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Classical Solutions and the Glassey-Strauss Theorem for the 3D Vlasov-Maxwell System

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Abstract

R. Glassey and W. Strauss have proved in [Arch. Rational Mech. Anal. **92** (1986), 59–90] that C^1 solutions to the relativistic Vlasov-Maxwell system in three space dimensions do not develop singularities as long as the support of the distribution function in the momentum variable remains bounded. The present paper simplifies their proof.

1. Introduction

1.1. The relativistic Vlasov-Maxwell system

The Vlasov-Maxwell system is a mean-field, kinetic model for plasmas. Within the formalism of kinetic theory, it describes the motion of a gas of charged, relativistic particles (e.g., electrons or ions). Each particle is subject to the electromagnetic field created by all the other particles, but not to its own self-consistent electromagnetic field which is neglected in this model.

For simplicity, we give up the constraint of global neutrality and consider only the case of a single species of charged particles, with distribution function denoted by f. Precisely, $f(t, x, \xi)$ is the phase space density of particles which at time $t > 0$ are located at the point $x \in \mathbf{R}^3$ and have momentum $\xi \in \mathbf{R}^3$. Let $E \equiv E(t, x)$ and $B \equiv B(t, x)$ be respectively the electric and magnetic fields. In dimensionless variables chosen so that the speed of light, the charge and the mass of the particles are all equal to unity, the unknown distribution function f and electromagnetic field (E, B) satisfy the Vlasov-Maxwell system

$$\begin{aligned} &\partial_t f + v(\xi) \cdot \nabla_x f = -\operatorname{div}_\xi[(E + v(\xi) \times B)f], \\ &\partial_t E - \operatorname{curl}_x B = -j_f, \qquad \operatorname{div}_x E = \rho_f, \\ &\partial_t B + \operatorname{curl}_x E = 0, \qquad \operatorname{div}_x B = 0. \end{aligned} \tag{1.1}$$

In this system, ρ_f and j_f denote respectively the charge and current densities

$$\rho_f(t,x) = \int_{\mathbf{R}^3} f(t,x,\xi)d\xi, \quad j_f(t,x) = \int_{\mathbf{R}^3} v(\xi) f(t,x,\xi)d\xi,$$

while $v(\xi)$ is the relativistic velocity corresponding to a momentum ξ for particles with mass 1 (in units such that the speed of light is 1):

$$v(\xi) = \frac{\xi}{\sqrt{1+|\xi|^2}}. \tag{1.2}$$

The system (1.1) is supplemented with the initial data

$$\begin{aligned} f(0,x,\xi) = f^{in}(x,\xi) \geq 0, &\qquad x \in \mathbf{R}^3,\ \xi \in \mathbf{R}^3, \\ (E,B)(0,x) = (E^{in}, B^{in})(x), &\qquad x \in \mathbf{R}^3. \end{aligned} \tag{1.3}$$

1.2. Glassey-Strauss' conditional result

In [5] GLASSEY & STRAUSS have proved the following remarkable result: any classical (C^1) solution of the system (1.1) does not develop singularities as long as the distribution function f has compact support in the momentum variable ξ. For each $f \equiv f(t,x,\xi)$, let

$$R_f(t) = \inf\{r > 0 \mid f(t,x,\xi) = 0 \text{ for each } x \in \mathbf{R}^3 \text{ and } |\xi| > r\}.$$

Theorem 1.1 (Glassey-Strauss [5]). *Let $\tau > 0$; let $f \in C^1([0,\tau) \times \mathbf{R}^3 \times \mathbf{R}^3)$ and $(E,B) \in C^1([0,\tau) \times \mathbf{R}^3)$ be a solution of (1.1) with initial data $f^{\text{in}} \in C_c^1(\mathbf{R}^3 \times \mathbf{R}^3)$ and $(E^{\text{in}}, B^{\text{in}}) \in C_c^2(\mathbf{R}^3)$ satisfying the compatibility condition*

$$\operatorname{div}_x E^{\text{in}} = \int_{\mathbf{R}^3} f^{\text{in}} d\xi, \quad \operatorname{div}_x B^{\text{in}} = 0. \tag{1.4}$$

If

$$\overline{\lim_{t\to\tau^-}} \left(\|f(t)\|_{W^{1,\infty}_{x,\xi}} + \|(E,B)(t)\|_{W^{1,\infty}_x} \right) = +\infty,$$

then

$$\overline{\lim_{t\to\tau^-}} R_f(t) = +\infty.$$

The proof in [5] is based on a very ingenious argument: derivatives of the fields (E,B) with respect to x are controlled in terms of $(\partial_t + v(\xi)\cdot\nabla_x)^m f$ for $m = 1, 2$ and traded for derivatives of f with respect to ξ only. When the charge and current densities are computed, these ξ-derivatives disappear after integration by parts in the variable ξ. However, this argument itself relies on rather formidable explicit computations, especially for the derivatives of the fields (E,B) in terms of f. In the present paper, we give a shorter proof of Theorem 1.1. The simplifications come mainly from (a) expressing the field (E,B) in terms of distributions of Lienard-Wiechert potentials and (b) a division lemma expressing second derivatives of the forward fundamental solution of the wave equation in terms of the first and second power of the streaming operator acting on that same fundamental solution. In particular (b) is done in a direct and intrinsic way that avoids the repeated use of Green's formula on the wave cone as in [5]. Moreover, (b) extends naturally to the two-dimensional case – which required a distinct treatment by the former method (see [3, 4]).

As in [1], the division lemma (Lemma 3.1) allows us to estimate the regularity of ξ-averages of $u \equiv u(t, x, \xi)$ satisfying a coupled wave-transport system of the form

$$\Box_{t,x} u = f, \qquad (\partial_t + v(\xi) \cdot \nabla_x) f = P(t, x, \xi, D_\xi) g,$$

where g is given. The pseudo-differential approach in [1] leads to L^p estimates ($1 < p < +\infty$) in a quite general framework, being essentially based on the gap between the characteristic speeds of $\Box_{t,x}$ and $(\partial_t + v(\xi) \cdot \nabla_x)$. The method described in the present work makes use, in a deeper way, of the structure of the D'Alembert operator $\Box_{t,x}$, as does [5], and especially of its (forward) fundamental solution Y in physical space (i.e., in the (t, x)-variables). The decomposition (3.5) below plus the fact that Y is a measure (in the case of space dimension 3) leads to L^∞ estimates – at the only expense of a logarithmic term in $\nabla_x f$ for the last piece $b_{ij}^2 Y$ in that decomposition: see Section 5.3. While these features were already present in [5], it seems that Lemma 3.1 is new; in any case, its proof is based solely on commutation properties of $\Box_{t,x}$ with the Lorentz boosts and not on the explicit form of Y.

