty\big(\mathbf{R}_+, L^1\big(\mathbf{T}^2_x \times S^1_\theta\big)\big), \tag{6.7}$$

so that the last term in the right-hand side of (6.1) also converges to 0.

The equivalence annouced in Theorem D(a) is established if we prove that the third term in the right-hand side of (6.1) also converges to 0. The same method as above does not apply. Indeed the second moment $\int |v|^2 f_\varepsilon\,\mathrm{d}v$ is only bounded in L^1_x, which would require a L^∞_x bound on E_ε. Unfortunately, we only have a H^1_x bound on E_ε by (6.4), and since we are in the limiting case of Sobolev injection, we cannot conclude by this method. Instead, we consider moments of the solution of the Vlasov equation of order slightly higher than 2:

LEMMA 6.1. – *Let* $f^{\mathrm{in}} \in L^\infty \cap L^1(\mathbf{T}^2 \times \mathbf{R}^2)$ *satisfy* (2.1) *and* (2.12), *and let* (f_ε) *be a family of weak solutions of* (1.9), (1.7b,c). *Then, for all* $\beta \in [0,\alpha] \cap [0,3[$, *the family*

$$\varepsilon \iint f_\varepsilon |v|^\beta\,\mathrm{d}x\,\mathrm{d}v \textit{ is uniformly bounded in } L^\infty_{\mathrm{loc}}\big(\mathbf{R}_+, L^1\big(\mathbf{T}^2\big)\big).$$

Proof. – The equation governing the propagation of moments can be written

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint f_\varepsilon(t,x,v)|v|^\beta\,\mathrm{d}x\,\mathrm{d}v = \frac{\beta}{\varepsilon}\iint E_\varepsilon(t,x)\cdot v|v|^{\beta-2} f_\varepsilon(t,x,v)\,\mathrm{d}x\,\mathrm{d}v. \tag{6.8}$$

As $\beta - 1 < 2$, applying Lemma 3.1, the L^∞ bound (3.5) and the energy inequality (3.6) leads to

$$\left\| \int f_\varepsilon |v|^{\beta-1}\,\mathrm{d}v \right\|_{L^\infty_t(L_x^{4/(\beta+1)})} \leqslant C\|f^{\mathrm{in}}\|_{L^\infty_{x,v}}^{(3-\beta)/4}\left(\iint f_\varepsilon |v|^2\,\mathrm{d}x\,\mathrm{d}v\right)^{(1+\beta)/4} \leqslant C \tag{6.9}$$

(denoting by C the various constants involved). By Sobolev embedding, the family (E_ε) is uniformly bounded in $L^\infty(\mathbf{R}_+, L^{4/(3-\beta)}(\mathbf{T}^2))$ so that

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\iint |v|^\beta f_\varepsilon\,\mathrm{d}x\,\mathrm{d}v\right| \leqslant \frac{C(\beta)}{\varepsilon}, \tag{6.10}$$

where the constant $C(\beta) \to +\infty$ as $\beta \to 3$. Integrating (6.10) with respect to the time variable gives the expected result. □

We now proceed estimate the third term of (6.1). Using Lemmas 3.1 and 6.1, we get the following estimate on the second moment of the solution of (1.9), (1.7b,c):

$$\begin{aligned}\left\|\int f_\varepsilon |v|^2\right\|_{L^\infty([0,T],L^{(2+\beta)/4}(\mathbf{R}^2))} &\leqslant C\|f^{\mathrm{in}}\|_{L^\infty}^{(\beta-2)/(\beta+2)} \sup_{t\in[0,T]}\left(\iint f_\varepsilon(t,x,v)|v|^\beta\,\mathrm{d}x\,\mathrm{d}v\right)^{4/(2+\beta)}\\ &\leqslant \frac{C(\beta,T)}{\varepsilon^{4/(2+\beta)}}\end{aligned} \tag{6.11}$$

for some $\beta \in]2, \inf(3,\alpha)[$. By (6.6), the family (E_ε) is bounded in $L^\infty(\mathbf{R}_+, L^{4/(\beta-2)}(\mathbf{T}^2))$ and thus

$$\varepsilon E_\varepsilon(t,x)\int_0^\infty |v|^2 f_\varepsilon\big(t,x,|v|(\cos\theta,\sin\theta)\big)|v|\,\mathrm{d}|v| \to 0 \quad \text{in } L^\infty_{\mathrm{loc}}\big(\mathbf{R}_+, L^1\big(\mathbf{T}_x^2\times S_\theta^1\big)\big). \tag{6.12}$$

It follows from (6.3), (6.7) and (6.12) that

$$\partial_\theta \nu(t,x,\theta) = \lim_{\varepsilon\to 0}\nabla_x\cdot\varepsilon\int_0^\infty v|v|^2 f_\varepsilon(t,x,v)|v|\,\mathrm{d}|v|,$$

thereby proving (a).

