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On the Distribution of Free Path Lengths for the Periodic Lorentz Gas

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Abstract: Consider the domain

$$Z_\varepsilon = \{x \in \mathbf{R}^n \,|\, \mathrm{dist}(x, \varepsilon\mathbf{Z}^n) > \varepsilon^\gamma\},$$

and let the free path length be defined as

$$\tau_\varepsilon(x,\omega) = \inf\{t > 0 \,|\, x - t\omega \in Z_\varepsilon\}\,.$$

The distribution of values of τ_ε is studied in the limit as $\varepsilon \to 0$ for all $\gamma \geq 1$. It is shown that the value $\gamma_c = \frac{n}{n-1}$ is critical for this problem: in other words, the limiting behavior of τ_ε depends only on whether γ is larger or smaller than γ_c.

1. Introduction

The Lorentz gas is a model system of Statistical Mechanics consisting of a large number of like point particles moving freely in a domain of the space where spherical obstacles are disposed with some given distribution. Collisions between two (or more) particles are rare events since these particles have diameter 0. Hence, only collisions involving one particle and one obstacle are taken into account. They are described by some adequate reflection law, the exact nature of which will be of no significance in the present work; the most classical example of such reflection law is of course the case of "specular reflection". The model considered in the present work is the case where the obstacles are periodically distributed; in other words, the centers of the obstacles form a lattice in the space $\mathbf{R}^n$, which, for simplicity, is assumed to be homothetic to $\mathbf{Z}^n$. Finally, each particle is assumed to move with speed 1 in the interval of time between two consecutive collisions with the obstacles. It is the purpose of the present work to study some aspects of the large scale dynamics of such a system.

Thus, let $n \in \mathbf{N}^*$ denote the space dimension and let

$$Z_\varepsilon = \{x \in \mathbf{R}^n \,|\, \mathrm{dist}(x, \varepsilon\mathbf{Z}^n) > \varepsilon^\gamma\}\,, \tag{1.1}$$

for all $0 < \varepsilon < \frac{1}{2}$ and $\gamma \geq 1$. The "free path length" (or equivalently "exit time", since the particles move with speed 1 between two consecutive collisions with the obstacles) is defined as follows, for all $x \in \overline{Z}_\varepsilon$ and $\omega \in S^{n-1}$:

$$\tau_\varepsilon(x,\omega) = \inf\{t > 0 \,|\, x + t\omega \in \partial Z_\varepsilon\}\,. \tag{1.2}$$

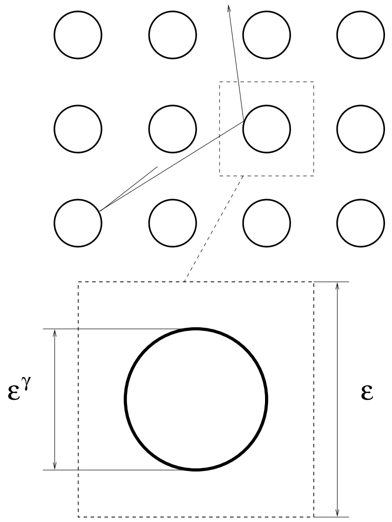

Fig. 1. The billiard table

Clearly τ_ε is a Borelian function for all $0 < \varepsilon < \frac{1}{2}$ and all $\gamma \geq 1$. The present paper studies the distribution of values of τ_ε as $\varepsilon \to 0$, which is one of the main features of the evolution of the Lorentz gas model associated to the domain Z_ε as explained above.

However, this problem is well posed only after a phase space equipped with a Borelian probability measure is defined. The most natural choice in this respect is the following one. Let $Y_\varepsilon = Z_\varepsilon/\varepsilon\mathbf{Z}^n$: topologically Y_ε is a punctured torus; let $Q_\varepsilon = dxd\omega-$meas $(Y_\varepsilon \times S^{n-1})$. Our choice of a phase space is $Y_\varepsilon \times S^{n-1}$ with the Borelian probability measure μ_ε defined by

$$d\mu_\varepsilon(x,\omega) = \frac{1}{Q_\varepsilon} dxd\omega\,. \tag{1.3}$$

Clearly $\tau_\varepsilon(x + \varepsilon k, \omega) = \tau_\varepsilon(x,\omega)$ for all $(x,\omega) \in \overline{Z}_\varepsilon \times S^{n-1}$ and all $k \in \mathbf{Z}^n$ so that τ_ε defines a Borelian function on $Y_\varepsilon \times S^{n-1}$. It is then natural to study the distribution of τ_ε with respect to the probability measure μ_ε. We recall its definition:

Definition. *The distribution ϕ_ε of τ_ε with respect to μ_ε is the push-forward of the measure μ_ε under τ_ε. In other words, ϕ_ε is the unique Borelian probability measure on $[0, +\infty[$ such that, for all $0 < a < b < +\infty$,*

$$\phi_\varepsilon(]a,b[) = \mu_\varepsilon(\{(x,\omega) \in Y_\varepsilon \times S^{n-1} \,|\, a < \tau_\varepsilon < b\})\,. \tag{1.4}$$

The main results in this paper bear on the limiting behavior of ϕ_ε as $\varepsilon \to 0$ and on how it depends on the parameter ε. These results are presented without proof in the next section (Sect. 2). The proofs are relegated to the subsequent sections (Sects. 3 to 5).

We shall conclude this section with a very elementary observation. In the case where particles impinging on the obstacles are specularly reflected, it is natural to consider the map which, to the position and velocity of any particle leaving the boundary of some obstacle associates its position and velocity immediately after the next collision with an obstacle. It is defined by

$$(x, \omega) \mapsto (x' = x + \tau_\varepsilon(x, \omega)\omega; \omega' = \omega - 2\omega \cdot n(x')\, n(x')), \tag{1.5}$$

where $n(x)$ denotes the inward unit normal at point $x \in \partial Z_\varepsilon$. Let then

$$\Sigma_\varepsilon^+ = \{(x, \omega) \in \partial Y_\varepsilon \times S^{n-1} \,|\, \omega \cdot n(x) > 0\} . \tag{1.6}$$

Since any two obstacles in Z_ε are congruent modulo $\varepsilon \mathbf{Z}^n$, the map (1.5) defines a map $B : \Sigma_\varepsilon^+ \to \Sigma_\varepsilon^+$ (sometimes called the billiard map: see for example [Ch1-2]). Let $\Gamma_\varepsilon = \omega \cdot n_x dS(x) d\omega -$ meas (Σ_ε^+); a Borelian probability measure ν_ε is defined on Σ_ε^+ by

$$d\nu_\varepsilon(x, \omega) = \frac{1}{\Gamma_\varepsilon} \omega \cdot n_x dS(x) d\omega\, . \tag{1.7}$$

The probability measure ν_ε is invariant under B, and hence a second choice of a phase space for the Lorentz gas is Σ_ε^+ equipped with the probability measure ν_ε, the dynamics being given by the iterates of the billiard map B. This is usually the phase space and dynamics studied in most of the literature devoted to billiards (see [Ch1-2] and the references therein). The first phase space $(Y_\varepsilon \times S^{n-1}, \mu_\varepsilon)$ is the suspension of $(\Sigma_\varepsilon^+, \nu_\varepsilon)$ under the function τ_ε and the Lorentz gas flow mod. $\varepsilon \mathbf{Z}^n$ (i.e. on $Y_\varepsilon \times S^{n-1}$) is the suspension flow of the map B under the function τ_ε.