Another, new proof of Glassey-Strauss' conditional theorem has been recently given by KLAINERMAN & STAFFILANI [7]. Their proof is rather different from either that in [5] or the present one. Maybe a combination of their arguments with the ones in the present work could help in proving the global existence of classical solutions without having to assume Glassey-Strauss' condition on R_f.

2. Distributions of Lienard-Wiechert potentials

The formulation of (1.1) involving distributions of Lienard-Wiechert potentials appeared first in [1] and is recalled below. Let $u \equiv u(t, x, \xi)$ solve

$$\Box_{t,x} u = f, \qquad u\big|_{t=0} = \partial_t u\big|_{t=0} = 0.$$

Choose a vector field $A_I \equiv A_I(t, x) \in \mathbf{R}^3$ such that

$$\operatorname{div}_x A_I = 0, \quad \operatorname{curl}_x A_I = B^{\text{in}}, \tag{2.1}$$

and solve for A^0 the wave equation

$$\Box A^0 = 0, \qquad A^0\big|_{t=0} = A_I, \quad \partial_t A^0\big|_{t=0} = -E^{\text{in}}. \tag{2.2}$$

Define the electromagnetic potential (ϕ, A) by

$$\phi(t, x) = \int_{\mathbf{R}^3} u(t, x, \xi) d\xi, \quad A(t, x) = A^0(t, x) + \int_{\mathbf{R}^3} v(\xi) u(t, x, \xi) d\xi, \tag{2.3}$$

and the electromagnetic field (E, B) by the usual formulas

$$E = -\partial_t A - \nabla_x \phi, \quad B = \mathrm{curl}_x\, A. \tag{2.4}$$

Then the fields (E, B) verify Maxwell's system of equations,

$$\begin{aligned}
&\partial_t E - \mathrm{curl}_x\, B = -\int_{\mathbf{R}^3} v(\xi) f\, d\xi, && \mathrm{div}_x\, E = \int_{\mathbf{R}^3} f\, d\xi,\\
&\partial_t B + \mathrm{curl}_x\, E = 0, && \mathrm{div}_x\, B = 0,\\
&E\big|_{t=0} = E^{\mathrm{in}}, && B\big|_{t=0} = B^{\mathrm{in}}.
\end{aligned}$$

Hence the relativistic Vlasov-Maxwell system (1.1) can be put in the equivalent form

$$\begin{aligned}
\partial_t f + v(\xi)\cdot\nabla_x f &= \mathrm{div}_\xi [K_u f],\\
\Box_{t,x} u &= f,
\end{aligned} \tag{2.5}$$

where K_u is minus the Lorentz force field given by the formula

$$\begin{aligned}
K_u(t,x,\xi) = {}& \partial_t A^0(t,x) - v(\xi)\times \mathrm{curl}_x\, A^0(t,x)\\
& + \partial_t \int_{\mathbf{R}^3} v(\xi) u(t,x,\xi) d\xi + \nabla_x \int_{\mathbf{R}^3} u\, d\xi\\
& - v(\xi)\times \mathrm{curl}_x \int_{\mathbf{R}^3} v(\xi) u(t,x,\xi)\, d\xi
\end{aligned} \tag{2.6}$$

with A^0 being defined by (2.2), (2.1). The initial conditions are

$$f\big|_{t=0} = f^{\mathrm{in}}, \quad u\big|_{t=0} = 0, \quad \partial_t u\big|_{t=0} = 0, \tag{2.7}$$

where f^{in}, E^{in} and B^{in} are assumed to satisfy the compatibility condition (1.4). Notice that the electromagnetic potential (2.3) satisfies the Lorentz gauge condition

$$\partial_t \phi + \mathrm{div}_x\, A = 0. \tag{2.8}$$

(Indeed, setting $G = \partial_t \phi + \mathrm{div}_x\, A$ and averaging Vlasov's equation in ξ, we have $\Box G = \partial_t \rho_f + \mathrm{div}_x\, j_f = 0$ with $G\big|_{t=0} = \partial_t G\big|_{t=0} = 0$).

3. A division lemma

Let $Y \in \mathcal{D}'(\mathbf{R}^4)$ be the forward fundamental solution of the d'Alembertian, which is characterized by

$$\Box_{t,x} Y = \delta_{(t,x)=(0,0)}, \qquad \mathrm{supp} Y \subset \{(t,x)\in\mathbf{R}^4 \mid |x| \le t\}.$$

Although most of the present section can be understood without using the explicit formula giving Y, we recall it below for convenience (with a slight abuse of notation):

$$Y(t,x) = \frac{\mathbf{1}_{t>0}}{4\pi t}\,\delta(|x|-t). \tag{3.1}$$

Notice that the distribution Y is homogeneous of degree -2 in $\mathbf{R}^4$. For $j = 1, 2, 3$, let $L_j = x_j \partial_t + t \partial_{x_j}$; it is easy to check that $[\Box, L_j] = 0$. On the other hand, $L_j \delta_{(t,x)=(0,0)} = 0$, and this relation, together with the fact that L_j commutes with $\Box$, implies that

$$L_j Y = 0, \qquad j = 1, 2, 3. \tag{3.2}$$

With each $v \in \mathbf{R}^3$ is associated the streaming operator $T = \partial_t + v \cdot \nabla_x$. Let $\mathcal{M}_m$ be the space of C^∞ homogeneous functions of degree m on $\mathbf{R}^4 \setminus 0$. Below, we use the notation

$$x_0 := t, \qquad \text{and } \partial_j := \partial_{x_j}, \quad j = 0, \dots, 3. \tag{3.3}$$

The main result in the present section is

Lemma 3.1 (Division lemma). *For each $v \in \mathbf{R}^3$ such that $|v| < 1$,*

- *there exists functions $a_i^k \equiv a_i^k(t, x)$ where $i = 0, \dots, 3$ and $k = 0, 1$, such that $a_i^k \in \mathcal{M}_{-k}$ and*

$$\partial_i Y = T(a_i^0 Y) + a_i^1 Y, \qquad i = 0, \dots, 3; \tag{3.4}$$

- *there exists functions $b_{ij}^k \equiv b_{ij}^k(t, x)$ with $i, j = 0, \dots, 3$, $k = 0, 1, 2$, such that $b_{ij}^k \in \mathcal{M}_{-k}$ and*

$$\partial_{ij}^2 Y = T^2(b_{ij}^0 Y) + T(b_{ij}^1 Y) + b_{ij}^2 Y, \qquad i, j = 0, \dots, 3; \tag{3.5}$$