Proof of (b). – The key point is to obtain an estimate of how $C(\beta)$ varies in (6.10). First, by Sobolev embedding

$$\|E_\varepsilon\|_{L^\infty_t(L^p_x)} \leqslant K(p)\|\rho_\varepsilon\|_{L^\infty_t(L^2_x)}, \tag{6.13}$$

where the constant $K(p)$ satisfies the asymptotic estimate

$$K(p) = \mathrm{O}\big(\sqrt{p}\big) \quad \text{as } p \to +\infty; \tag{6.14}$$

(we refer to the Appendix for a quick proof of (6.14) based on Fourier series). We use this inequality in (6.8) together with (6.4), to obtain

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\iint f_\varepsilon(t,x,v)|v|^\beta\,\mathrm{d}x\,\mathrm{d}v\right| \leqslant \frac{C(\|f^{\mathrm{in}}\|_{L^\infty},\mathcal{E}(f^{\mathrm{in}}))}{\varepsilon}\sqrt{\frac{4}{3-\beta}}. \tag{6.15}$$

We enhance (6.15) by Hölder's inequality

$$\begin{aligned}\iint f_\varepsilon(t,x,v)|v|^\beta\,\mathrm{d}x\,\mathrm{d}v &\leqslant \left(\iint f_\varepsilon(t,x,v)|v|^2\,\mathrm{d}x\,\mathrm{d}v\right)^{\frac{3-\beta}{\beta-1}}\left(\iint f_\varepsilon(t,x,v)|v|^{\frac{\beta+3}{2}}\,\mathrm{d}x\,\mathrm{d}v\right)^{\frac{2\beta-4}{\beta-1}}\\ &\leqslant \frac{C(\|f^{\mathrm{in}}\|_{L^\infty},\||v|^3 f^{\mathrm{in}}\|_{L^1},\mathcal{E}(f^{\mathrm{in}}))}{\varepsilon}\sqrt{\frac{1}{3-\beta}}\left(\frac{t+1}{\varepsilon}\right)^{\frac{2\beta-4}{\beta-1}}.\end{aligned} \tag{6.16}$$

We finally estimate the 3rd moment as follows, using again the interpolation inequality in Lemma 3.1: with the usual notation $p'=p/(p-1)$

$$\begin{aligned}&\left|\frac{\mathrm{d}}{\mathrm{d}t}\iint f_\varepsilon(t,x,v)|v|^3\,\mathrm{d}x\,\mathrm{d}v\right|\\ &\quad\leqslant \frac{3}{\varepsilon}\|E_\varepsilon\|_{L_t^\infty(L_x^p)}\left\|\int f_\varepsilon(t,x,v)|v|^2\,\mathrm{d}v\right\|_{L_x^{p'}}\\ &\quad\leqslant \frac{C(\|f^{\mathrm{in}}\|_{L^\infty},\mathcal{E}(f^{\mathrm{in}}))}{\varepsilon}\sqrt{p}\|f^{\mathrm{in}}\|_{L^\infty}^{\frac{\beta-2}{\beta+2}}\left(\iint f_\varepsilon(t,x,v)|v|^\beta\,\mathrm{d}x\,\mathrm{d}v\right)^{\frac{4}{\beta+2}}\end{aligned} \tag{6.17}$$

where $\beta=4p'-2$. Inequalities (6.16) and (6.17) give

$$\begin{aligned}&\left|\frac{\mathrm{d}}{\mathrm{d}t}\iint f_\varepsilon(t,x,v)|v|^3\,\mathrm{d}x\,\mathrm{d}v\right|\\ &\quad\leqslant C\big(t,\big\|f^{\mathrm{in}}\big\|_{L^\infty},\big\||v|^3 f^{\mathrm{in}}\big\|_{L^1},\mathcal{E}\big(f^{\mathrm{in}}\big)\big)\varepsilon^{-\frac{3\beta-5}{\beta-1}}\sqrt{\frac{\beta+2}{(\beta-2)(3-\beta)}}\end{aligned} \tag{6.18}$$

where β is any element of $]2,3[$. Setting $\beta=2+|\log\varepsilon|^{-1}$ establishes (b). □

7. The 3D results

7.A. Magnetic field of constant direction but variable intensity

In this subsection, we prove Theorem F. We shall adopt the following notations:

$$x=(x',x_3),\ x'=(x_1,x_2),\quad v=(v',v_3),\ v'=(v_1,v_2),\quad v^\perp=(-v_2,v_1,0).$$

The magnetic field is of the form $B(x)=(0,0,b(x'))$ with $b\in C^1(\mathbf{T}^2)$ such that $b\neq 0$ on $\mathbf{T}^2$.

We begin with the following lemma, which is the 3D analogue of Lemma 3.3.

LEMMA 7.1. – *Let* $f^{\mathrm{in}}\in L^\infty\cap L^1(\mathbf{T}^3\times\mathbf{R}^3)$ *satisfy* (2.1). *Let* $b\in C^1(\mathbf{T}^2)$ *such that* $b\neq 0$ *everywhere on* $\mathbf{T}^2$, *and let* (f_ε) *be a family of weak solutions of* (1.7a–c). *The family* (f_ε) *is relatively compact in* w^*-$L^\infty(\mathbf{R}_+\times\mathbf{T}^3\times\mathbf{R}^3)$ *and any of its limit points* f *is of the form*

$$f(t,x,v)=F\big(t,x,|v'|,v_3\big) \tag{7.1}$$

for some $F\in L^\infty(\mathbf{R}_+\times\mathbf{T}^3\times\mathbf{R}_+\times\mathbf{R})$.