In [Ch1-2], the following quantity, called the "geometric mean free path" in [DDG2], is considered:

$$l_\varepsilon = \int_{\Sigma_\varepsilon^+} \tau_\varepsilon(x, \omega) d\nu_\varepsilon(x, \omega)\, . \tag{1.8}$$

As explained in [Ch2] (Sect. 2), it is a natural notion of mean free path because it is the time average of free paths lengths along typical trajectories whenever the map B is ergodic. There is an explicit formula for it, (see [Ch1] Sect. 3.2 or [DDG2] for a quick proof):

$$l_\varepsilon = \frac{Q_\varepsilon}{\Gamma_\varepsilon} = \frac{1}{|B^{n-1}|} \varepsilon^{n-\gamma(n-1)} + O(\varepsilon^\gamma)\, . \tag{1.9}$$

This formula clearly points at the special value

$$\gamma_c = \frac{n}{n-1} \tag{1.10}$$

as being critical. Indeed, as $\varepsilon \to 0$,

- if $\gamma > \gamma_c$, $l_\varepsilon \to +\infty$ as $\varepsilon \to 0$, which seems to indicate a purely ballistic behavior for the Lorentz gas;
- if $1 \leq \gamma < \gamma_c$, $l_\varepsilon \to 0$ as $\varepsilon \to 0$, corresponding to a hydrodynamic limit;
- if $\gamma = \gamma_c$, $l_\varepsilon \to |B^{n-1}|^{-1} > 0$ as $\varepsilon \to 0$, corresponding to the so-called "Boltzmann-Grad limit".

However, it does not seem possible to extract any information about the distribution of free path lengths ϕ_ε defined in (1.4), which is our main object of study here, from the explicit formula (1.9). This simply reflects the fact that the billiard under consideration in this paper does not have the "finite horizon property" (the function τ_ε is not uniformly bounded on Σ_ε) and hence the first phase space $(Y_\varepsilon \times S^{n-1}, \mu_\varepsilon)$ contains more information than the second phase space $(\Sigma_\varepsilon^+, \nu_\varepsilon)$.

Let us close this introductory section with some references. In the case $\gamma = 1$, Bunimovich, Sinai and later Chernov ([BS1-2, BSC1-2]) established the diffusion limit for the Lorentz gas with finite horizon. If the specular reflection condition is replaced by an accommodation reflection condition, a simpler proof, based on PDE methods, leads to a similar diffusion limit: see [BDG]. The Boltzmann-Grad limit ($\gamma = \gamma_c$) has been studied by many authors, in the case where the distribution of obstacles is not periodic as considered here but random: see [Gal, Sp, BBS]. These papers prove that the limiting number density f of gas particles satisfies a linear transport equation with absorption and scattering of the form

$$\partial_t f(t,x,\omega) + \omega \cdot \nabla_x f(t,x,\omega) + \sigma f(t,x,\omega) = \sigma \int_{S^{n-1}} k(\omega,\omega') f(t,x,\omega') d\omega' \,. \quad (1.11)$$

The methods developed in these papers do not apply to the periodic case under consideration in this paper. In fact the limiting behavior of the periodic Lorentz gas in the critical scaling $\gamma = \gamma_c$ is qualitatively different from the one described by (1.11): see Sect. 2, Remark 2.

2. Main Results

With the definitions and notations of Sect. 1, we first state the main theorem in this paper:

Theorem A. *1) If $\gamma > \gamma_c$, $\phi_\varepsilon \to 0$ vaguely as $\varepsilon \to 0$;*

2) If $1 \le \gamma < \gamma_c$, $\phi_\varepsilon \to \delta_0$ weakly as $\varepsilon \to 0$;

3) If $\gamma = \gamma_c$, any vague limit point ϕ of the family (ϕ_ε) is a probability measure and satisfies

$$\limsup_{t\to+\infty} t\phi([t,+\infty[) < +\infty \,;$$

4) If $\gamma = \gamma_c$ and $n = 2$, any vague limit point ϕ of the family (ϕ_ε) satisfies

$$\liminf_{t\to+\infty} t\phi([t,+\infty[) > 0 \,.$$

We recall the terminology for the various topologies on the space of Borelian probability measures on $\mathbf{R}^+$ (see [Bil]). The weak topology is the topology defined by the family of seminorms $\mu \mapsto |\langle \mu, f\rangle|$ for all bounded continuous f's, while the vague topology is the one defined by the subfamily of these same seminorms corresponding to continuous f's with compact support.

Point 1) in Theorem A was proved in [DDG2] (see [G, DDG1] for an alternative proof). Point 2] was essentially proved in [DDG1] (although stated in a different manner there; see [DDG2]) when $n = 2$. It then remains to prove point 2) for all $n > 2$ and points 3) and 4).

Remark 1. When $n > 2$ and $\gamma = \gamma_c$, we can prove that

$$\liminf_{\varepsilon \to 0} t^{n-1}\phi([t, +\infty[) > 0\,,$$

but we don't know whether this or point 3) in Theorem A is optimal: see [GW].

Remark 2. Point 4) in Theorem A or Remark 1 show the difference between the limiting dynamics of the Lorentz gas in the periodic and the random cases. In the random cases studied in [Gal, Sp and BBS], the limiting number density is proved to satisfy an equation of the type (1.11); in particular, the free path length is exponentially distributed (σ being the parameter in the exponential law). In the periodic case, the distribution of free path lengths has only algebraic decay, as shown by Theorem A 4) or Remark 1.

Theorem A depends essentially on the following technical estimates. Before stating them, we need some notations. Let $r \in]0, 1/2[$ and consider

$$Z = \{x \in \mathbf{R}^n \,|\, \text{dist}\,(x, \mathbf{Z}^n) > r\}\,; \quad Y = Z/\mathbf{Z}^n\,; \tag{2.1}$$

$$Q = dxd\omega - \text{meas}\,(Y \times S^{n-1})\,; \quad d\mu(x, \omega) = \frac{1}{Q}dxd\omega\,; \tag{2.2}$$

$$\tau(x, \omega, r) = \inf\{t > 0 \,|\, x + t\omega \in \partial Z\}\,; \quad T(\omega, r) = \sup_{x \in Y} \tau(x, \omega, r)\,. \tag{2.3}$$

Clearly τ is $\mathbf{Z}^n$-periodic and can be considered either as defined for $x \in Z$ or for $x \in Y$.

Remark 3. $T(\omega, r)$ is the quantity referred to as the "ergodization time" in [D1].