- *moreover, the functions b_{ij}^2 satisfy the conditions*

$$\int_{\mathbf{S}^2} b_{ij}^2(1, y) d\sigma(y) = 0, \qquad i, j = 0, \dots, 3, \tag{3.6}$$

where $d\sigma(y)$ is the rotation-invariant surface element on the unit sphere $\mathbf{S}^2$ of $\mathbf{R}^3$. In both formulas (3.4) *and* (3.5)*, $a_i^0 Y$, $a_i^1 Y$, $b_{ij}^0 Y$ and $b_{ij}^1 Y$ designate, for each $i, j = 0, \dots, 3$, the unique extensions*[1] *as homogeneous distributions on $\mathbf{R}^4$ of those same expressions – which are a priori only defined on $\mathbf{R}^4 \setminus 0$. Likewise, $b_{ij}^2 Y$ designates, for $i, j = 0, \dots, 3$ the unique extension as a homogeneous distribution of degree -4 on $\mathbf{R}^4$ of that same expression for which the relation* (3.5) *holds in the sense of distributions on $\mathbf{R}^4$.*

Remark. The first and second statements in Lemma 3.1 hold verbatim in space dimension 2. As for the third statement, the degree of homogeneity of $b_{ij}^2 Y$ is -3 in $\mathbf{R}^3$ in the case of space dimension 2, and the condition (3.6) becomes

$$\int_{|y|<1} \frac{b_{ij}^2(1, y)}{\sqrt{1 - |y|^2}} dy = 0, \qquad i, j = 0, \dots, 2. \tag{3.7}$$

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[1] We abandon in the main body of the text the notation $\dot{f}$ for the unique homogeneous extension to $\mathbf{R}^4$ of a distribution f on $\mathbf{R}^4 \setminus 0$ that is homogeneous of degree > -4; this notation is used in the appendix only for the sake of clarity.

Proof. Observe that

$$\sum_{j=1}^{3} v_j L_j = (x \cdot v - t)\partial_t + tT,$$

$$(t - x \cdot v)L_i + x_i \sum_{j=1}^{3} v_j L_j = t[(t - x \cdot v)\partial_i + x_i T], \quad i = 1, 2, 3.$$

These relations and (3.2) imply that

$$(t - x \cdot v)\partial_t Y = tTY, \quad t(x \cdot v - t)\partial_i Y = t x_i TY, \quad i = 1, 2, 3. \tag{3.8}$$

Set

$$\alpha_0(t, x) = \frac{t}{t - x \cdot v}, \qquad \alpha_i(t, x) = \frac{x_i}{x \cdot v - t}, \quad i = 1, 2, 3.$$

Since $|v| < 1$, the functions α_i are C^∞ near the support of the restriction of Y to $\mathbf{R}^4 \setminus 0$: hence $\alpha_i TY$ defines, for $i = 0, \ldots, 3$, a distribution on $\mathbf{R}^4 \setminus 0$ that is homogeneous of degree -3. It has a unique extension as a homogeneous distribution of degree -3 on $\mathbf{R}^4$, still denoted by $\alpha_i TY$. Because of (3.8), the distribution $\partial_i Y - \alpha_i TY$ has support in the set $\{(t, x) \in \mathbf{R}^4 \mid x \cdot v = t\} \cup \{(t, x) \in \mathbf{R}^4 \mid t = 0\}$; since Y is supported in the wave cone $\{(t, x) \in \mathbf{R}^4 \mid |x| \le t\}$, it follows that

$$\begin{aligned} \operatorname{supp}(\partial_i Y - \alpha_i TY) \subset &\{(t, x) \in \mathbf{R}^4 \mid x \cdot v = t \text{ or } t = 0\} \\ &\cap \{(t, x) \in \mathbf{R}^4 \mid |x| \le t\} = \{(0, 0)\}. \end{aligned}$$

Thus $\partial_i Y - \alpha_i TY$ is both a homogeneous distribution on $\mathbf{R}^4$ of degree -3 and a finite linear combination of $\delta_{(t,x)=(0,0)}$ and of its derivatives: hence

$$\partial_i Y - \alpha_i TY = 0, \quad i = 0, \ldots, 3.$$

The same holds if we replace α_i by a smooth truncation a_i^0 of it near its singular set: indeed, as observed above, this singular set $\{(t, x) \in \mathbf{R}^4 \mid x \cdot v = t\}$ does not intersect the support of Y restricted to $\mathbf{R}^4 \setminus 0$. Hence $\partial_i Y = a_i^0 TY = T(a_i^0 Y) - T(a_i^0)Y$ and formula (3.4) holds with

$$a_i^0(t, x) = \alpha_i(t, x)\chi\left(\frac{|x|}{t}\right), \qquad \text{and } a_i^1 = -Ta_i^0, \quad i = 0, \ldots, 3, \tag{3.9}$$

where $\chi \in C_c^\infty(\mathbf{R}_+)$ satisfies

$$0 \le \chi \le 1, \quad \chi\big|_{\left[0, \frac{1}{2} + \frac{1}{2|v|}\right]} \equiv 1, \quad \operatorname{supp}\chi \subset \left[0, \tfrac{1}{|v|}\right[.$$

By the same argument, the equality

$$\partial_i(mY) = T(ma_i^0 Y) + \left(\partial_i m - T(ma_i^0)\right) Y, \qquad i = 0, \ldots, 3 \tag{3.10}$$

holds in the sense of distributions on $\mathbf{R}^4$ for each $m \in \mathcal{M}_0$ – where mY, $ma_i^0 Y$ and $(\partial_i m - T(ma_i^0))Y$ designate the homogeneous extensions to $\mathbf{R}^4$ of these same distributions that are defined and homogeneous of degree > -4 on $\mathbf{R}^4 \setminus 0$.