Proof. – As in dimension 2, for any $\varepsilon > 0$, f_ε satisfies the following estimates:

$$f_\varepsilon \geqslant 0 \text{ a.e.}; \quad \forall t > 0 \iint f_\varepsilon(t,x,v)\,\mathrm{d}x\,\mathrm{d}v = \iint f^{\mathrm{in}}(x,v)\,\mathrm{d}x\,\mathrm{d}v, \quad \|f_\varepsilon\|_{L^\infty} \leqslant \|f^{\mathrm{in}}\|_{L^\infty} \tag{7.2}$$

and the energy inequality

$$\forall t \geqslant 0, \mathcal{E}\big(f_\varepsilon(t)\big) \leqslant \mathcal{E}\big(f^{\mathrm{in}}\big). \tag{7.3}$$

The family (f_ε) is therefore relatively compact in w-$L^\infty(\mathbf{R}_+ \times \mathbf{T}^3 \times \mathbf{R}^3)$; let f be one of its limit points, the limit of a subsequence of (f_ε) (still abusively denoted (f_ε)) as $\varepsilon \to 0$. Multiplying (2.17) by ε leads to

$$v^\perp \cdot \nabla_v f_\varepsilon = -\frac{\varepsilon}{b}\partial_t f_\varepsilon - \frac{\varepsilon}{b} v \cdot \nabla_x f_\varepsilon - \frac{\varepsilon}{b} E_\varepsilon \cdot \nabla_v f_\varepsilon. \tag{7.4}$$

By our assumption on b, $1/b$ is bounded in $L^\infty(\mathbf{T}^2)$. By the energy inequality (7.3), the family $((1+|v|^2)f_\varepsilon)$ is bounded in $L^\infty(\mathbf{R}_+, L^1(\mathbf{T}^3 \times \mathbf{R}^3))$, so that the first two terms of the right-hand side of (7.4) converge to 0 in the sense of distributions. The energy inequality implies moreover that the family (E_ε) is bounded in $L^\infty(\mathbf{R}_+, L^2(\mathbf{T}^3))$. Combining this with the L^∞ bound (7.2) on f_ε shows that the last term in the right-hand side of (7.4) also converges to 0 in the sense of distributions. Thus

$$v^\perp \cdot \nabla_v f = 0. \tag{7.5}$$

Since the operator $v^\perp \cdot \nabla_v$ generates the group of rotations of axis B in the velocity space, (7.5) implies that the limiting density f depends only on the length of v'. □

Integrating (2.17) with respect to the polar angle of $v' = r\omega$ with $r = |v'|$ leads to

$$\begin{aligned}(\partial_t + v_3\partial_{x_3} + E_{\varepsilon 3}\partial_{v_3}) \int f_\varepsilon(t,x,r\omega,v_3)\,\mathrm{d}\omega + \partial_r \int (E_\varepsilon \cdot \omega) f_\varepsilon(t,x,r\omega,v_3)\,\mathrm{d}\omega \\ = -\partial_{x_1} \int f_\varepsilon(t,x,r\omega,v_3) r\omega_1\,\mathrm{d}\omega - \partial_{x_2} \int f_\varepsilon(t,x,r\omega,v_3) r\omega_2\,\mathrm{d}\omega.\end{aligned} \tag{7.6}$$

First of all, it is easy to see that the right-hand side of (7.6) converges to 0 in the sense of distributions. By the energy inequality (7.3) the family $((1+|v|^2)f_\varepsilon)$ is bounded in $L^\infty(\mathbf{R}_+; L^1(\mathbf{T}^3 \times \mathbf{R}^3))$ so that

$$\int f_\varepsilon(t,x,r\omega,v_3) r\omega\,\mathrm{d}\omega \to \int f(t,x,r\omega,v_3) r\omega\,\mathrm{d}\omega = F(t,x,r,v_3) r \int \omega\,\mathrm{d}\omega = 0 \tag{7.7}$$

by Lemma 7.1.

Next we use the Poisson equation to show some compactness on the electric fields.

LEMMA 7.2. – *Let* $f^{\mathrm{in}} \in L^\infty \cap L^1(\mathbf{T}^3 \times \mathbf{R}^3)$ *satisfy* (2.1). *Let* $b \in C^1(\mathbf{T}^2)$ *such that* $b \neq 0$ *everywhere on* $\mathbf{T}^2$*, and let* (f_ε) *be a family of weak solutions of* (2.17), (1.7b,c). *There exists a positive constant* C *such that*

$$\|E_\varepsilon\|_{L^\infty(\mathbf{R}_+; W^{1,5/3}(\mathbf{T}^3))} \leqslant C, \qquad \|\partial_t E_\varepsilon\|_{L^\infty(\mathbf{R}_+; L^{5/4}(\mathbf{T}^3))} \leqslant C. \tag{7.8}$$

In particular, the family (E_ε) *is relatively compact in* $L^\infty([0,T]; L^p(\mathbf{T}^3))$ *for all* $T > 0$ *and* $p \in [1, 2[$.