With these definitions and notations, we can state

Theorem B. *For all $n \in \mathbf{N}^*$ there exists $C(n) > 0$ such that*

$$d\omega - \textit{meas}\,(\{\omega \in S^{n-1} \,|\, T(\omega, r) > t\}) \leq \frac{C(n)}{r^{n-1}t}\,. \tag{2.4}$$

This estimate is sharp in the case $n = 2$: indeed

Theorem C. *Let $n = 2$. There exists $C' > 0$ such that, for all $t > \frac{1}{r}$.*

$$\mu(\{(x, \omega) \in Y \times S^1 \,|\, \tau(x, \omega, r) > t\}) \geq \frac{C'}{rt}\,. \tag{2.5}$$

Remark 4. Theorem B shows that $\tau(\cdot, \cdot, r) \in L^\infty(Y; L^{1,\infty}(S^{n-1}))$. Theorem C shows that, at least if $n = 2$, $\tau(\cdot, \cdot, r) \notin L^1(Y \times S^{n-1})$. Hence the mean free path in the sense of the first phase space considered in Sect. 1 (that is, $(Y \times S^{n-1}, \mu)$), defined as

$$\int_{Y \times S^{n-1}} \tau(x, \omega, r) d\mu(x, \omega) = +\infty$$

does not contain any information on the Lorentz gas, being infinite for all $r \in]0, 1/2[$.

As an aside result, we improve an upper bound for T due to H.S. Dumas [D1]. We first recall the notations for diophantine vectors: for all $K > 0$, $s \in \mathbf{R}$, let

$$\mathcal{D}(s,K) = \{\omega \in S^{n-1} \,|\, \forall k \in \mathbf{Z}^n \setminus \{0\}\,,\ |\omega \cdot k| \geq K|k|^{-s}\}\,. \tag{2.6}$$

We recall that

$$\forall K > 0\,,\ \forall s < n-1\,,\quad \mathcal{D}(s,K) = \emptyset\,, \tag{2.7}$$

(which is a variant of a result due to Dirichlet, see [Ca] chapter I, Theorem VI), and that

$$\forall s > n-1\,,\quad d\omega - \text{meas}\,(\mathcal{D}(s,K)^c) = O(K)\,. \tag{2.8}$$

Theorem D. *For all $n \in \mathbf{N}^*$ and $s > n-1$, there exists $C(n,s) > 0$ such that, for all $K > 0$ and all $\omega \in \mathcal{D}(s,K)$,*

$$T(\omega,r) \leq \frac{C''(n,s)}{Kr^s}\,.$$

We refer to [D2], [ChGa] for an application of this type of estimate.

3. Proof of Theorem B

Formulation of the ergodization time problem. For all $x \in \mathbf{R}$, let $\|x\| = \inf_{k\in\mathbf{Z}} |x-k|$. Let

$$\mathcal{F} = \{\omega \in S^{n-1} \,|\, \forall 1 \leq i \leq n\,,\ \omega_n \geq |\omega_i|\}\,; \tag{3.1}$$

later, we shall need the following mapping:

$$A:\ \mathcal{F} \to [-1,1]^{n-1}\,,\quad \omega \mapsto A(\omega) = \left(\frac{\omega_i}{\omega_n}\right)_{1\leq i\leq n-1}\,. \tag{3.2}$$

Let $\Omega \in [-1,1]^{n-1}$ and $R \in]0,1/2[$; define $N(\Omega,R)$ as the smallest positive integer N such that

$$\forall z \in [0,1]^{n-1}\,,\quad \min_{l\in\mathbf{Z}\,,\ |l|\leq N}\ \max_{1\leq i\leq n-1} \|z_i - l\Omega_i\| \leq R\,. \tag{3.3}$$

Clearly, if $\Omega \in A(\mathcal{F})$ and if $N \geq N(\Omega,R)$, one has

$$\forall x \in [0,1]^{n-0}\,,\quad \min_{l\in\mathbf{Z}\,,\ |l|\leq N}\ \max_{1\leq i\leq n-1} \left\| x_i - x_n\frac{\omega_i}{\omega_n} - l\frac{\omega_i}{\omega_n}\right\| \leq R\,. \tag{3.4}$$

If $\omega \in \mathcal{F}$, then $\omega_n \geq \frac{1}{\sqrt{n}}$. Therefore, if $T \geq \sqrt{n}(N+1)$,

$$\forall x \in [0,1]^{n-0}\,,\quad \min_{|t|\leq T}\ \max_{1\leq i\leq n} \|x_i - t\omega_i\| \leq R\,, \tag{3.5}$$

by specializing t to be of the form $t = \frac{x_n+l}{\omega_n}$. This argument shows that

$$\forall \omega \in \mathcal{F}\,,\quad T(\omega,r) \leq 2\sqrt{n}N\left(A(\omega), \frac{r}{\sqrt{n}}\right)\,. \tag{3.6}$$

Let ϕ be a nonnegative C^∞ function on $\mathbf{R}^{n-1}$ supported in $[-1,1]^{n-1}$, positive on $]-1,1[^{n-1}$ and let $\phi_R(z) = \sum_{k\in\mathbf{Z}^{n-1}} \phi\frac{z+k}{R}$ for all $R \in]0,1/2[$. Let $(\sigma_l)_{l\in\mathbf{Z}}$ be a nonnegative doubly infinite sequence such that $\sigma_l > 0$ if and only if $|l| < N$. Then $N \geq N(\Omega, R)$ if and only if

$$\forall z \in [0,1]^{n-1}, \quad S_N(z) = \sum_{l \in \mathbf{Z}} \sigma_l \phi_R(z - l\Omega) > 0 . \tag{3.7}$$

For $S_n(z) = 0$ if and only if $\phi_R(z - l\Omega) = 0$ for all $l \in \mathbf{Z}$ such that $|l| \leq N$; obviously $\phi_R(z - l\Omega) = 0$ if and only if $max_{1 \leq i \leq n-1} \|z_i - l\Omega_i\| > R$.

It is then convenient to express S_N in terms of the Fourier coefficients of ϕ_R:

$$\forall z \in [0,1]^{n-1}, \quad S_N(z) = \sum_{\xi \in \mathbf{Z}^{n-1}} \widehat{\phi}_R(\xi) e^{i2\pi\langle \xi, z\rangle} \sum_{|l| \leq N} \sigma_l e^{-i2\pi l \langle \xi, \Omega\rangle}, \tag{3.8}$$

In particular, if one takes $\sigma_l = (1 - \frac{|l|}{N})$ for $|l| \leq N$ and $\sigma_l = 0$ if $|l| > N$, the inner sum in (3.8) is a Fejer kernel, that is to say

$$S_N(z) = \sum_{\xi \in \mathbf{Z}^{n-1}} \widehat{\phi}_R(\xi) e^{i2\pi\langle \xi, z\rangle} F_N(\langle \xi, \Omega\rangle), \tag{3.9}$$

with

$$F_N(z) = \sum_{|l| \leq N} \sigma_l e^{-i2\pi l z} = \frac{1}{N} \frac{\sin^2 \pi N z}{\sin^2 \pi z} . \tag{3.10}$$

Suppose now that $N \leq N(\Omega, R)$; then there exists $z_0 \in [0,1]^{n-1}$ such that $S_N(z_0) = 0$. Hence

$$\widehat{\phi}_R(0) F_N(0) = - \sum_{\xi \in \mathbf{Z}^{n-1} \setminus \{0\}} \widehat{\phi}_R(\xi) e^{i2\pi\langle \xi, z_0\rangle} F_N(\langle \xi, \Omega\rangle),$$

which implies

$$N R^{n-1} \leq \frac{1}{\hat{\phi}(0)} \sum_{\xi \in \mathbf{Z}^{n-1} \setminus \{0\}} |\widehat{\phi}_R(\xi)| F_N(\langle \xi, \Omega\rangle). \tag{3.11}$$

Using the Fejer kernel as above is reminiscent of [M] (chapter 5, Theorem 9).