If $m \in \mathcal{M}_{-1}$, the distributions mY and $ma_i^0 Y$ for $i = 0, \ldots, 3$ are homogeneous of degree -3 in $\mathbf{R}^4 \setminus 0$ and thus have unique extensions to $\mathbf{R}^4$ as homogeneous distributions of degree -3 (see appendix). Since

$$\partial_i(mY) - T(ma_i^0 Y) = \left(\partial_i m - T(ma_i^0)\right) Y, \qquad i = 0, \ldots, 3$$

in the sense of distributions on $\mathbf{R}^4 \setminus 0$, the right-hand side of the above equality extends as a homogeneous distribution of degree -4 on $\mathbf{R}^4$. Hence (see appendix)

$$\mathrm{Res}_0 \left(\partial_i m - T(ma_i^0)\right) Y = 0.$$

Using (6.3) and the formula for Y in the case of space dimension 3, we can write this condition in the following, more explicit form: for each $\chi \in C_c^\infty((0, +\infty))$,

$$\int_{\mathbf{R}^4} \left(\partial_i m - T(ma_i^0)\right)(t, x)\chi(t^2 + |x|^2)Y(t, dx)dt$$
$$= \int_0^{+\infty} \frac{\chi(2t^2)}{4\pi t} \int_{|x|=t} \left(\partial_i m - T(ma_i^0)\right)(t, x)d\sigma_t(x)dt = 0.$$

Here, $d\sigma_t(x)$ designates the surface element on the sphere of equation $|x| = t$. Also, in the case of space dimension 3, the distribution Y is in fact a measure – we recall from (3.1) that $Y(t, dx) \equiv \frac{\mathbf{1}_{t>0}}{4\pi t} d\sigma_t(x)$ – which makes it legitimate to write the left-hand side of the equality above as an integral.

The function $\partial_i m - T(ma_i^0)$ is homogeneous of degree -2 on $\mathbf{R}^4 \setminus 0$, so that, in terms of the new variable $y = x/t$, the last integral in the relation above becomes

$$\int_0^{+\infty} \frac{\chi(2t^2)}{4\pi t} \int_{|x|=t} \left(\partial_i m - T(ma_i^0)\right)(t, x)d\sigma_t(x)dt$$
$$= \int_0^{+\infty} \frac{\chi(2t^2)}{4\pi t} dt \int_{|y|=1} \left(\partial_i m - T(ma_i^0)\right)(1, y)d\sigma(y).$$

Eventually

$$\int_{\mathbf{S}^2} \left(\partial_i m - T(ma_i^0)\right)(1, x)d\sigma(x) = 0. \tag{3.11}$$

In order to obtain (3.5), we apply ∂_j to both sides of the relation (3.4) so as to obtain

$$\partial_{ij}^2 Y = T(\partial_i(a_j^0 Y)) + \partial_i(a_j^1 Y), \quad i, j = 0, \ldots, 3.$$

Then we apply (3.10), first with $m = a_j^0 \in \mathcal{M}_0$, then with $m = a_j^1 = -Ta_j^0 \in \mathcal{M}_{-1}$: this leads to (3.5) with

$$\begin{aligned} b_{ij}^0 &= a_i^0 a_j^0, \\ b_{ij}^1 &= \partial_i a_j^0 - T(a_i^0 a_j^0) + a_i^0 a_j^1, \quad i, j = 0, \ldots, 3, \\ b_{ij}^2 &= \partial_i a_j^1 - T(a_i^0 a_j^1), \end{aligned} \tag{3.12}$$

the functions a_i^k being defined in (3.9).

Finally, the condition (3.11) with $m = a_j^1$ is equivalent to (3.6).

4. Bounds on the electromagnetic field

After these preparations, we give the proof of Theorem 1.1. Since the solution (f, E, B) belongs to $C^1([0,\tau)\times\mathbf{R}^3\times\mathbf{R}^3)\times C^1([0,\tau)\times\mathbf{R}^3)^2$, the distribution function f is constant along the characteristic curves of the vector field $(v(\xi), -K_u(t,x,\xi))$ – observe that $\operatorname{div}_\xi K_u(t,x,\xi)=0$ (see (2.5), (2.6)). In particular

$$\|f\|_{L^\infty([0,\tau]\times\mathbf{R}^3\times\mathbf{R}^3)} = \|f^{in}\|_{L^\infty(\mathbf{R}^3\times\mathbf{R}^3)}, \tag{4.1}$$

and, since f^{in} has compact support,

$$\sup_{t\in[0,\tau']} R_f(t) < +\infty \text{ for each } \tau'\in[0,\tau).$$

Hence proving Theorem 1.1 amounts to proving the implication

$$\sup_{t\in[0,\tau)} R_f(t) < +\infty \Longrightarrow \sup_{t\in[0,\tau)} \left(\|f(t)\|_{W^{1,\infty}_{x,\xi}} + \|(E,B)(t)\|_{W^{1,\infty}_x} \right) < +\infty. \tag{4.2}$$

From now on, assume that

$$\sup_{t\in[0,\tau)} R_f(t) = r^*. \tag{4.3}$$

In other words

$$f(t,x,\xi)\equiv 0 \text{ and } u(t,x,\xi)\equiv 0, \quad t\in[0,\tau),\ x\in\mathbf{R}^3,\ |\xi|>r^*. \tag{4.4}$$

Next we want to estimate the electromagnetic field in $L^\infty([0,\tau)\times\mathbf{R}^3)$. Start from the relation[2] $u = Y\star(\mathbf{1}_{t\geq 0}f)$ and use Lemma 3.1 to compute, for each $m\equiv m(\xi)$ in $C(\mathbf{R}^3)$,

$$\begin{aligned}\partial_j\int m(\xi)u(t,x,\xi)d\xi &= \int m(\xi)\left(\partial_j Y\star(\mathbf{1}_{t\geq 0}f)\right)(t,x,\xi)d\xi \\ &= \int m(\xi)\left((a_j^0 Y)\star T(\mathbf{1}_{t\geq 0}f)\right)(t,x,\xi)d\xi \\ &\quad + \int m(\xi)\left((a_j^1 Y)\star(\mathbf{1}_{t\geq 0}f)\right)(t,x,\xi)d\xi\end{aligned}$$

for $j=0,\ldots,3$, with $a_j^k\equiv a_j^k(t,x,\xi)$ given by (3.9) for $v\equiv v(\xi)$ as in (1.2). First, $a_j^k\in C^\infty((\mathbf{R}^4\setminus 0)\times\mathbf{R}^3)$; also $\partial_\xi^\beta a_j^k(\cdot,\cdot,\xi)$ is an element of $\mathcal{M}_{-k}$ for each $\xi\in\mathbf{R}^3$ and each multi-index $\beta\in\mathbf{N}^3$. By the first equation in (2.5),

$$T(\mathbf{1}_{t\geq 0}f) = \delta_{t=0}f^{\mathrm{in}} + \mathbf{1}_{t\geq 0}\operatorname{div}_\xi(K_u f);$$

[2] In what follows the notation $f\star g$ always means convolution in $\mathbf{R}^4_{t,x}$; the symbol $\star_x$ designates the convolution in the variable $x\in\mathbf{R}^3$ only.