Proof. – Lemma 3.1 coupled with the estimates (7.2), (7.3) gives the following bound:

$$\|\rho_\varepsilon\|_{L^\infty(\mathbf{R}^+,L^{5/3}(\mathbf{T}^3))} \leqslant C\|f^{\text{in}}\|_{L^\infty}^{2/5}\mathcal{E}(f^{\text{in}})^{3/5}. \tag{7.9}$$

By the Poisson equation (1.7b),

$$E_\varepsilon = \nabla_x \Delta_x^{-1}\left(\rho_\varepsilon - \iint f^{\text{in}}\,\mathrm{d}x\,\mathrm{d}v\right). \tag{7.10}$$

The first estimate in (7.8) follows directly from (7.9), (7.10). By the continuity equation (i.e., the relation obtained after integrating (2.17) in v) and (7.10)

$$\partial_t E_\varepsilon = \nabla_x \Delta_x^{-1}\nabla_x \cdot \int f_\varepsilon v\,\mathrm{d}v. \tag{7.11}$$

Applying again Lemma 3.1 to control the momentum density $\int f_\varepsilon v\,\mathrm{d}v$ leads to the second estimate in (7.8). The announced compactness property follows from (7.8) and the energy inequality (7.3) by an easy interpolation argument (see [2]). □

Proof of Theorem F. – Let f be a limit point of (f_ε) in w-$L^\infty(\mathbf{R}_+ \times \mathbf{T}^3 \times \mathbf{R}^3)$, the limit of a subsequence of (f_ε) (still abusively denoted (f_ε)) as $\varepsilon \to 0$. By the compactness of the electric fields proved in Lemma 7.2, we have:

$$f_\varepsilon E_\varepsilon \to f E \tag{7.12}$$

in w-$L^\infty([0,T] \times \mathbf{R}^3; L^p(\mathbf{T}^3))$ for all $T > 0$ and all $p \in [1,2[$, with

$$E = \nabla_x \Delta_x^{-1}\left(\rho - \iint f^{\text{in}}\,\mathrm{d}x\,\mathrm{d}v\right). \tag{7.13}$$

In particular, the following convergences hold in the sense of distributions on $\mathbf{R}_+ \times \mathbf{T}^3 \times \mathbf{R}^3$ as $\varepsilon \to 0$:

$$\partial_{v_3}\left(E_{\varepsilon 3}\int f_\varepsilon(t,x,r\omega,v_3)\,\mathrm{d}\omega\right) \to \partial_{v_3}\left(E_3\int f(t,x,r\omega,v_3)\,\mathrm{d}\omega\right) = \partial_{v_3}(E_3 f)\int \mathrm{d}w \tag{7.14}$$

because of the rotational invariance in Lemma 7.1, and

$$\partial_r \int (E_\varepsilon \cdot \omega) f_\varepsilon(t,x,r\omega,v_3)\,\mathrm{d}\omega \to \partial_r \int (E \cdot \omega) f(t,x,r\omega,v_3)\,\mathrm{d}\omega = 0, \tag{7.15}$$

again because of the rotational invariance in Lemma 7.1.

Taking limits as $\varepsilon \to 0$ in (7.6) leads, on account of (7.7), (7.14) and (7.15) to the limiting system (2.18) announced in Theorem F. □

7.B. Case of a magnetic field of constant modulus

In this last case, we conjugate the Vlasov equation by the local rotation generated by the magnetic field. This technique is standard in the theory of averaging of perturbations of ODEs (see for example [18]); for its application to PDEs, we refer for example to [19].

In this last subsection, we use the following notation. Let $u \in \mathbf{R}^3 \setminus \{0\}$ and $s \in \mathbf{R}$; we designate by $R(u,s)$ the rotation of angle $+s$ in $\mathbf{R}^3$ around the axis $\mathbf{R}u$ oriented by u. With this notation, the local rotation generated by the operator $-\frac{1}{\varepsilon} v \wedge B \cdot \nabla_v$ in (1.7a) is therefore $R(B, -t/\varepsilon)$. We therefore change variables in (1.7a) and consider a new unknown function g_ε, as follows:

$$w = R\left(B, \frac{t}{\varepsilon}\right) v, \qquad g_\varepsilon(t,x,w) = f_\varepsilon(t,x,v). \tag{7.16}$$

A straightforward computation shows that f_ε solves (1.7a–d) if and only if g_ε satisfies:

$$\partial_t g_\varepsilon + R\left(B, -\frac{t}{\varepsilon}\right) w \cdot \nabla_x g_\varepsilon + R\left(B, \frac{t}{\varepsilon}\right) E_\varepsilon \cdot \nabla_w g_\varepsilon$$

$$= -\left[\left(R\left(B, -\frac{t}{\varepsilon}\right) w \cdot \nabla_x\right) R\left(B, \frac{t}{\varepsilon}\right)\right] R\left(B, -\frac{t}{\varepsilon}\right) w \cdot \nabla_w g_\varepsilon, \tag{7.17a}$$

$$E_\varepsilon = \nabla_x \Delta_x^{-1}\left(\int g_\varepsilon \,\mathrm{d}w - \iint f^{\mathrm{in}} \,\mathrm{d}x\,\mathrm{d}v\right), \tag{7.17b}$$

$$g_\varepsilon(0,x,w) = f^{\mathrm{in}}(x,w). \tag{7.17c}$$

(Notice that the local densities $\int g_\varepsilon \,\mathrm{d}w$ and $\int f_\varepsilon \,\mathrm{d}v$ are equal since $R(B(x), -t/\varepsilon)$ is an isometry for all x, which entails that the change of variables (7.16) leaves the Lebesgue measure $\mathrm{d}v$ invariant.)