The weak L^1 type estimate. We now come to the main result of this section, Theorem B' below. It is a slight generalization of Theorem B to the case where the probability measure on S^{n-1} is not the normalized Lebesgue measure.

Let m be a probability measure on S^{n-1}. We assume the existence of $0 < c \leq 1$ and $K > 0$ such that

$$(H) \qquad m^*(r) = \sup_{e \in S^{n-1}} m(\{\alpha \in S^{n-1} \,|\, |\langle \alpha, e\rangle| \leq r\}) \leq K r^c .$$

Obviously, the Lebesgue measure on S^{n-1} satisfies (H) with $c = 1$.

Theorem B'. *B'. Let m be a probability measure on S^{n-1} satisfying the assumption (H) above. Then there exists a constant $C(m,n) > 0$ (depending only on the dimension n and the measure m) such that*

$$m(\{\omega \in S^{n-1} \,|\, T(\omega, r) > t\}) \leq \frac{C(m,n)}{t^c r^{n-c}} .$$

Proof. Let us first restrict our attention to $\omega \in \mathcal{F}$. This can be done without loss of generality: indeed, S^{n-1} can be covered by the images of $\mathcal{F}$ under finitely many elements of the orthogonal group $O_n(\mathbf{R})$; moreover, if a probability measure satisfies (H), its push-forward under an element of the orthogonal group still satisfies (H).

If $\omega \in \mathcal{F}$ and $N \leq N(A(\omega), R)$, then $\Omega = A(\omega)$ must satisfy (3.11)). Hence, applying Chebyshev's inequality shows that

$$m(\{\omega \in \mathcal{F} \,|\, T(\omega, \sqrt{n}R) \geq 2\sqrt{n}N\}) \leq$$

$$m\left(\left\{\omega \in \mathcal{F} \,|\, \frac{1}{\hat{\phi}(0)} \sum_{\xi \in \mathbf{Z}^{n-1}\setminus\{0\}} |\widehat{\phi}_R(\xi)| F_N(\langle \xi, A(\omega)\rangle) \geq NR^{n-1}\right\}\right)$$

$$\leq \frac{1}{\hat{\phi}(0)\, NR^{n-1}} \sum_{\xi \in \mathbf{Z}^{n-1}\setminus\{0\}} |\widehat{\phi}_R(\xi)| \int_{\mathcal{F}} F_N(\langle \xi, A(\omega)\rangle) dm(\omega)\,. \tag{3.12}$$

This shows that

$$m(\{\omega \in \mathcal{F} \,|\, T(\omega, \sqrt{n}R) \geq 2\sqrt{n}N\})$$

$$\leq \frac{1}{\hat{\phi}(0)\, NR^{n-1}} \sum_{\zeta \in \mathbf{Z}^{n-1}\setminus\{0\}} |\widehat{\phi}_R(\zeta)| \cdot \sup_{\xi \in \mathbf{Z}^{n-1}\setminus\{0\}} \int_{\mathbf{T}^1} F_N(z) dm_\xi(z)\,, \tag{3.13}$$

where, for any measurable subset U of $\mathbf{T}^1$,

$$m_\xi(U) = m(\{\omega \in \mathcal{F} \,|\, \langle A(\omega), \xi\rangle \in U \text{ mod. } \mathbf{Z}\})\,. \tag{3.14}$$

In other words, m_ξ is the push-forward of m under the map $\mathcal{F} \to \mathbf{T}^1$ defined by

$$\omega \mapsto \langle A(\omega), \xi\rangle \text{ mod. } \mathbf{Z} = \frac{1}{\alpha_n} \sum_{i=1}^{n-1} \alpha_i \xi_i \text{ mod. } \mathbf{Z}\,.$$

We shall appeal to the next lemma to estimate the integrals appearing in the right-hand side of (3.13).

Lemma 1. *Let m be a probability measure on S^{n-1} satisfying (H), and let m_ξ be associated to m as in (3.14). Then there exists a positive constant $C_0(m, n)$ depending only on the dimension n and the measure m such that*

$$0 \leq \int_{\mathbf{T}^1} F_N(z) dm_\xi(z) \leq C_0(m, n) N^{1-c} |\xi|^{1-c}\,. \tag{3.15}$$

We defer the proof of Lemma 1 to after that of Theorem B'. It follows from (3.15) and the estimate (3.12) that

$$m(\{\omega \in \mathcal{F} \,|\, T(\omega, \sqrt{n}R) \geq 2\sqrt{n}N\}) \leq \frac{C_0(m, n)}{N^c R^{n-1}} \frac{1}{\widehat{\phi}_{(0)}} \sum_{\xi \in \mathbf{Z}^{n-1}\setminus\{0\}} |\widehat{\phi}_R(\xi)||\xi|^{1-c}\,. \tag{3.16}$$

But then, the function ϕ being smooth, one has, for all $l > 0$, the existence of $K_l > 0$ such that

$$|\widehat{\phi}_R(\xi)| \leq K_l R^{n-1} (1 + |R\xi|)^{-l}\,. \tag{3.17}$$

Hence, choosing $l > n - c$ and observing that

$$R^{n-1} \sum_{\xi \in \mathbf{Z}^{n-1}} |R\xi|^{1-c}(1 + |R\xi|)^{-m} \sim \int_{\mathbf{R}^{n-1}} |x|^{1-c}(1 + |x|)^{-m} dx < +\infty, \tag{3.18}$$

we obtain that

$$m(\{\omega \in \mathcal{F} \mid T(\omega, \sqrt{n}R) \geq 2\sqrt{n}N\}) \leq \frac{C'(m, n)}{N^c R^{n-c}}$$

which completes the proof of Theorem B'. □

Proof of Lemma 1. We proceed as in [GLPS]

$$I = \int_{\mathbf{T}^1} F_N(z) dm_\xi(z) \leq N \int_{\|z\| \leq \delta} dm_\xi(z) + \frac{C_1}{N} \int_\delta^{1-\delta} \frac{dm_\xi(z)}{z^2(1-z)^2} \tag{3.19}$$

with $C_1 = \sup_{z \in [0,1]} z^2(1-z)^2 / \sin^2 \pi z$. Then, the definition (3.14) and the assumption (H) on the measure m show that

$$m_\xi(\{z \in \mathbf{T}^1 \mid \|z\| \leq \delta\}) \leq \sum_{|k| \leq \sqrt{n}|\xi|+1} m(\{\omega \in \mathcal{F} \mid |\sum_{i=1}^{n-1} \omega_i \xi_i - k\omega_n| \leq \omega_n \delta\})$$

$$\leq \sum_{|k| \leq \sqrt{n}|\xi|+1} K \left(\frac{\delta \omega_n}{\sqrt{|\xi|^2 + k^2}} \right)^c \leq K' \delta^c |\xi|^{1-c}, \tag{3.20}$$

for some $K' > 0$. Hence,

$$I \leq K_2' N \delta^c |\xi|^{1-c} + \frac{2C_1}{N} \int_\delta^{1-\delta} \left(\frac{1}{z^2} + \frac{1}{(1-z)^2} \right) dm_\xi(z). \tag{3.21}$$