hence, if $m \in W^{1,\infty}$, we find that

$$\begin{aligned}\int m(\xi)\left((a_j^0 Y)\star T(\mathbf{1}_{t\geq 0}f)\right)(t,x,\xi)d\xi \\ = \int \left(\left(-\nabla_\xi(ma_j^0)Y\right)\star(\mathbf{1}_{t\geq 0}K_u f)\right)(t,x,\xi)d\xi \\ + \int m(\xi)\left(\left(a_j^0(t,\cdot,\cdot)Y(t,\cdot)\right)\star_x f^{\text{in}}\right)(x,\xi)d\xi.\end{aligned}$$

Let $\phi \in C_c^\infty(\mathbf{R}^3)$ satisfy

$$\phi \geq 0, \quad \phi(\xi) = 1 \text{ for } |\xi| \leq r^*, \quad \phi(\xi) = 0 \text{ for } |\xi| \geq 2r^*. \tag{4.5}$$

By the support condition (4.4), for each $m \in C(\mathbf{R}^3)$, we have

$$\begin{aligned}\int_{\mathbf{R}^3} m(\xi)f(t,x,\xi)d\xi = \int_{\mathbf{R}^3}\phi(\xi)m(\xi)f(t,x,\xi)d\xi, \\ \int_{\mathbf{R}^3} m(\xi)u(t,x,\xi)d\xi = \int_{\mathbf{R}^3}\phi(\xi)m(\xi)u(t,x,\xi)d\xi.\end{aligned}\tag{4.6}$$

Since $Y(t,\cdot)$ is a positive measure with total mass t, it follows from (4.6) that

$$\begin{aligned}&\left|\partial_j \int m(\xi)u(t,x,\xi)d\xi\right| \\ &\leq \|m\|_{W^{1,\infty}}\|\phi a_j^0\|_{L^\infty_{t,x}(W^{1,\infty}_\xi)}\tfrac{4}{3}\pi r^{*3}\int_0^t (t-s)\|fK_u(s,\cdot,\cdot)\|_{L^\infty}\,ds \\ &\quad +\|m\|_{L^\infty}\|\phi t a_j^1\|_{L^\infty}\tfrac{4}{3}\pi r^{*3}\int_0^t \|f(s,\cdot,\cdot)\|_{L^\infty}ds \\ &\quad +\|m\|_{L^\infty}\|\phi a_j^0\|_{L^\infty}\tfrac{4}{3}\pi r^{*3} t\|f^{\text{in}}\|_{L^\infty}.\end{aligned}\tag{4.7}$$

Without loss of generality, we only consider the case where $B^{\text{in}} = 0$; hence $A^0 = -Y(t,\cdot)\star_x E^{\text{in}} \in C_t(W_x^{2,\infty})$. We recall at this point the elementary estimates that hold for $k = 0, 1, 2$:

$$\begin{aligned}\|A^0(t)\|_{W_x^{k,\infty}} &\leq t\|E^{\text{in}}\|_{W^{k,\infty}} \\ \|\partial_t A^0(t)\|_{W_x^{k-1,\infty}} &\leq (1+t)\|E^{\text{in}}\|_{W^{k,\infty}}.\end{aligned}\tag{4.8}$$

Define

$$I_m(t) = \sup_{j=0,\dots,3}\left\|\partial_j \int m(\xi)u(t,\cdot,\xi)d\xi\right\|_{L^\infty};$$

by using (4.8), (4.7) and (2.6), we see, for each $t \in [0,\tau)$ and some positive constant $C(\tau, r^*, \|m\|_{W^{1,\infty}}, \|f^{\text{in}}\|_{L^\infty}) > 0$, that

$$I_m(t) \leq C(\tau, r^*, \|m\|_{W^{1,\infty}}, \|f^{\text{in}}\|_{L^\infty})\left(1 + \int_0^t (I_1(s) + I_v(s))\,ds\right). \tag{4.9}$$

Using (4.9) for $m \equiv 1$ and $m = v$ and applying Gronwall's inequality, we find that

$$\sup_{t\in[0,\tau)} I_m(t) < +\infty \quad \text{for each } m \in W^{1,\infty}(\mathbf{R}^3). \tag{4.10}$$

In particular, using again (2.6), we find eventually that

$$\|K_u\|_{L^\infty([0,\tau)\times\mathbf{R}^3;W^{k,\infty}_\xi)} < +\infty \quad \text{for each } k \geq 0. \tag{4.11}$$

5. Bounds on first derivatives

For each $m \in C(\mathbf{R}^3)$, we have, by using Lemma 3.1 and the support condition (4.4),

$$\begin{aligned}\partial_{ij}\int m(\xi)u(t,x,\xi)d\xi =& \int m(\xi)\partial_{ij}Y\star(\mathbf{1}_{t\geq 0}f)(t,x,\xi)d\xi\\ =& \int m(\xi)\left((b^0_{ij}Y)\star T^2(\mathbf{1}_{t\geq 0}f)\right)(t,x,\xi)d\xi\\ &+\int m(\xi)\left((b^1_{ij}Y)\star T(\mathbf{1}_{t\geq 0}f)\right)(t,x,\xi)d\xi\\ &+\int m(\xi)\left((b^2_{ij}Y)\star (\mathbf{1}_{t\geq 0}f)\right)(t,x,\xi)d\xi\\ =&S_1+S_2+S_3\end{aligned}$$

for $j = 0,\dots,3$, where $b^k_{ij} \equiv b^k_{ij}(t,x,\xi)$ is given by (3.12). In the second integral appearing in the right-hand side of the relation above, $T(\mathbf{1}_{t\geq 0}f)$ is replaced by $\delta_{t=0}f^{\text{in}} + \mathbf{1}_{t\geq 0}\,\text{div}_\xi(K_u f)$, in view of the first equation in (2.5) in the previous section. Likewise, in the first integral in the right-hand side of the equality above, $T^2(\mathbf{1}_{t\geq 0}f)$ is expressed as

$$\begin{aligned}T^2(\mathbf{1}_{t\geq 0}f) =&T(\delta_{t=0}f^{\text{in}}) + T\left(\mathbf{1}_{t\geq 0}\,\text{div}_\xi(K_u f)\right)\\ =&\delta'_{t=0}f^{\text{in}} + \delta_{t=0}\left(v\cdot\nabla_x f^{\text{in}} + \text{div}_\xi(K^{\text{in}}_u f^{\text{in}})\right)\\ &+\mathbf{1}_{t\geq 0}\,\text{div}_\xi\left(fTK_u + K_u\,\text{div}_\xi(K_u f)\right) + \mathbf{1}_{t\geq 0}[T,\text{div}_\xi](K_u f)\\ =&\delta'_{t=0}f^{\text{in}} + \delta_{t=0}\left(v\cdot\nabla_x f^{\text{in}} + \text{div}_\xi(K^{\text{in}}_u f^{\text{in}})\right)\\ &+\mathbf{1}_{t\geq 0}\nabla^{\otimes 2}_\xi : (fK^{\otimes 2}_u) + \mathbf{1}_{t\geq 0}\,\text{div}_\xi(fTK_u - fK_u\cdot\nabla_\xi K_u)\\ &-\left(\nabla_\xi v\right)^T : \nabla_x\left(\mathbf{1}_{t\geq 0}fK_u\right).\end{aligned}$$