The physical a priori estimates (7.2) and (7.3) (conservation of mass and energy, and Maximum Principle) are still satisfied by any family (f_ε) of weak solutions of (1.7a–d), and therefore by g_ε, again because the change of variables (7.16) leaves the Lebesgue measure $\mathrm{d}v$ invariant. The family (g_ε) is therefore relatively compact in w-$L^\infty(\mathbf{R}_+ \times \mathbf{T}^3 \times \mathbf{R}^3)$; let g be one of its limit points, the limit of a subsequence of (g_ε) still abusively denoted by (g_ε). In the sequel, we shall restrict our attention to such a subsequence.

First, we control the fast time oscillations of the locally rotated number density g_ε, as follows:

LEMMA 7.3. – *Let* $f^{\mathrm{in}} \in L^\infty \cap L^1(\mathbf{T}^3 \times \mathbf{R}^3)$ *satisfy* (2.1). *Let* (f_ε) *be a family of weak solutions of* (1.7a–d) *with B a $C^1(\mathbf{T}^3)$ divergence-free magnetic field of constant strength* $|B| \equiv 1$. *Then, the family g_ε defined by* (7.16) *satisfies the following bounds*: *for all* $\phi \in C_c^1(\mathbf{T}^3 \times \mathbf{R}^3)$:

$$\left|\partial_t \iint \phi g_\varepsilon \,\mathrm{d}x\,\mathrm{d}w\right| \leqslant C\left(f^{\mathrm{in}}\right)\left(1 + \|\nabla_x B\|_{L^\infty(\mathbf{T}^3)}\right)\left[\left\|\left(1+|w|^2\right)\phi\right\|_{W^{1,1}(\mathbf{T}^3\times\mathbf{R}^3)} + \|\nabla_w \phi\|_{L^1(\mathbf{R}^3;L^2(\mathbf{T}^3))}\right]; \tag{7.18}$$

$$\left|\partial_t \iint \phi E_\varepsilon g_\varepsilon \,\mathrm{d}x\,\mathrm{d}w\right| \leqslant C\left(f^{\mathrm{in}}\right)\left(1 + \|\nabla_x B\|_{L^\infty(\mathbf{T}^3)}\right)\left[\left\|\left(1+|w|^2\right)\phi\right\|_{L^1(\mathbf{R}^3;W^{1,5/2}(\mathbf{T}^3))} + \left\|\left(1+|w|^2\right)\nabla_w \phi\right\|_{L^1(\mathbf{R}^3;L^\infty(\mathbf{T}^3))} + \|\phi\|_{L^1(\mathbf{R}^3;L^\infty(\mathbf{T}^3))}\right]. \tag{7.19}$$

Proof. – Multiplying (7.17a) by ϕ and integrating in (x,v) leads to:

$$\partial_t \iint \phi g_\varepsilon \,\mathrm{d}x\,\mathrm{d}w = \iint g_\varepsilon R\left(B, -\frac{t}{\varepsilon}\right) w \cdot \nabla_x \phi \,\mathrm{d}x\,\mathrm{d}w$$
$$+ \iint g_\varepsilon \phi \nabla_x \cdot R\left(B, -\frac{t}{\varepsilon}\right) w \,\mathrm{d}x\,\mathrm{d}w + \iint g_\varepsilon R\left(B, -\frac{t}{\varepsilon}\right) E_\varepsilon \cdot \nabla_w \phi \,\mathrm{d}x\,\mathrm{d}w$$
$$+ \iint g_\varepsilon \phi \nabla_w \cdot \left(\left[\left(R\left(B, -\frac{t}{\varepsilon}\right) w\right) \cdot \nabla_x R\left(B, -\frac{t}{\varepsilon}\right)\right] R\left(B, -\frac{t}{\varepsilon}\right) w\right) \mathrm{d}x\,\mathrm{d}w$$

$$(7.20) \qquad + \iint g_\varepsilon \left[\left(R\left(B, -\frac{t}{\varepsilon}\right) w \right) \cdot \nabla_x R\left(B, -\frac{t}{\varepsilon}\right) \right] R\left(B, -\frac{t}{\varepsilon}\right) w \cdot \nabla_w \phi \, \mathrm{d}x \, \mathrm{d}w.$$

By the Maximum Principle (7.2), the first term in the right-hand side of (7.20) is bounded by a constant, depending on f^{in} only, times the L^1-norm of $|w||\nabla_x \phi|$.

By the energy inequality (7.3), the third term is bounded by a constant depending on f^{in} only times the $L^1_w(L^2_x)$-norm of $\nabla_w \phi$.