Then, integrating by parts and using (H) leads to

$$\int_\delta^1 \frac{dm_\xi(z)}{z^2} = \left[\frac{1}{z^2} \int_0^z dm_\xi(t) \right]_\delta^1 + \int_\delta^1 \frac{2}{z^3} \left(\int_0^z dm_\xi(t) \right) dz \leq C_2 |\xi|^{1-c} \delta^{c-2}. \tag{3.22}$$

Proceeding in the same manner with the other integral in the right hand side of (3.21) leads to

$$I \leq K_2' N \delta^c |\xi|^{1-c} + C_3 N^{-1} \delta^{c-2} |\xi|^{1-c}. \tag{3.23}$$

Optimizing in δ leads to the choice of $\delta = 1/N$ and hence,

$$I \leq C_0(m, n) N^{1-c} |\xi|^{1-c} \tag{3.24}$$

as announced. □

The following bound for averages of sufficiently small powers of the ergodization time follows from Theorem B' by using a classical interpolation argument.

Corollary B". *Under the assumptions of Theorem B', one has, for all* $0 < \beta < 1$

$$\int_{S^{n-1}} T(\omega, r)^{c\beta} dm(\omega) \leq \frac{2C(m, n)^\beta}{(1-\beta) r^{\beta(n-c)}}.$$

4. Proof of Theorem C

In this section, only the case of $n = 2$ is considered. Let $r \in]0, 1/2[$; the notations Z, Y, μ and τ are as in (2.1)-(2.3). A unit vector $\omega \in \mathbf{R}^2$ will be called irrational if and only if $\omega_1/\omega_2 \in \mathbf{R} \setminus \mathbf{Q}$.

Definition. *An open strip S of $\mathbf{R}^2$ of width l is a subset of $\mathbf{R}^2$ which can be mapped onto $\mathbf{R}\times]0, l[$ with $l > 0$ by a displacement D (i.e. a rotation composed with a translation). The middle third of S is the open strip of $\mathbf{R}^2$ which the same displacement D maps onto $\mathbf{R}\times]\frac{1}{3}l, \frac{2}{3}l[$. The boundary ∂S consists of two parallel straight lines whose direction is determined by a unit vector V of $\mathbf{R}^2$; $\pm V$ will be called the direction of the strip S.*

Channels.

Definition. *A channel in Z is an open strip included in Z of maximal width.*

The idea of considering channels in the context of the periodic Lorentz gas seems to be due to Bleher [Bl] (who used instead the term "corridor") — see also [Da].

It is well-known that, if $\omega \in S^1$ is irrational, for all $x \in \mathbf{R}^2$, the set $x + \mathbf{R}\omega + \mathbf{Z}^2$ is dense in $\mathbf{R}^2$. Hence a channel in Z must have a rational direction. For, if $\mathcal{C}$ is a channel in Z with direction ω, any point $x \in \mathcal{C}$ must satisfy the condition $x + \mathbf{R}\omega \subset Z$, implying that dist $(\mathbf{Z}^2, x + \mathbf{R}\omega) > r$, which is obviously not satisfied if $x + \mathbf{R}\omega + \mathbf{Z}^2$ is dense in $\mathbf{R}^2$. However any rational unit vector is not necessarily a direction of a channel in Z as shown by the next lemma.

Lemma 2. *Let $(p, q) \in \mathbf{Z}^2 \setminus \{0\}$ with p and q coprime, and let $\omega_0 = \frac{1}{\sqrt{p^2+q^2}}(p, q)$. A necessary and sufficient condition for a channel of direction ω_0 to exist is that*

$$\sqrt{p^2 + q^2} < \frac{1}{2r} \, . \tag{4.1}$$

The set of all such unit vectors is denoted by $\mathcal{A}_r$. If $\omega_0 \in \mathcal{A}_r$, the width of any channel of direction ω_0 is

$$W(\omega_0, r) = \frac{1}{\sqrt{p^2 + q^2}} - 2r \, . \tag{4.2}$$

Proof. Let S be a channel with direction ω_0 included in $\mathbf{R}^2 \setminus \mathbf{Z}^2$; its boundary ∂S is the union of two parallel lines L and L', each of which contains infinitely many integer points. Let $qx - py = a$ be an equation for L and $qx - py = a'$ an equation for L'. Since both $L \cap \mathbf{Z}^2$ and $L' \cap \mathbf{Z}^2$ are non-empty, both a and a' belong to $p\mathbf{Z} + q\mathbf{Z} = \mathbf{Z}$ since p and q are coprime. Then

$$\text{dist}\,(L, L') = \frac{|a - a'|}{\sqrt{p^2 + q^2}} \in \frac{1}{\sqrt{p^2 + q^2}}\mathbf{N}^* \, . \tag{4.3}$$

But since S must not contain any integer point, the distance between L and L' must be minimal among the distances between lines of direction ω_0 containing infinitely many integer points, which means that

$$\text{dist}\,(L, L') = \inf \frac{1}{\sqrt{p^2 + q^2}}\mathbf{N}^* = \frac{1}{\sqrt{p^2 + q^2}} \, . \tag{4.4}$$

The same argument shows that if L and L' are two distinct parallel lines tangent to ∂Z,

$$\text{dist}\,(L, L') \in \frac{1}{\sqrt{p^2+q^2}}\mathbf{N}^* \cup \left(\frac{1}{\sqrt{p^2+q^2}} - 2r\right)\mathbf{N}^*\,. \tag{4.5}$$

If $\mathcal{C}$ is now a channel in Z, its width is

$$\inf \frac{1}{\sqrt{p^2+q^2}}\mathbf{N}^* \cup \left(\frac{1}{\sqrt{p^2+q^2}} - 2r\right)\mathbf{N}^* = W(\omega_0, r)$$

as predicted by (4.2). Since a channel in Z has positive width, p and q must satisfy (4.1) if there exists a channel with direction ω_0. Conversely it is easily seen that if L and L' are two distinct parallel lines tangent to ∂Z with dist (L, L') minimal, then L and L' define an open channel in Z. $\square$

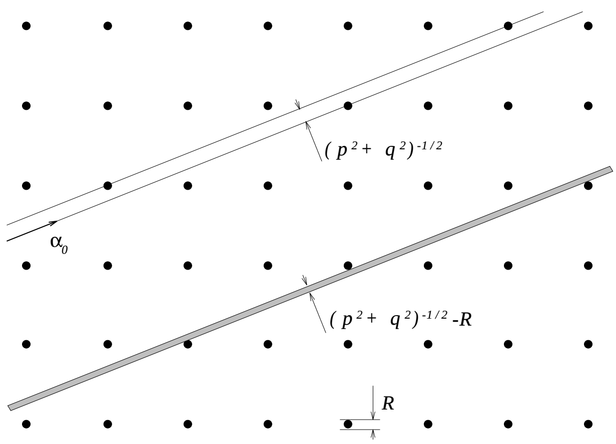

Fig. 2. Channels in O_R with $(p, q) = (2, 5)$

The Lower Bound for the Tail of the Distribution of Exit Times. The main result of this section is the