Below, we shall use the notation

$$J_m(t) = \sup_{i,j=0,\dots,3}\left\|\partial_{ij}\int m(\xi)u(t,\cdot,\xi)d\xi\right\|_{L^\infty}.$$

5.1. Estimating S_1

We decompose further:

$$
\begin{aligned}
S_1 = &\int m(\xi)(b^0_{ij}Y) \star \left(\delta'_{t=0} f^{\text{in}} + \delta_{t=0}(v \cdot \nabla_x f^{\text{in}} + \text{div}_\xi (K^{\text{in}}_u f^{\text{in}}))\right) d\xi \\
&+ \int \left(\left(\nabla^{\otimes 2}_\xi (mb^0_{ij})Y\right) \star \left(\mathbf{1}_{t\geq 0} f K^{\otimes 2}_u\right)\right)(t,x,\xi)\, d\xi \\
&+ \int \left(\left(-\nabla_\xi (mb^0_{ij})Y\right) \star \left(\mathbf{1}_{t\geq 0}(f T K_u - f K_u \cdot \nabla_\xi K_u)\right)\right)(t,x,\xi) d\xi \\
&+ \int m(\xi) \left(\left(\nabla_\xi v \cdot \nabla_x (b^0_{ij}Y)\right) \star \left(\mathbf{1}_{t\geq 0} f K_u\right)\right)(t,x,\xi) d\xi \\
=& S_{11} + S_{12} + S_{13} + S_{14}.
\end{aligned}
$$

By using the classical estimates (4.8) for the wave equation, together with the support condition (4.4) and the definition of ϕ in (4.5), we get

$$
\begin{aligned}
|S_{11}| \leq &\|\phi m b^0_{ij}\|_{L^\infty_x (W^{1,\infty}_{t,\xi})} \\
&\times \tfrac{4}{3}\pi r^{*3}(1+\tau)^2 \|f^{\text{in}}\|_{W^{1,\infty}} \left(1 + \|K^{\text{in}}_u\|_{L^\infty([0,\tau)\times \mathbf{R}^3; W^{1,\infty}_\xi)}\right).
\end{aligned} \tag{5.1}
$$

By the same argument as in Section 4

$$
|S_{12}| \leq \|\phi m b^0_{ij}\|_{L^\infty_{t,x}(W^{2,\infty}_\xi)} \tfrac{4}{3}\pi r^{*3} \tfrac{1}{2}\tau^2 \|f^{\text{in}}\|_{L^\infty} \|K_u\|^2_{L^\infty([0,\tau)\times \mathbf{R}^3 \times \mathbf{R}^3)}. \tag{5.2}
$$

Likewise

$$
\begin{aligned}
|S_{13}| \leq &\|\phi m b^0_{ij}\|_{L^\infty_{t,x}(W^{1,\infty}_\xi)} \tfrac{4}{3}\pi r^{*3} \|f^{\text{in}}\|_{L^\infty} \\
&\times \left(\int_0^t (t-s)(J_1(s) + J_v(s)) ds + \|K\|^2_{L^\infty([0,\tau)\times \mathbf{R}^3; W^{1,\infty}_\xi)}\right).
\end{aligned} \tag{5.3}
$$

In S_{14}, we apply once more Lemma 3.1 – or (3.10) with $m = b^0_{ij} \in \mathcal{M}_0$ and $j = 1,2,3$ – so as to write

$$
\partial_k (b^0_{ij}Y) = T(b^0_{ij}a^0_k Y) - \left(\partial_k b^0_{ij} - T(b^0_{ij}a^0_k)\right) Y;
$$

substituting this in the expression giving S_{14} and proceeding as in Section 4 leads to

$$
\begin{aligned}
|S_{14}| \leq \tfrac{4}{3}\pi r^{*3} \|f^{in}\|_{L^\infty} \Bigg[&\sup_{k=0,\dots,3} \|\phi m a^0_k b^0_{ij}\|_{L^\infty_{t,x}(W^{1,\infty}_\xi)} \\
&\times \left(1 + \|K_u\|_{L^\infty([0,\tau)\times \mathbf{R}^3; W^{1,\infty}_\xi)}\right) \int_0^t (t-s)(J_1(s) + J_v(s)) ds \\
&+ \sup_{k=0,\dots,3} \left\|\phi m \left(t\partial_k b^0_{ij} - tT(b^0_{ij}a^0_k)\right)\right\|_{L^\infty} \|K_u\|_{L^\infty([0,\tau)\times \mathbf{R}^3 \times \mathbf{R}^3)} \\
&+ \sup_{k=0,\dots,3} \|\phi m a^0_k b^0_{ij}\|_{L^\infty} \tfrac{1}{2} t^2 \|K^{\text{in}}\|_{L^\infty} \Bigg].
\end{aligned} \tag{5.4}
$$

5.2. *Estimating* S_2

This part of the argument follows Section 4, except $b^1_{ij} \in \mathcal{M}_{-1}$ while $a^0_i \in \mathcal{M}_0$. Thus

$$S_2 = \int \left(\left(-\nabla_\xi (m b^1_{ij}) Y\right) \star (\mathbf{1}_{t\geq 0} K_u f)\right)(t,x,\xi) d\xi$$
$$+ \int m(\xi) \left((b^1_{ij}(t,\cdot,\cdot) Y(t,\cdot)) \star_x f^{\text{in}}\right)(x,\xi) d\xi,$$

so that, by the same estimates as those leading to the last two terms in the right-hand side of (4.7), we arrive at

$$|S_2| \leq \|\phi m t b^1_{ij}\|_{L^\infty_{t,x}(W^{1,\infty}_\xi)} \tfrac{4}{3}\pi r^{*3} \left(\tau \|K_u\|_{L^\infty([0,\tau)\times\mathbf{R}^3\times\mathbf{R}^3)} + \|f^{\text{in}}\|_{L^\infty}\right). \quad (5.5)$$