The other terms are controlled in terms of the spatial derivatives of the local rotation $R(B, t/\varepsilon)$. We start with the formula

$$(7.21) \qquad R\left(B, \frac{t}{\varepsilon}\right) u = (B \cdot u) B + \big(u - (u \cdot B) B\big) \cos\left(\frac{t}{\varepsilon}\right) + u \wedge B \sin\left(\frac{t}{\varepsilon}\right).$$

It shows that, for all $u \in \mathbf{R}^3$,

$$(7.22) \qquad \left| \nabla_x \left[R\left(B, \frac{t}{\varepsilon}\right) u \right] \right| \leqslant C |\nabla_x B| |u|.$$

Therefore, the second and fourth terms in the right-hand side of (7.20) are bounded by a constant depending on f^{in} times $\|\nabla_x B\|_{L^\infty} \| |w| \phi \|_{L^1}$. By the same token, the last term is bounded by a constant depending on f^{in} times $\|\nabla_x B\|_{L^\infty} \| |w|^2 \nabla_w \phi \|_{L^1}$. This proves (7.18).

In order to prove (7.19), we apply estimate (7.18) with ϕE_ε in the place of ϕ. The same argument as in Lemma 7.2 shows that the family (E_ε) is bounded in $L^\infty(\mathbf{R}_+; W^{1,5/3}(\mathbf{T}^3))$, so that

$$\begin{aligned} \left| \iint \phi E_\varepsilon \partial_t g_\varepsilon \, \mathrm{d}x \, \mathrm{d}w \right| &\leqslant C\big(f^{\mathrm{in}}\big)\big(1 + \|\nabla_x B\|_{L^\infty(\mathbf{T}^3)}\big)\big[\big\| \big(1 + |w|^2\big)\phi \big\|_{L^1(\mathbf{R}^3; W^{1,5/2}(\mathbf{T}^3))} \\ (7.23) \qquad &\quad + \big\| \big(1 + |w|^2\big) \nabla_w \phi \big\|_{L^1(\mathbf{R}^3; L^\infty(\mathbf{T}^3))} + \|\phi\|_{L^1(\mathbf{R}^3; L^\infty(\mathbf{T}^3))}\big]. \end{aligned}$$

As in Lemma 7.2, the family $(\partial_t E_\varepsilon)$ is bounded in $L^\infty(\mathbf{R}_+; W^{1,5/4}(\mathbf{T}^3))$, so that

$$(7.24) \qquad \left| \iint \phi (\partial_t E_\varepsilon) g_\varepsilon \, \mathrm{d}x \, \mathrm{d}w \right| \leqslant C\big(f^{\mathrm{in}}\big) \|\phi\|_{L^1(\mathbf{R}^3; L^4(\mathbf{T}^3))}.$$

Both inequalities (7.23) and (7.24) entail (7.19). □

An immediate consequence of Lemma 7.3 is

COROLLARY 7.4. – *With the same assumptions and notations as in Lemma* 7.3, *for any smooth, zero mean, periodic function a on* $\mathbf{R}$ *and any* $\psi \in C^1_c(\mathbf{T}^3 \times \mathbf{R}^3)$,

$$(7.25) \qquad a(t/\varepsilon)\psi(x, w) g_\varepsilon(t, x, w) \to 0, \qquad a(t/\varepsilon) E_\varepsilon(t, x) \psi(x, w) g_\varepsilon(t, x, w) \to 0$$

as $\varepsilon \to 0$, *in the sense of distributions on* $\mathbf{R}^*_+ \times \mathbf{T}^3 \times \mathbf{R}^3$.

Proof. – This is an instance of "nonstationary phase": to prove it, integrate by parts in t and apply (7.18) and (7.19). □

After these lengthy but necessary preparations, we are ready to give the:

Proof of Theorem G. – As explained before the statement of Lemma 7.3, we restrict our attention to a subsequence (g_ε) converging to g in $L^\infty(\mathbf{R}_+ \times \mathbf{T}^3 \times \mathbf{R}^3)$ weak-*. Define E by:

$$(7.26) \qquad E = \nabla_x \Delta_x^{-1} \left(\int g \, \mathrm{d}w - \iint f^{\mathrm{in}} \, \mathrm{d}x \, \mathrm{d}v \right) = \lim_{\varepsilon \to 0} E_\varepsilon.$$

Notice that, as in Lemma 7.2, $E_\varepsilon \to E$ in $L^\infty([0,T]; L^p(\mathbf{T}^3))$ for all $p \in [1,2[$ and $T > 0$. This and Corollary 7.4 show, in view of formula (7.21) giving the expression of the local rotation, that

$$R\left(B, -\frac{t}{\varepsilon}\right) w \cdot \nabla_x g_\varepsilon \to (B \cdot w)(B \cdot \nabla_x g), \tag{7.27}$$

while

$$R\left(B, \frac{t}{\varepsilon}\right) E_\varepsilon \cdot \nabla_w g_\varepsilon \to (B \cdot E)(B \cdot \nabla_w g). \tag{7.28}$$