Theorem C'. *There exists a positive constant C such that for all $A >> 1$, $0 < r < 1$ and $t > 1/r$,*

$$dxd\omega\text{-meas}\,(\{(x, \omega) \in Z \times S^1 \,|\, |x| < \tfrac{1}{\sqrt{2}}A \text{ and } \tau(x, \omega, r) \geq t\}) \geq C\frac{A^2}{rt}\,.$$

Proof. Let $\omega_0 \in \mathcal{A}_r$, and consider a channel $\mathcal{C}$ with direction ω_0. Then let $\mathcal{C}'$ be the middle third of $\mathcal{C}$. The forward or backward trajectory of any point belonging to $\mathcal{C}'$ in a direction $\omega \in S^1$ cannot exit $\mathcal{C}$ in time less than t provided that ω belongs to an arc of S^1 centered at α_0 and of length

$$\theta_0 = 2\arcsin\left(\frac{1}{3t}\left(\frac{1}{\sqrt{p^2+q^2}} - 2r\right)\right) \geq \frac{2}{3t}\left(\frac{1}{\sqrt{p^2+q^2}} - 2r\right)\,. \tag{4.6}$$

Therefore the set $E'(\alpha_0) = \left(\mathcal{C}' \cap \{x \mid |x \cdot \omega_0| < \frac{1}{2}A\}\right) \times \,]\omega_0 - \theta_0/2, \omega_0 + \theta_0/2[$ (i.e., only segments of $\mathcal{C}'$ of length A are considered) has the following properties:

$$\text{if } (x, \omega) \in E'(\omega_0), \text{ then } \tau(x, \omega, r) \geq t; \tag{4.7}$$

and

$$dxd\omega - \text{meas } (E'(\omega_0)) = A \cdot \tfrac{1}{3} W(\omega_0, r) \cdot \theta_0\,. \tag{4.8}$$

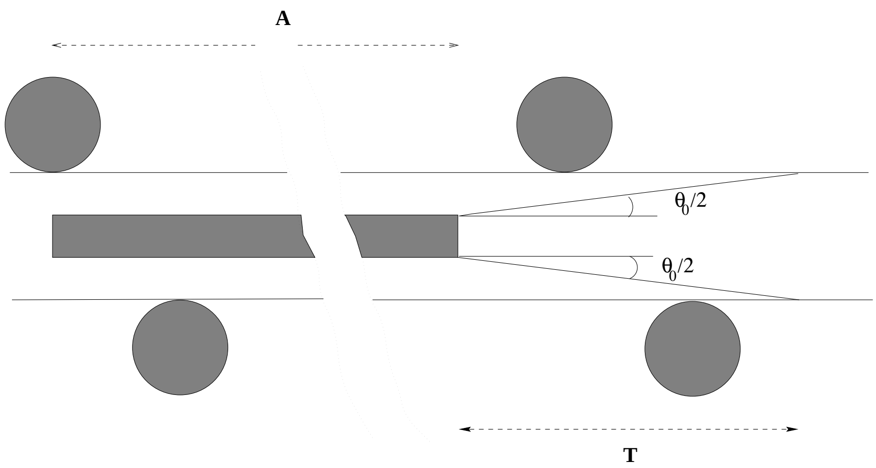

Fig. 3. Construction of the set $E(\omega_0)$

Given ω_0, there are infinitely many strips $\mathcal{C}'$, (and correspondingly $E'(\omega_0)$'s), all being $\mathbf{Z}^n$-translates of each other; consider next a square $Q \subset \mathbf{R}^2$ of side $A >> 1$ centered at the origin, with one side parallel to the direction ω_0, and define

$$E(\omega_0) = \left(\bigcup_{k \in \mathbf{Z}^n} E'(\omega_0) + k \right) \cap (Q \times S^1)\,, \tag{4.9}$$

(that is, the union is taken over all such translates). Let $N(A, \omega_0, r)$ be the number of channels of direction ω_0 intersecting with the square Q; since

$$N(A, \omega_0, r) \geq \tfrac{1}{4} A \sqrt{p^2 + q^2}\,, \tag{4.10}$$

any set $E(\omega_0)$ corresponding to $\omega_0 \in \mathcal{A}_r$ satisfies

$$dxd\omega - \text{meas } (E(\omega_0)) \simeq N(A, \omega_0, r) \cdot A \cdot \tfrac{1}{3} W(\omega_0, r) \cdot \theta_0 \geq A^2 m(p, q, r) \tag{4.11}$$

with

$$m(p, q, r) = \frac{1}{18t} (1 - 2r\sqrt{p^2 + q^2}) \left(\frac{1}{\sqrt{p^2 + q^2}} - 2r \right)\,, \tag{4.12}$$

according to the inequalities (4.6) and (4.10).

The result will now follow by summing over all $\omega_0 \in \mathcal{A}_r$, at least if it can be established that the corresponding sets $E'(\omega_0)$ are disjoint. To this end, consider another rational direction $\omega_1 = \frac{1}{\sqrt{p'^2+q'^2}}(p', q') \in \mathcal{A}_r$. The angle between ω_0 and ω_1 is given by the expression

$$\arcsin\left(\frac{|qp' - pq'|}{\sqrt{p^2+q^2}\sqrt{p'^2+q'^2}}\right) \geq \arcsin\left(\frac{2r}{\sqrt{p^2+q^2}}\right). \tag{4.13}$$

Thus, for $t > 1/3r$, the arc of S^1 centered at ω_0 and of length θ_0 cannot intersect the arc of S^1 of the same length centered at ω_1, for any rational direction $\omega_1 \in \mathcal{A}_r$ different from ω_0. Now, if one varies the direction ω_0 in the class $\mathcal{A}_r$, it follows that

$$\bigcup_{\omega_0 \in \mathcal{A}_r} E(\omega_0) \subset \{(x, \omega) \in Z \times S^1 \,|\, |x| < \tfrac{1}{\sqrt{2}}A \text{ and } \tau(x, \omega, r) \geq t\}, \tag{4.14}$$

and the left side of the inclusion above is a finite disjoint union. Hence

$$dxd\alpha\text{-meas}(\{(x, \omega) \in Z \times S^1 \,|\, |x| < \tfrac{1}{\sqrt{2}}A \text{ and } \tau(x, \omega, r) \geq t\}) \tag{4.15}$$

$$\geq \sum_{\omega_0 \in \mathcal{A}_r} dxd\omega - \text{meas}\,(E(\omega_0)) \geq S \tag{4.16}$$

with

$$S = \sum_{\substack{(p,q) \in B(0, \frac{1}{4r}) \\ G.C.D.(p,q) = 1}} A^2 m(p, q, r)\,. \tag{4.17}$$