5.3. *Estimating* S_3

Let $\phi \in C^\infty_c(\mathbf{R}^4 \setminus 0)$; since b^2_{ij} is homogeneous of degree -2 (we recall from Lemma 3.1 that $b^2_{ij}(\cdot,\cdot,\xi) \in \mathcal{M}_{-2}$), we have

$$\langle b^2_{ij} Y, \phi\rangle = \int_0^{+\infty} \int_{\mathbf{S}^2} \frac{1}{4\pi t} b^2_{ij}(t, t\omega, \xi)\phi(t, t\omega) t^2 d\sigma(\omega) dt$$
$$= \int_0^{+\infty} \int_{\mathbf{S}^2} \frac{1}{4\pi t} b^2_{ij}(1, \omega, \xi)\phi(t, t\omega) d\sigma(\omega) dt$$

(where $d\sigma(\omega)$ is the rotation-invariant surface element on $\mathbf{S}^2$). Further, the relation (3.6) shows that, for each $\psi \in C^\infty_c(\mathbf{R}^4)$, the quantity

$$\langle \text{p.v.}(b^2_{ij} Y), \psi\rangle = \int_\theta^{+\infty} \int_{\mathbf{S}^2} b^2_{ij}(1,\omega,\xi) \frac{\psi(t, t\omega)}{4\pi t} d\sigma(\omega)\, dt$$
$$+ \int_0^\theta \int_{\mathbf{S}^2} b^2_{ij}(1,\omega,\xi) \frac{\psi(t,t\omega) - \psi(t,0)}{4\pi t} d\sigma(\omega)\, dt \quad (5.6)$$

is independent of $\theta \in \mathbf{R}_+$. This defines $\text{p.v.}(b^2_{ij} Y)$ as a homogeneous distribution of degree -4 on $\mathbf{R}^4$ that extends $b^2_{ij} Y\big|_{\mathbf{R}^4\setminus 0}$. Hence (see appendix)

$$b^2_{ij}(\cdot,\cdot,\xi) Y - \text{p.v.}\left(b^2_{ij}(\cdot,\cdot,\xi) Y\right) = c_{ij}(\xi)\delta_{(t,x)=(0,0)}, \quad (5.7)$$

where $c_{ij} \in C^\infty(\mathbf{R}^3)$ – we recall that the left-hand side of the equality above is of class C^∞ in ξ – and p.v. stands for "principal value".

Therefore, for $\theta_t \in (0,t)$ to be chosen later,

$$S_3 - \int m(\xi) c_{ij}(\xi) f(t,x,\xi)\, d\xi$$
$$= \int m(\xi) \left(\text{p.v.}(b^2_{ij} Y) \star (\mathbf{1}_{t\geq 0} f)\right)(t,x,\xi)\, d\xi$$
$$= \int m(\xi) \int_{\theta_t}^t \int_{\mathbf{S}^2} b^2_{ij}(1,\omega,\xi) f(t-s, x-s\omega, \xi) \frac{d\sigma(\omega) ds}{4\pi s}\, d\xi$$
$$+ \int m(\xi) \int_0^{\theta_t} \int_{\mathbf{S}^2} b^2_{ij}(1,\omega,\xi) \frac{f(t-s, x-s\omega,\xi) - f(t-s,x,\xi)}{4\pi s}\, d\sigma(\omega)\, ds d\xi.$$

The first integral in the right-hand side of the relation above is estimated by

$$\left|\int_{\theta_t}^{t}\int_{\mathbf{S}^2} b_{ij}^2(1,\omega,\xi) f(t-s,x-s\omega,\xi)\frac{d\sigma(\omega)ds}{4\pi s}\right| \le \ln(t/\theta_t)\|b_{ij}^2(1,\cdot,\xi)\|_{L^\infty(\mathbf{S}^2)}\|f\|_{L^\infty},$$

while the second is estimated by

$$\left|\int_0^{\theta_t}\int_{\mathbf{S}^2} b_{ij}^2(1,\omega,\xi)\frac{f(t-s,x-s\omega,\xi)-f(t-s,x,\xi)}{4\pi s}d\sigma(\omega)ds\right| \le \theta_t\|b_{ij}^2(1,\cdot,\xi)\|_{L^\infty(\mathbf{S}^2)}\|\nabla_x f\|_{L^\infty([0,t]\times\mathbf{R}^3\times\mathbf{R}^3)}.$$

Choosing

$$\theta_t = \inf\left(\frac{1}{\|\nabla_x f\|_{L^\infty([0,t]\times\mathbf{R}^3\times\mathbf{R}^3)}}, t\right)$$

we find that

$$|S_3| \le Cr^{*3}\|m\|_{L^\infty}\Big[\|c_{ij}\|_{L^\infty(B(0,r^*))}\|f\|_{L^\infty} + \|b_{ij}^2(1,\cdot,\cdot)\|_{L^\infty(\mathbf{S}^2\times\mathbf{R}^3)} \times\left(1+\|f\|_{L^\infty}\ln_+\left(t\|\nabla_x f\|_{L^\infty([0,t]\times\mathbf{R}^3\times\mathbf{R}^3)}\right)\right)\Big], \tag{5.8}$$

where $\ln_+ z = \sup(\ln z, 0)$.

Remark. In the case of space dimension 2, a similar argument, based on (3.7) instead of (3.6), leads to an estimate that involves a logarithmic term just as in (5.8). The condition (3.7) is not apparent in [3, 4], which uses instead the fact that $b_{ij}^2 Y$ is a linear combination of derivatives of distributions that are homogeneous of degree ≥ -2 in $\mathbf{R}^3$ – see p. 344 of [3, 4]. As explained in the appendix, this is equivalent to (3.7).

5.4. *Proof of Theorem 1.1*

The estimates (5.1)–(5.5) and (5.8) show that, for each $m \in W^{2,\infty}(\mathbf{R}^3)$, there exists a positive constant $C_2 \equiv C_2(\tau, r^*, \|m\|_{W^{2,\infty}}, \|f^{\text{in}}\|_{W^{1,\infty}})$ such that

$$J_m(t') \le C_2(\tau, r^*, \|m\|_{W^{2,\infty}}, \|f^{\text{in}}\|_{W^{1,\infty}}) \times\left(1+\int_0^{t'}(J_1(s)+J_m(s))ds + \ln_+\left(\|\nabla_x f\|_{L^\infty([0,t]\times\mathbf{R}^3\times\mathbf{R}^3)}\right)\right) \tag{5.9}$$

for each t and t' such that $0 < t' < t < \tau$. Using (5.9) with $m \equiv 1$ and $m = v$ and applying Gronwall's inequality shows that, for each $t \in [0, \tau)$,

$$J_1(t) + J_v(t) \le 2C_2 e^{2C_2\tau}\left(1+\ln_+\left(\|\nabla_x f\|_{L^\infty([0,t]\times\mathbf{R}^3\times\mathbf{R}^3)}\right)\right). \tag{5.10}$$