It remains to compute the limit of the term in the right-hand side of (7.17a). We recast it in the form

$$\begin{aligned}
&\left[\left(R\left(B, -\frac{t}{\varepsilon}\right) w \cdot \nabla_x\right) R\left(B, \frac{t}{\varepsilon}\right)\right] R\left(B, -\frac{t}{\varepsilon}\right) w \cdot \nabla_w g_\varepsilon \\
&\quad = \left[(B \cdot w)\nabla_B + \cos\left(\frac{t}{\varepsilon}\right)\big(\nabla_w - (B \cdot w)\nabla_B\big) + \sin\left(\frac{t}{\varepsilon}\right)\nabla_{B\wedge w}\right] \\
&\qquad \left[\left(1 - \cos\left(\frac{t}{\varepsilon}\right)\right)(B.U_\varepsilon)B + \sin\left(\frac{t}{\varepsilon}\right)(U_\varepsilon \wedge B)\right] \cdot \nabla_w g_\varepsilon
\end{aligned} \tag{7.29}$$

with

$$U_\varepsilon = \left[(B \cdot w)B + \cos\left(\frac{t}{\varepsilon}\right)\big(w - (B \cdot w)B\big) + \sin\left(\frac{t}{\varepsilon}\right) B \wedge w\right]. \tag{7.30}$$

The general term in (7.29), (7.30) is of the form

$$\partial_{w_i}^\alpha \partial_{x_j}^\beta \left(\cos^m\left(\frac{t}{\varepsilon}\right)\sin^n\left(\frac{t}{\varepsilon}\right)\psi(x,w) g_\varepsilon\right), \tag{7.31}$$

where α, β, m and n are integers while $\psi \in C^1(\mathbf{T}^3 \times \mathbf{R}^3)$. A straightforward application of Corollary 7.4 shows that

$$\partial_{w_i}^\alpha \partial_{x_j}^\beta \left(\cos^m\left(\frac{t}{\varepsilon}\right)\sin^n\left(\frac{t}{\varepsilon}\right)\psi g_\varepsilon\right) \to \partial_{w_i}^\alpha \partial_{x_j}^\beta \big(\langle \cos^m \sin^n \rangle \psi g\big) \tag{7.32}$$

in the sense of distributions on $\mathbf{R}_+^* \times \mathbf{T}^3 \times \mathbf{R}^3$ as $\varepsilon \to 0$, where

$$\langle \cos^m \sin^n \rangle = \int_0^1 \cos^m(2\pi x)\sin^n(2\pi x)\,\mathrm{d}x. \tag{7.33}$$

Thus the right-hand side of (7.17a) converges in the sense of distributions on $\mathbf{R}_+^* \times \mathbf{T}^3 \times \mathbf{R}^3$ to

$$\begin{aligned}
-\Big\{&\big[(B \cdot w)^2(\nabla_B B \cdot B)B + (B \cdot w)^2 \nabla_B B\big] \\
&+ \tfrac{1}{2}\big[(B \cdot w)\nabla_B B \wedge (w \wedge B) - (B \cdot w)\big(\nabla_B B \cdot (w - (B \cdot w)B)\big)B\big] \\
&+ \tfrac{1}{2}\big[\big((\nabla_w - (B \cdot w)\nabla_B\big)B \cdot \big(w - 2(B \cdot w)B\big)\big)B - (w \cdot B)\big(\nabla_w - (B \cdot w)\nabla_B\big)B\big] \\
&+ \tfrac{1}{2}\big[-(B \cdot w)(\nabla_{B\wedge w}B) \wedge B + \big(\nabla_{B\wedge w}B \cdot (B \wedge w)\big)B\big]\Big\} \cdot \nabla_w g.
\end{aligned} \tag{7.34}$$

Since B has constant length, (7.34) can be rewritten as:

$$\Big\{-\tfrac{3}{2}\big[(B\cdot w)^2\nabla_B B-(B\cdot w)(\nabla_B B\cdot w)B\big]-\tfrac{1}{2}\big[(\nabla_w B\cdot w)B-(w\cdot B)\nabla_w B\big]$$
$$-\tfrac{1}{2}\big[(B\cdot w)B\wedge\nabla_{B\wedge w}B-\big((B\wedge\nabla_{B\wedge w}B)\cdot w\big)B\big]\Big\}\cdot\nabla_w g$$
$$=\Big\{+\tfrac{3}{2}(B\cdot w)w\wedge(B\wedge\nabla_B B)-\tfrac{1}{2}w\wedge(B\wedge\nabla_w B)-\tfrac{1}{2}w\wedge\nabla_{B\wedge w}B\Big\}\cdot\nabla_w g. \tag{7.35}$$

Combining (7.27), (7.28) and (7.35) leads to the announced result. □

Appendix

We recall the behavior of the Sobolev constant for the embedding $H^1(\mathbf{T}^2)\subset L^p(\mathbf{T}^2)$ as p tends to its critical value which, in dimension 2, is $p=+\infty$. Let $\alpha>0$ and let $u\in H^{1+\alpha}(\mathbf{T}^2)$; then

$$u(x)=\sum_{k\in\mathbf{Z}^2}\hat{u}(k)\mathrm{e}^{\mathrm{i}k\cdot x},\quad \text{with}\ \sum_{k\in\mathbf{Z}^2}\big(1+|k|\big)^{2+2\alpha}\big|\hat{u}(k)\big|^2=\|u\|^2_{H^{1+\alpha}}<+\infty. \tag{A.1}$$