Observe that if $p^2 + q^2 \leq \frac{1}{16r^2}$, then

$$m(p, q, R) \geq \frac{1}{72t}\frac{1}{\sqrt{p^2+q^2}} \tag{4.18}$$

so that

$$S \geq \frac{A^2}{72t} \sum_{\substack{(p,q) \in B(0, \frac{1}{4r}) \setminus \{0\} \\ G.C.D.(p,q) = 1}} \frac{1}{\sqrt{p^2+q^2}}\,. \tag{4.19}$$

We interrupt the course of the proof to recall the following very simple

Lemma 3. *Let f be a homogeneous function of degree 0. Then*

$$\sum_{\substack{(p,q) \in B(0,\rho) \setminus \{0\} \\ G.C.D.(p,q) = 1}} f(p, q)\left[\frac{\rho}{\sqrt{p^2+q^2}}\right] = \sum_{(p,q) \in B(0,\rho) \setminus \{0\}} f(p, q) \tag{4.20}$$

(we recall the notation $[\cdot]$ *for "integer part of").*

Applying this to the case where f is the constant function equal to 1, one sees that

$$\sum_{\substack{(p,q)\in B(0,\frac{1}{4r})\setminus\{0\}\\ G.C.D.(p,q)=1}} \frac{1}{\sqrt{p^2+q^2}} \geq 4r\cdot\sharp\{(p,q)\in B(0,\tfrac{1}{4r})\setminus\{0\}\}\,. \tag{4.21}$$

Putting all this together shows that

$$\begin{aligned} dxd\omega - \text{meas}\,(\{(x,\omega)\in Z\times S^1\,|\,|x|<\tfrac{1}{\sqrt{2}}A \text{ and } \tau(x,\omega,r)\geq t\} \\ \geq \frac{A^2 r}{18t}\cdot\sharp\{(p,q)\in B(0,\tfrac{1}{4r})\setminus\{0\}\} \end{aligned} \tag{4.22}$$

from which Theorem C' easily follows. □

Finally, Theorem C' clearly implies Theorem C of Sect. 2.

5. Proof of Theorem A

As we said in Sect. 2, we only have to prove points 2), 3) and 4). An obvious scaling argument shows that

$$\tau_\varepsilon(\varepsilon x,\omega) = \varepsilon\tau(x,\omega,\varepsilon^{\gamma-1})\,. \tag{5.1}$$

Therefore, applying Theorem B and (5.1) leads to

$$\begin{aligned} \phi_\varepsilon([t,+\infty[) &\leq d\omega - \text{meas}\,\left\{\omega\in S^{n-1}\,|\,T(\omega,\varepsilon^{\gamma-1})\geq\frac{t}{\varepsilon}\right\} \\ &\leq \frac{C(n)}{\frac{t}{\varepsilon}(\varepsilon^{\gamma-1}r)^{n-1}} = \frac{C(n)}{tr^{n-1}}\varepsilon^{n-\gamma(n-1)}\,. \end{aligned} \tag{5.2}$$

This proves immediately the inequality in point 3]. Likewise, if $n=2$ and $\gamma=2$, for all $t>1$,

$$\phi_\varepsilon([t,+\infty[) = \mu\left(\left\{(x,\omega)\in Y\times S^1\,|\,\tau(x,\omega,\varepsilon)>\frac{t}{\varepsilon}\right\}\right)\geq\frac{C'}{t}\,, \tag{5.3}$$

which establishes point 4].

If ϕ is a vague limit point of (ϕ_ε) and if χ is a bounded continuous function on $\mathbf{R}^+$, one has, for all $t>0$,

$$\int_{\mathbf{R}^+}\chi(z)d\phi_\varepsilon(z) = \int_{\mathbf{R}^+}a(tz)\chi(z)d\phi_\varepsilon(z) + \int_{\mathbf{R}^+}(1-a(tz))\chi(z)d\phi_\varepsilon(z), \tag{5.4}$$

where a is a continuous function supported in $[0,2]$, equal to 1 on $[0,1]$, and such that $0\leq a\leq 1$. Hence

$$\left|\int_{\mathbf{R}^+}(1-a(tz))\chi(z)d\phi_\varepsilon(z)\right| \leq \|\chi\|_{L^\infty}\phi_\varepsilon([t,+\infty[) \leq \|\chi\|_{L^\infty}\frac{C(n)}{tr^{n-1}}\varepsilon^{n-\gamma(n-1)}. \tag{5.5}$$

If $\gamma=\frac{n}{n-1}$, (5.5) shows that

$$\int_{\mathbf{R}^+} (1 - a(tz))\chi(z)d\phi_\varepsilon(z) \to 0 \text{ as } t \to +\infty, \text{ uniformly in } \varepsilon. \tag{5.6}$$

Hence, for all bounded continuous function χ on $\mathbf{R}^+$, one has

$$\int_{\mathbf{R}^+} \chi(z)d\phi_\varepsilon(z) \to \int_{\mathbf{R}^+} \chi(z)d\phi(z), \tag{5.7}$$

which shows that ϕ is a limit point of (ϕ_ε) in the weak topology; applying (5.7) to $\chi \equiv 1$ shows that ϕ is a probability measure. This completes the proof of point 3).

As for point 2), if $1 \le \gamma < \frac{n}{n-1}$, (5.4) shows that

$$\int_{\mathbf{R}^+} (1 - a(tz))\chi(z)d\phi_\varepsilon(z) \to 0 \text{ as } \varepsilon \to 0, \text{ for all } t > 0. \tag{5.8}$$

Hence, for all $t > 0$,

$$\limsup_{\varepsilon\to 0} \left| \int_{\mathbf{R}^+} \chi(z)d\phi_\varepsilon(z) \right| \le \sup_{z\ge 0} |a(tz)\chi(z)| \le \sup_{0\le z\le 1/t} |\chi(z)| \,. \tag{5.9}$$

If f is a bounded continuous function on $\mathbf{R}^+$ and if, for all $z \ge 0$, one sets $\chi(z) = f(z) - f(0)$, (5.9) shows that

$$\int_{\mathbf{R}^+} f(z)d\phi_\varepsilon(z) \to f(0)\phi_\varepsilon(\mathbf{R}^+) = f(0)\,, \quad \text{as } \varepsilon \to 0. \tag{5.10}$$

This proves point 2].

6. Proof of Theorem D

We consider a fixed direction $\omega \in S^{n-1}$ and let $\Omega := A(\omega)$ as in (3.2). Without loss of generality, we can assume that $\omega \in \mathcal{F}$ (see (3.1) for a definition of $\mathcal{F}$). Assume that $0 < T < T(\omega, R)$; hence, if $R = \frac{r}{\sqrt{n}}$, $N = [T/2\sqrt{n}] < N(\Omega, R)$, which, according to (3.11), implies

$$1 \le \frac{1}{\hat{\phi}(0)} \sum_{\xi\in\mathbf{Z}^{n-1}\setminus\{0\}} \frac{|\widehat{\phi}_R(\xi)|}{R^{n-1}} \frac{F_N(\langle\xi,\Omega\rangle)}{N} \,. \tag{6.1}$$

There exists $C_0 > 0$ such that

$$\frac{\sin^2(\pi N z)}{N^2 \sin^2(\pi z)} \le \frac{C_0}{1 + N^2\|z\|^2} \,. \tag{6.2}$$

On the other hand, we recall that ϕ_R satisfies the regularity estimate (3.17). Putting together (3.17), (6.2) and (6.1), if $N < N(\Omega, R)$, for all $l > 0$ there exists a constant C_l such that

$$1 \le \sum_{\xi\in\mathbf{Z}^{n-1}\setminus\{0\}} \frac{C_l}{(1 + R|\xi|)^l(1 + N^2\|\langle\xi,\Omega\rangle\|^2)} \,. \tag{6.3}$$

Let now

$$E_{M,\delta} = \{\xi \in \mathbf{Z}^{n-1} \setminus \{0\} \,\big|\, |\xi| \le M \,,\ \|\langle\xi,\Omega\rangle\| \le \delta\} \,. \tag{6.4}$$

In order to analyze the sum in the right hand side of (6.3), it will be useful to estimate $\sharp E_{M,\delta}$ (the number of elements in $E_{M,\delta}$).