In particular, this implies the existence of yet another positive constant $C_3 \equiv C_3(\tau, r^*, \|m\|_{W^{2,\infty}}, \|f^{\text{in}}\|_{W^{1,\infty}})$ such that

$$\|K_u(t)\|_{W^{1,\infty}_{x,\xi}} \le C_3 e^{2C_2\tau}\left(1+\ln_+\left(\|\nabla_x f\|_{L^\infty([0,t]\times\mathbf{R}^3\times\mathbf{R}^3)}\right)\right). \tag{5.11}$$

Finally, differentiating in (x, v) the transport equation (2.5) and integrating in t

shows that

$$\|\nabla_{x,\xi} f(t)\|_{L^\infty_{x,\xi}} \leq \|\nabla_{x,\xi} f^{\text{in}}\|_{L^\infty_{x,\xi}} + \int_0^t \left(\|\nabla_\xi v\|_{L^\infty} + \|\nabla_{x,\xi} K_u(s)\|_{L^\infty_{x,\xi}}\right) \|\nabla_{x,\xi} f(s)\|_{L^\infty_{x,\xi}} ds. \tag{5.12}$$

The estimates (5.11) and (5.12) show that the Lipschitz semi-norm of f, i.e., $N(t) = \sup_{s\in[0,t]} \|\nabla_{x,\xi} f(s)\|_{L^\infty_{x,\xi}}$ satisfies a logarithmic Gronwall inequality of the form

$$N(t) \leq N(0) + C \int_0^t (1 + \ln_+ N(s))N(s)ds, \quad t \in [0, \tau].$$

This implies that $N \in L^\infty([0, \tau])$. Inserting this in (5.10) and using (2.4) shows that $(E, B) \in L^\infty([0, \tau], W^{1,\infty}(\mathbf{R}^3))$, which in turn implies Theorem 1.1.

6. Appendix: Homogeneous distributions

This section recalls some classical material from [2] (Chapter III, Section 3.3) and [6] (pp. 75–79 and Theorem 3.2.3).

A distribution f on $\mathbf{R}^N$ (or $\mathbf{R}^N \setminus 0$) is homogeneous of degree α if $\langle f, M_\lambda \phi\rangle = \lambda^{\alpha+N}\langle f, \phi\rangle$ (where $M_\lambda\phi(x) = \phi(x/\lambda)$) for each $\lambda > 0$ and each $\phi \in C_c^\infty(\mathbf{R}^N)$ (resp. $\phi \in C_c^\infty(\mathbf{R}^N \setminus 0)$). Equivalently, $f \in \mathcal{D}'(\mathbf{R}^N)$ (or in $\mathcal{D}'(\mathbf{R}^N \setminus 0)$) is homogeneous of degree α if and only if

$$\operatorname{div}_x(xf) = (\alpha + N)f \text{ on } \mathbf{R}^N \text{ (resp. on } \mathbf{R}^N \setminus 0) \tag{6.1}$$

in the sense of distributions (Euler's relation in conservation form).

For $\alpha > -N$, each homogeneous distribution f of degree α on $\mathbf{R}^N \setminus 0$ has a unique extension $\dot{f}$ that is a homogeneous distribution on $\mathbf{R}^N$.

If $f \in \mathcal{D}'(\mathbf{R}^N \setminus 0)$ is homogeneous of degree $-N$, $\operatorname{div}_x((xf)^\cdot)$ is a homogeneous distribution of degree $-N$ on $\mathbf{R}^N$ supported in $\{0\}$, hence there exists $c \in \mathbf{R}$

$$\operatorname{div}_x((xf)^\cdot) = c\delta_{x=0} \text{ in the sense of distributions on } \mathbf{R}^N. \tag{6.2}$$

The constant c in the right-hand side of (6.2) is called *the residue of f at* 0 and denoted $\operatorname{Res}_0 f$. Equivalently, the residue of f at 0 can be defined by the relation

$$\langle f, \Phi\rangle = \operatorname{Res}_0 f \, \frac{1}{|S^{N-1}|} \int_{\mathbf{R}^N} \Phi(x) \frac{dx}{|x|^N}, \tag{6.3}$$

whenever $\Phi(x) = \phi(|x|)$ with $\phi \in C_c^\infty((0, +\infty))$.

Any $f \in \mathcal{D}'(\mathbf{R}^N \setminus 0)$ which is homogeneous of degree $-N$ can be extended as a homogeneous distribution $\dot{f}$ of degree $-N$ on $\mathbf{R}^N$ if and only if $\mathrm{Res}_0 f = 0$. For each $\chi \in C_c^\infty(\mathbf{R}^N)$, set $X(x) = \int_0^1 \nabla_x \chi(tx)dt$; we have $\chi(x) = \chi(0) + x \cdot X(x)$ and $X \in C^\infty(\mathbf{R}^N)$. Given $f \in \mathcal{D}'(\mathbf{R}^N \setminus 0)$ that is homogeneous of degree $-N$ with $\mathrm{Res}_0 f = 0$ and $\phi \equiv \phi(|x|)$ in $C_c^\infty(\mathbf{R}^N)$ such that $\phi \equiv 1$ near 0, the linear functional

$$\chi \mapsto \langle f, (1-\phi)\chi \rangle + \langle (xf)^{\cdot}, \phi X \rangle$$

is a homogeneous extension of f to $\mathbf{R}^N$. Two homogeneous extensions of f may differ by a multiple of $\delta_{x=0}$.

References

1. Bouchut, F., Golse, F., Pallard, C.: Nonresonant smoothing for coupled wave + transport equations and the Vlasov-Maxwell system. To appear in *Revista Mat. Iberoamericana*.
2. Gelfand, I., Shilov, G.: *Generalized functions*, Vol. I. Academic Press, New York-London, 1964
3. Glassey, R., Schaeffer, J.: The relativistic Vlasov-Maxwell system in two space dimensions, I. *Arch. Rational Mech. Anal.* **141**, 331–354 (1998)
4. Glassey, R., Schaeffer, J.: The relativistic Vlasov-Maxwell system in two space dimensions, II. *Arch. Rational Mech. Anal.* **141**, 355–374 (1998)
5. Glassey, R., Strauss, W.: Singularity formation in a collisionless plasma could occur only at high velocities. *Arch. Rational Mech. Anal.* **92**, 59–90 (1986)
6. Hörmander, L.: *The analysis of linear partial differential operators. I. Distribution theory and Fourier analysis.* Springer-Verlag, Berlin, 1983
7. Klainerman, S., Staffilani, G.: A new approach to study the Vlasov-Maxwell system. *Commun. Pure Appl. Anal.* **1**, 103–125 (2002)

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