Thus, for all $x\in\mathbf{T}^2$

$$\big|u(x)\big|\leqslant\sum_{k\in\mathbf{Z}^2}\big|\hat{u}(k)\big|\leqslant C_\alpha\|u\|^2_{H^{1+\alpha}}, \tag{A.2}$$

with

$$C_\alpha=\bigg(\sum_{k\in\mathbf{Z}^2}\frac{1}{(1+|k|)^{2+2\alpha}}\bigg)^{1/2}=\mathrm{O}\bigg(\int_{\mathbf{R}^2}\frac{\mathrm{d}z}{(1+|z|)^{2+2\alpha}}\bigg)^{1/2}=\mathrm{O}\big(\sqrt{1/\alpha}\big), \tag{A.3}$$

as $\alpha\to 0$. By interpolation, $H^1(\mathbf{T}^2)$ embeds into $L^p(\mathbf{T}^2)$ with

$$\|u\|_{L^p}\leqslant C_\alpha^{1/(1+\alpha)}\|u\|_{H^1},\quad \text{for}\ p=\frac{2(1+\alpha)}{\alpha}. \tag{A.4}$$

In (6.14), it suffices to take

$$K(p)=C_\alpha^{1/(1+\alpha)}\quad \text{for}\ p=\frac{2(1+\alpha)}{\alpha}$$

or, in other words,

$$K(p)=C_{2/(p-2)}^{(p-2)/p}=\mathrm{O}\bigg[\bigg(\frac{p}{2}-1\bigg)^{(p-2)/p}\bigg]=\mathrm{O}\big(\sqrt{p}\big). \tag{A.5}$$

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REFERENCES

[1] A.A. ARSEN'EV, Existence in the large of a weak solution of Vlasov's system of equations, *Ž. Vyčisl. Mat. i Mat. Fiz.* 15 (1975) 136–147, 276 (Russian).

[2] J.-P. AUBIN, Un théorème de compacité, *C. R. Acad. Sci. Paris* 256 (1963) 5042–5044.

[3] K. ASANO and S. UKAI, On the Vlasov–Poisson limit of the Vlasov–Maxwell equation. Pattern and waves. Qualitative analysis of nonlinear differential equations, *Stud. Math. Appl*. 18 (1986) 369–383.

[4] Y. BRENIER, Convergence of the Vlasov–Poisson system to the incompressible Euler equations, to appear in *Comm. in Partial Diff. Eq.*

[5] J.-Y. CHEMIN, *Fluides Parfaits Incompressibles*, Astérisque 230 (1995), pp. 177.

[6] P. DEGOND, Local existence of solutions of the Vlasov–Maxwell equations and convergence to the Vlasov–Poisson equations for infinite light velocity, *Math. Methods Appl. Sci*. 8 (1986) 533–558.

[7] P. DEGOND, Global existence of smooth solutions for the Vlasov–Fokker–Planck equation in 1 and 2 space dimensions, *Ann. Sci. Ec. Norm. Super., IV. Ser*. 19 (1986) 519–542.

[8] J.-M. DELORT, Existence de nappes de tourbillon en dimension deux, *J. Amer. Math. Soc.* 4 (3) (1991) 553–586.

[9] R. DIPERNA and P.-L. LIONS, Global weak solutions of Vlasov–Maxwell systems, *Commun. Pure Appl. Math.* 42 (6) (1989) 729–757.

[10] E. FRÉNOD and E. SONNENDRÜCKER, Homogenization of the Vlasov equation and of the Vlasov–Poisson system with a strong external magnetic field, *Asympt. Anal.* 18 (3–4) (1998) 193–213.

[11] E. FRÉNOD and E. SONNENDRÜCKER, Long time behavior of the two-dimensional Vlasov equation with a strong external magnetic field, Report RR-3428, INRIA Lorraine, May 1998.

[12] H. GRAD, The guiding center plasma, in: *Proc. Sympos. Appl. Math.*, Vol. 18, Amer. Math. Soc., Providence, RI, 1967, pp. 162–248.

[13] P. GÉRARD, Résultats récents sur les fluides parfaits incompressibles bidimensionnels (d'après J.-Y. Chemin et J.-M. Delort), *Séminaire Bourbaki*, *1991/92*, Astérisque 206 (1992), Exp. no. 757, 5, 411–444.

[14] F. GOLSE and L. SAINT-RAYMOND, L'approximation centre-guide pour l'équation de Vlasov–Poisson 2D, *C. R. Acad. Sci. Paris, Série I* 327 (1998) 865–870.

[15] E. GRENIER, Pseudo-differential energy estimates of singular perturbations, *Commun. Pure Appl. Math.* 50 (9) (1997) 821–865.

[16] E.M. LIFSHITZ and L.P. PITAEVSKIĬ, *Course of Theoretical Physics (*"*Landau–Lifshitz*"*)*, Vol. 10, Pergamon Press, Oxford, 1981.

[17] P.-L. LIONS, *Mathematical Topics in Fluid Mechanics, Vol. 1: Incompressible Models*, Clarendom Press, Oxford.

[18] P. LOCHAK and C. MEUNIER, *Multiphase Averaging for Classical Systems. With Applications to Adiabatic Theorems*, Springer, Berlin, 1988.

[19] S. SCHOCHET, Fast singular limits of hyperbolic PDEs, *J. Differential Equations* 114 (2) (1994) 476–512.

[20] S. UKAI and T. OKABE, On classical solutions in the large in time of two-dimensional Vlasov's equation, *Osaka J. Math*. 15 (1978) 245–261.