At this point, we need to introduce the assumption that the direction ω satisfies a Diophantine condition. Let $s > n-1$ and $K > 0$ be such that $\omega \in \mathcal{D}(s, K)$ (see Sect. 2 for the definition of $\mathcal{D}(s, K)$).

Lemma 4. *Let $\omega \in \mathcal{D}(s, K)$, and consider the associated $E_{M,\delta}$.*

i. If $\delta M^s < (4n)^{-s/2} K$, then $E_{M,\delta} = \emptyset$.

ii. There exists a constant $C_2 > 0$ such that for all $\delta > 0$ satisfying $\delta M^s \geq (4n)^{-s/2} K$, one has

$$\sharp E_{M,\delta} \leq C_2 M^{n-1} K^{-\frac{(n-1)}{s}} \delta^{\frac{(n-1)}{s}} .$$

Proof of Lemma 4. We shall only prove ii]; the proof of i] follows the same lines but is slightly simpler. Let ξ_1 and ξ_2 be two different vectors in $E_{M,\delta}$. Then

$$\|\langle \xi_1 - \xi_2, \Omega\rangle\| \leq \|\langle \xi_1, \Omega\rangle\| + \|\langle -\xi_2, \Omega\rangle\| \leq 2\delta . \tag{6.5}$$

On the other hand, suppose that

$$\|\langle \xi_1 - \xi_2, \Omega\rangle\| = |\langle \xi_1 - \xi_2, \Omega\rangle - k| ;$$

in addition one has the identity resulting from the definition (3.2) of Ω:

$$|\langle \xi_1 - \xi_2, \Omega\rangle - k| = \frac{1}{|\alpha_n|} |\langle (\xi_1 - \xi_2, -k), \alpha\rangle| .$$

Then,

$$\|\langle \xi_1 - \xi_2, \Omega\rangle\| \geq \frac{K}{|\alpha_n|} |(|\xi_1 - \xi_2|^2 + k^2)^{-s/2} \geq \frac{K}{(\sqrt{n}+2)^s} |\xi_1 - \xi_2|^{-s} . \tag{6.6}$$

Therefore, putting together (6.5) and (6.6) leads to

$$|\xi_1 - \xi_2| \geq \frac{1}{\sqrt{n}+2} \left(\frac{K}{2\delta}\right)^{1/s} .$$

The conclusion follows easily from the pigeonhole principle. $\square$

Next we estimate the right-hand side of (6.3). Set $K' = \frac{1}{4}(4n)^{-s/2} K$; one has then

$$\begin{aligned} \Sigma &= \sum_{\xi \in \mathbf{Z}^{n-1} \setminus \{0\}} \frac{C_l}{(1 + R|\xi|)^l (1 + N^2 \|\langle \xi, \Omega\rangle\|^2)} \\ &= \sum_{0 < R|\xi| < 1} \frac{C_l}{(1 + R|\xi|)^l (1 + N^2 \|\langle \xi, \Omega\rangle\|^2)} \\ &\quad + \sum_{i \geq 0} \sum_{2^i \leq R|\xi| < 2^{i+1}} \frac{C_l}{(1 + R|\xi|)^l (1 + N^2 \|\langle \xi, \Omega\rangle\|^2)} . \end{aligned}$$

These sums are estimated by using the distribution of values of $\langle \Omega, \xi\rangle$, as follows:

$$\Sigma \leq \sum_{K'R^s<2^{-j}<1} C_l(N2^{-j})^{-2} \cdot \sharp E_{R^{-1},2^{1-j}}$$
$$+ \sum_{i\geq 0} C_l 2^{-il} \sum_{2^{-(i+1)s}K'R^s<2^{-j}<1} (N2^{-j})^{-2} \cdot \sharp E_{2^{i+1}R^{-1},2^{1-j}} .$$

Observe that these summations are truncated according to the first statement in Lemma 4 above. Now we inject in the sums above the estimate provided in point ii] of Lemma 4:

$$\Sigma \leq C_l C_2 K^{-\frac{n-1}{s}} N^{-2} 2^{\frac{n-1}{s}} R^{1-n} \Big(\sum_{K'R^s<2^{-j}<1} 2^{j(2-\frac{n-1}{s})}$$
$$+ 2^{n-1} \sum_{i\geq 0} 2^{i(n-1-l)} \sum_{2^{-(i+1)s}K'R^s<2^{-j}<1} 2^{j(2-\frac{n-1}{s})}\Big)$$
$$\leq C_l C_2 K^{-\frac{n-1}{s}} N^{-2} 2^{\frac{n-1}{s}} R^{1-n}$$
$$\left((K'^{1/s}R)^{-s(2-\frac{n-1}{s})} + 2^{n-1} \sum_{i\geq 0} 2^{i(n-1-l)} (2^{(i+1)s} K'^{-1} R^{-s})^{(2-\frac{n-1}{s})} \right)$$
$$\leq C_l C_2 K^{-\frac{n-1}{s}} N^{-2} 2^{\frac{n-1}{s}} R^{-2s} \left(1 + 2^{2s} \sum_{i\geq 0} 2^{i(2s-l)} \right) K'^{-(2-\frac{n-1}{s})}$$
$$= C_l C_2 K^{-\frac{n-1}{s}} \left(1 + \frac{2^{2s}}{1-2^{2s-l}} \right) N^{-2} 2^{\frac{n-1}{s}} R^{-2s} K'^{-(2-\frac{n-1}{s})} .$$

To summarize: let $s > n-1$ and $K > 0$ be chosen small enough that $\mathcal{D}(s,K) \neq \emptyset$. Let $l > 2s$; for example choose $l = 3s$. There exists a constant $C(n,s) > 0$ such that if $N < N(\Omega, R)$ with $\Omega = A(\omega)$ and $\omega \in \mathcal{D}(s,K)$, then

$$1 \leq \Sigma \leq C(n,s)^2 K^{-2} N^{-2} R^{-2s} . \tag{6.7}$$

Inequality (3.7) can be recast as

$$N \leq \frac{C(n,s)}{KR^s},$$

which, together with (3.6), establishes Theorem D. □

Remark. This method of proof gives the same order of magnitude for the ergodization time as the method based on approximating rotation angles by their continued fraction expansion which was used in [D1,DDG1] to treat the two dimensional case. This observation could lead to the belief that the result of Theorem D is sharp.

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